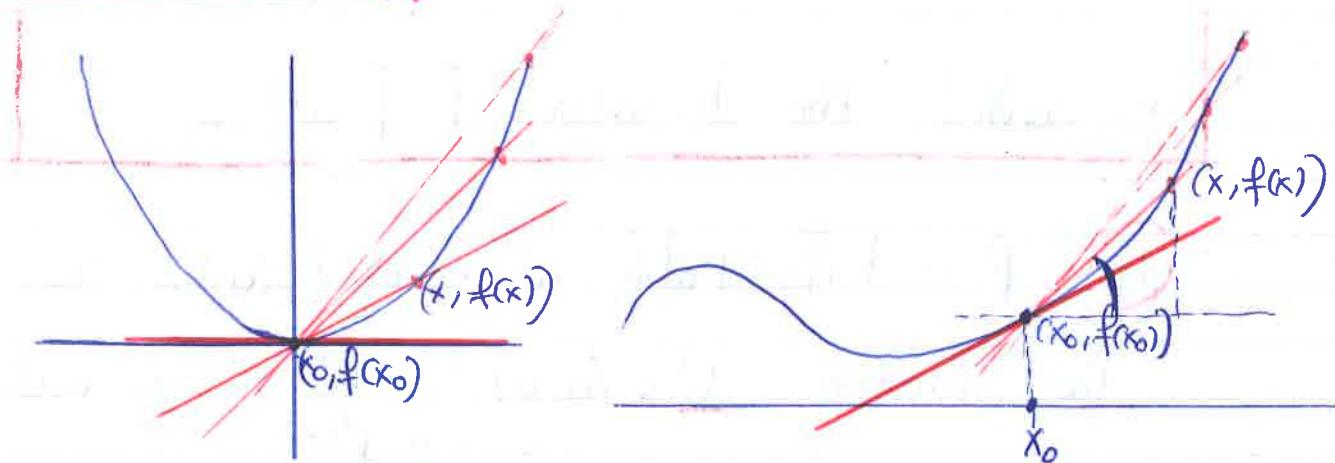


## Differentiability



→ Def: Let  $f: (a, b) \rightarrow \mathbb{R}$ , and  $x_0 \in (a, b)$

(⚠ To talk about differentiability of  $f$  at  $x_0$ ,  $f$  needs to be defined in some neighbourhood of  $x_0$ . That is what  $(a, b)$  is about.)

for each  $x \in (a, b)$ ,  $x \neq x_0$ , we consider the line  $l_{x, x_0}$

passing through  $(x_0, f(x_0))$  and  $(x, f(x))$ . This line has slope  $\frac{f(x)-f(x_0)}{x-x_0}$  (i.e., this is the tangent of the angle this line forms with the x-axis).

We say that  $f$  is differentiable at  $x_0$  if

$$\lim_{x \rightarrow x_0} \frac{f(x)-f(x_0)}{x-x_0} \text{ exists in } \mathbb{R}.$$

(2)

In that case,  $f'(x_0) := \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$

is called the derivative of  $f$  at  $x_0$ .

- If  $f$  is differentiable at every  $x \in (a, b)$ , then

the function  $\begin{cases} f' : (a, b) \rightarrow \mathbb{R} \\ x \mapsto f'(x) \end{cases}$  is well-defined,

and is called the derivative of  $f$ .

Note that, if  $f'(x_0)$  exists, then

$$f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$



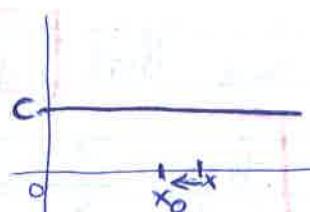
As  $x \rightarrow x_0$ , the lines  $l_{x, x_0}$  tend to the tangent line of the graph of  $f$  at  $x_0$ , and  $f'(x_0)$  is the slope of that tangent line.

→ Examples:

①  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = c \forall x \in \mathbb{R}$ .

Let  $x_0 \in \mathbb{R}$ :

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{c - c}{x - x_0} = \frac{0}{x - x_0} = 0,$$



(3)

$$\text{so } \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} 0 = 0 \in \mathbb{R}.$$

Thus,  $f'(x_0)$  exists, and  $f'(x_0) = 0$ .

Since  $x_0 \in \mathbb{R}$  was arbitrary,  $f$  is differentiable everywhere, and  $f' \equiv 0$ .

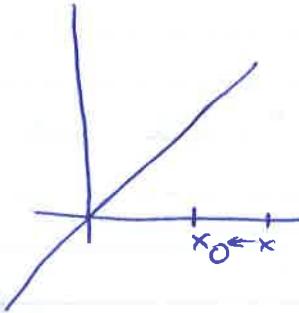
(2)  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x \quad \forall x \in \mathbb{R}$ .

Let  $x_0 \in \mathbb{R}$ .

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} =$$

$$= \lim_{h \rightarrow 0} \frac{x_0 + h - x_0}{h} = \lim_{h \rightarrow 0} 1 = 1 \in \mathbb{R}. \quad \text{Since } x_0 \text{ was arbitrary,}$$

$f'$  exists, and  $f' \equiv 1$ .



(3)  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x^2, \quad \forall x \in \mathbb{R}$ .

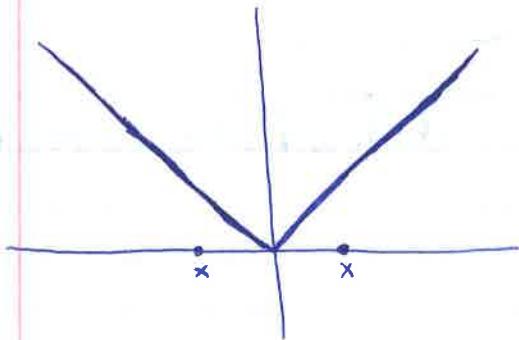
$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{(x_0 + h)^2 - x_0^2}{h} = \lim_{h \rightarrow 0} \frac{x_0^2 + 2x_0 h + h^2 - x_0^2}{h} =$$

$$= \lim_{h \rightarrow 0} \frac{2x_0 h + h^2}{h} = \lim_{h \rightarrow 0} (2x_0 + h) = 2x_0 \in \mathbb{R}.$$

So,  $f'(x) = 2x, \quad \forall x \in \mathbb{R}$ .

(4)

$$④ \quad f: \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = |x| \quad \forall x \in \mathbb{R}.$$



That is,  $f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$ .

- Let  $x > 0$ .  $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$   $x+h > 0$   
when  $h$  small  
enough

$$= \lim_{h \rightarrow 0} \frac{x+h-x}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = \lim_{h \rightarrow 0} 1 = 1,$$

so  $f'(x) = 1$ .

- Let  $x < 0$ .  $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$   $x+h < 0$  when  
 $h$  is small enough

$$= \lim_{h \rightarrow 0} \frac{-(x+h) - (-x)}{h} = \lim_{h \rightarrow 0} \frac{-x-h+x}{h} = \lim_{h \rightarrow 0} \frac{-h}{h} =$$

$$= \lim_{h \rightarrow 0} (-1) = -1, \quad \text{so } f'(x) = -1.$$

- $f'(0)$  doesn't exist:

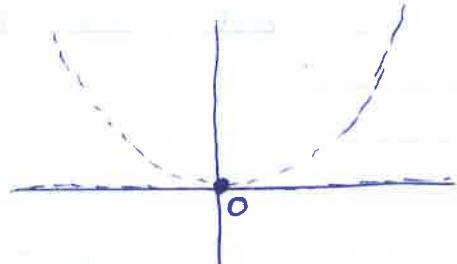
(5)

We have that  $\frac{f(0+h) - f(0)}{h} = \frac{|h|}{h} = \begin{cases} 1, & \text{if } h > 0 \\ -1, & \text{if } h < 0. \end{cases}$

So,  $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$  doesn't exist.

$\left( \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = 1, \text{ and } \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = -1 \right)$

$$\textcircled{3} \quad f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$



Then,  $f'(0)$  exists, and  $f'(0) = 0$ :

$$\frac{f(0+h) - f(0)}{h} = \begin{cases} \frac{h^2}{h} = h, & \text{if } h \in \mathbb{Q} \setminus \{0\} \\ 0, & \text{if } h \notin \mathbb{Q} \end{cases}$$

We have:  $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = 0$ :

1st proof: Let  $\varepsilon > 0$ . We want to show that  $\exists \delta > 0$  s.t. if  $|h| < \delta$ , then  $\left| \frac{f(0+h) - f(0)}{h} - 0 \right| < \varepsilon$ .

Indeed,  $\delta = \varepsilon$  will do.

2nd proof:  $0 \leq \left| \frac{f(0+h) - f(0)}{h} \right| \leq |h|$ ,  $\forall h \neq 0$ .

By the sandwich lemma,  $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = 0$ .

(6)

→ Thm: Let  $f: (a,b) \rightarrow \mathbb{R}$ , and  $x_0 \in (a,b)$ .

If  $f$  is differentiable at  $x_0$ , then  $f$  is continuous at  $x_0$ .

Proof: Since  $x_0$  is an accumulation point of  $(a,b)$ ,

all we need to show is that  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ .

$$\Leftrightarrow \lim_{x \rightarrow x_0} (f(x) - f(x_0)) = 0.$$

Indeed:  $f(x) - f(x_0) = \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0)$

$\uparrow$   
 $x \neq x_0$

$$\xrightarrow{x \rightarrow x_0} f'(x_0) \cdot (x_0 - x_0) = 0.$$

## Rules of differentiation

→ Thm: Let  $f, g : (a,b) \rightarrow \mathbb{R}$ ,  $x_0 \in (a,b)$ . If  $f'(x_0), g'(x_0)$  both exist, then:

(a)  $(f+g)'(x_0) = f'(x_0) + g'(x_0)$ .

(b)  $(\lambda \cdot f)'(x_0) = \lambda \cdot f'(x_0)$ ,  $\forall \lambda \in \mathbb{R}$ .

(c)  $(f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0) \cdot g'(x_0)$ .

(d) If  $g(x_0) \neq 0$   $\forall x \in (a,b)$ , then  $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0) \cdot g'(x_0)}{(g(x_0))^2}$ .

(7)

Proof:

$$(a) \lim_{h \rightarrow 0} \frac{(f+g)(x_0+h) - (f+g)(x_0)}{h} =$$

$$= \lim_{h \rightarrow 0} \frac{f(x_0+h) + g(x_0+h) - f(x_0) - g(x_0)}{h} =$$

$$= \lim_{h \rightarrow 0} \left( \frac{f(x_0+h) - f(x_0)}{h} + \frac{g(x_0+h) - g(x_0)}{h} \right) = \hookrightarrow \text{since both limits exist}$$

$$= \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} + \lim_{h \rightarrow 0} \frac{g(x_0+h) - g(x_0)}{h} =$$

$= f'(x_0) + g'(x_0)$ . So,  $(f+g)'(x_0)$  exists, and

$$(f+g)'(x_0) = f'(x_0) + g'(x_0).$$

$$(b) \lim_{h \rightarrow 0} \frac{(\lambda f)(x_0+h) - (\lambda f)(x_0)}{h} = \lim_{h \rightarrow 0} \frac{\lambda \cdot f(x_0+h) - \lambda \cdot f(x_0)}{h} =$$

$$= \lambda \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} = \lambda \cdot f'(x_0).$$

So,  $(\lambda f)'(x_0)$  exists, and  $(\lambda f)'(x_0) = \lambda \cdot f'(x_0)$ .

$$(c) \frac{f(x_0+h)g(x_0+h) - f(x_0)g(x_0)}{h} =$$

$$= \frac{f(x_0+h)g(x_0+h) - f(x_0+h)g(x_0) + f(x_0+h)g(x_0) - f(x_0)g(x_0)}{h} =$$

$$= \underbrace{f(x_0+h)}_{\substack{\downarrow h \rightarrow 0 \\ f(x_0)}} \cdot \underbrace{\frac{g(x_0+h) - g(x_0)}{h}}_{\substack{\downarrow h \rightarrow 0 \\ g'(x_0)}} + g(x_0) \cdot \underbrace{\frac{f(x_0+h) - f(x_0)}{h}}_{\substack{\downarrow h \rightarrow 0 \\ f'(x_0)}}$$

as, since  $f$   
is differentiable  
at  $x_0$ ,  $f$  is continuous  
at  $x_0$

$$\xrightarrow[h \rightarrow 0]{} f(x_0) \cdot g'(x_0) + g(x_0) \cdot f'(x_0).$$

So,  $(f \cdot g)'(x_0)$  exists, and  $(f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$ .

$$(d) \frac{\frac{1}{g(x_0+h)} - \frac{1}{g(x_0)}}{h} = \frac{g(x_0) - g(x_0+h)}{g(x_0+h)g(x_0) \cdot h} =$$

$$= \underbrace{\frac{1}{g(x_0+h)}}_{\substack{\downarrow h \rightarrow 0 \\ \frac{1}{g(x_0)}, \text{ as } g \text{ cont.}}} \cdot \frac{1}{g(x_0)} \cdot \frac{g(x_0) - g(x_0+h)}{h} \xrightarrow[h \rightarrow 0]{} \frac{1}{g(x_0)} \cdot \frac{1}{g(x_0)} (g'(x_0)) = - \frac{g'(x_0)}{g^2(x_0)}.$$

at  $x_0$ , thus  $\frac{1}{g}$  cont. at  $x_0$

(9)

i.e.,  $\left(\frac{1}{g}\right)'(x_0)$  exists, and  $\left(\frac{1}{g}\right)'(x_0) = -\frac{g'(x_0)}{g^2(x_0)}$

So, by (i),  $\frac{f}{g} = f \cdot \left(\frac{1}{g}\right)$  is differentiable at  $x_0$ , and,

$$\begin{aligned} \left(\frac{f}{g}\right)'(x_0) &= \left(f \cdot \frac{1}{g}\right)'(x_0) = f'(x_0) \cdot \frac{1}{g(x_0)} + f(x_0) \cdot \left(\frac{1}{g}\right)'(x_0) = \\ &= \frac{f'(x_0)}{g(x_0)} - f(x_0) \cdot \frac{g'(x_0)}{g^2(x_0)} = \frac{f'(x_0) \cdot g(x_0) - f(x_0) \cdot g'(x_0)}{g^2(x_0)} \end{aligned}$$

(1)

i.e.,  $\left(\frac{1}{g}\right)'(x_0)$  exists, and  $\left(\frac{1}{g}\right)'(x_0) = -\frac{g'(x_0)}{g^2(x_0)}$ .

So, by(c),  $\frac{f}{g} = f \cdot \left(\frac{1}{g}\right)$  is differentiable at  $x_0$ , and,

$$\begin{aligned} \left(\frac{f}{g}\right)'(x_0) &= \left(f \cdot \frac{1}{g}\right)'(x_0) = f'(x_0) \cdot \frac{1}{g(x_0)} + f(x_0) \cdot \left(\frac{1}{g}\right)'(x_0) = \\ &= \frac{f'(x_0)}{g(x_0)} - f(x_0) \cdot \frac{g'(x_0)}{g^2(x_0)} = \frac{f'(x_0) \cdot g(x_0) - f(x_0) \cdot g'(x_0)}{g^2(x_0)}. \end{aligned}$$

## Lecture 23:

→ Thm: (Chain rule):

Let  $f: (a, b) \rightarrow (c, d)$ ,  $g: (c, d) \rightarrow \mathbb{R}$ . Let  $x_0 \in (a, b)$ .

Suppose that  $f'(x_0)$  exists, and  $g'(f(x_0))$  exists.

Then,  $gof: (a, b) \rightarrow \mathbb{R}$  is differentiable at  $x_0$ ,

and  $(gof)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$ .

Wrong "proof": for  $x \neq x_0$ ,

$$\frac{g(f(x)) - g(f(x_0))}{x - x_0} = \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0}$$

(2)

When  $x \rightarrow x_0$ ,  $\frac{f(x) - f(x_0)}{x - x_0} \rightarrow f'(x_0)$   
 (as  $f'(x_0)$  exists).

Also, when  $x \rightarrow x_0$ ,  $f(x) \rightarrow f(x_0)$  ( $f$  differentiable at  $x_0$   
 $\rightarrow f$  continuous at  $x_0$ )

So, if  $f(x) \neq f(x_0)$  as  $x \rightarrow x_0$ , then

$$\frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \xrightarrow{x \rightarrow x_0} g'(f(x_0)).$$

However, this is a big if; it may well be that,

arbitrarily close to  $x_0$ , there are  $x$  with  $f(x) = f(x_0)$   
 (imagine, for instance,  $f$  constant around  $x_0$ ).

So,  $\frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)}$  may not even be defined for  
 $x$  close to  $x_0$ .

Ideal: The above is really an artificial problem. As our intuition tells us, we should consider the fact

that  $\frac{g(f(x)) - g(f(x_0))}{x - x_0} = \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0}$  \*

when  $f(x) \neq f(x_0)$ . Not only do these quantities approach what we want when  $f(x)$  close to  $f(x_0)$  (but  $\neq f(x_0)$ ), but also, when  $f(x) = f(x_0)$ , both  $\frac{g(f(x)) - g(f(x_0))}{x - x_0}$  and  $\frac{f(x) - f(x_0)}{x - x_0}$  equal 0, while the

slopes  $\frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)}$   
 as  $f(x)$  "hits"  $f(x_0)$ .

actually "hit" the desired  $g'(f(x_0))$

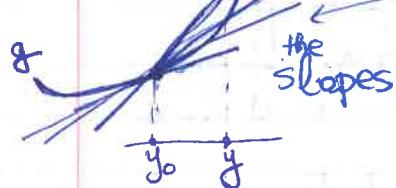
(3)

In other words, (3) really holds even when  $f(x)=f(x_0)$ , just with the (even better!)  $g'(f(x_0))$  in the place of the problematic  $\frac{g(f(x))-g(f(x_0))}{x-x_0}$ .

Let's make the above rigorous:

→ Observation: Let  $g: I \rightarrow \mathbb{R}$ ,  $y_0 \in I$ ,  $g$  differentiable at  $y_0$ .

Then, the function  $\tilde{g}: I \rightarrow \mathbb{R}$ , with



$$\tilde{g}(y) = \begin{cases} \frac{g(y)-g(y_0)}{y-y_0}, & y \neq y_0 \\ g'(y_0), & y = y_0 \end{cases}$$

is continuous at  $y_0$ .

Proof:  $x_0$  is an accumulation point of  $I$ .

So,  $\tilde{g}$  continuous at  $y_0$

$$\Rightarrow \lim_{y \rightarrow y_0} \tilde{g}(y) = \tilde{g}(y_0) \iff \lim_{\substack{y \rightarrow y_0 \\ y \neq y_0 \text{ when } y \rightarrow y_0}} \frac{g(y)-g(y_0)}{y-y_0} = g'(y_0),$$

which is true.

[~~The above didn't really tell us something new. But it makes it obvious that we should be using  $\tilde{g}(f(x_0))$  instead of  $\frac{g(f(x))-g(f(x_0))}{f(x)-f(x_0)}$ !~~]

(4)

→ Proper proof of Chain Rule:

Since  $g$  is differentiable at  $f(x_0)$ ,

the function  $\tilde{g}(y) = \begin{cases} \frac{g(y) - g(f(x_0))}{y - f(x_0)}, & y \in (c, d) \setminus \{f(x_0)\} \\ g'(f(x_0)), & y = f(x_0) \end{cases}$

is continuous at  $f(x_0)$ . And :

$$\text{for all } x \neq x_0, \quad \frac{g(f(x)) - g(f(x_0))}{x - x_0} = \tilde{g}(f(x)) \cdot \frac{f(x) - f(x_0)}{x - x_0}$$

Indeed, let  $x \neq x_0$ . If  $f(x) \neq f(x_0)$ , then

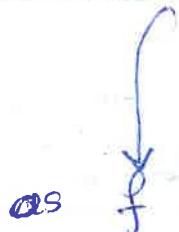
$$\begin{aligned} \frac{g(f(x)) - g(f(x_0))}{x - x_0} &= \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0} = \\ &= \tilde{g}(f(x)) \cdot \frac{f(x) - f(x_0)}{x - x_0} \end{aligned}$$

If  $f(x) = f(x_0)$ , then

$$\frac{g(f(x)) - g(f(x_0))}{x - x_0} = 0 = \tilde{g}(f(x_0)) \cdot \frac{f(x) - f(x_0)}{x - x_0}$$

Thus:  $\lim_{x \rightarrow x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} = \lim_{x \rightarrow x_0} \left( \tilde{g}(f(x)) \cdot \frac{f(x) - f(x_0)}{x - x_0} \right) =$

(5)



as  $f$  cont. at  $x_0$  and  $f'(x_0)$  exists

and  $\tilde{g}$  cont. at  $f(x_0)$ ,

we have:  $(\tilde{g} \circ f)$

cont. at  $x_0$ , thus

$$x \rightarrow x_0 \Rightarrow (\tilde{g} \circ f)(x) \rightarrow (\tilde{g} \circ f)(x_0),$$

$$\text{i.e. } \lim_{x \rightarrow x_0} \tilde{g}(f(x)) = \tilde{g}(f(x_0)).$$

$$= \lim_{x \rightarrow x_0} \tilde{g}(f(x)) \cdot \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \tilde{g}(f(x_0)) \cdot f'(x_0) =$$

$$= g'(f(x_0)) \cdot f'(x_0).$$

→ Thm: (Derivative of inverse function):

Let  $f: (a, b) \rightarrow \mathbb{R}$  1-1 and continuous.

Let  $x_0 \in (a, b)$ . If  $f'(x_0)$  exists and is  $\neq 0$ ,

then  $f^{-1}$  is differentiable at  $f(x_0)$ , and

$$(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}.$$

(6)

"Wrong proof":  $f^{-1} \circ f: (a, b) \rightarrow (a, b)$  is the

identity map: it sends  $x$  to  $x$ ,  $\forall x \in (a, b)$ .

Thus,  $(f^{-1} \circ f)'(x) = 1$ ,  $\forall x \in (a, b)$ .

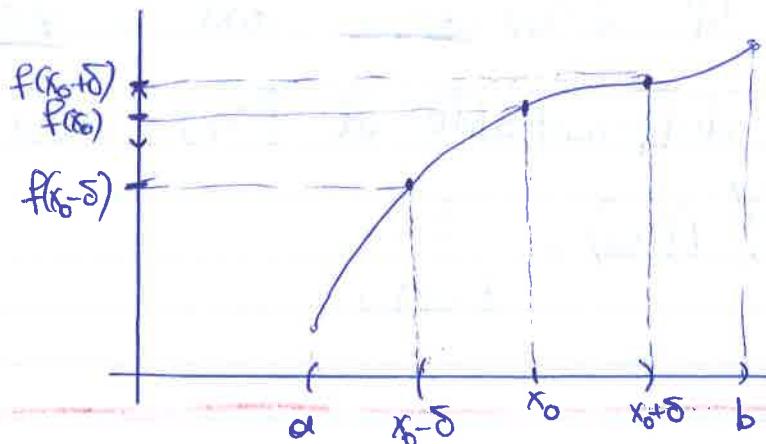
Thus, by the chain rule,

$$\Rightarrow (f^{-1})'(f(x)) \cdot f'(x) = 1,$$

i.e.  $(f^{-1})'(f(x)) = \frac{1}{f'(x)}$

This step is wrong: we can't apply the chain rule, unless we know that  $(f^{-1})'(f(x))$  exists.

Proper proof: Since  $f: (a, b) \rightarrow \mathbb{R}$  is 1-1 and continuous, it is strictly monotonic. Let us assume it is strictly increasing.



Idea:  
We somehow want to make  $\frac{1}{f'(x_0)}$  appear  
Here's how:

(7)

We know that  $f'(x_0) \neq 0$ .

Thus:

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \neq 0, \text{ in } \mathbb{R}$$

$\Rightarrow$

$$\boxed{\lim_{x \rightarrow x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)}} \quad (*)$$

Note that we are indeed allowed to put  $f(x) - f(x_0)$  in the denominator:  $f$  is 1-1, so, since  $x \neq x_0$  when we consider a limit as  $x \rightarrow x_0$ , we have  $f(x) \neq f(x_0)$

Our goal is to show that

$$\boxed{\lim_{y \rightarrow f(x_0)} \frac{f^{-1}(y) - f^{-1}(f(x_0))}{y - f(x_0)} = \frac{1}{f'(x_0)}} \quad (**)$$

(Note that  $*$  is essentially  $**$ ; at least the quantity whose limit we are considering!).

Let  $\epsilon > 0$ . We want to show that there exists $\delta > 0$  s.t. if  $y \in (f(x_0) - \delta, f(x_0) + \delta) \setminus \{f(x_0)\}$ ,

then  $\left| \frac{f^{-1}(y) - f^{-1}(f(x_0))}{y - f(x_0)} - \frac{1}{f'(x_0)} \right| < \epsilon$ , i.e.

(8)

$$\left| \frac{\frac{f^{-1}(y) - x_0}{y - f(x_0)} - \frac{1}{f'(x_0)}}{\underbrace{f(f^{-1}(y)) - f(x_0)}_{=y}} \right| < \varepsilon$$

$\rightarrow$  small enough so that  $(x_0 - \delta, x_0 + \delta) \subseteq (a, b)$

By (\*), there exists  $\delta > 0$  s.t. :

if  $x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$ ,

then  $\left| \frac{x - x_0}{f(x) - f(x_0)} - \frac{1}{f'(x_0)} \right| < \varepsilon$ .

\*''

Now,  $f((x_0 - \delta, x_0 + \delta)) = (f(x_0 - \delta), f(x_0 + \delta))$

$f([x_0 - \delta, x_0 + \delta])$  is a closed interval, as  $f$  continuous.

So,  $f([x_0 - \delta, x_0 + \delta]) = [f(x_0 - \delta), f(x_0 + \delta)]$ , as  $f$  is increasing.

So,  $f((x_0 - \delta, x_0 + \delta)) = (f(x_0 - \delta), f(x_0 + \delta))$ , as  $f$  is 1-1.

Thus, if  $y \in (f(x_0 - \delta), f(x_0 + \delta)) \setminus \{f(x_0)\}$

then  $f^{-1}(y) \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$  (as  $f$  is 1-1),

and therefore, by (\*''),  $\left| \frac{\frac{f^{-1}(y) - x_0}{y - f(f^{-1}(y))} - \frac{1}{f'(x_0)}}{\underbrace{f(f^{-1}(y)) - f(x_0)}_{=y}} \right| < \varepsilon$

(9)

In particular, let  $\delta_1 = \min \{ f(x_0 + \delta) - f(x_0), f(x_0) - f(x_0 - \delta) \} > 0$

Then,  $(f(x_0) - \delta_1, f(x_0) + \delta_1) \subseteq (f(x_0 - \delta), f(x_0 + \delta))$ .

Therefore, if  $y \in (f(x_0) - \delta_1, f(x_0) + \delta_1) \setminus \{f(x_0)\}$ ,

then  $\left| \frac{f^{-1}(y) - x_0}{y - f(x_0)} - \frac{1}{f'(x_0)} \right| < \varepsilon$ .

Since  $\varepsilon > 0$  was arbitrary,

$$\lim_{y \rightarrow f(x_0)} \frac{f^{-1}(y) - f^{-1}(f(x_0))}{y - f(x_0)} = \frac{1}{f'(x_0)},$$

thus  $(f^{-1})'(f(x_0))$  exists, and

$$(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}.$$

■

What follows aims to help us understand a function via properties of its derivative.

## Local extrema

→ **Lemma:** Let  $f: (a, b) \rightarrow \mathbb{R}$  differentiable.

(i) If  $f \uparrow$ , then  $f'(x) \geq 0$ ,  $\forall x \in (a, b)$ .

(ii) If  $f \downarrow$ , then  $f'(x) \leq 0$ ,  $\forall x \in (a, b)$ .

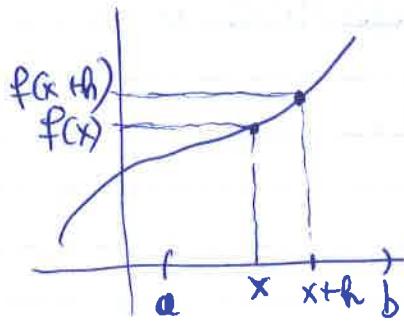
**⚠** This will be trivial. The converse holds,

and the proof of that is not trivial;  
it involves the mean value theorem,  
and we will see it soon.

Proof: We assume that  $f \uparrow$ .  
Let  $x \in (a, b)$ . We will show that  $f'(x) \geq 0$ .

Indeed,  $f'(x)$  exists, thus there exists the limit

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h} \quad \left( = \lim_{h \rightarrow 0^-} \frac{f(x+h) - f(x)}{h} \right)$$



for all  $h > 0$ , with  $x+h \in (a, b)$ , we have:

$$x < x+h$$

$$\stackrel{f \uparrow}{\Rightarrow} f(x) \leq f(x+h)$$

$$\stackrel{h > 0}{\Rightarrow} \frac{f(x+h) - f(x)}{h} \geq 0 \Rightarrow \lim_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h} \geq 0,$$

(11)

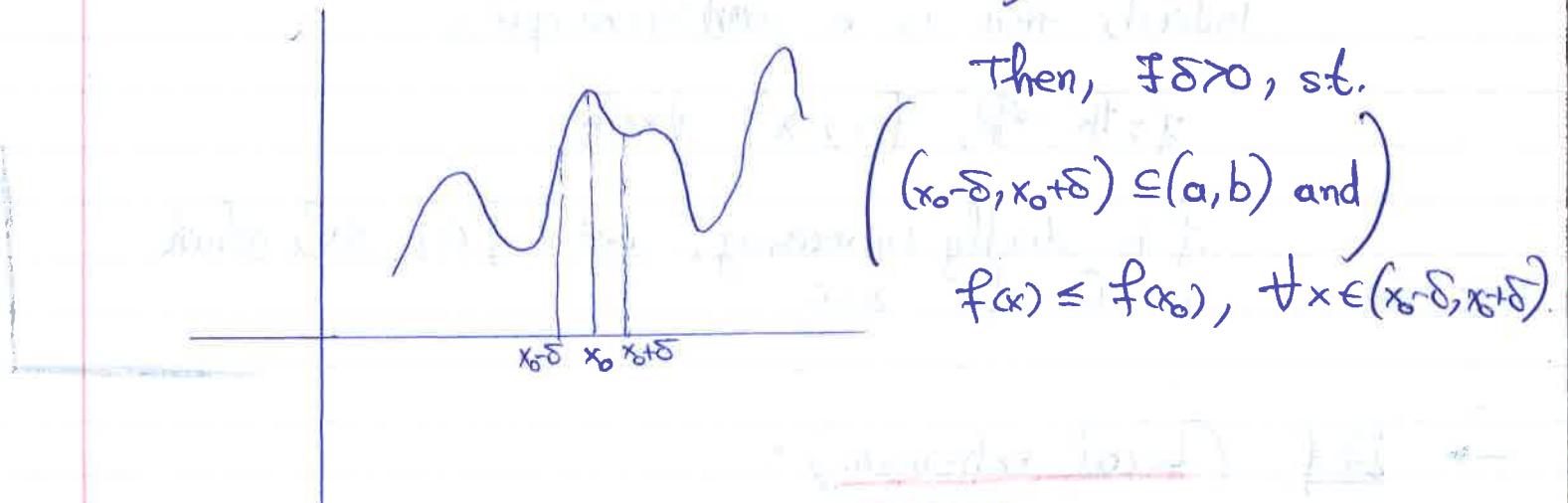
i.e.  $f'(x) \geq 0$ .

(1)

Lecture 24:

→ Prop. (Fermat). Let  $f: (a, b) \rightarrow \mathbb{R}$ . We assume that  
 $f$  has a local extremum at  $x_0 \in (a, b)$ , and  
 $f'(x_0)$  exists. Then:  $f'(x_0) = 0$ .

Proof: Let us work in the case where  $f$  has  
a local maximum at  $x_0$  (we work similarly  
if it has a local minimum  
at  $x_0$ ).



$$\text{We know that } f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} =$$

$$= \lim_{h \rightarrow 0^+} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \rightarrow 0^-} \frac{f(x_0 + h) - f(x_0)}{h}$$

- If  $0 < h < \delta$ , then  $x_0 < x_0 + h < x_0 + \delta$ , thus  $f(x_0 + h) \leq f(x_0)$ ,  
therefore  $\frac{f(x_0 + h) - f(x_0)}{h} \leq 0$  (as  $f(x_0 + h) - f(x_0) \leq 0$ ),  
and  $h > 0$ .

(2)

i.e.  $f'(x) \geq 0$ .

We CANNOT say anything better if  $f$  is strictly increasing. I.e. :

$$f' \nearrow \neq f' > 0 \text{ on } (a, b).$$

Indeed, here is a counterexample:

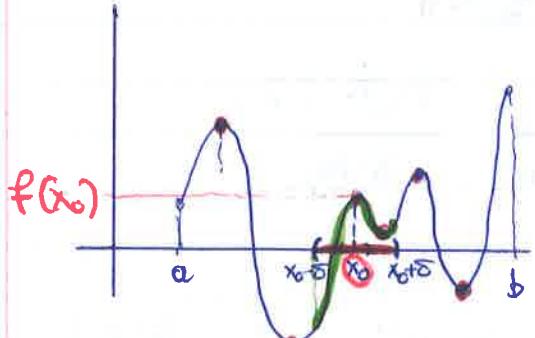
$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^3 \quad \forall x \in \mathbb{R}.$$

$f$  is strictly increasing, yet  $f'(x) = 3x^2$ , which is  $> 0$  for  $x \neq 0$ .

→ Def (Local extremum):

Let  $f: (a, b) \rightarrow \mathbb{R}$ , and let  $x_0 \in (a, b)$ .

We say that:



- $f$  has a local maximum at  $x_0$  if  $\exists \delta > 0$  s.t.

$$f(x) \leq f(x_0), \quad \forall x \in (x_0 - \delta, x_0 + \delta).$$

- $f$  has a local minimum at  $x_0$  if  $\exists \delta > 0$  s.t.

$$f(x) \geq f(x_0), \quad \forall x \in (x_0 - \delta, x_0 + \delta).$$



(3)

thus  $\lim_{h \rightarrow 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \leq 0$ , i.e.  $f'(x_0) \leq 0$ .  $\textcircled{*}_1$

- If  $-\delta < h < 0$ , then  $x_0 - \delta < x_0 + h < x_0$ , thus

$$f(x_0 + h) \leq f(x_0),$$

therefore  $\frac{f(x_0 + h) - f(x_0)}{h} \geq 0$  (as  $f(x_0 + h) - f(x_0) \leq 0$  and  $h < 0$ ),

thus  $\lim_{h \rightarrow 0^-} \frac{f(x_0 + h) - f(x_0)}{h} \geq 0$ , i.e.  $f'(x_0) \geq 0$ .  $\textcircled{*}_2$

By  $\textcircled{*}_1$ ,  $\textcircled{*}_2$ ,  $f'(x_0) = 0$ .

**!** This Proposition doesn't hold for local extrema achieved at the endpoints of  $(a, b)$ . Find a counterexample.

→ Def: (Critical point): Let  $f: I \rightarrow \mathbb{R}$ . Let  $I$  be an interval

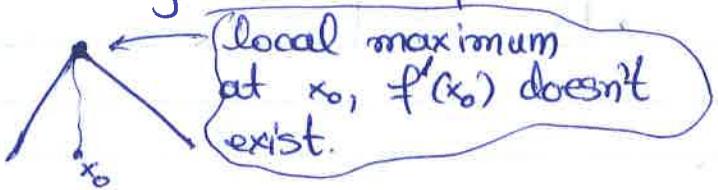
$x_0 \in I$ , not an endpoint. We say that  $x_0$  is in the domain!

$x_0$  is a critical point of  $I$  if  $f'(x_0) = 0$ .

**!** By Fermat's proposition, the critical points of  $(a, b)$  are the only potential candidates for

(4)

local extrema of  $f: (a, b) \rightarrow \mathbb{R}$ , as long as  $f$  is differentiable. If  $f$  not differentiable everywhere, the points of non-differentiability are also potential candidates: imagine



Thus:

→ Application: Let  $f: [a, b] \rightarrow \mathbb{R}$  be continuous. We

know that  $f$  takes a maximal and a minimal value in  $[a, b]$

At which  $x_0 \in [a, b]$  can the maximal (or minimal) value be attained?

Answer: ① At  $a$  or  $b$ .

② At some  $x_0 \in (a, b)$ , where

$f'(x_0)$  exists  
and  $f'(x_0) = 0$

$f'(x_0)$  doesn't exist.

(5)

We will now prepare to prove the Mean Value Theorem.

We first prove a special case: Rolle's theorem.

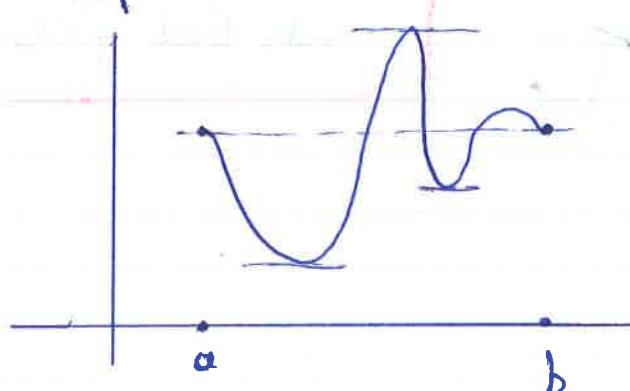
**Rolle's theorem:** Let  $f : [a, b] \rightarrow \mathbb{R}$ , with:

- $f$  continuous on  $[a, b]$ .
- $f$  differentiable on  $(a, b)$ .
- $f(a) = f(b)$ .

Then,  $\exists \gamma \in (a, b)$  such that  $f'(\gamma) = 0$ .

Proof:

Idea: It suffices to show that  $f$  has a local extremum in  $(a, b)$  (irrelevantly of potential extrema at the endpoints  $a, b$ ). Then, Fermat's proposition will imply that  $f'$  vanishes there.



- If  $f$  is constant on  $[a, b]$ , then  $f'(\gamma) = 0 \forall \gamma \in (a, b)$
- If  $f$  is not constant on  $[a, b]$ , then

$\exists x_0 \in (a, b)$  with  $f(x_0) \neq f(a)$ .

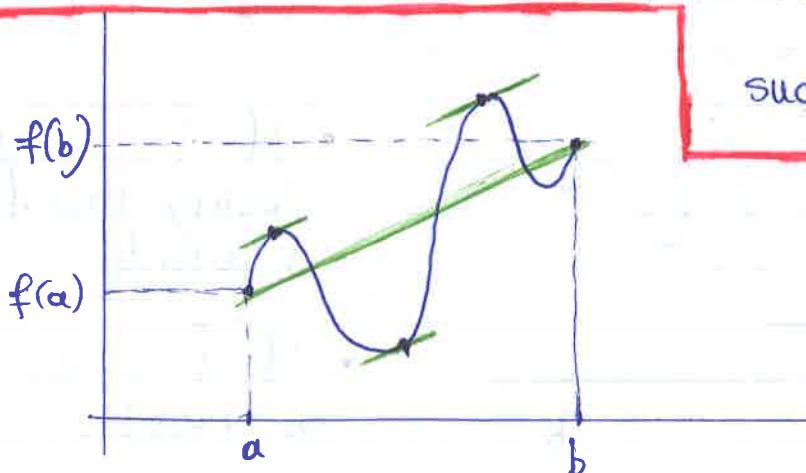
- Suppose that  $f(x_0) > f(a)$ . Since  $f$  is continuous on the closed interval  $[a, b]$ , we know that

(6)

$f$  takes a maximal value in  $[a, b]$ . This maximal value cannot be attained at  $a$  or  $b$ , as  $f(a) > f(a) = f(b)$ . So,  $f$  takes a maximal value at some  $y \in (a, b)$ . By Fermat's proposition,  $f'(y) = 0$ . (as  $f$  is differentiable at  $y$ ).

→ Mean value theorem: Let  $f: [a, b] \rightarrow \mathbb{R}$ , with:

- $f$  continuous on  $[a, b]$ .
- $f$  differentiable on  $(a, b)$ .



Then, there exists  $y \in (a, b)$

$$\text{such that } f'(y) = \frac{f(b) - f(a)}{b - a}$$

(Idea: This looks like a "slid" version of Rolle's theorem. So, we'll just change our  $f$ , to get a new function  $g$  for which we can apply Rolle's theorem)

(7)

Proof: We define  $g: [a,b] \rightarrow \mathbb{R}$ , with

$$g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a} \cdot (x-a).$$

the difference between  $f$   
and the line connecting  $(a, f(a)), (b, f(b))$ .

We have:

- $g$  continuous on  $[a,b]$ , as a sum of continuous functions.
- $g$  differentiable on  $(a,b)$ , as a sum of differentiable functions on  $(a,b)$ .
- $g(a) = 0 = g(b)$ .

By Rolle's theorem, there exists  $\bar{y} \in (a,b)$  with

$$g'(\bar{y}) = 0.$$

Now,  $\forall x \in (a,b)$ ,  $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$ .

Therefore,  $g'(\bar{y}) = 0$  means that

$$f'(\bar{y}) = \frac{f(b)-f(a)}{b-a}.$$

(8)

## Application to monotonicity study

→ Theorem: Let  $f: (a, b) \rightarrow \mathbb{R}$  be differentiable.

- If  $f'(x) \geq 0 \quad \forall x \in (a, b)$ , then  $f$  is increasing.
- If  $f'(x) > 0 \quad \forall x \in (a, b)$ , then  $f$  is strictly increasing.
- If  $f'(x) \leq 0 \quad \forall x \in (a, b)$ , then  $f$  is decreasing.
- If  $f'(x) < 0 \quad \forall x \in (a, b)$ , then  $f$  is strictly decreasing.

while  
 $f' \neq 0$   
 $f' > 0$ !

Proof: Let's prove the first bullet point, the rest follow similarly.

Suppose  $f'(x) \geq 0, \quad \forall x \in (a, b)$ .

Let  $x_1 < x_2$  in  $(a, b)$ . By the mean value theorem, there exists  $\bar{y} \in (x_1, x_2)$ , with

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\bar{y}) \geq 0$$

$$\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq 0 \quad \xrightarrow{x_2 - x_1 > 0} \quad f(x_2) - f(x_1) \geq 0$$

$$\Rightarrow f(x_2) \geq f(x_1).$$

(9)

(Note that we can apply the mean value theorem

because

$f$  is continuous on  $[x_1, x_2]$

and differentiable on  $(x_1, x_2)$ .

### L'Hôpital's rule

Thm: Let  $f, g : (a, x_0) \cup (x_0, b) \rightarrow \mathbb{R}$ , both differentiable, such that:

- $\lim_{x \rightarrow x_0} f(x) = 0, \lim_{x \rightarrow x_0} g(x) = 0.$
- $g'(x) \neq 0$ , for all  $x \in (a, x_0) \cup (x_0, b).$

If  $\lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)} = l \in \mathbb{R}$ , then  $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = l.$

for the proof of L'Hôpital's rule, we need a generalised version of the mean value theorem:

## Cauchy's generalised mean value theorem:

Let  $f, g : [a, b] \rightarrow \mathbb{R}$ , that are:

- continuous on  $[a, b]$
- differentiable on  $(a, b)$ .

Then, there exists  $y \in (a, b)$ , with

$$(f(b) - f(a)) g'(y) = (g(b) - g(a)) f'(y).$$



See how, for  $g(x) = x$ ,  $\forall x \in [a, b]$ , we get the usual mean value theorem from the above.

Proof: Let  $h : [a, b] \rightarrow \mathbb{R}$ , with

$$h(x) = (f(x) - f(a))(g(b) - g(a)) - (f(b) - f(a))(g(x) - g(a)).$$

- $h$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ .
- $h(a) = 0 = h(b)$ .

By Rolle's theorem,  $\exists y \in (a, b)$  s.t.  $h'(y) = 0$ .

$$\text{Now, } h'(x) = f'(x) \cdot (g(b) - g(a)) - (f(b) - f(a)) \cdot g'(x), \\ \forall x \in (a, b).$$

Thus,  $h'(y)=0$  means that

$$f'(y) \cdot (g(b)-g(a)) = (f(b)-f(a)) \cdot g'(y).$$

→ Proof of L'Hopital's rule:

Step 1: We extend  $f, g$  on the whole of  $(a, b)$ , by setting

$$f(x_0) = \lim_{x \rightarrow x_0} f(x) = 0 \text{ and } g(x_0) = \lim_{x \rightarrow x_0} g(x) = 0$$

These functions  $f, g$  are continuous on the whole of  $(a, b)$ . Indeed, by our hypothesis they are continuous on  $(a, b) \setminus \{x_0\}$ . And, since

$f(x_0) = \lim_{x \rightarrow x_0} f(x)$  and  $g(x_0) = \lim_{x \rightarrow x_0} g(x)$ ,  $f$  and  $g$  are also continuous at  $x_0$ .

Step 2: Since  $f(x_0) = g(x_0) = 0$ , we have:

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)}, \quad \forall x \neq x_0.$$

- Let  $x > x_0$ . By Cauchy's mean value theorem,

(12)

$$\frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f'(\beta_x)}{g'(\beta_x)}, \text{ for some } \beta_x \in (x_0, x).$$

- Let  $x < x_0$ . By Cauchy's mean value theorem,

$$\frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f'(\gamma_x)}{g'(\gamma_x)}, \text{ for some } \gamma_x \in (x, x_0).$$

In any case,

$\forall x \neq x_0$  in  $(a, b)$ ,

$$\frac{f(x)}{g(x)} = \frac{f'(\beta_x)}{g'(\beta_x)}, \text{ for some } \beta_x \text{ strictly between } x \text{ and } x_0.$$

Idea: When  $x$  is close to  $x_0$ , then  $\beta_x$  is even closer, thus  $\frac{f'(\beta_x)}{g'(\beta_x)}$  must be close to  $l$ , thus so is  $\frac{f(x)}{g(x)}$ .

Step 3: Let  $\epsilon > 0$ .

Since  $\lim_{y \rightarrow x_0} \frac{f'(y)}{g'(y)} = l$ , we know

that, for this  $\epsilon > 0$ , there exists  $\delta > 0$

(small enough for  $(x_0 - \delta, x_0 + \delta)$  to be inside  $(a, b)$ ),

$$\text{s.t. : } \left\{ \begin{array}{l} \text{if } y \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}, \\ \text{then } \left| \frac{f'(y)}{g'(y)} - l \right| < \epsilon. \end{array} \right\} \quad \text{④}$$

Now, for this same  $\delta$ , let  $x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$ .

We know that, by Step 2, there exists  $\beta_x$ , strictly between  $x$  and  $x_0$ , such that  $\frac{f(x)}{g(x)} = \frac{f'(\beta_x)}{g'(\beta_x)}$ .

(13)

Thus,  $\exists x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$ , which implies that

$$\left| \frac{f'(g_x)}{g'(g_x)} - l \right| < \epsilon, \text{ by } \star.$$

$\parallel$

$$\frac{f(x)}{g(x)}$$

Therefore, for this arbitrary  $x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$ ,

$$\left| \frac{f(x)}{g(x)} - l \right| < \epsilon. \quad \text{Since } \epsilon > 0 \text{ was arbitrary,}$$

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = l.$$



L'Hôpital's rule generalises for  $x_0 = \pm \infty$ ,

and  $l = \pm \infty$  (Exercise)