

SPECIALIZATION MAPS FOR SCHOLZE’S CATEGORY OF DIAMONDS

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ABSTRACT. We introduce the specialization map in Scholze’s theory of diamonds. We consider v -sheaves that “behave like formal schemes” and call them kimberlites. We attach to them: a reduced special fiber, an analytic locus, a specialization map, a Zariski site, and an étale site. When the kimberlite comes from a formal scheme, our sites recover the classical ones. We prove that unramified p -adic Beilinson–Drinfeld Grassmannians are kimberlites with finiteness and normality properties.

CONTENTS

Introduction	1
Acknowledgements	6
1. The v -topology	6
1.1. Recollections on diamonds and small v -sheaves	6
1.2. Spectral spaces and locally spatial diamonds	8
1.3. Pre-adic spaces as v -sheaves	9
2. The olivine spectrum	9
2.1. Review, terminology and conventions	10
2.2. Definitions and basic properties	10
2.3. Olivine Huber pairs	13
2.4. Some open and closed subsheaves	16
2.5. Discrete Huber pairs in characteristic p	18
3. The reduction functor	21
3.1. The v -topology for perfect schemes	21
3.2. Reduction functor and formal adicness	24
4. Specialization	27
4.1. Specialization for Tate Huber pairs	27
4.2. Specializing v -sheaves	27
4.3. Pre-kimberlites, formal schemes and formal neighborhoods	29
4.4. Heuer’s specialization map and étale formal neighborhoods	31
4.5. Kimberlites and smelted kimberlites	33
4.6. Finiteness and normality	35
5. The specialization map for unramified p -adic Beilinson–Drinfeld Grassmannians	37
5.1. Twisted loop sheaves	37
5.2. Demazure resolution	39
References	44

INTRODUCTION

As Fargues–Scholze [FS21] and Scholze–Weinstein [SW20] show, Scholze’s theory of diamonds and v -sheaves [Sch17] is a powerful geometric framework to study (among other things) the local Langlands correspondence and the theory of local Shimura varieties. One of the early milestones of Scholze’s theory roughly says that diamonds capture correctly the étale site of an analytic adic space ([Sch17, §15]). Moreover, the category of diamonds contains many geometric objects of arithmetic interest that do not come from an analytic adic space, and for these spaces one still gets a well-behaved étale site.

Although Scholze’s theory of v -sheaves is also useful to study adic spaces that are not analytic (like the ones coming from a formal scheme), some complications arise. For example, the v -sheaf associated to a non-analytic adic space has more open subsets than one would expect [SW20, §18]. In particular, a comparison of étale sites can’t hold since even the site of open subsets do not coincide. Despite this

complications, it is still profitable to understand the behavior of the v-sheaves associated to non-analytic adic spaces. The main motivation to work this out is because there are v-sheaves of arithmetic interest that do not come from an adic space, but “resemble” the behavior of a formal scheme. The main examples to keep in mind are the integral models of moduli spaces of p -adic shtukas proposed in [SW20, §25] or the p -adic Beilinson–Drinfeld Grassmannians of [SW20, §21].

In rough terms this article does the following:

- (1) Study the topological space $|\mathrm{Spd}(A, A^+)|$ for a general Huber pair (A, A^+) over \mathbb{Z}_p . We overcome many of the technical difficulties of working with these spaces.
- (2) Propose a rigorous definition of what it means for a v-sheaf to “resemble” a formal scheme. We call these v-sheaves *kimberlites*.
- (3) Construct specialization maps attached to kimberlites. This recovers the classical specialization maps of formal schemes.
- (4) Attach “Zariski” and “étale” sites to a kimberlite. This allow us to recover the Zariski and étale sites of a formal scheme intrinsically from the v-sheaf attached to it.
- (5) Verify that the p -adic Beilinson–Drinfeld Grassmannians attached to reductive groups over \mathbb{Z}_p are interesting examples of kimberlites that do not come from formal schemes.

Although this work (admittedly of technical nature) is far from being a robust theory, we think it makes appreciable progress in our understanding of this kind of v-sheaves and sets a stepping stone for future investigations. For instance, the constructions and techniques discussed here have already found applications in the following works:

- (1) In our work on geometric connected components of local Shimura varieties [Gle22].
- (2) In our collaborative work with Anschütz, Lourenço and Richarz on the Scholze–Weinstein conjecture [AGLR22].
- (3) In the representability results of Pappas and Rapoport [PR21].

We find it reasonable to expect that our considerations will play a role in more general representability results of integral local Shimura varieties, and a role in the study of “the nearby cycles functor” (Remark 4.29).

Let us give a more detailed account of our results. In [Sch17], Scholze sets foundations for the theory of diamonds which can be defined as certain sheaves on the category of characteristic p perfectoid spaces endowed with a Grothendieck topology called the v-topology. He associates to any pre-adic space X over \mathbb{Z}_p (not necessarily analytic) a small v-sheaf X^\diamond , and whenever X is analytic he proves that X^\diamond is a locally spatial diamond. If $X = \mathrm{Spa}(B, B^+)$ for a Huber pair (B, B^+) , then X^\diamond is denoted $\mathrm{Spd}(B, B^+)$. Moreover, Scholze assigns to any small v-sheaf \mathcal{F} an underlying topological space $|\mathcal{F}|$ and whenever $\mathcal{F} = X^\diamond$ he constructs a functorial surjective and continuous map $|\mathcal{F}| \rightarrow |X|$. When X is analytic it is proven in [SW20] that this map is a homeomorphism, but this map fails to be injective almost always for pre-adic spaces that have non-analytic points. To tackle this difficulty, we associate to a Huber pair (B, B^+) what we call below its *olivine spectrum*, which we denote $\mathrm{Spo}(B, B^+)$. The definition of this topological space is concrete enough to allow computations to take place. Moreover, in most cases of interest $\mathrm{Spo}(B, B^+)$ recovers $|\mathrm{Spd}(B, B^+)|$.

Theorem 1. *Let (B, B^+) be a complete Huber pair over \mathbb{Z}_p , there is a functorial bijective and continuous map $|\mathrm{Spd}(B, B^+)| \rightarrow \mathrm{Spo}(B, B^+)$. Moreover, it is a homeomorphism if (B, B^+) is topologically of finite type over (B_0, B_0) for $B_0 \subseteq B^+$ a ring of definition.*

In [SW20, §18] Scholze and Weinstein attach to a perfect scheme X in characteristic p a v-sheaf denoted X^\diamond . Moreover, they prove that the functor $X \mapsto X^\diamond$ is fully-faithful. It will be clear to the reader that many of our techniques used to prove Theorem 1 are borrowed from Scholze and Weinstein’s approach to their full-faithfulness result, but our perspective allow us to go farther.

Theorem 2. *Let Y be a perfect non-analytic adic space over \mathbb{F}_p and let X be a pre-adic space over \mathbb{Z}_p . The natural map $\mathrm{Hom}_{\mathrm{PreAd}}(Y, X) \rightarrow \mathrm{Hom}(Y^\diamond, X^\diamond)$ is bijective. In particular, $(-)^\diamond$ is fully faithful when restricted to the category of perfect non-analytic adic spaces over \mathbb{F}_p .*

Theorem 2 allow us to recover Scholze and Weinstein’s result as a particular case. Also, the olivine spectrum allows us to show that Scholze and Weinstein’s functor $X \mapsto X^\diamond$ is continuous for the v-topology and admits a right adjoint $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{red}}$ at the level of topoi. We call this the *reduction functor* and it plays a key role in the rest of our theory. In general, the objects obtained from the reduction functor might not be perfect schemes, but they are “scheme theoretic v-sheaves” and they come equipped with

an underlying topological space that agrees with the Zariski topology whenever they are representable.

Let us describe our approach to study v-sheaves that “behave like” formal schemes. We consider three layers, in each layer we get closer to capture the behavior of formal schemes. We first recall a more classical case. Let \mathcal{X} be a p -adic separated formal scheme topologically of finite type over \mathbb{Z}_p . One can associate to \mathcal{X} a rigid analytic space over \mathbb{Q}_p , that we will denote by X_η . We can also associate to \mathcal{X} a finite type reduced scheme over \mathbb{F}_p , that we denote by \overline{X} . Huber’s theory of adic spaces allows us to consider X_η as an adic space and assign to it a topological space $|X_\eta|$. Moreover, one can construct a continuous map $\mathrm{sp}_\mathcal{X} : |X_\eta| \rightarrow |\overline{X}|$, where $|\overline{X}|$ is the usual Zariski space underlying \overline{X} [Bha17, Remark 7.4.12] or [Lou17, Definition 6.4]. A theorem of Lourenço ([SW20, Theorem 18.4.2]) says that “nice enough” formal schemes can be recovered from the triple $(X_\eta, \overline{X}, \mathrm{sp}_\mathcal{X})$. We propose that v-sheaves that “resemble” formal schemes should be those for which a specialization map can be constructed.

Consider the following. Given a Tate Huber pair (A, A^+) with pseudo-uniformizer $\varpi \in A^+$ the specialization map $\mathrm{sp}_A : \mathrm{Spa}(A, A^+) \rightarrow \mathrm{Spec}(A^+/\varpi)$ assigns to $x \in \mathrm{Spa}(A, A^+)$ the prime ideal \mathfrak{p}_x of those elements $a \in A^+$ for which $|a|_x < 1$. Observe that this construction is functorial in the category of Tate Huber pairs. We wish to exploit functoriality to descend this specialization map to more general v-sheaves. The first question is: What should the target of the specialization map be?

One can compute directly that if (A, A^+) is a uniform Tate Huber pair, then $\mathrm{Spd}(A^+)^{\mathrm{red}}$ is the perfection of $\mathrm{Spec}(A^+/\varpi)$. This suggests that if we want to attach a specialization map to a v-sheaf \mathcal{F} the target of this map should be $|\mathcal{F}^{\mathrm{red}}|$.

A key aspect that makes the specialization map for Tate Huber pairs functorial is that every map of Tate Huber pairs $\mathrm{Spa}(A, A^+) \rightarrow \mathrm{Spa}(B, B^+)$ automatically upgrades “integrally” to a map $\mathrm{Spa}(A^+) \rightarrow \mathrm{Spa}(B^+)$. This motivates the following definition:

Definition 3. Let \mathcal{F} be a small v-sheaf, (A, A^+) be a Tate Huber pair and $f : \mathrm{Spd}(A, A^+) \rightarrow \mathcal{F}$ a map.

- (1) We say that \mathcal{F} *formalizes* f (or that f is *formalizable*) if there is $t : \mathrm{Spd}(A^+) \rightarrow \mathcal{F}$ factoring f .
- (2) We say that \mathcal{F} *v-formalizes* f if there is a v-cover $g : \mathrm{Spa}(B, B^+) \rightarrow \mathrm{Spa}(A, A^+)$ such that \mathcal{F} formalizes $f \circ g$.
- (3) We say that \mathcal{F} is *v-formalizing* if it v-formalizes any f as above.

Given a v-formalizing v-sheaf \mathcal{F} one could define the specialization map $\mathrm{sp}_\mathcal{F} : |\mathcal{F}| \rightarrow |\mathcal{F}^{\mathrm{red}}|$ so that for any “formalized” map $f : \mathrm{Spd}(A^+) \rightarrow \mathcal{F}$ the following diagram is commutative:

$$\begin{array}{ccccc} |\mathrm{Spa}(A, A^+)| & \longrightarrow & |\mathrm{Spd}(A^+)| & \xrightarrow{|f|} & |\mathcal{F}| \\ \downarrow \mathrm{sp}_A & & & & \downarrow \mathrm{sp}_\mathcal{F} \\ |\mathrm{Spec}(A^+/\varpi)^{\mathrm{perf}}| & \xrightarrow{|\mathcal{F}^{\mathrm{red}}|} & & & |\mathcal{F}^{\mathrm{red}}| \end{array}$$

The recipe to compute the specialization map would then be as follows: given $x \in |\mathcal{F}|$ represent it by a map $\iota_x : \mathrm{Spa}(C, C^+) \rightarrow \mathcal{F}$, find a formalization $f_x : \mathrm{Spd}(C^+) \rightarrow \mathcal{F}$ of ι_x . Apply the reduction functor to f_x to obtain a map $f_x^{\mathrm{red}} : \mathrm{Spec}(C^+/\varpi)^{\mathrm{perf}} \rightarrow \mathcal{F}^{\mathrm{red}}$. Look at the image of the closed point in $\mathrm{Spec}(C^+/\varpi)$ under $|f_x^{\mathrm{red}}|$. This is $\mathrm{sp}_\mathcal{F}(x)$.

The natural question is whether or not this is well defined. The problem being that the map $\iota_x : \mathrm{Spa}(C, C^+) \rightarrow \mathcal{F}$ might have more than one formalization. The naive guess is that this doesn’t happen if \mathcal{F} is separated as a v-sheaf. Unfortunately, this is false. At the heart of the problem is the following pathology: although $|\mathrm{Spa}(C, C^+)|$ is dense within $|\mathrm{Spa}(C^+)|$ it is not true that $|\mathrm{Spd}(C, C^+)|$ is dense within $|\mathrm{Spd}(C^+)|$. It is this key subtlety that requires sufficient understanding of the olivine spectrum of Huber pairs.

Definition 4. Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be a map of v-sheaves.

- (1) We say f is *formally adic* if the following diagram induced by adjunction is Cartesian:

$$\begin{array}{ccc} (\mathcal{F}^{\mathrm{red}})^\diamond & \longrightarrow & (\mathcal{G}^{\mathrm{red}})^\diamond \\ \downarrow & & \downarrow \\ \mathcal{F} & \longrightarrow & \mathcal{G} \end{array}$$

- (2) If \mathcal{F} comes with a formally adic map to $\mathrm{Spd}(\mathbb{Z}_p)$ we say that \mathcal{F} is *p-adic*.
- (3) We say f is *formally closed* if it is a formally adic closed immersion.
- (4) We say \mathcal{F} is *formally separated* if the diagonal $\mathcal{F} \rightarrow \mathcal{F} \times \mathcal{F}$ is formally closed.

Using the olivine spectrum we prove that $|\mathrm{Spd}(C, C^+)|$ is “formally dense” in $|\mathrm{Spd}(C^+)|$. The main feature of a formally separated v-sheaf \mathcal{F} is that a map $\iota : \mathrm{Spa}(A, A^+) \rightarrow \mathcal{F}$ has at most one formalization (if any).

Combining the two inputs we say that a v-sheaf \mathcal{F} is *specializing* if it is v-formalizing and formally separated, this is the first layer of approximation to the definition. We attach functorially to such \mathcal{F} a continuous specialization map $|\mathcal{F}| \rightarrow |\mathcal{F}^{\mathrm{red}}|$.

Now, specializing v-sheaves produce all the specialization maps we are interested in, but they are still too general to capture the behavior of formal schemes.

Definition 5. Let \mathcal{F} be a specializing v-sheaf. We say \mathcal{F} is a *prekimberlite* if:

- a) $\mathcal{F}^{\mathrm{red}}$ is represented by a scheme.
- b) The map $(\mathcal{F}^{\mathrm{red}})^\diamond \rightarrow \mathcal{F}$ coming from adjunction is a closed immersion.

If \mathcal{F} is a prekimberlite, we let the *analytic locus* be $\mathcal{F}^{\mathrm{an}} = \mathcal{F} \setminus (\mathcal{F}^{\mathrm{red}})^\diamond$.

We can attach étale and Zariski sites to a prekimberlite as follows:

Definition 6. Suppose \mathcal{F} is a prekimberlite, we let $(\mathcal{F})_{\mathrm{qc}, \mathrm{for}\text{-}\acute{\mathrm{e}}\mathrm{t}}$ be the category that has as objects maps $f : \mathcal{G} \rightarrow \mathcal{F}$ where \mathcal{G} is a prekimberlite and f is formally adic, étale and quasicompact. Morphisms are maps of v-sheaves commuting with the structure map. We call objects in this category the *étale formal neighborhoods* of \mathcal{F} . If f is also injective we call them open formal neighborhoods of \mathcal{F} .

Theorem 7. For \mathcal{F} a prekimberlite, the reduction functor $(-)^{\mathrm{red}} : (\mathcal{F})_{\mathrm{qc}, \mathrm{for}\text{-}\acute{\mathrm{e}}\mathrm{t}} \rightarrow (\mathcal{F}^{\mathrm{red}})_{\mathrm{qc}, \acute{\mathrm{e}}\mathrm{t}, \mathrm{sep}}$ is an equivalence. Here, the target category are the maps of perfect schemes $f : Y \rightarrow \mathcal{F}^{\mathrm{red}}$ that are quasi-compact, étale and separated. Moreover, this functor restricts to an equivalence between open formal neighborhoods and quasicompact open immersions.

Now, if \mathfrak{X} is a separated formal scheme locally admitting a finitely generated ideal of definition (see [Convention 1](#) below for details), then \mathfrak{X}^\diamond is a prekimberlite and $(\mathfrak{X}^\diamond)^{\mathrm{red}}$ is the perfection of the reduced subscheme of \mathfrak{X} . In particular, one can recover the étale site of \mathfrak{X} from $(\mathfrak{X}^\diamond)_{\mathrm{qc}, \mathrm{for}\text{-}\acute{\mathrm{e}}\mathrm{t}}$ through [Theorem 7](#).

Let us describe the inverse functor, for this we consider the following construction due to Heur [[Heu21](#)]. For a perfect scheme X in characteristic p we let $X^{\diamond/\circ}$ denote the v-sheaf given by the analytic sheafification of the rule $(R, R^+) \mapsto X(\mathrm{Spec}(R^+/\varpi)^{\mathrm{perf}})$ where (R, R^+) is affinoid perfectoid and $\varpi \in R^+$ is a pseudo-uniformizer. For a prekimberlite \mathcal{F} we get a map of v-sheaves $\mathrm{SP}_{\mathcal{F}} : \mathcal{F} \rightarrow (\mathcal{F}^{\mathrm{red}})^{\diamond/\circ}$. If $f : V \rightarrow \mathcal{F}^{\mathrm{red}}$ is étale, quasicompact and separated then $\widehat{\mathcal{F}}_V := \mathcal{F} \times_{(\mathcal{F}^{\mathrm{red}})^{\diamond/\circ}} V^{\diamond/\circ}$ is the étale formal neighborhood of \mathcal{F} with $(\widehat{\mathcal{F}}_V)^{\mathrm{red}} = V$. The two key ingredients are that for perfect schemes X we have an identification $(X^{\diamond/\circ})^{\mathrm{red}} = X$, and if $V \rightarrow X$ is étale then $V^{\diamond/\circ} \rightarrow X^{\diamond/\circ}$ is formally adic and étale. This later statement in turn reduces to the invariance of étale sites under nilpotent thickenings and perfection. Heur’s construction also allows us to consider what we call *formal neighborhoods*. If \mathcal{F} is a prekimberlite and $S \subseteq \mathcal{F}^{\mathrm{red}}$ is a locally closed subscheme we can consider $\widehat{\mathcal{F}}_S := \mathcal{F} \times_{(\mathcal{F}^{\mathrm{red}})^{\diamond/\circ}} S^{\diamond/\circ}$. We always have $\widehat{\mathcal{F}}_S \subseteq \mathcal{F}$ and when S is constructible this is even an open immersion.¹ This construction generalizes “completion” of a formal scheme along a locally closed immersion.

We are ready for the third approximation.

Definition 8. Let \mathcal{F} be a prekimberlite.

- (1) We say \mathcal{F} is *valuative* if $\mathrm{SP}_{\mathcal{F}} : \mathcal{F} \rightarrow (\mathcal{F}^{\mathrm{red}})^{\diamond/\circ}$ is partially proper.
- (2) A *smelted kimberlite* is a pair $\mathcal{K} = (\mathcal{F}, \mathcal{D})$ where \mathcal{F} is a valuative prekimberlite, \mathcal{D} is a quasi-separated locally spatial diamond and $\mathcal{D} \subseteq \mathcal{F}^{\mathrm{an}}$ is open. The main cases of interest are when $\mathcal{D} = \mathcal{F}^{\mathrm{an}}$ or when $\mathcal{D} = \mathcal{F} \times_{\mathrm{Spd}(\mathbb{Z}_p)} \mathrm{Spd}(\mathbb{Q}_p)$.
- (3) We define the specialization map $\mathrm{sp}_{\mathcal{K}} : |\mathcal{D}| \rightarrow |\mathcal{F}^{\mathrm{red}}|$ as the composition $|\mathcal{D}| \rightarrow |\mathcal{F}| \xrightarrow{\mathrm{sp}_{\mathcal{F}}} |\mathcal{F}^{\mathrm{red}}|$. If the context is clear we write $\mathrm{sp}_{\mathcal{D}}$ instead of $\mathrm{sp}_{\mathcal{K}}$.
- (4) We say \mathcal{F} is a *kimberlite* if $(\mathcal{F}, \mathcal{F}^{\mathrm{an}})$ is a smelted kimberlite and $\mathrm{sp}_{\mathcal{F}^{\mathrm{an}}}$ is quasicompact.

The specialization map for kimberlites and smelted kimberlites is better behaved since it is even continuous for the constructible topology.

Theorem 9. Let $\mathcal{K} = (\mathcal{F}, \mathcal{D})$ be a smelted kimberlite and \mathcal{G} be a kimberlite, the following hold:

- (1) $\mathrm{sp}_{\mathcal{D}} : |\mathcal{D}| \rightarrow |\mathcal{F}^{\mathrm{red}}|$ is a specializing, spectral map of locally spectral spaces.
- (2) $\mathrm{sp}_{\mathcal{G}^{\mathrm{an}}} : |\mathcal{G}^{\mathrm{an}}| \rightarrow |\mathcal{G}^{\mathrm{red}}|$ is also a closed map.

¹These open immersions are only formally adic when S is an open immersion.

If $(\mathcal{F}, \mathcal{D})$ is a smelted kimberlite and $S \subseteq |\mathcal{F}^{\text{red}}|$ we can define analogs of Berthelot tubes, by letting $\mathcal{D}_{/S}^{\circ} = \widehat{\mathcal{F}}_{/S} \times_{\mathcal{F}} \mathcal{D}$. We call these spaces the *tubular neighborhood* of \mathcal{D} around S .

Finally, to study the p -adic Beilinson–Drinfeld Grassmannians we introduce some “finiteness” and “normality” conditions.

Definition 10. Let $\mathcal{K} = (\mathcal{F}, \mathcal{D})$ a smelted kimberlite and \mathcal{G} a kimberlite.

- (1) We say \mathcal{D} is a *cJ-diamond* (constructibly Jacobson) if rank 1 points are dense in the constructible topology of \mathcal{D} .
- (2) We say that \mathcal{K} is *rich* if: \mathcal{D} is a cJ-diamond, $|\mathcal{F}^{\text{red}}|$ is locally Noetherian and $\text{sp}_{\mathcal{D}} : |\mathcal{D}| \rightarrow |\mathcal{F}^{\text{red}}|$ is surjective.
- (3) We say that \mathcal{G} is *rich* if: $(\mathcal{G}, \mathcal{G}^{\text{an}})$ is rich.
- (4) If \mathcal{K} is rich we say it is *topologically normal* if for every closed point $x \in |\mathcal{F}^{\text{red}}|$ the tubular neighborhood $\mathcal{D}_{/x}^{\circ}$ is connected.²

Let G denote a reductive over \mathbb{Z}_p and let $T \subseteq B \subseteq G$ denote integrally defined maximal torus and Borel subgroups respectively. Let $\mu \in X_*^+(T_{\overline{\mathbb{Q}}_p})$ be a dominant cocharacter with reflex field $E \subseteq \overline{\mathbb{Q}}_p$. Let O_E denote the ring of integers of E and let k_E denote the residue field. Let $\text{Gr}_{O_E}^{G, \leq \mu}$ denote the v -sheaf parametrizing B_{dR}^+ -lattices with G -structure whose relative position is bounded by μ as in [SW20, Definition 20.5.3] and let $\text{Gr}_{\mathcal{W}, k_E}^{G, \leq \mu}$ denote the Witt vector affine Grassmannian [Zhu17], [BS17]. Here is our result:

Theorem 11. $\text{Gr}_{O_E}^{G, \leq \mu}$ is a topologically normal rich p -adic kimberlite with $(\text{Gr}_{O_E}^{G, \leq \mu})^{\text{red}} = \text{Gr}_{\mathcal{W}, k_E}^{G, \leq \mu}$. In particular, the specialization map is a closed, surjective and spectral map of spectral topological spaces.

This result has partially been generalized in our collaboration [AGLR22]. There, we prove that the local models for parahoric groups are rich p -adic kimberlites. Nevertheless, we only improve the “normality” part of the result if we assume that μ is minuscule and outside certain cases in small characteristic.

In [Gle22], we use normality of $\text{Gr}_{O_E}^{G, \leq \mu}$ to prove normality of moduli spaces of p -adic shtukas which is a key step to prove the main theorem of [Gle22] for the following reason. Classically, normality of formal scheme ensures that the generic fiber and special fibers have the same connected components. This also happens for rich smelted kimberlites.

We prove **Theorem 11** by using a Demazure resolution. Our key observation is that one can do the Demazure resolution using either B_{dR}^+ -coefficients or A_{inf} -coefficients. The use of A_{inf} -coefficients makes it clear that $\text{Gr}_{O_E}^{G, \leq \mu}$ is v -formalizing. Normality of $\text{Gr}_{O_E}^{G, \leq \mu}$ can be deduced from the normality of the source in the Demazure resolution, which in turn can be deduced inductively from its expression as iterated $(\mathbb{P}^1)^{\diamond}$ -bundles.

Let us comment on the organization of the paper.

- I) In the first section, we give a short review of the theory of diamonds, the v -topology and some facts about spectral topological spaces. We also review Scholze’s \diamond functor that takes as input a pre-adic space over \mathbb{Z}_p and returns as output a v -sheaf.
- II) In the second section, we introduce and study the olivine spectrum of a Huber pair. We prove **Theorem 1** and **Theorem 2**.
- III) In the third section, we review the small diamond functor \diamond . We prove the continuity of \diamond . We introduce the reduction functor as the right adjoint to \diamond . We introduce and study “formally adic” maps.
- IV) In the fourth section, we develop our theory of specialization maps. We introduce specializing v -sheaves and prekimberlites. We introduce formal neighborhoods, étale formal neighborhoods and we prove **Theorem 7**. We introduce kimberlites, and smelted kimberlites and prove **Theorem 9**. We prove that formal schemes give rise to kimberlites. Finally, we introduce the finiteness and normality conditions.
- V) In the fifth section, we study the specialization map for p -adic Beilinson–Drinfeld Grassmannians. We review the construction of twisted loop groups with B_{dR}^+ and A_{inf} coefficients. We construct the two versions of the “integral” Demazure resolution. We prove **Theorem 11**.

²In a previous version of this article we had already introduced this notions, but we hadn’t realized the connection to normality. Our motivation to call this “topologically normal” comes from [AGLR22, Proposition 2.38].

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1. THE v -TOPOLOGY

We assume familiarity with the theory of perfectoid spaces and diamonds as discussed in [\[SW20, §7\]](#) or [\[Sch17, §3\]](#). For the most part the reader can ignore the set-theoretic subtleties that arise from the theory. Nevertheless, for some of our constructions set-theoretic carefullness is necessary.

1.1. Recollections on diamonds and small v -sheaves. We let Perfd denote the category of perfectoid spaces and Perf the subcategory of perfectoid spaces in characteristic p . Recall that we can endow Perfd with two Grothendieck topologies, called the pro-étale topology and v -topology respectively [\[Sch17, Definition 7.8, Definition 8.1\]](#). The following example of a cover for the v -topology will be used repeatedly.

Example 1.1. Let $\text{Spa}(A, A^+)$ be an affinoid perfectoid space, with pseudo-uniformizer $\varpi \in A^+$. Given $x \in |\text{Spa}(A, A^+)|$ let $\iota_x : \text{Spa}(k(x), k(x)^+) \rightarrow \text{Spa}(A, A^+)$ be the residue field. By [\[Sch12, Corollary 6.7\]](#), each $\text{Spa}(k(x), k(x)^+)$ is perfectoid. Let $R^+ := \prod_{x \in |\text{Spa}(A, A^+)|} k(x)^+$ endowed with the ϖ -adic topology and let $R = R^+[\frac{1}{\varpi}]$. Then $\text{Spa}(R, R^+)$ is perfectoid and $\text{Spa}(R, R^+) \rightarrow \text{Spa}(A, A^+)$ is a v -cover.

Definition 1.2. Let I be a set and $\{(C_i, C_i^+), \varpi_i\}_{i \in I}$ a collection of tuples where each C_i is an algebraically closed nonarchimedean field, the C_i^+ are open and bounded valuation subrings of C_i , and ϖ_i is a pseudo-uniformizer. Let $R^+ := \prod_{i \in I} C_i^+$, let $\varpi = (\varpi_i)_{i \in I}$, endow R^+ with the ϖ -adic topology and let $R := R^+[\frac{1}{\varpi}]$. Any space of the form $\text{Spa}(R, R^+)$ constructed in this way will be called a product of points.

Remark 1.3. Different choices of pseudo-uniformizers $(\varpi_i)_{i \in I}$ give rise to different adic spaces. Also, [Example 1.1](#) proves that products of points form a basis for the v -topology in the category of perfectoid spaces. Recall the notion of totally disconnected spaces.

Definition 1.4. ([\[Sch17, Definition 7.1, Definition 7.15, Lemma 7.5\]](#)) An affinoid perfectoid space $\text{Spa}(R, R^+)$ is totally disconnected if it splits every open cover. Moreover, it is strictly totally disconnected if it splits every étale cover.

Proposition 1.5. ([\[Sch17, Lemma 7.3, Proposition 7.16, Lemma 11.27\]](#)) *Let Y be an affinoid perfectoid space. Y is represented by a strictly totally disconnected space if and only if every connected component of Y is represented by $\text{Spa}(C, C^+)$ for C an algebraically closed field and C^+ an open and bounded valuation subring.*

Proposition 1.6. *Product of points are strictly totally disconnected perfectoid space.*

Proof. Fix notation as in [Definition 1.2](#). The closed-opens subsets of $\text{Spa}(R, R^+)$ are given by subsets of I . Every $x \in \pi_0(\text{Spa}(R, R^+))$ is computed as $\bigcap_{U \in \mathcal{U}_x} U$ for some ultrafilter. This is a Zariski closed subsets cut out by an ideal of idempotents $\mathcal{I}_x = \langle 1_V \rangle$, with $V \subseteq I$ and $V \notin \mathcal{U}$. The \mathcal{O}^+ -structure sheaf of x is the ϖ -completion of R^+/\mathcal{I}_x . Let $V = R^+/\mathcal{I}_x$ and V' the ϖ -adic completion of V . By [Proposition 1.5](#), it suffices to prove that V' is a valuation ring with algebraically closed fraction field. This easily reduces to the same claim on V . It is not hard to see that $\text{Frac}(V) = (\prod_{i \in I} C_i)/\mathcal{I}_x$. Moreover, the properties of

being a valuation ring or being an algebraically closed field can be expressed in first order logic so these properties pass to ultraproducts. \square

The v -topology on Perfd is subcanonical [Sch17, Corollary 8.6]. We denote a perfectoid space and the sheaf it represents with the same letter. When a distinction is needed, if X denotes a perfectoid space we denote by h_X the sheaf it represents. Let Y be a diamond [Sch17, Definition 11.1], we recall the definition of its associated underlying topological space $|Y|$.

Definition 1.7. A map $p : \text{Spa}(K, K^+) \rightarrow Y$ is a point if K is a perfectoid field in characteristic p and K^+ is an open and bounded valuation subring of K . Two points $p_i : \text{Spa}(K_i, K_i^+) \rightarrow Y$, $i \in \{1, 2\}$, are equivalent if there is a third point $p_3 : \text{Spa}(K_3, K_3^+) \rightarrow Y$, and surjective maps $q_i : \text{Spa}(K_3, K_3^+) \rightarrow \text{Spa}(K_i, K_i^+)$ making the following commutative diagram:

$$\begin{array}{ccccc}
 & & \text{Spa}(K_1, K_1^+) & & \\
 & \nearrow^{q_1} & & \searrow_{p_1} & \\
 \text{Spa}(K_3, K_3^+) & & & & Y \\
 & \searrow_{q_2} & & \nearrow_{p_2} & \\
 & & \text{Spa}(K_2, K_2^+) & & \\
 & & & \xrightarrow{p_3} &
 \end{array}$$

We let $|Y|$ denote the set of equivalence classes of points of Y .

Scholze proves that if Y has a presentation X/R with X and R perfectoid, then there is canonical bijection between $|Y|$ and $|X|/|R|$. Moreover, the quotient topology on $|Y|$ coming from the surjection $|X| \rightarrow |Y|$ doesn't depend on the presentation [Sch17, Proposition 11.13]. Also, if X is a perfectoid space, then h_X is a diamond and $|h_X|$ is canonically homeomorphic to $|X|$.

We refer to sheaves on Perf for the v -topology as v -sheaves. Recall that a v -sheaf is said to be small if it admits a surjection from a representable sheaf. We denote by $\widehat{\text{Perf}}$ the category of small v -sheaves. There's a more explicit way of defining this. Given a cut-off cardinal κ ([Sch17, §4, §8] for details) denote by Perf_κ the category of κ -small perfectoid spaces in characteristic p and by $\widehat{\text{Perf}}_\kappa$ the topos of sheaves for the v -topology on this category. Objects in this topos are called κ -small v -sheaves. We have natural fully-faithful embeddings $\widehat{\text{Perf}}_\kappa \subseteq \widehat{\text{Perf}}_\lambda$ for $\kappa < \lambda$ and $\widehat{\text{Perf}} = \bigcup_\kappa \widehat{\text{Perf}}_\kappa$ as a big filtered colimit over cut-off cardinals κ .

Scholze associates to any small v -sheaf a topological space. The definition is similar to Definition 1.7, with the role of perfectoid spaces exchanged by diamonds. The key point being that if $X \rightarrow Y$ is a map of small v -sheaves with X a diamond then $R = X \times_Y X$ is also a diamond and $Y = X/R$ [Sch17, Proposition 12.3]. Scholze then defines $|Y|$ as $|X|/|R|$ with the quotient topology and by [Sch17, Proposition 12.7] this is well defined. Given a topological space T we can consider a presheaf on $\widehat{\text{Perf}}$, denoted \underline{T} , defined as

$$\underline{T}(R, R^+) = \{f : |\text{Spa}(R, R^+)| \rightarrow T \mid f \text{ is continuous}\}$$

This is a v -sheaf but it might not be small. There is a natural transformation, $X \rightarrow \underline{|X|}$ of v -sheaves. A morphism of small v -sheaves $j : U \rightarrow X$ is open if it is relatively representable in perfectoid spaces and after basechange it becomes an open embedding of perfectoid spaces. Open subsheaves of X are uniquely determined by open subsets of $|X|$ ([Sch17, Proposition 11.15, Proposition 12.9]). The concept of closed immersion is a little more subtle. It is not a purely topological condition in the sense that closed subsheaves of \mathcal{F} are not in bijection with closed subsets of $|\mathcal{F}|$. Indeed, there are more closed subsets than closed immersions.

Definition 1.8. ([Sch17, Definition 10.7, Proposition 10.11, Definition 5.6]) A map of sheaves $\mathcal{F} \rightarrow \mathcal{G}$ is a closed immersion if for every $X = \text{Spa}(R, R^+)$ a strictly totally disconnected space and a map $X \rightarrow \mathcal{G}$ the pullback $X \times_{\mathcal{F}} \mathcal{G} \subseteq X$ is representable by a closed immersion of perfectoid spaces.

The following result characterizes closed immersions.

Proposition 1.9. ([AGLR22]) For a v -sheaf \mathcal{F} we say a subset $X \subseteq |\mathcal{F}|$ is weakly generalizing if for any geometric point $f : \text{Spa}(C, C^+) \rightarrow \mathcal{F}$ we have that $f^{-1}(X) \subseteq |\text{Spa}(C, C^+)|$ is stable under generization. For any v -sheaf \mathcal{F} the rule

$$X \mapsto \mathcal{F} \times_{|\mathcal{F}|} \underline{X} \subseteq \mathcal{F}$$

gives a bijection between weakly generalizing closed subsets of $|\mathcal{F}|$ and closed subsheaves of \mathcal{F} .

1.2. Spectral spaces and locally spatial diamonds. We recall the basic theory of spectral topological spaces. This material is taken from section [Sch17, §2] where most of the proofs can be found.

Definition 1.10. Let S, T be topological spaces, and $f : S \rightarrow T$ a continuous map.

- (1) T is spectral if it is quasicompact, quasiseparated, and it has a basis of open neighborhoods stable under intersection that consists of quasicompact and quasiseparated subsets.
- (2) T is locally spectral if it admits an open cover by spectral spaces.
- (3) f is a spectral map of spectral spaces if S and T are spectral and f is quasicompact.
- (4) f is a spectral map of locally spectral spaces if for every quasicompact open $U \subseteq S$ and quasicompact open $V \subseteq T$ with $f(U) \subseteq V$ $f|_U : U \rightarrow V$ is spectral.

Theorem 1.11. (Hochster) *For a topological space T the following are equivalent:*

- (1) T is spectral.
- (2) T is homeomorphic to the spectrum of a ring.
- (3) T is a projective limit of finite T_0 topological spaces.

Moreover, the category of spectral topological spaces with spectral maps is equivalent to the pro-category of finite T_0 topological spaces.

Given a spectral space T , we say that a subset S is constructible if it lies in the Boolean algebra generated by quasicompact open subsets of T . For a locally spectral space T , a subset S is constructible if for every quasicompact open subset $U \subseteq T$ the subset $S \cap U$ is constructible in U . The patch (or constructible) topology on T is the one in which constructible subsets form a basis for the topology. A spectral space is Hausdorff and profinite for its patch topology and a locally spectral space is locally profinite for the patch topology.

Proposition 1.12. *A continuous map of locally spectral spaces $f : S \rightarrow T$ is spectral if and only if it is continuous for the patch topology.*

Definition 1.13. Let $f : S \rightarrow T$ be a continuous map of topological spaces.

- (1) We say f is generalizing if given $t_1, t_2 \in T$ and $s_1 \in S$ with $f(s_1) = t_1$ and such that t_2 generalizes t_1 , then there exists an element s_2 generalizing s_1 with $f(s_2) = t_2$.
- (2) We say f is specializing if given $t_1, t_2 \in T$ and $s_1 \in S$ with $f(s_1) = t_1$ and such that t_2 specializes from t_1 , then there exists an element s_2 specializing from s_1 with $f(s_2) = t_2$.

For a locally spectral space T we say that a subset is pro-constructible if it is closed for the patch topology, or equivalently if it is an arbitrary intersection of constructible subsets.

Proposition 1.14. ([Sch17, Lemma 2.4]) *Let T be a spectral space and $S \subseteq T$ a pro-constructible subset. The closure \bar{S} of S in T consists of the points that specialize from a point in S .*

Corollary 1.15. *Let $f : S \rightarrow T$ be a spectral map of spectral spaces. If f is specializing then it is also a closed map.*

We warn the reader that the analogue of **Corollary 1.15** for locally spectral spaces does not hold.

Proposition 1.16. ([Sch17, Lemma 2.5]) *Let $f : S \rightarrow T$ be a spectral map of spectral topological spaces. Assume f is surjective and generalizing, then it is a quotient map.*

Definition 1.17. ([Sch17, Definition 11.17]) Let X be a diamond. We say that X is a spatial diamond if it is quasicompact, quasiseparated and $|X|$ has a basis of open neighborhoods of the form $|U|$ where $U \subseteq X$ is a quasicompact open embedding. We say that X is locally spatial if it has an open cover by spatial diamonds.

The topology of spatial diamonds is spectral. Nevertheless, a diamond that has a spectral underlying topological space might not necessarily be spatial since the quasicompactness and quasiseparatedness conditions of **Definition 1.17** are imposed on the topos-theoretic sense.

Proposition 1.18. ([Sch17, Proposition 11.18, Proposition 11.19]) *Let X and Y be locally spatial diamonds and $f : X \rightarrow Y$ a morphism of v -sheaves. The following assertions hold:*

- (1) $|X|$ is a locally spectral topological space.
- (2) Any open subfunctor $U \subseteq X$ is a locally spatial diamond.

- (3) $|X|$ is quasicompact (respectively quasiseparated) as a topological space if and only if X is quasicompact (respectively quasiseparated) as a v -sheaf.
- (4) The topological map $|f|$ is spectral and generalizing. In particular, if $|X|$ is quasicompact and $|f|$ is surjective then by [Proposition 1.16](#) it is also a quotient map.

1.3. Pre-adic spaces as v -sheaves. The theory of diamonds is mainly of “analytic” nature. On the other hand, we wish to consider spaces that are closer to schemes or formal schemes. The category of v -sheaves allows us to consider these three types of spaces at the same time. Recall that to any Huber pair (A, A^+) we can associate a pre-adic space, $\mathrm{Spa}^{\mathrm{ind}}(A, A^+)$, as in [\[SW20, Appendix to Lecture 3\]](#). One then constructs pre-adic spaces by appropriately glueing along rational covers.³ Every pre-adic space X has an underlying topological space, and we can define the open analytic locus $|X|^{\mathrm{an}}$ and the non-analytic locus $|X|^{\mathrm{na}}$ in the naive way. That is, a point $x \in |X|$ is analytic if for every open affinoid $x \in \mathrm{Spa}^{\mathrm{ind}}(A, A^+) \subseteq |X|$ (equivalently one affinoid) x is analytic in $\mathrm{Spa}(A, A^+)$.

Proposition 1.19. *Given a pre-adic space X there is a reduced non-analytic adic space X^{na} and a map $X^{\mathrm{na}} \rightarrow X$ which is final in the category of maps $Y \rightarrow X$ with Y a reduced non-analytic adic space. Moreover, the map $|X^{\mathrm{na}}| \rightarrow |X|^{\mathrm{na}}$ is a homeomorphism.*

Proof. In the affinoid case $\mathrm{Spa}^{\mathrm{ind}}(A, A^+)^{\mathrm{na}} = \mathrm{Spa}(A/A^{\circ\circ} \cdot A, A^+/A^{\circ\circ} \cdot A^+)$. Since $A/(A^{\circ\circ} \cdot A)$ is discrete it is sheafy. Moreover, if (B, B^+) is discrete then $\mathrm{Hom}(\mathrm{Spa}^{\mathrm{ind}}(B, B^+), \mathrm{Spa}^{\mathrm{ind}}(A, A^+))$ is in bijection with maps $(A, A^+) \rightarrow (B, B^+)$. Topological nilpotents map to 0 in B which proves the universal property. The claim of topological spaces is clear. For general pre-adic space X we define X^{na} to have underlying topological space $|X|^{\mathrm{na}}$ and if $V \subseteq |X^{\mathrm{na}}|$ is of the form $U \cap |X|^{\mathrm{na}}$ for $U \subseteq |X|$ open and of the form $U = \mathrm{Spa}^{\mathrm{ind}}(A, A^+)$ we let $\mathcal{O}_{X^{\mathrm{na}}}^{\mathrm{ind}}(V) := \mathcal{O}_X^{\mathrm{ind}}(U)/A^{\circ\circ} \cdot \mathcal{O}_X^{\mathrm{ind}}(U)$. Since the construction $A \mapsto A/A^{\circ\circ}$ is compatible with rational localization $\mathcal{O}_{X^{\mathrm{na}}}^{\mathrm{ind}}(V)$ is well-defined and glues to a sheaf of ind-topological rings on X^{na} . Moreover, locally the ind-topological rings come from a topological ring because $A/A^{\circ\circ}$ is sheafy. This implies X^{na} is an adic space. \square

Recall that to any pre-adic space X over \mathbb{Z}_p one can associate a small v -sheaf X^\diamond over $\mathrm{Spd}(\mathbb{Z}_p)$. This is done by letting $\mathrm{Spd}(\mathbb{Z}_p)(Y) = \{(Y^\sharp, \iota)\}/\cong$ and letting $X^\diamond(Y) = \{(Y^\sharp, \iota, f)\}/\cong$, where $Y^\sharp \in \mathrm{Perfd}$, $\iota : (Y^\sharp)^\flat \rightarrow Y$ is an isomorphism, and $f : Y^\sharp \rightarrow X$ is a morphism of pre-adic spaces.

Proposition 1.20. ([\[SW20, Lemma 18.1.1\]](#)) *For any pre-adic space X over \mathbb{Z}_p (not necessarily analytic), the presheaf X^\diamond is a small v -sheaf.*

From now on, given a Huber pair (A, A^+) we denote $(\mathrm{Spa}(A, A^+)^{\mathrm{ind}})^\diamond$ by $\mathrm{Spd}(A, A^+)$. If (R, R^+) is a Huber pair for which the R^+ “guessed” from R we will abbreviate $\mathrm{Spa}(R, R^+)$ and $\mathrm{Spd}(R, R^+)$ by $\mathrm{Spa}(R)$ and $\mathrm{Spd}(R)$. For example, $\mathrm{Spd}(\mathbb{F}_p)$, $\mathrm{Spd}(\mathbb{Z}_p)$, $\mathrm{Spd}(\mathbb{Q}_p)$. When R is adic the natural guess is $\mathrm{Spa}(R) = \mathrm{Spa}(R, R)$, and when R is an analytic the natural guess $\mathrm{Spa}(R) = \mathrm{Spa}(R, R^\circ)$. Given an I -adic ring R with $I \subseteq R$ finitely generated, we will say that (R, R) is a “formal” Huber pair.

Proposition 1.21. *We collect some facts about \diamond , that are either in the literature or not hard to prove. Let $\mathrm{PreAd}_{\mathbb{Z}_p}$ denote the category of pre-adic spaces over \mathbb{Z}_p and let $X \in \mathrm{PreAd}_{\mathbb{Z}_p}$.*

- (1) If X is perfectoid, then $X^\diamond \cong h_{X^\flat}$ [\[Sch17, Lemma 15.2\]](#).
- (2) There is a surjective map of topological spaces $|X^\diamond| \rightarrow |X|$ [\[SW20, Proposition 18.2.2\]](#).
- (3) If X is analytic, then X^\diamond is a locally spatial diamond and $|X^\diamond| \cong |X|$, [\[Sch17, Lemma 15.6\]](#).
- (4) The functor $\diamond : \mathrm{PreAd}_{\mathbb{Z}_p} \rightarrow \mathrm{Perf}$ commutes with limits and colimits. More precisely, if X_i is a family of pre-adic spaces and $\varinjlim_{i \in I} X_i$ (respectively $\varprojlim_{i \in I} X_i$) is represented by a pre-adic space X then $X^\diamond = \varinjlim_{i \in I} X_i^\diamond$ (respectively $X^\diamond = \varprojlim_{i \in I} X_i^\diamond$).
- (5) The structure map $\mathrm{Spd}(B, B^+) \rightarrow \mathrm{Spd}(\mathbb{Z}_p)$ is separated.
- (6) The map $(X^{\mathrm{na}})^\diamond \rightarrow X^\diamond$ is a closed immersion and $X^\diamond \setminus (X^{\mathrm{na}})^\diamond = (X^{\mathrm{an}})^\diamond$.

2. THE OLIVINE SPECTRUM

As we will see below, the map $|X^\diamond| \rightarrow |X|$ of [item 2](#) in [Proposition 1.21](#) is usually not injective when X has non-analytic points. Although the map is always surjective, it might not be a quotient map in pathological and drastic non-Noetherian situations. To develop a theory of specialization maps for v -sheaves, we need better understanding of $|\mathrm{Spd}(A)|$ when A is an I -adic ring over \mathbb{Z}_p . To tackle this difficulty, we introduce what we call the *olivine spectrum* of a Huber pair. It is a small variation

³One has to do this carefully since sheafiness doesn’t hold in this generality.

of Huber's adic spectrum with a diamond-like twist. Under some mild "finiteness" conditions we prove that the olivine spectrum recovers $|\mathrm{Spd}(B, B^+)|$. For the rest of the section we fix (B, B^+) a Huber pair (not necessarily over \mathbb{Z}_p and not necessarily complete).

2.1. Review, terminology and conventions. We assume familiarity with the construction of Huber's adic spectrum, $\mathrm{Spa}(B, B^+)$, but we review some definitions, some key facts, and we fix some terminology. Let $x \in \mathrm{Spa}(B, B^+)$ and fix a representative $|\cdot|_x : B \rightarrow \Gamma_x \cup \{0\}$.

- (1) The support $\mathbf{supp}(x) \subseteq B$ is the prime ideal of $b \in B$ with $|b|_x = 0$.
- (2) We say x is non-analytic if $\mathbf{supp}(x)$ is open in B , we say it is analytic otherwise.
- (3) Let $H \subseteq \Gamma_x$ be a convex subgroup. We let $|\cdot|_y : B \rightarrow (\Gamma_x/H) \cup \{0\}$ with $|b|_y = |b|_x + H \in \Gamma_x/H$ when $|b|_x \neq 0$ and $|b|_y = 0$ when $|b|_x = 0$. Equivalence classes of valuations constructed this way are called a *vertical generalizations* of x .
- (4) There is a residue field map of complete Huber pairs $\iota_x^* : (B, B^+) \rightarrow (K_x, K_x^+)$, where K_x is either a discrete field or a complete nonarchimedean field. In both cases, K_x^+ is an open and bounded valuation subring of K_x . The induced map $\iota_x : \mathrm{Spa}(K_x, K_x^+) \rightarrow \mathrm{Spa}(B, B^+)$ is a homeomorphism onto the subspace of $\mathrm{Spa}(B, B^+)$ of continuous vertical generalizations of x and satisfies the universal property of maps that factor through this locus.
- (5) Residue fields relate to vertical generalizations as follows. Let K_x° be the subring of power-bounded elements in K_x . If y is a continuous vertical generalizations of x we let $K_y^+ = \{b \in K_x \mid |b|_y \leq 1\}$. This gives a bijection between the set of continuous vertical generalizations of x and valuation rings of K_x with $K_x^\circ \supseteq K_y^+ \supseteq K_x^+$. Moreover, the residue field at y is (K_x, K_y^+) .
- (6) We say x is *trivial* if $\Gamma_x = \{1\}$. In this case, K_x is discrete.
- (7) We say that a valuation is *microbial* if it has a non-trivial rank 1 vertical generalization.
- (8) For technical reasons we take the convention that trivial valuations have rank 0.
- (9) The *characteristic subgroup* of $|\cdot|_x$, denoted by $c\Gamma_x$, is the smallest convex subgroup of Γ_x containing $\gamma = |b|_x$ for all $b \in B$ with $1 \leq \gamma$.
- (10) Given a convex subgroup $H \subseteq \Gamma_x$ containing $c\Gamma_x$, we define $|\cdot|_y : B \rightarrow H \cup \{0\}$ with $|b|_y = |b|_x$ if $|b|_x \in H$ and $|b|_y = 0$ otherwise. Equivalence classes of valuations constructed in this way are called *horizontal specializations* of x .
- (11) Residue fields relate to horizontal specializations as follows. Let K_B be the subring of K_x containing K_x^+ and the image of B in K_x . Consider the induced map $f : \mathrm{Spec}(K_B) \rightarrow \mathrm{Spec}(B)$. Horizontal specializations of x are in bijection with prime ideals of B that are in the image of f . For a convex subgroup H containing $c\Gamma_x$ and inducing y , the associated prime ideal $\mathfrak{p}_y = \{b \in B \mid |b|_x < \gamma \text{ for } \gamma \in H\}$. We sometimes denote $|\cdot|_y$ by $|\cdot|_{x/\mathfrak{p}_y}$.
- (12) Given a topological space T we construct a partial order on T by letting $t_1 \preceq_T t_2$ if $t_1 \in \overline{\{t_2\}}$. We call this partial order the *generalization pattern* of T . We use \preceq_B instead when $T = \mathrm{Spa}(B, B^+)$.
- (13) The generalization pattern of $\mathrm{Spa}(B, B^+)$ is determined by vertical generalizations and horizontal specializations. More precisely, letting $(y, z) \in R$ if z is a vertical generalization of y or if y is horizontal specialization of z . Then \preceq_B is the transitive closure of R .

2.2. Definitions and basic properties.

Definition 2.1. We define a topological space $\mathrm{Spo}(B, B^+)$ which we call the *olivine spectrum* of B .

- (1) Let $\mathrm{Spo}(B, B^+) \subseteq \mathrm{Spa}(B, B^+)^2$ consist of pairs, $x := (|\cdot|_x^h, |\cdot|_x^a)$, such that $|\cdot|_x^a$ has rank 1 or 0 and a vertical generalization of $|\cdot|_x^h$.
- (2) Pick $b_1, b_2 \in B$ and let $U_{b_1 \leq b_2 \neq 0} = \{x \in \mathrm{Spo}(B, B^+) \mid |b_1|_x^h \leq |b_2|_x^h \neq 0\}$, we call such subsets classical localizations.
- (3) Pick $b_1, b_2 \in B$ and let $N_{b_1 \ll b_2} = \{x \in \mathrm{Spo}(B, B^+) \mid |b_1|_x^a < |b_2|_x^a \neq 0\}$, we call such subsets analytic localizations.
- (4) We endow $\mathrm{Spo}(B, B^+)$ with the topology generated by classical and analytic localizations.

We denote by $h : \mathrm{Spo}(B, B^+) \rightarrow \mathrm{Spa}(B, B^+)$ the first coordinate projection, this map is continuous and surjective. Moreover, both Spo and h are functorial.

Definition 2.2. Let $x \in \mathrm{Spo}(B, B^+)$.

- (1) We say that x is *discrete* if $|\cdot|_x^a$ is trivial. We say that a discrete point is *microbial* if $h(x)$ is microbial. We say that a discrete point is *algebraic* if $|\cdot|_x^h$ is trivial.
- (2) We say that x is *d-analytic* if $|\cdot|_x^a$ is non-trivial. Suppose that x is d-analytic, we say that it is *analytic* if $h(x)$ is analytic and we say it is *meromorphic* otherwise.

- (3) We say that x is *bounded* if $|B|_x^a \leq 1$.
(4) We say that x is *formal* if it is bounded and d-analytic.

For $x \in \text{Spo}(B, B^+)$ the set $h^{-1}(h(x))$ has at most one d-analytic point and at most one discrete point. Consequently, $\mathbf{Card}(h^{-1}(h(x))) \in \{1, 2\}$. Moreover, $\mathbf{Card}(h^{-1}(h(x))) = 2$ if and only if $h(x)$ is discrete and $h(x)$ is microbial. The definitions are made so that x is analytic if and only if $h(x)$ is, and in this way we can talk about the analytic locus. Nevertheless, with our terminology, there is no longer a dichotomy since meromorphic points are not analytic but they are also not discrete.

Definition 2.3. Let $x \in \text{Spo}(B, B^+)$, we let $\mathbf{supp}(x) := \mathbf{supp}(h(x))$, this is the *support ideal*. We let $\mathbf{sp}(x) = \{b \in B^+ \mid |b|_x^h < 1\}$, this is the *specialization ideal*. If x is bounded, we let $\mathbf{def}(x) = \{b \in B \mid |b|_x^a < 1\}$, this is the *deformation ideal*.

Notice that x is bounded if and only if $c\Gamma_{x^a} = \{1\}$, this only happens if x is either discrete or formal. If x is bounded, it is discrete whenever $\mathbf{supp}(x) = \mathbf{def}(x)$ and it is formal otherwise. We define the bounded locus, and denote it $\text{Spo}(B, B^+)^\dagger$, as the subset of bounded points. This is a closed subset with complement of $\cup_{b \in B} N_{1 \ll b}$.

Definition 2.4. Let x and y be two points in $\text{Spo}(B, B^+)$.

- (1) y is a *vertical generization* of x (x a *vertical specialization* of y respectively) if $|\cdot|_x^a = |\cdot|_y^a$ and $|\cdot|_y^h$ is a vertical generization of $|\cdot|_x^h$ in $\text{Spa}(B, B^+)$. We abbreviate this as y is v.g. of x (x is v.s. of y respectively).
- (2) y is a *meromorphic generization* of x (x a *meromorphic specialization* of y respectively) if y is meromorphic, x is discrete and $h(x) = h(y)$. We abbreviate this as y is m.g. of x (x is m.s. of y respectively).
- (3) y is a *formal generization* of x (x a *formal specialization* of y respectively) if y is formal, x is discrete $\mathbf{def}(y) = \mathbf{supp}(x)$ and $|\cdot|_x^h = |\cdot|_y^h / \mathbf{def}(y)$. We abbreviate this as y is f.g. of x (x is f.s. of y respectively).

Given $x \in \text{Spo}(B, B^+)$ let $\mathcal{I}^\preceq(x)$ denote the set of generizations of x in $\text{Spo}(B, B^+)$ and let $\mathcal{I}_{\text{ver}}^\preceq(x)$ denote the set of vertical generizations of x . If the context is clear, for a point $y \in \text{Spa}(B, B^+)$ we will also use $\mathcal{I}_{\text{ver}}^\preceq(y)$ to denote the vertical generizations of y in $\text{Spa}(B, B^+)$. Let us make some easy observations and set some convenient notation:

- (1) If x is discrete it has a meromorphic generization (necessarily unique) if and only if x is microbial. We denote this generization by x^{mer} .
- (2) If x is meromorphic it has a unique meromorphic specialization, we denote it by x_{mer} .
- (3) If x is formal it has a unique formal specialization, we denote it by x_{for} . If x is discrete, we let x^{For} denote the set of formal generizations of x .

We recommend the reader to work through the following example confirming all of the claims.

Example 2.5. Let $B = \mathbb{F}_p[[u]]$ endowed with the discrete topology, then $\text{Spa}(B)$ consists of 3 points:

$$\left\{ \eta = |\cdot|_\eta, s = |\cdot|_s, t = |\cdot|_t \right\}$$

Here $|\cdot|_\eta$ is the trivial valuation with residue field $\mathbb{F}_p((u))$, $|\cdot|_s$ is the trivial valuation with residue field \mathbb{F}_p and $|\cdot|_t$ is the (u) -adic valuation on $\mathbb{F}_p[[u]]$ with residue affinoid field $(\mathbb{F}_p((u)), \mathbb{F}_p[[u]])$. All valuations have rank 1 or 0. The only non-trivial vertical generization in $\text{Spa}(B)$ goes from $|\cdot|_t$ to $|\cdot|_\eta$.

On the other hand, $\text{Spo}(B)$ has 4 points:

$$\left\{ \eta := (|\cdot|_\eta, |\cdot|_\eta), s := (|\cdot|_s, |\cdot|_s), t^f := (|\cdot|_t, |\cdot|_t), t^d := (|\cdot|_t, |\cdot|_\eta) \right\}$$

Now, $\{\eta\} = U_{1 \leq u \neq 0}$, $\{\eta, t^d, t^f\} = U_{0 \leq u \neq 0}$, $\{t^f\} = N_{u^2 \ll u}$ and $\{t^f, s\} = N_{u \ll 1}$, and these are the only proper open subsets. Here s , η and t^d are discrete. Moreover, t^d is microbial, and t^f is both a meromorphic and formal d-analytic point. The generization pattern is: η is a vertical generization of t^d , t^d is the meromorphic specialization of t^f , and s is the formal specialization of t^f . We have $\text{Spo}(\mathbb{F}_p[[u]])^\dagger = \text{Spo}(\mathbb{F}_p[[u]])$.

The following shows that the v.g., f.s. and m.s. determine the generization pattern in $\text{Spo}(B, B^+)$.

Proposition 2.6. Let $x \in \text{Spo}(B, B^+)$.

- (1) If x is d-analytic then $\mathcal{I}^\preceq(x) = \mathcal{I}_{\text{ver}}^\preceq(x)$.
- (2) If x is discrete then $\mathcal{I}^\preceq(x) = \mathcal{I}_{\text{ver}}^\preceq(x) \cup \mathcal{I}_{\text{ver}}^\preceq(x^{\text{mer}}) \cup (\cup_{z \in \mathcal{I}_{\text{ver}}^\preceq(x)} z^{\text{For}})$.

Proof. We prove the right to left side. Let $y \in \mathcal{I}_{\text{ver}}^{\leq}(x) \cup \mathcal{I}_{\text{ver}}^{\leq}(x^{\text{mer}}) \cup (\bigcup_{z \in \mathcal{I}_{\text{ver}}^{\leq}(x)} z^{\text{For}})$ if x is discrete and let $y \in \mathcal{I}_{\text{ver}}^{\leq}(x)$ otherwise. Since h is continuous, y is in every classical localization of x , so it suffices to check analytic localizations. Suppose $x \in N_{b_1 \ll b_2}$, if y is a v.g. of x then $|\cdot|_y^a = |\cdot|_x^a$ so $y \in N_{b_1 \ll b_2}$. If x is discrete, then $|b_1|_x^a = 0$ and $|b_2|_x^a = 1$, this implies $|b_1|_{x^{\text{mer}}}^a = 0$ and that $|b_2|_{x^{\text{mer}}}^a \neq 0$, so $x^{\text{mer}} \in N_{b_1 \ll b_2}$ in case x^{mer} exists. Moreover, if $y \in x^{\text{For}}$ then $\mathbf{def}(y) = \mathbf{supp}(x)$ so that $|b_1|_y^a < 1$, $|b_2|_y^a = 1$, and $x^{\text{For}} \in N_{b_1 \ll b_2}$.

We prove the left to right side, let $y \in \mathcal{I}^{\leq}(x)$. With classical localizations we deduce $\mathbf{supp}(y) \subseteq \mathbf{supp}(x)$, and if x is d-analytic we claim that $\mathbf{supp}(y) = \mathbf{supp}(x)$. Indeed, let $b \in B$ such that $|b|_x^a \notin \{0, 1\}$, and let $b_1 \in \mathbf{supp}(x)$. If $|b|_x^a < 1$ then $|b|_y^a < 1$, which implies that y is d-analytic. Additionally, the inequalities $|b_1|_y^a < |b^n|_y^a$ hold for all n since $x \in N_{b_1 \ll b^n}$. Similarly, if $1 < |b|_x^a$ then $1 < |b|_y^a$ and $|b_1 \cdot b^n|_y^a < |b|_y^a$ hold instead. In both cases, the archimedean property of rank 1 valuations prove $b_1 \in \mathbf{supp}(y)$. Since the only generalizations of $h(x)$ in $\text{Spa}(B, B^+)$ with the same support are v.g. we get $h(y) \in \mathcal{I}_{\text{ver}}^{\leq}(h(x))$ and $y \in \mathcal{I}_{\text{ver}}^{\leq}(x)$ for x d-analytic.

Suppose x is discrete, if $\mathbf{supp}(x) = \mathbf{supp}(y)$ we can reason as above. Let $b \in \mathbf{supp}(x) \setminus \mathbf{supp}(y)$. Since $x \in N_{b \ll 1}$ we have $0 < |b|_y^a < 1$ and that y is d-analytic. For all $b_1 \in B$ $|b \cdot b_1|_y^a < 1$ holds and y is formal with $\mathbf{supp}(x) \subseteq \mathbf{def}(y)$. If $b_2 \notin \mathbf{supp}(x)$ then $x \in U_{b \leq b_2^n \neq 0}$ for all n , giving $|b_2|_y^a = 1$ and $\mathbf{def}(y) = \mathbf{supp}(x)$. Let $z = y_{\text{for}}$ then $\mathbf{supp}(z) = \mathbf{supp}(x)$ and it follows from the construction of horizontal specializations that $z \in \mathcal{I}^{\leq}(x)$. As above, $h(z)$ is a v.g. of $h(x)$, and since both z and x are discrete then z is a v.g. of x . In other words, $z \in \mathcal{I}_{\text{ver}}^{\leq}(x)$ and $y \in z^{\text{For}}$. \square

The olivine spectrum is compatible with completion and rational localization.

Proposition 2.7. *If (\hat{B}, \hat{B}^+) denotes the completion of (B, B^+) , then $\text{Spo}(\hat{B}, \hat{B}^+) = \text{Spo}(B, B^+)$.*

Proof. Since $\text{Spa}(\hat{B}, \hat{B}^+) = \text{Spa}(B, B^+)$ the map $\text{Spo}(\hat{B}, \hat{B}^+) \rightarrow \text{Spo}(B, B^+)$ is bijective and classical localizations of $\text{Spa}(\hat{B}, \hat{B}^+)$ are open in $\text{Spa}(B, B^+)$. It suffices to prove $N_{g \ll f}$ is open in $\text{Spo}(B, B^+)$ for $f, g \in \hat{B}$. Let $x \in N_{g \ll f}$ and $f_x \in B$ with $|f_x|_x^h = |f|_x^h$. We have $U_{f_x \leq f \neq 0} \cap U_{f \leq f_x \neq 0} \cap N_{g \ll f} = U_{f_x \leq f \neq 0} \cap U_{f \leq f_x \neq 0} \cap N_{g \ll f_x}$, so we may assume $f \in B$. Take a ring of definition $B_0 \subseteq B$ and an ideal of definition $I \subseteq B_0$ with $|i_k|_x^h \leq |f|_x^h$ for a finite set of generators $\{i_1 \dots i_m\} \subseteq I$. Let $g_x \in B$ such that $g - g_x \in I^2 \cdot \hat{B}_0$. Then $(\bigcap_i U_{i \leq f \neq 0}) \cap N_{g_x \ll f} = (\bigcap_i U_{i \leq f \neq 0}) \cap N_{g \ll f}$ so the left hand side is open in $\text{Spo}(B, B^+)$. \square

Proposition 2.8. *Let $s, t_1, \dots, t_n \in B$ defining a rational localization $\text{Spa}(R, R^+) := U(\frac{t_1, \dots, t_n}{s}) \subseteq \text{Spa}(B, B^+)$. Then $\text{Spo}(R, R^+) \rightarrow \text{Spo}(B, B^+)$ is a homeomorphism onto $h^{-1}(U(\frac{t_1, \dots, t_n}{s}))$.*

Proof. It suffices to check $N_{r_1 \ll r_2} \subseteq \text{Spo}(R, R^+)$ is open in $\text{Spo}(B, B^+)$ for $r_1, r_2 \in R$. By **Proposition 2.7** and the construction of rational localizations we may assume $r_1, r_2 \in B[\frac{1}{s}] \subseteq R$. Write $r_1 = \frac{b_1}{s^{n_1}}$, $r_2 = \frac{b_2}{s^{n_2}}$ and let $m = n_1 - n_2$. Then $N_{r_1 \ll r_2} = N_{b_1 \ll b_2 \cdot s^m} \cap \text{Spo}(R, R^+)$ when $m \geq 0$ and $N_{r_1 \ll r_2} = N_{b_1 \cdot s^m \ll b_2} \cap \text{Spo}(R, R^+)$ otherwise. \square

The following example is key to the prove **Lemma 2.30** and **Theorem 2**. We encourage the reader to workout this example carefully. Recalling **Example 2.5** under this light might be helpful.

Example 2.9. Suppose $B^+ \subseteq B$ are valuation rings with $\text{Frac}(B) = \text{Frac}(B^+)$ both with the discrete topology. We describe $\text{Spo}(B, B^+)$ in two steps: first observe that $\text{Spo}(B, B^+) \subseteq \text{Spo}(B^+, B^+)$ and that it acquires the subspace topology. Indeed, this follows from **Proposition 2.8** and the identification $\text{Spo}(B, B^+) = \bigcap_{b \in B \setminus B^+} U_{0 \leq b \neq 0} \subseteq \text{Spo}(B^+, B^+)$.

Second, we describe $\text{Spo}(B^+, B^+)$ explicitly using that all points are bounded and admit a deformation ideal. Since B^+ is a valuation ring elements of $\text{Spa}(B^+, B^+)$ are determined by their support and specialization ideals. Consider the map $\text{Spo}(B^+, B^+) \rightarrow \text{Spec}(B^+)^3$ with $q \mapsto (\mathbf{supp}(q), \mathbf{def}(q), \mathbf{sp}(q))$. The following hold:

- The map is injective with image those triples (q_1, q_2, q_3) with $q_1 \subseteq q_2 \subseteq q_3$, and such that $[q_1, q_2] = \{q_1, q_2\}$ where the left term is an interval for the order defined by containment.
- A triple $q = (q_1, q_2, q_3)$ is meromorphic if and only if $q_1 \neq q_2$. In this case, there is $b \in B^+$ with $b \in q_2 \setminus q_1$ and $q_2 = \sqrt{(b)}$.
- $q \in \mathcal{I}_{\text{ver}}^{\leq}(r)$ if $q_1 = r_1$, $q_2 = r_2$ and $q_3 \subseteq r_3$.
- $q = r^{\text{mer}}$ if $r_1 = q_1 = r_2$, $q_2 \neq q_1$ and $r_3 = q_3$.
- F.g. are unique and r is the f.s. of q if $q_1 \neq q_2$, $r_1 = q_2 = r_2$ and $r_3 = q_3$.

We describe the open subsets.

- If $\frac{f}{g} \in B^+$ then $U_{f \leq g \neq 0} = U_{0 \leq g \neq 0}$ and consists of triples (q_1, q_2, q_3) with $g \notin q_1$.
- If $\frac{g}{f} \in B^+$ we can let $b = \frac{g}{f}$ then $U_{f \leq g \neq 0} = U_{1 \leq b \neq 0}$ and it consists of triples with $b \notin q_3$.
- The families $\{U_{1 \leq b \neq 0}\}_{b \in B^+}$ and $\{U_{0 \leq g \neq 0}\}_{g \in B^+}$ are nested. In particular, finite intersections of classical localizations have the form $U_{1 \leq b \neq 0} \cap U_{0 \leq g \neq 0}$ for some $b, g \in B^+$.
- When $n = \frac{f}{g} \in B^+$ then $N_{g \ll f}$ is empty and $N_{f \ll g} = U_{0 \leq g \neq 0} \cap N_{n \ll 1}$.
- The set $N_{n \ll 1}$ consists of the triples $q = (q_1, q_2, q_3)$ such that $n \in q_2$.
- The family of sets $\{N_{n \ll 1}\}_{n \in B^+}$ is nested.

In summary, if $x \in U \subseteq \text{Spo}(B^+, B^+)$ for U an open subset there are elements $g, b, n \in B^+$ with $x \in U_{0 \leq g \neq 0} \cap U_{1 \leq b \neq 0} \cap N_{n \ll 1} \subseteq U$. Moreover, elements of $U_{0 \leq g \neq 0} \cap U_{1 \leq b \neq 0} \cap N_{n \ll 1} \subseteq U$ are explicitly described by the constraints: $g \notin q_1$, $b \notin q_3$ and $n \in q_2$.

2.3. Olivine Huber pairs. For the rest of the section (B, B^+) denotes a complete Huber pair over \mathbb{Z}_p .

Proposition 2.10. *If R is a Tate Huber pair, then $h : \text{Spo}(R, R^+) \rightarrow \text{Spa}(R, R^+)$ is a homeomorphism.*

Proof. Since (R, R^+) is Tate, $\text{Spa}(R, R^+)$ has no trivial continuous valuations and h is injective. If x^a is the maximal generization of x in $\text{Spa}(R, R^+)$ then $h^{-1}(x) = \{(x, x^a)\}$. It suffices to prove that $h(N_{r_1 \ll r_2})$ is open. But if $\varpi \in R$ is a topologically nilpotent unit, then $h(N_{r_1 \ll r_2}) = \bigcup_{0 < n} \{z \in \text{Spa}(R, R^+) \mid |r_1^n|_z \leq |r_2^n \varpi|_z \neq 0\}$. \square

We define a canonical map $\pi : |\text{Spd}(B, B^+)| \rightarrow \text{Spo}(B, B^+)$.⁴ Let $[x] \in |\text{Spd}(B, B^+)|$ be given by $x : \text{Spa}(C_x, C_x^+) \rightarrow \text{Spd}(B, B^+)$, and let $s \in \text{Spa}(C_x, C_x^+)$ be the closed point. The map is given by an untilt C_x^\sharp and a map $f_x^* : (B, B^+) \rightarrow (C_x^\sharp, C_x^{\sharp,+})$. We define $\pi([x])$ to be $\text{Spo}(f_x)(b^{-1}(s))$ with b being the tilting homeomorphism. This doesn't depend of the representative picked.

Proposition 2.11. *The map $\pi : |\text{Spd}(B, B^+)| \rightarrow \text{Spo}(B, B^+)$ defined above is continuous and bijective.*

Proof. Continuity follows from **Definition 1.7** and **Proposition 2.10**. For injectivity, take points $y_1, y_2 : \text{Spa}(C_i, C_i^+) \rightarrow \text{Spd}(B, B^+)$ with $\pi(y_1) = \pi(y_2) =: x$. Let $(K_{h(x)}, K_{h(x)}^+)$ be the residue field $\text{Spa}(B, B^+)$. The maps $(B, B^+) \rightarrow (C_i^\sharp, C_i^{\sharp,+})$ factor through $(B, B^+) \rightarrow (K_{h(x)}, K_{h(x)}^+)$. Let s_i be the closed point of $\text{Spa}(C_i, C_i^+)$, we show that the s_i define the same point. We split our analysis in three cases.

Case 1: Suppose that x is analytic. In this case, s_1 and s_2 map to $h(\pi(x))$ in $\text{Spa}(B, B^+)^{\text{an}}$. This case follows from the bijectivity of $|X^\diamond| \rightarrow |X|$ for analytic pre-adic spaces (**Proposition 1.21**).

Case 2: Suppose that x is meromorphic, then $h(x)$ is non-analytic in $\text{Spa}(B, B^+)$. Let $K_{h(x)}^\circ := \{k \in K_{h(x)} \mid |k|_x^a \leq 1\}$ since $|\cdot|_x^a$ is non-trivial $K_{h(x)}^\circ \neq K_{h(x)}$. Choose $b \in B$ with $0 < |b|_x^a < 1$ or $|b|_x^a > 1$. The subspace topology of $(K_{h(x)}^\circ) \subseteq_{y_i^*} O_{C_i^\sharp}$ is either the (b) -adic topology or the $(\frac{1}{b})$ -adic topology. After taking completion we get a commutative diagram:

$$\begin{array}{ccc}
& \text{Spa}(C_1, C_1^+) & \\
& \downarrow p'_1 & \searrow y_1 \\
\text{Spa}(C_2, C_2^+) & \xrightarrow{p'_2} \text{Spd}(\hat{K}_{h(x)}, \hat{K}_{h(x)}^+) & \xrightarrow{\iota_x} \text{Spd}(K_{h(x)}, K_{h(x)}^+) \\
& \searrow y_2 & \nearrow
\end{array}$$

Now, $p'_1(s_1) = p'_2(s_2)$ in $\text{Spd}(\hat{K}_{h(x)}, \hat{K}_{h(x)}^+)$. Since $\text{Spa}(\hat{K}_{h(x)}, \hat{K}_{h(x)}^+)$ is analytic we may conclude as in the first case.

Case 3: Suppose that x is discrete, in this case $h(x)$ is non-analytic in $\text{Spa}(B, B^+)$ and $(K_{h(x)}, K_{h(x)}^+)$ has the discrete topology. Since $|\cdot|_x^a$ is trivial, $y_i^*(K_{h(x)}) \subseteq O_{C_i^\sharp}^{\sharp, \times}$. After choosing pseudo-uniformizers $\varpi_i \in O_{C_i^\sharp}$ we may extend the y_i to continuous adic maps of topological rings $p_i'^* : K_{h(x)}[[t]] \rightarrow O_{C_i^\sharp}$ where $K_{h(x)}[[t]]$ has the (t) -adic topology. These induce the following commutative diagram:

⁴The topological considerations in what follows can be done purely in the context of adic spaces without the theory of perfectoid spaces. To do this one substitutes $|\text{Spd}(B, B^+)|$ by $\text{Spo}(B, B^+)$ ' where this second space has $\text{Spo}(B, B^+)$ as underlying set, but has the strongest topology making maps coming from Tate Huber pairs continuous

$$\begin{array}{ccc}
& \text{Spa}(C_1, C_1^+) & \\
& \downarrow p'_1 & \searrow y_1 \\
\text{Spa}(C_2, C_2^+) \xrightarrow{p'_2} & \text{Spd}(K_{h(x)}((t)), K_{h(x)}^+ + t \cdot K_{h(x)}[[t]]) & \\
& \searrow \iota_x & \\
& & \text{Spd}(K_{h(x)}, K_{h(x)}^+) \\
& \nearrow y_2 & \\
& &
\end{array}$$

Again, $p'_1(s_1) = p'_2(s_2)$ in $\text{Spd}(K_{h(x)}((t)), K_{h(x)}^+ + t \cdot K_{h(x)}[[t]])$ which is also analytic.

The case by case study given above also shows that π is surjective. Indeed, we can take a completed algebraic closures of $K_{h(x)}$ ($\hat{K}_{h(x)}$, or $K_{h(x)}((t)$ respectively) when x is analytic (meromorphic or discrete respectively). \square

Definition 2.12. Whenever x is d-analytic we let (K_x, K_x^+) denote $(\hat{K}_{h(x)}, \hat{K}_{h(x)}^+)$, and if x is discrete we let (K_x, K_x^+) denote $(K_{h(x)}((t)), K_{h(x)}^+ + t \cdot K_{h(x)}[[t]])$ as in the proof of [Proposition 2.11](#). In both cases we call (K_x, K_x^+) the *pseudo-residue field* at x .

Remark 2.13. The pseudo-residue field map $\text{Spo}(K_x, K_x^+) \rightarrow \text{Spo}(B, B^+)$ is a homeomorphism onto its image. The functor $\text{Spd}(K_x, K_x^+) \rightarrow \text{Spd}(B, B^+)$ surjects onto the subsheaf of $\text{Spd}(B, B^+)$ consisting of maps that factor through $\mathcal{I}_{\text{ver}}^{\leq}(x)$, but when x is discrete the map $\text{Spd}(K_x, K_x^+) \rightarrow \text{Spd}(B, B^+)$ is not injective. Actually, when x is discrete and $|\cdot|_x^h$ is non-trivial the subsheaf of points that factor through $\mathcal{I}_{\text{ver}}^{\leq}(x)$ is not representable by an adic space.

Corollary 2.14. For any map of Huber pairs $m^* : (B_1, B_1^+) \rightarrow (B_2, B_2^+)$ the map $\text{Spo}(m)$ is compatible with v.g.. More precisely, if $x \in \text{Spo}(B_2, B_2^+)$, $y = \text{Spo}(m)(x)$ and y' is a v.g. of y then there exist x' , a v.g. of x , with $\text{Spo}(m)(x') = y'$.

Proof. Given $x \in \text{Spo}(B_2, B_2^+)$ and $y \in \text{Spo}(B_1, B_1^+)$ as in the statement we may, after making some choices if necessary, construct the following commutative diagram of pseudo-residue fields:

$$\begin{array}{ccc}
\text{Spd}(K_x, K_x^+) & \longrightarrow & \text{Spd}(K_y, K_y^+) \\
\downarrow & & \downarrow \\
\text{Spd}(B_2, B_2^+) & \longrightarrow & \text{Spd}(B_1, B_1^+)
\end{array}$$

Since the map $\text{Spd}(K_x, K_x^+) \rightarrow \text{Spd}(K_y, K_y^+)$ is a map of locally spatial diamonds it is generalizing and consequently surjective. But $|\text{Spd}(K_x, K_x^+)| = \mathcal{I}_{\text{ver}}^{\leq}(x)$ and analogously for y . \square

Lemma 2.15. The topological spaces $\text{Spo}(B, B^+)$ and $|\text{Spd}(B, B^+)|$ have the same generization pattern.

Proof. Since $|\text{Spd}(B, B^+)| \rightarrow \text{Spo}(B, B^+)$ is continuous the generization pattern of $|\text{Spd}(B, B^+)|$ is smaller than that of $\text{Spo}(B, B^+)$, it suffices by [Proposition 2.6](#) to prove that formal, meromorphic and vertical specializations are specializations in $|\text{Spd}(B, B^+)|$. For $x \in \text{Spo}(B, B^+)$ the pseudo-residue field map $\iota_x : \text{Spd}(K_x, K_x^+) \rightarrow \text{Spd}(B, B^+)$ is a bijection onto $\mathcal{I}_{\text{ver}}^{\leq}(x)$ so v.s. are specializations in $\text{Spd}(B, B^+)$. Let $x \in \text{Spo}(B, B^+)$ be d-analytic and let b such that $|b|_x^a \notin \{0, 1\}$. Let $p : \text{Spa}(C, C^+) \rightarrow \text{Spa}(B, B^+)$ be a geometric point mapping to x and let $\varpi \in C^{\circ\circ}$ be either $p^*(b)$ or $\frac{1}{p^*(b)}$. To this choice we will associate two product of points as follows. Let $R^+ = \prod_{i=1}^{\infty} C^+$, let $\varpi_0 = (\varpi^{\frac{1}{n}})_{n=1}^{\infty}$ and $\varpi_{\infty} = (\varpi^n)_{n=1}^{\infty}$. Let R_0^+ (R_{∞}^+ respectively) be R^+ endowed with the ϖ_0 -topology (ϖ_{∞} -topology respectively), and let $R_0 = R_0^+[\frac{1}{\varpi_0}]$ ($R_{\infty} = R_{\infty}^+[\frac{1}{\varpi_{\infty}}]$ respectively). We have diagonal maps of rings $C^+ \rightarrow R_{\infty}^+$ and $C \rightarrow R_{\infty}$, but we warn the reader that these maps are not continuous. On the other hand, the map $C^+ \rightarrow R_0^+$ is continuous but ϖ is not invertible in R_0 so the map does not extend to a map $C \rightarrow R_0$.

Suppose that x is meromorphic. The diagonal map $f : B \rightarrow K_{h(x)} \rightarrow R_{\infty}$ becomes continuous giving a map $\text{Spa}(R_{\infty}, R_{\infty}^+) \rightarrow \text{Spa}(B, B^+)$. The space $\pi_0(|\text{Spa}(R_{\infty}, R_{\infty}^+)|)$ is the Stone-Ćech compactification of \mathbb{N} whose elements are ultrafilters of \mathbb{N} . Principal ultrafilters $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$ define inclusions $\iota_n : \text{Spa}(C, C^+) \rightarrow \text{Spa}(R_{\infty}, R_{\infty}^+)$ that correspond to the n th-projection in the coordinate rings. The closed point of a principal connected component maps to x under $\text{Spo}(f)$. We claim that the closed point of a non-principal connected component maps to x_{mer} . It suffices to construct a commutative diagram as below:

$$\begin{array}{ccccc}
\mathrm{Spa}(C_{\mathcal{U}}, C_{\mathcal{U}}^+) & \longrightarrow & \mathrm{Spa}(K_{x_{\mathrm{mer}}}, K_{x_{\mathrm{mer}}}^+) & & \\
\downarrow & & \downarrow & & \\
\mathrm{Spa}(R_{\infty}, R_{\infty}^+) & \longrightarrow & \mathrm{Spa}(K_{h(x)}, K_{h(x)}^+) & \longrightarrow & \mathrm{Spa}(B, B^+)
\end{array}$$

We claim that the natural map $K_{h(x)} \rightarrow C_{\mathcal{U}}$ maps to $O_{C_{\mathcal{U}}}$. It suffices to prove $\varpi_{\infty} \cdot K_{h(x)} \subseteq O_{C_{\mathcal{U}}}$, and since $K_{h(x)} = K_{h(x)}^+[b, \frac{1}{b}]$ it suffices to prove that $\frac{\varpi_{\infty}}{\varpi_0^n} \in O_{C_{\mathcal{U}}}$ for $n \in \mathbb{N}$. Clearly $\frac{\varpi_{\infty}}{\varpi_0^n} \in \prod_{i=n+1}^{\infty} O_C$ and since the ultrafilter is non-principal complements of finite sets are in \mathcal{U} , which proves the claim.

By letting t map to ϖ_{∞} we get a map $K_{h(x)}(\!(t)\!) \rightarrow C_{\mathcal{U}}$, the intersection of $K_{h(x)}(\![t]\!)$ with $C_{\mathcal{U}}^+$ in $O_{C_{\mathcal{U}}}$ is $K_{h(x)}^+ + t \cdot K_{h(x)}(\![t]\!) = K_{x_{\mathrm{mer}}}^+$ which gives our factorization. The set of closed points contained in a principal component are dense within the set of closed points of $|\mathrm{Spa}(R_{\infty}, R_{\infty}^+)|$. This gives that m.s. in $\mathrm{Spo}(B, B^+)$ are specializations in $|\mathrm{Spd}(B, B^+)|$.

Suppose that x is formal. Since $|B|_x^a \leq 1$ the map $(B, B^+) \rightarrow (C, C^+)$ factors through (O_C, C^+) and $\mathrm{def}(x) = B \cap C^{\circ\circ}$. This allows us to define a map $\mathrm{Spa}(R_0, R_0^+) \rightarrow \mathrm{Spa}(B, B^+)$. As above, we prove that principal components of $\pi_0(\mathrm{Spa}(R_0, R_0^+))$ map to x in $\mathrm{Spo}(B, B^+)$ while the non-principal ones map to x_{for} , which proves that f.s. are specializations. Let $k = O_C/C^{\circ\circ}$ and $k^+ = C^+/C^{\circ\circ}$, it suffices to prove that $(O_C, C^+) \rightarrow (C_{\mathcal{U}}, C_{\mathcal{U}}^+)$ factors as:

$$(O_C, C^+) \rightarrow (k, k^+) \rightarrow (k(\!(t)\!), k^+ + t \cdot k(\![t]\!)) \rightarrow (C_{\mathcal{U}}, C_{\mathcal{U}}^+)$$

Now, $\frac{\varpi_{\infty}}{\varpi_0^n} \in \prod_{i=n+1}^{\infty} O_C$ which implies that $|\varpi|_{\mathcal{U}} \leq |\varpi_0^n|_{\mathcal{U}}$. Since ϖ_0 is a pseudo-uniformizer in $C_{\mathcal{U}}$ this implies $|\varpi|_{\mathcal{U}} = 0$. Clearly $k \subseteq O_{C_{\mathcal{U}}}$ and we may send t to ϖ_0 to construct our factorization. \square

Proposition 2.16. *Let (B, B) be a formal Huber pair then $|\mathrm{Spd}(B)| \rightarrow \mathrm{Spo}(B, B)$ is a homeomorphism.*

Proof. By Proposition 2.11 the map is a continuous bijection. Let $Y = \mathrm{Spa}(B(\![t]\!))^{t \neq 0}$ and recall that $|Y| = |Y^{\diamond}|$ since this is an analytic pre-adic space. Let U be open in $|\mathrm{Spd}(B)|$, let $x \in U$ and let $y \in Y$ mapping to x . We construct a neighborhood of x in U open in $\mathrm{Spo}(B, B)$. Let $f : (B, B) \rightarrow (B(\![t]\!), B(\![t]\!))$ be the canonical map. For $U_{b_1 \leq b_2 \neq 0}$ or $N_{b_1 \ll b_2}$ containing x we choose quasicompact neighborhoods of y in $\mathrm{Spa}(B(\![t]\!))$, that we denote $U_{b_1, b_2, y}$ and $N_{b_1, b_2, y}$, whose image in $\mathrm{Spo}(B, B)$ are contained in $U_{b_1 \leq b_2 \neq 0}$ and $N_{b_1 \ll b_2}$ respectively. For $U_{b_1 \leq b_2 \neq 0}$ pick a finite set $S \subseteq B$ and $n \in \mathbb{N}$ such that $|s|_y \leq |b_2|_y$ for $s \in S$, that $|t^n|_y \leq |b_2|_y$, and that the ideal generated by S is open in B . We let $U_{b_1, b_2, y} = U(\frac{S, t^n, b_1}{b_2}) \subseteq \mathrm{Spa}(B(\![t]\!))$. Rational localizations are quasicompact and clearly $\mathrm{Spo}(f)(h^{-1}(U_{b_1, b_2, y})) \subseteq U_{b_1 \leq b_2 \neq 0}$. For $N_{b_1 \ll b_2}$ pick a finite set S and $n_1, n_2 \in \mathbb{N}$, such that $|b_1^{n_1}|_y \leq |b_2^{n_1} \cdot t|_y$, that $|s|_y \leq |b_2^{n_1} \cdot t|_y$ for $s \in S$, that $|t^{n_2}|_y \leq |t \cdot b_2^{n_1}|_y$ and that S generates an open ideal in B . We let $N_{b_1, b_2, y} = U(\frac{S, t^{n_2}, b_1^{n_1}}{b_2^{n_1} \cdot t})$. Since t is topologically nilpotent in $B(\![t]\!)$, if $z \in \mathrm{Spa}(B(\![t]\!))$ then $|t|_z < 1$ and $\mathrm{Spo}(f)(h^{-1}(N_{b_1, b_2, y})) \subseteq N_{b_1 \ll b_2}$. Notice that $N_{b_1, b_2, y} \subseteq \mathrm{Spa}(B(\![t]\!))^{t \neq 0}$.

Let $X = (\bigcap N_{b_1, b_2, y}) \cap (\bigcap U_{b_1, b_2, y})$, then $\mathrm{Spo}(f)(X) \subseteq \mathcal{I}^{\leq}(x)$ and by Lemma 2.15, also $\mathrm{Spo}(f)(X) \subseteq U$. Now, $\mathrm{Spo}(f)^{-1}(U)$ is open in $\mathrm{Spa}(B(\![t]\!))^{t \neq 0}$ and the families, $\{U_{b_1, b_2, y} \cap N_{0, 1, y}\}$ and $\{N_{b_1, b_2, y}\}$, consist of quasicompact open subsets of $\mathrm{Spa}(B(\![t]\!))^{t \neq 0}$. A compactness argument in the patch topology of $\mathrm{Spa}(B(\![t]\!))$ proves that a finite intersection is contained in $\mathrm{Spo}(f)^{-1}(U)$. We prove that the image under $\mathrm{Spo}(f)$ of such a finite intersection is open in $\mathrm{Spo}(B, B)$. More generally, let $Z = \bigcap_{i=1}^n V_i$ with V_i of the form $V_{b_{i,1}, b_{i,2}} := \{z \in \mathrm{Spa}(B(\![t]\!))^{t \neq 0} \mid |b_{i,1}|_z \leq |b_{i,2}|_z \neq 0\}$ where $b_{i,1} \in B \cup \{t^n\}_{n \in \mathbb{N}}$ and $b_{i,2} \in B \cup t \cdot B$, we claim that $\mathrm{Spo}(f)(Z)$ is open in $\mathrm{Spo}(B, B)$. If $b_{i,1}, b_{i,2} \in B$ then $V_{b_{i,1}, b_{i,2}} = \mathrm{Spo}(f)^{-1}(U_{b_{i,1} \leq b_{i,2} \neq 0})$ and for Z as above we have $\mathrm{Spo}(f)(Z \cap V_{b_{i,1}, b_{i,2}}) = \mathrm{Spo}(f)(Z) \cap U_{b_{i,1} \leq b_{i,2} \neq 0}$, so we can reduce to the case where each $V_i = V_{b_{i,1}, b_{i,2}}$ satisfy that either $b_{i,1} \in \{t^n\}_{n \in \mathbb{N}}$ or $b_{i,2} = b_2 \cdot t$. Let $T_Z^n \subseteq B$ with $b \in T_Z^n$ if either $b_{i,1} = t^n$ and $b = b_{i,2}$ or $b_{i,1} = t^{n+1}$ and $b_{i,2} = b \cdot t$ for some i . Let $T_Z^{\leq} \subseteq B \times B$ with $(b_1, b_2) \in T_Z^{\leq}$ if $(b_1, b_2) = (b_{i,1}, b_{i,2})$ for some i , and let T_Z^- and T_Z^+ denote the image of T_Z^{\leq} under the first and second projection maps. We prove that $\mathrm{Spo}(f)(Z)$ is the intersection of all the sets of the form $U_{b_1 \leq b_2 \neq 0}$ where $(b_1, b_2) \in T_Z^{\leq}$ and $b_3 \in T_Z^n$ and all the sets of the form $N_{b_1 \ll b_2}$, with $(b_1, b_2) \in T_Z^{\leq}$, which proves $\mathrm{Spo}(f)(Z)$ is open.

It is not hard to see $\mathrm{Spo}(f)(Z)$ is contained in this intersection. To prove the converse, let w be in the intersection, we construct a lift in Z . Pick a point $q : \mathrm{Spa}(C, C^+) \rightarrow \mathrm{Spa}(B)$ over w , the choice of $\varpi \in C^{\circ\circ, \times}$ defines a lift of q to $\mathrm{Spa}(C, C^+) \rightarrow \mathrm{Spa}(B(\![t]\!))^{t \neq 0}$. If w is discrete then $|b_1|_w^a = 0$ for every $b_1 \in T_Z^-$ and $|b_2|_w^a = |b_3|_w^a = 1$ for every $b_2 \in T_Z^+$ and $b_3 \in T_Z^n$. In this case, any choice of ϖ defines a lift landing inside of Z . If w is d-analytic ϖ must be chosen more carefully. Since C is algebraically closed we may choose n th-roots of (b_3) for all $b_3 \in T_Z^n$. For a lift of q to land in Z , ϖ must satisfy the following:

$|\varpi|_q \leq |(b_3)^{\frac{1}{n}}|_q$ for all $b_3 \in T_Z^n$ and $\frac{|(b_1)|_q}{|(b_2)|_q} \leq |\varpi|_q$ for all $(b_1, b_2) \in T_Z^{\ll}$. We let m be the smallest of the values in Γ_q of the form $|b_3^{\frac{1}{n}}|_q$ with $b_3 \in T_Z^n$ and we let M be the largest of the values of the form $\frac{|b_1|_q}{|b_2|_q}$ with $(b_1, b_2) \in T_Z^{\ll}$. Since $w \in U_{b_1^n \leq b_2^n \cdot b_3 \neq 0}$ we have $M \leq m$. Since $w \in N_{b_1 \ll b_2}$ for all pairs $(b_1, b_2) \in T_Z^{\ll}$ we also have $M < 1$. Any $\varpi \in C$ with $|\varpi|_q < 1$ and $M \leq |\varpi|_q \leq m$ defines a lift of q in Z . \square

Definition 2.17. Let (B, B^+) be a complete Huber pair over \mathbb{Z}_p , we say that (B, B^+) is *olivine* if the map $|\mathrm{Spd}(B, B^+)| \rightarrow \mathrm{Spo}(B, B^+)$ is a homeomorphism.

Question 2.18. *Is every complete Huber pair over \mathbb{Z}_p an olivine Huber pair?*

We have enough partial progress answering this question. Although we do not know what to expect in full generality, for the Huber pairs that we consider this is true. Let us clarify. By [Proposition 1.21](#) Tate Huber pairs are olivine. By [Proposition 2.16](#) formal Huber pairs are olivine. By [Proposition 2.8](#) if $(B, B^+) \rightarrow (R, R^+)$ induces a locally closed immersion $\mathrm{Spd}(R, R^+) \subseteq \mathrm{Spd}(B, B^+)$ and (B, B^+) is olivine then (R, R^+) is olivine. Moreover, being olivine can be verified locally in the analytic topology of $\mathrm{Spa}(B, B^+)$. The following criterion can be used in most circumstances of interest.

Proposition 2.19. *Let (B, B^+) be a complete Huber pair over \mathbb{Z}_p , suppose it is topologically of finite type over a formal Huber pair (B_0, B_0) . Then (B, B^+) is olivine.*

Proof. By definition, there is $M = \{M_i\}_{i=1}^n$ with $B_0 \cdot M_i \subseteq B_0$ open and a strict surjection $f : B_0 \langle T_1 \dots, T_n \rangle_{M_1, \dots, M_n} \rightarrow B$. Let C be the ring of integral elements of $B_0 \langle T_1 \dots, T_n \rangle_{M_1, \dots, M_n}$, then B^+ is the integral closure of $f(C)$ in B . Since $\mathrm{Spd}(B, B^+) \rightarrow \mathrm{Spd}(B_0 \langle T_1 \dots, T_n \rangle_{M_1, \dots, M_n}, C)$ is a closed immersion it suffices to prove the claim for $(B_0 \langle T_1 \dots, T_n \rangle_{M_1, \dots, M_n}, C)$. We proceed by induction the base case being [Proposition 2.16](#). Let $\mathrm{Spa}(R, R^+)$ be the rational localization corresponding to $\{x \in \mathrm{Spa}(B, B^+) \mid |T_1|_x \leq |1|_x \neq 0\}$, then (R, R^+) is olivine by induction. Indeed, $(R, R^+) = (A_0 \langle T_2, \dots, T_n \rangle_{M_2, \dots, M_n}, C')$ for $(A_0, A_0) = (B_0 \langle T_1 \rangle_{\{1\}}, B_0 \langle T_1 \rangle_{\{1\}})$ which is a formal. Let $\mathrm{Spa}(S, S^+) = \{x \in \mathrm{Spa}(B, B^+) \mid |1|_x \leq |T_1|_x \neq 0\}$. If we let $A_0 = B_0 \langle \frac{1}{T_1} \rangle_{\{1\}}$ then we may rewrite $\mathrm{Spa}(S, S^+)$ as the locus of points in

$$\mathrm{Spa}(A_0 \langle T_2, \dots, T_n \rangle_{M_2, \dots, M_n}, C'')$$

such that $m \leq \frac{1}{T_1} \neq 0$ for $m \in M_1$. By induction $(A_0 \langle T_2, \dots, T_n \rangle_{M_2, \dots, M_n}, C'')$ is olivine, and since rational localizations preserve olivine Huber pairs we conclude (S, S^+) is olivine. \square

Remark 2.20. For an arbitrary Huber pair (B, B^+) with B_0 a ring of definition we can consider the commutative diagram

$$\begin{array}{ccc} |\mathrm{Spd}(B, B^+)| & \longrightarrow & \mathrm{Spo}(B, B^+) \\ \downarrow & & \downarrow \\ \varprojlim_i |\mathrm{Spd}(B_i, B_i^+)| & \longrightarrow & \varprojlim_i \mathrm{Spo}(B_i, B_i^+) \end{array}$$

where (B_i, B_i^+) ranges over all subrings of B that are topologically of finite type over B_0 . By [Proposition 2.19](#) the bottom horizontal arrow is a homeomorphism and one can verify directly that the right vertical arrow is also a homeomorphism. It is not clear to us if the left vertical arrow is a homeomorphism or not since taking limits of v-sheaf does not necessarily commute with taking underlying topological spaces. Adding to the complexity of the situation the transition maps $\mathrm{Spd}(B_i, B_i^+) \rightarrow \mathrm{Spd}(B_j, B_j^+)$ might not be quasicompact. Counterexample to [Question 2.18](#) should come from this failure. We do not know if letting $B = \mathbb{F}_p[T_1, \dots, T_n, \dots]$ and $B^+ = \mathbb{F}_p$ with the discrete topology gives an olivine Huber pair.

2.4. Some open and closed subsheaves. By [[Sch17](#), Proposition 12.9] open subsets of $\mathrm{Spo}(B, B^+)$ define open subsheaves of $\mathrm{Spd}(B, B^+)$, and when (B, B^+) is olivine this association is bijective. Since the formation of $\mathrm{Spd}(B, B^+)$ commutes with localization in $\mathrm{Spa}(B, B^+)$, one can compute the open subsheaf corresponding to classical localizations. The following lemma describes, in some cases, the open subsheaf associated to analytic localizations.

Lemma 2.21. *Suppose that B^+ is I -adic and that $B \subseteq \mathrm{Frac}(B^+)$. Let $b \in B$, let B_b^+ be the (b, I) -adic completion of B^+ and let $B_b = B \otimes_{B^+} B_b^+$. If (B_b, B_b^+) is Huber, then, $N_{b \ll 1} \subseteq \mathrm{Spd}(B, B^+)$ is represented by $\mathrm{Spd}(B_b, B_b^+)$. This condition is satisfied if $B^+ = B$ or if B and B^+ are valuation rings.*

Proof. Since $B \subseteq B_b$ is dense, $\mathrm{Spd}(B_b, B_b^+) \rightarrow \mathrm{Spd}(B, B^+)$ is injective. If $f : \mathrm{Spa}(R, R^+) \rightarrow \mathrm{Spd}(B, B^+)$ factors through $\mathrm{Spd}(B_b, B_b^+)$ then $f^*(b)$ is topologically nilpotent in R^\sharp . This gives $\mathrm{Spo}(f)(\mathrm{Spa}(R, R^+)) \subseteq$

$N_{b \ll 1}$ and since this happens for all $(R, R^+) \in \text{Perf}$, $\text{Spd}(B_b, B_b^+) \rightarrow \text{Spd}(B, B^+)$ factors through $N_{b \ll 1}$. Conversely, pick $f : \text{Spa}(R^\sharp, R^{\sharp,+}) \rightarrow \text{Spa}(B, B^+)$ with $\text{Spo}(f)(\text{Spa}(R^\sharp, R^{\sharp,+})) \subseteq N_{b \ll 1}$. Let $x \in \text{Spa}(R^\sharp, R^{\sharp,+})$, let $\varpi \in R^{\sharp,+}$ be a pseudo-uniformizer, then $|f^*b^n|_x \leq |\varpi|_x$ for some n . Let $\text{Spa}(R_1, R_1^+) = U(\frac{f^*b^n}{\varpi}) \subseteq \text{Spa}(R^\sharp, R^{\sharp,+})$. Now, $B^+ \rightarrow R_1^+$ is continuous for the (I, b) -topology so we get a map $(B_b, B_b^+) \rightarrow (R_1, R_1^+)$. This proves f factors locally, and by injectivity it also does globally. \square

In general the subsheaf $N_{b \ll 1}$ is not of the form $\text{Spd}(R, R^+)$. Recall that $\text{Spo}(B, B^+)^\dagger \subseteq \text{Spo}(B, B^+)$ is the closed subset of bounded points. Observe that $\text{Spo}(B, B^+)^\dagger$ is stable under v.g. and by [Proposition 1.9](#) it defines a closed subsheaf of $\text{Spd}(B, B^+)$. Let $\text{Spd}(B, B^+)^\dagger$ denote this closed subsheaf.

Proposition 2.22. *Let $\mathcal{F} : \text{Perf} \rightarrow \text{Sets}$ parametrize triples (R^\sharp, ι, f) where (R^\sharp, ι) is an untilt of R and $f : \text{Spa}(R^{\sharp,\circ}, R^{\sharp,+}) \rightarrow \text{Spa}(B, B^+)$ is a morphism of pre-adic spaces. Then $\mathcal{F} = \text{Spd}(B, B^+)^\dagger$.*

Proof. We prove $\mathcal{F} \rightarrow \text{Spd}(B, B^+)$ is a closed immersion. Let $\mathbb{A}_{\mathbb{Z}_p}^{|B|}$ parametrize tuples (R^\sharp, ι, x) with (R^\sharp, ι) an untilt and $x : B \rightarrow R^\sharp$ a map of sets. Define $\mathbb{A}_{\mathbb{Z}_p}^{|B|, \dagger}$ similarly with $x : B \rightarrow R^{\sharp,\circ}$. We have a basechange identity $\mathcal{F} = \mathbb{A}_{\mathbb{Z}_p}^{|B|, \dagger} \times_{\mathbb{A}_{\mathbb{Z}_p}^{|B|}} \text{Spd}(B, B^+)$. Since limits preserve closed immersions it suffices to prove $\mathbb{A}_{\mathbb{Z}_p}^{|B|, \dagger} \rightarrow \mathbb{A}_{\mathbb{Z}_p}^{|B|}$ is a closed immersion, which can be checked after basechange. Let $f_r : \text{Spa}(R, R^+) \rightarrow \mathbb{A}_{\mathbb{Z}_p}^{|B|}$ defined by $r \in R^\sharp$. Then $\mathbb{A}^{1, \dagger} \times_{\mathbb{A}_{\mathbb{Z}_p}^{|B|}} \text{Spa}(R, R^+)$ is the complement in $\text{Spa}(R, R^{\sharp,+})$ of $\bigcup_{\varpi \in R^{\sharp,\circ}} \{x \in \text{Spa}(R^\sharp, R^{\sharp,+}) \mid |1|_x \leq |r \cdot \varpi|_x \neq 0\}$. This is and stable under v.g. as we wanted to show.

Since $\text{Spd}(B, B^+)^\dagger$ and \mathcal{F} are closed immersions it suffices to prove they coincide on geometric points. This follows from the definition of the bounded locus. \square

Lemma 2.23. *Let (A, A^+) and (B, B^+) be complete Huber pairs over \mathbb{Z}_p and $(B, B^+) \rightarrow (A, A^+)$ be an adic morphism. Then $\text{Spd}(A, A^+)^\dagger \rightarrow \text{Spd}(B, B^+)^\dagger$ is representable in spatial diamonds. In particular, it is qcqs.*

Proof. Since the map $(B, B^+) \rightarrow (A, A^+)$ is adic we can write (A, A^+) as a (completion of a) filtered colimit $\varinjlim_{i \in I} (A_i, A_i^+)$ where each (A_i, A_i^+) is topologically of finite type over (B, B^+) , and the transition maps realize $A_i \rightarrow A_j$ as a topological subring for $i < j$. One can see that $\text{Spd}(A, A^+)^\dagger = \varprojlim_i \text{Spd}(A_i, A_i^+)^\dagger$ and by [\[Sch17, Lemma 12.17\]](#) it suffices to prove that $\text{Spd}(A_i, A_i^+)^\dagger \rightarrow \text{Spd}(B, B^+)^\dagger$ is representable in spatial diamonds. A presentation of A_i as a topologically of finite type B -algebra gives a closed immersion $\text{Spd}(A_i, A_i^+)^\dagger \rightarrow \text{Spd}(B \langle \langle T_k \rangle_{k=1}^n \rangle_{M_k})^\dagger$. Since closed immersions are representable in spatial diamonds we may assume $A_i = B \langle T_1 \rangle_{M_1}$. There is an open immersion $\text{Spd}(B \langle T_1 \rangle_{M_1}) \rightarrow \mathbb{A}_B^1$ and $\text{Spd}(B \langle T_1 \rangle_{M_1}, B \langle T_1 \rangle_{M_1}^+) \cap \mathbb{A}_B^{1, \dagger} = \text{Spd}(B \langle T_1 \rangle_{M_1}, B \langle T_1 \rangle_{M_1}^+)^\dagger$. Clearly, $\mathbb{A}_B^{1, \dagger} \rightarrow \text{Spd}(B, B^+)^\dagger$ is representable in locally spatial diamonds, we need to verify it is quasicompact. This can be done after basechanges by affinoid perfectoid. But the basechange by a map $\text{Spa}(R, R^+) \rightarrow \text{Spd}(B, B^+)^\dagger$ is representable by $\text{Spd}(R^\sharp \langle T \rangle, R')$ where R' is the minimal ring of integral elements containing $R^{\sharp,+}$. \square

The following statement says that at least the bounded locus of a Huber pair is always olivine.

Proposition 2.24. *Suppose that (B, B^+) is a complete Huber pair over \mathbb{Z}_p . The natural map*

$$|\text{Spd}(B, B^+)^\dagger| \rightarrow \text{Spo}(B, B^+)^\dagger$$

is a homeomorphism.

Proof. Let $B_0 \subseteq B^+$ be a ring of definition and express (B, B^+) as a filtered colimit $\varinjlim_{i \in J} (B_i, B_i^+)$ with both B_i and B_i^+ of finite type over B_0 , then $\text{Spd}(B, B^+)^\dagger = \varprojlim \text{Spd}(B_i, B_i^+)^\dagger$. By [Proposition 2.19](#) each (B_i, B_i^+) is olivine and by [Lemma 2.23](#) the transition maps are representable in spatial diamonds. Let $\pi_i : \mathbb{D}_{B_i^+}^\times \rightarrow \text{Spd}(B_i, B_i^+)^\dagger$ denote the punctured open unit disc over $\text{Spd}(B_i, B_i^+)^\dagger$. Observe that π_i is open. Now, $\mathbb{D}_{B_0^+}^\times$ is a locally spatial diamond represented by $(\text{Spa}(B_0[[t]])^{t \neq 0})^\diamond$. In particular, $\mathbb{D}_{B_i^+}^\times$ is also a locally spatial diamond and since the transition maps $\mathbb{D}_{B_i^+}^\times \rightarrow \mathbb{D}_{B_j^+}^\times$ are qcqs we see that by [\[Sch17, Lemma 12.17\]](#) $|\mathbb{D}_{B^+}^\times| = \varprojlim |\mathbb{D}_{B_i^+}^\times|$. It suffices to prove that $\pi : |\mathbb{D}_{B^+}^\times| \rightarrow \text{Spo}(B, B^+)^\dagger$ is a quotient map. Let $S \subseteq \text{Spo}(B, B^+)^\dagger$ with $\pi^{-1}(S)$ open. For every point $y \in \pi^{-1}(S)$ there is an index $j_y \in J$ and an open subset of $U_y \subseteq \mathbb{D}_{B_j^+}^\times$ whose preimage in $\mathbb{D}_{B^+}^\times$ is contained in $\pi^{-1}(S)$ and contains y . Now, $\pi_{j_y}(U_y)$

is open in $|\mathrm{Spd}(B_i, B_i^+)^\dagger|$ and since (B_i, B_i^+) is olivine it is also open in $\mathrm{Spo}(B_i, B_i^+)^\dagger$. The preimage of $\pi_{j_y}(U_y)$ in $\mathrm{Spo}(B, B^+)^\dagger$ contains $\pi(y)$, is open and it is contained in $h^{-1}(S)$. \square

2.5. Discrete Huber pairs in characteristic p . For the rest of the subsection A denotes a discrete perfect ring in characteristic p and $A^+ \subseteq A$ is integrally closed.

Proposition 2.25. *Let (A, A^+) be as above. The projection map $\mathrm{Spo}(A, A^+)^\dagger \rightarrow \mathrm{Spa}(A, A^+)$ is surjective. Moreover, if $\mathfrak{S} \subseteq \mathrm{Spa}(A, A^+)$ is stable under arbitrary generization and $h^{-1}(\mathfrak{S})$ is open in $\mathrm{Spo}(A, A^+)^\dagger$ then \mathfrak{S} is open in $\mathrm{Spa}(A, A^+)$.*

Proof. The complement of the bounded locus consists of d-analytic points. Since A has the discrete topology every d-analytic point is meromorphic. If $x \in \mathrm{Spo}(A, A^+)^\dagger$ is meromorphic, then $y := x_{\mathrm{mer}}$ is bounded and satisfies $h(x) = h(y)$. Consequently, $h(\mathrm{Spo}(A, A^+)^\dagger) = h(\mathrm{Spo}(A, A^+)^\dagger)$.

Now, observe that $\mathrm{Spd}(A((t)), A^+ + t \cdot A[[t]]) \rightarrow \mathrm{Spd}(A, A^+)$ surjects onto $\mathrm{Spd}(A, A^+)^\dagger$ and represents the punctured open unit ball over it. Consider, $f : \mathrm{Spa}(A((t)), A^+ + t \cdot A[[t]]) \rightarrow \mathrm{Spa}(A, A^+)$, it suffices to prove that if \mathfrak{S} is stable under generization and $f^{-1}(\mathfrak{S})$ is open, then \mathfrak{S} is open. The rest of the argument is a variant of the proof of [Proposition 2.16](#), using only classical localizations. In this case, one exploits the constructible topology of $\mathrm{Spa}(A((t)), A^+ + t \cdot A[[t]])$. We omit the details. \square

Proposition 2.26. *If A and A^+ are valuation rings with the same fraction field then (A, A^+) is olivine.*

Proof. If $\mathrm{Spo}(A, A^+)^\dagger = \mathrm{Spo}(A, A^+)$, then [Proposition 2.24](#) proves that (A, A^+) is olivine. Suppose $x \in \mathrm{Spo}(A, A^+) \setminus \mathrm{Spo}(A, A^+)^\dagger$, then x is meromorphic and there is $\pi \in A$ with $1 < |\pi|_x^a$. We must have $\frac{1}{\pi} \in A^+$ since A^+ is a valuation ring and $\pi \notin A^+$. Let $b = \frac{1}{\pi}$, we claim that $A = A^+[\frac{1}{b}]$. By the archimedean property of $|\cdot|_x^a$ for every $a' \in A$ there is a big enough $n \in \mathbb{N}$ with $|b^n \cdot a'|_x^a < 1$. Since A^+ is a valuation ring either $a' \cdot b^n \in A^+$ or $\frac{1}{a' \cdot b^n} \in A^+$, but the second case contradicts that $|\cdot|_x^a \in \mathrm{Spa}(A, A^+)$.

By [Proposition 2.16](#), (A^+, A^+) is olivine and since $\mathrm{Spo}(A, A^+) \subseteq \mathrm{Spo}(A^+, A^+)$ is the open locus in which $b \neq 0$ we conclude by [Proposition 2.8](#) that (A, A^+) is also olivine. \square

Lemma 2.27. *There is a unique map $\mathrm{Spd}(A, A^+) \rightarrow \mathrm{Spd}(\mathbb{Z}_p)$, it is given by $\mathrm{Spd}(A, A^+) \rightarrow \mathrm{Spd}(\mathbb{F}_p) \rightarrow \mathrm{Spd}(\mathbb{Z}_p)$.*

Proof. It suffices to prove that $\mathrm{Spa}(C, C^+) \rightarrow \mathrm{Spd}(A, A^+) \rightarrow \mathrm{Spd}(\mathbb{Z}_p)$ factors through $\mathrm{Spd}(\mathbb{F}_p)$ for geometric points. Consider, $\mathrm{Spa}(R_\infty, R_\infty^+) \rightarrow \mathrm{Spd}(A, A^+)$ as in the proof of [Lemma 2.15](#), with $R_\infty^+ = \prod_{i=1}^\infty C^+$ and $\varpi_\infty = (\varpi^{p^i})$. The map $\mathrm{Spa}(R_\infty, R_\infty^+) \rightarrow \mathrm{Spd}(A, A^+) \rightarrow \mathrm{Spd}(\mathbb{Z}_p)$ defines an untilt of R_∞ given by $\xi = p + (\varpi_\infty)^{\frac{1}{p^k}} \cdot \alpha$ with $\alpha \in W(R_\infty^+)$. For any $i \in \mathbb{N}$ the projection $\iota_i : R_\infty \rightarrow C$ gives an untilt of C . Since $(A, A^+) \rightarrow (R_\infty, R_\infty^+) \xrightarrow{\iota_i} (C, C^+)$ is independent of the projection, all of these untilts agree. This says that the ideal I_i generated by $\iota_i(\xi)$ in $W(C^+)$ agree, we call this ideal I .

Since $\iota_i(\xi) = p - \varpi^{\frac{i}{p^k}} \iota_i(\alpha)$ the sequence $\iota_i(\xi)$ converges to p in the (p, ϖ) -adic topology. But the ideal associated to an untilt is closed, so $p \in I$ and $\mathrm{Spa}(C, C^+) \rightarrow \mathrm{Spd}(\mathbb{Z}_p)$ factors through $\mathrm{Spd}(\mathbb{F}_p)$. \square

Lemma 2.28. *Let (A, A^+) be as above and let (B, B^+) be a complete Huber pairs over \mathbb{Z}_p . Then every morphism of v -sheaves $\mathrm{Spd}(A, A^+) \rightarrow \mathrm{Spd}(B, B^+)$ comes from a unique morphism of Huber pairs $(B, B^+) \rightarrow (A, A^+)$.*

Proof. Given a map $g : \mathrm{Spd}(A, A^+) \rightarrow \mathrm{Spd}(B, B^+)$ we first construct a map $m : \mathrm{Spa}(A, A^+) \rightarrow \mathrm{Spa}(B, B^+)$, and then prove that $m^\diamond = g$. Let $R = A((t^{\frac{1}{p^\infty}}))$, $R^+ = A^+ + (t^{\frac{1}{p^\infty}})A[[t^{\frac{1}{p^\infty}}]]$ and $X = \mathrm{Spa}(R, R^+)$. The natural map $X \rightarrow \mathrm{Spd}(A, A^+)$ surjects onto $\mathrm{Spd}(A, A^+)^\dagger$. Since X representable, any $f : X \rightarrow \mathrm{Spd}(B, B^+)$ is given by an untilt R^\sharp and a map $f^* : (B, B^+) \rightarrow (R^\sharp, R^{\sharp,+})$. Let f be induced by g , by [Lemma 2.27](#) the untilt must be R . Since f factors through g , $f^*(B)$ is invariant under automorphism of R over A . Take $b \in B$, we show $f^*(b) \in A \subseteq R$. Now, $t^{p^n} \cdot f^*(b)$ is topologically nilpotent for big enough n . Replacing by $t \mapsto t^{\frac{1}{p^m}}$ we conclude that $t^{p^n} f^*(b)$ is topologically nilpotent for $n \in \mathbb{Z}$. This proves that $f^*(b)$ is power-bounded so that $f^*(b) \in A[[t^{\frac{1}{p^\infty}}]]$. Write $f^*(b) = a_0 + t^{\frac{1}{p^m}} q$ with $a_0 \in A$ and $q \in A[[t^{\frac{1}{p^\infty}}]]$. Now, $t^{\frac{1}{p^m}} q$ converges to 0 under $t \mapsto t^{p^n}$ so $f^*(b) = a_0$. We get a ring map $m^* : B \rightarrow A$. The subspace topology of A in R is discrete, this gives continuity of m^* . Moreover, $R^+ \cap A = A^+$. We have constructed $m : \mathrm{Spa}(A, A^+) \rightarrow \mathrm{Spa}(B, B^+)$ with $m^\diamond = g$ over $\mathrm{Spd}(A, A^+)^\dagger$.

Consider $(g, m^\diamond) : \mathrm{Spd}(A, A^+) \rightarrow \mathrm{Spd}(B, B^+) \times \mathrm{Spd}(B, B^+)$ we show that $\mathrm{Spd}(A, A^+)$ factors through the diagonal $\Delta : \mathrm{Spd}(B, B^+) \rightarrow \mathrm{Spd}(B, B^+) \times \mathrm{Spd}(B, B^+)$. We can check this on geometric points $x : \mathrm{Spd}(C, C^+) \rightarrow \mathrm{Spd}(A, A^+)$. Since the maps agree on $\mathrm{Spd}(A, A^+)^\dagger$ we can assume that x is meromorphic.

Pick a pseudo-uniformizer $\varpi \in C$ and consider R_∞ as in [Lemma 2.15](#). Consider $\mathrm{Spa}(R_\infty, R_\infty^+) \rightarrow \mathrm{Spd}(A, A^+)$ given by the diagonal morphism $A \rightarrow \prod_{i=1}^\infty C$. Recall, $C \subseteq_\Delta R_\infty \subseteq \prod_{i=1}^\infty C$, and that although $C \subseteq_\Delta R_\infty$ is not continuous the composition $A \rightarrow R_\infty$ is. Now, $\mathrm{Spa}(R_\infty, R_\infty^+) \rightarrow \mathrm{Spd}(A, A^+) \rightarrow \mathrm{Spd}(B, B^+) \times \mathrm{Spd}(B, B^+)$ gives two maps $f_1, f_2 : B \rightarrow R_\infty$, and both have to factor through the diagonal $C \subseteq_\Delta R_\infty$. By the proof of [Lemma 2.15](#) the residue field at a non-principal ultrafilter \mathcal{U} maps to x_{mer} . Since $\mathrm{Spa}(C_{\mathcal{U}}, C_{\mathcal{U}}^+) \rightarrow \mathrm{Spd}(B, B^+) \times \mathrm{Spd}(B, B^+)$ factors through $\mathrm{Spd}(A, A^+)^{\dagger}$ (being discrete in $\mathrm{Spo}(A, A^+)$), it also factors through the diagonal. These ring maps are the compositions $f_i : B \rightarrow C \rightarrow R_\infty \rightarrow C_{\mathcal{U}}$. We can conclude $f_1 = f_2$ since $C \rightarrow C_{\mathcal{U}}$ is injective. \square

Theorem 2.29. *Let Y be a perfect discrete adic space over \mathbb{F}_p and let X be a pre-adic space over \mathbb{Z}_p . The natural map $\mathrm{Hom}_{\mathrm{PreAd}}(Y, X) \rightarrow \mathrm{Hom}(Y^\diamond, X^\diamond)$ is bijective. In particular, \diamond is fully faithful when restricted to the category of perfect discrete adic spaces over \mathbb{F}_p .*

This theorem says, intuitively speaking, that (up to perfection) one does not get new morphisms of v-sheaves when the source is a discrete adic space.

Proof. It is not hard to prove injectivity. For surjectivity, the hard part is to prove that morphisms $g : Y^\diamond \rightarrow X^\diamond$ induce a map of topological spaces $f : |Y| \rightarrow |X|$ making the following diagram commute:

$$\begin{array}{ccc} |Y^\diamond| & \xrightarrow{g} & |X^\diamond| \\ \downarrow & & \downarrow \\ |Y| & \xrightarrow{f} & |X| \end{array}$$

Indeed, if this holds true one can reduce to [Lemma 2.28](#) by standard glueing arguments. Verifying that $g : |Y^\diamond| \rightarrow |X^\diamond|$ descends to $f : |Y| \rightarrow |X|$ can be done locally on $|Y|$, we may assume $Y = \mathrm{Spa}(A, A^+)$. Let $y \in |Y|$ and $z \in \mathrm{Spo}(A, A^+)$ with $h(z) = y$, we define $f(y) := h(g(z))$. We must verify that this doesn't depend on z and that it is continuous. The map f is well defined if and only if $h(g(z)) = h(g(z_{\mathrm{mer}}))$ when z is meromorphic, and by [Proposition 2.25](#) to prove continuity it suffices to prove that if $\mathfrak{S} \subseteq |X|$ is open then $f^{-1}(\mathfrak{S})$ is stable under arbitrary generization in $\mathrm{Spa}(A, A^+)$. Let $w \in \mathrm{Spa}(A, A^+)$ be a horizontal generization of y . Let (k_y, k_y^+) and (k_w, k_w^+) denote the affinoid residue fields of w and y and let K_w denote the smallest ring containing k_w^+ and A as in [item 11](#). It suffices to prove that $|\mathrm{Spd}(K_w, k_w^+)| \rightarrow |X^\diamond|$ and $|\mathrm{Spd}(k_y, k_y^+)| \rightarrow |X^\diamond|$ descend to continuous maps $|\mathrm{Spa}(K_w, k_w^+)| \rightarrow |X|$ and $|\mathrm{Spa}(k_y, k_y^+)| \rightarrow |X|$. In summary, we have reduced to the case where $Y = \mathrm{Spa}(A, A^+)$ with $A^+ \subseteq A$ two valuation rings with the same fraction field. We deal with this case in [Lemma 2.30](#) below. \square

Lemma 2.30. *Let X and $Y = \mathrm{Spa}(A, A^+)$ as above, with $A^+ \subseteq A \subseteq \mathrm{Frac}(A^+)$ both valuation rings. Let $g : \mathrm{Spd}(A, A^+) \rightarrow X^\diamond$ be a map. Let $h(c_{\min}) \in \mathrm{Spa}(A, A^+)$ denote the unique closed point and let $c_{\min} \in \mathrm{Spo}(A, A^+)$ denote the unique discrete point mapping to $h(c_{\min})$. If $h(g(c_{\min})) \in |X|$ lies in $\mathrm{Spa}(B_1, B_1^+) \subseteq X$ then g factors through a map $\mathrm{Spd}(A, A^+) \rightarrow \mathrm{Spd}(B_1, B_1^+) \subseteq X^\diamond$. In particular, g is coming from a map of pre-adic spaces $\mathrm{Spa}(A, A^+) \rightarrow \mathrm{Spa}(B_1, B_1^+) \subseteq X$.*

Proof. Suppose to get a contradiction that there is an “exotic” g that does not satisfy this property. By [Proposition 2.26](#), $|\mathrm{Spd}(A, A^+)| = \mathrm{Spo}(A, A^+)$ and we work with the later. Let $U_1 \subseteq \mathrm{Spo}(A, A^+)$ be pullback of $\mathrm{Spd}(B_1, B_1^+)$, this a proper open subset. Let $Z = \mathrm{Spo}(A, A^+) \setminus U_1$, it is quasicompact and by [[SW20](#), Lemma 18.3.2] we may find the largest prime $\mathfrak{p}_m \in \mathrm{Spec}(A)$ of the form $\mathrm{supp}(z)$ with $z \in Z$. Replacing A and A^+ by A/\mathfrak{p}_m and A^+/\mathfrak{p}_m we may assume that if $z \in Z$ then $\mathrm{supp}(z) = 0$. Let $K = \mathrm{Frac}(A)$, then $Z \subseteq \mathrm{Spo}(K, A^+)$. Since Z is a closed it contains the unique closed point $q_{\min} \in \mathrm{Spo}(K, A^+)$.⁵ Now, U_1 contains all analytic localizations $N_{n \ll 1}$ with $n \neq 0$ and $n \in \mathrm{supp}(c_{\min})$. Indeed, if $z \in Z \cap N_{n \ll 1}$ then $|n|_z^a < 1$ and either z or z_{for} have non-trivial support giving a contradiction.

Also, $\mathrm{Spd}(K, A^+) \rightarrow X^\diamond$ factors through another open affine subsheaf $\mathrm{Spd}(B_2, B_2^+)$, since it has a unique closed point. Let U_2 be the pullback of $\mathrm{Spd}(B_2, B_2^+)$. By [Example 2.9](#), there is an open with $q_{\min} \in U_{0 \leq b \neq 0} \cap U_{1 \leq b' \neq 0} \cap N_{n \ll 1} \subseteq U_2$. Moreover, in the notation of [Example 2.9](#) $q_{\min} = (0, 0, \mathfrak{m})$ with \mathfrak{m} the maximal ideal of A^+ . This gives that $n = 0$ and that $b' \in A^+ \setminus \mathfrak{m}$ so $U_{1 \leq b' \neq 0} = \mathrm{Spo}(A, A^+)$ and $N_{n \ll 1} = \mathrm{Spo}(A, A^+)$. In summary, $q_{\min} \in U_{0 \leq b \neq 0} \subseteq U_2$.

We have found neighborhoods $N_{b \ll 1} \subseteq U_1$ and $U_{0 \leq b \neq 0} \subseteq U_2$. Observe that $\mathrm{Spo}(A, A^+) = N_{b \ll 1} \cup U_{0 \leq b \neq 0}$. Let A_b^+ denote the (b) -adic completion and $A_b = A_b^+ \otimes_{A^+} A$. [Lemma 2.21](#) shows that $N_{b \ll 1}$ is represented by $\mathrm{Spd}(A_b, A_b^+)$. Also, $U_{0 \leq b \neq 0}$ is represented by $\mathrm{Spd}(A[\frac{1}{b}], A^+)$ and $N_{b \ll 1} \cap U_{0 \leq b \neq 0}$ is

⁵This is the unique discrete point such that $h(q_{\min})$ is closed in $\mathrm{Spa}(K, A^+)$.

represented by $\mathrm{Spa}(A_b[\frac{1}{b}], A_b^+)$. Notice that this is a perfectoid field. Let q_b be the closed point of $N_{b \ll 1} \cap U_{0 \leq b \neq 0}$. Since the morphisms glue, there is $\mathrm{Spa}(B_3, B_3^+) \subseteq \mathrm{Spa}(B_1, B_1^+) \times_X \mathrm{Spa}(B_2, B_2^+)$ and $\mathrm{Spa}(A_b[\frac{1}{b}], A_b^+) \rightarrow \mathrm{Spd}(B_3, B_3^+)$ making the following diagram commute:

$$\begin{array}{ccccc}
\mathrm{Spa}(A_b[\frac{1}{b}], A_b^+) & \longrightarrow & \mathrm{Spd}(A[\frac{1}{b}], A^+) & & \\
\downarrow & \searrow & \downarrow & & \\
& & \mathrm{Spd}(B_3, B_3^+) & \longrightarrow & \mathrm{Spd}(B_2, B_2^+) \\
& & \downarrow & & \downarrow \\
\mathrm{Spd}(A_b, A_b^+) & \longrightarrow & \mathrm{Spd}(B_1, B_1^+) & \longrightarrow & X^\diamond
\end{array}$$

By [Lemma 2.28](#) the map $\mathrm{Spd}(A[\frac{1}{b}], A^+) \rightarrow \mathrm{Spd}(B_2, B_2^+)$ is given by a map of Huber pairs $(B_2, B_2^+) \rightarrow (A[\frac{1}{b}], A^+)$. The pullback of $\mathrm{Spd}(B_3, B_3^+)$ to $\mathrm{Spd}(A[\frac{1}{b}], A^+)$ has the form $h^{-1}(U_3)$ for some $U_3 \subseteq \mathrm{Spa}(A[\frac{1}{b}], A^+)$. Moreover, $h(q_b)$ is the closed point of $\mathrm{Spa}(A[\frac{1}{b}], A^+)$. This proves that $\mathrm{Spd}(A[\frac{1}{b}], A^+)$ factors through $\mathrm{Spd}(B_3, B_3^+)$ and consequently through $\mathrm{Spd}(B_1, B_1^+)$ contradicting our assumption. \square

We now study perfect discrete Huber pairs of the form (A, A) .

Proposition 2.31. *Let A be discrete ring and $f^* : (B, B^+) \rightarrow (A, A)$ a map. The following hold:*

- (1) $f(\mathrm{Spo}(A, A)) = h^{-1}(f(\mathrm{Spa}(A)))$.
- (2) $f(\mathrm{Spa}(A))$ is stable under horizontal specializations in $\mathrm{Spa}(B, B^+)$.
- (3) $f(\mathrm{Spa}(A))$ is stable under vertical generizations in $\mathrm{Spa}(B, B^+)$.

Proof. For the first claim let $y \in \mathrm{Spo}(A, A)$ and let $x = \mathrm{Spo}(f)(y)$. If x is d-analytic, y is meromorphic and y_{mer} maps to x_{mer} , giving $h^{-1}(h(x)) \subseteq \mathrm{Im}(\mathrm{Spo}(f))$. Suppose that x is discrete and that x^{mer} exists. In this case, y^{mer} might not exist and even if it does it might not map to x^{mer} . Consider instead $h(y) \in \mathrm{Spa}(A)$ and its residue field map $\iota_{h(y)} : \mathrm{Spa}(K_{h(y)}, K_{h(y)}^+) \rightarrow \mathrm{Spa}(A)$. Notice that $\iota_{h(y)}$ factors through $g : \mathrm{Spa}(K_{h(y)}^+) \rightarrow \mathrm{Spa}(A)$. We prove that x^{mer} is in the image of $\mathrm{Spo}(f \circ g)$. Take $b \in B$ with $|b|_{x^{\mathrm{mer}}} \notin \{0, 1\}$ and replace it with its inverse in $K_{h(y)}$, if necessary, so that $b \in K_{h(y)}^+$. Define K^+ as the (b) -adic completion of $K_{h(y)}^+$, and let $K = K^+[\frac{1}{b}]$. We get a map $\mathrm{Spa}(K, K^+) \rightarrow \mathrm{Spo}(K_{h(y)}^+, K_{h(y)}^+) \rightarrow \mathrm{Spo}(B, B^+)$. One can verify that the closed point of $\mathrm{Spa}(K, K^+)$ maps to x^{mer} .

The proof of the second claim also follows from observing that the residue field map $\iota_{h(y)}$ factors through g . Indeed, we get the following commutative diagram of adic spaces:

$$\begin{array}{ccccc}
\mathrm{Spa}(K_{h(y)}, K_{h(y)}^+) & \longrightarrow & \mathrm{Spa}(K_{h(y)}^+) & \xrightarrow{g} & \mathrm{Spa}(A) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Spa}(K_{h(x)}, K_{h(x)}^+) & \longrightarrow & \mathrm{Spa}(K_{h(x)}^+) & \xrightarrow{g'} & \mathrm{Spa}(B, B^+)
\end{array}$$

Where we use that $K_{h(x)}^+ = K_{h(y)}^+ \cap K_{h(x)}$ to define g' . Moreover, the vertical map on the left is surjective since $h(x) = f(h(y))$ and one can deduce that the vertical map in the middle column is also surjective because the map of valuation rings is local. A prime ideal of $J \subseteq K_{h(x)}^+$ determines a horizontal specializations of $|\cdot|_{h(x)}$, namely $|\cdot|_{h(x)}/J$, and every horizontal specialization of $h(x)$ can be constructed in this way. For J as above we let $K_J^+ = K_{h(x)}^+/J$ and $K_J = \mathrm{Frac}(K_J^+)$, we get the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Spa}(K_J, K_J^+) & \longrightarrow & \mathrm{Spa}(K_J^+) & \longrightarrow & \mathrm{Spa}(K_{h(y)}^+) \\
& \searrow & \downarrow & \swarrow & \downarrow \\
& & \mathrm{Spa}(B, B^+) & &
\end{array}$$

The closed point of $\mathrm{Spa}(K_J, K_J^+)$ maps to the horizontal specialization of $h(x)$ associated to the ideal J .

The third claim follows from [Corollary 2.14](#) and from the first claim. \square

Definition 2.32. We say that a subset of $\mathrm{Spo}(B, B^+)$ is a *schematic subset* if it is a union of sets of the form $\mathrm{Spo}(m)(\mathrm{Spo}(A, A))$ where (A, A) is given the discrete topology and $m^* : (B, B^+) \rightarrow (A, A)$ is a map of Huber pairs.

The following statement is the key result that allow us to construct a well-defined specialization map.

Proposition 2.33. *Suppose that $Z \subseteq \mathrm{Spo}(B, B^+)$ is a schematic closed subset. Let $\sigma : \mathrm{Spo}(B, B^+) \rightarrow \mathrm{Spec}(B)$ denote the map $x \mapsto \mathbf{supp}(x)$ attaching to every point of $\mathrm{Spo}(B, B^+)$ its support ideal. Notice that $\sigma = \mathbf{supp} \circ h$ where $\mathbf{supp} : \mathrm{Spa}(B, B^+) \rightarrow \mathrm{Spec}(B)$ also attaches the support ideal. Then $Z = \sigma^{-1}(V(I))$ for some prime ideal $I \subseteq B$ open for the topology in B .*

Proof. Any map $m^* : (B, B^+) \rightarrow (A, A)$ with A a discrete ring factors through $(B/B^{\circ\circ}, B^+/B^{\circ\circ})$, so we may assume that B has the discrete topology. By [Proposition 2.31](#), $Z = h^{-1}(h(Z))$ and by [Corollary 2.14](#), Z is closed under v.g. Moreover, since Z is closed in $\mathrm{Spo}(B, B^+)$ it is also stable under vertical specialization. This implies that $Z = \sigma^{-1}(\sigma(Z))$. We prove $\sigma(Z)$ is closed. Since B has the discrete topology, the support map admits a continuous section $\mathrm{Triv} : \mathrm{Spec}(B) \rightarrow \mathrm{Spa}(B, B^+)$ that assigns to a prime ideal $\mathfrak{p} \subseteq B$ the trivial valuation with support \mathfrak{p} . We have $\sigma(Z) = \mathrm{Triv}^{-1}(h(Z))$ so we may prove $h(Z)$ is closed instead. By [Proposition 2.31](#), $h(Z)$ is also closed under horizontal specialization, this gives that the complement of $h(Z)$ in $\mathrm{Spa}(B, B^+)$ is stable under (arbitrary) generization. Now, $\mathrm{Spo}(B, B^+) \setminus Z = h^{-1}(\mathrm{Spa}(B, B^+) \setminus h(Z))$ and by [Proposition 2.25](#) the set $\mathrm{Spa}(B, B^+) \setminus h(Z)$ is open. \square

3. THE REDUCTION FUNCTOR

3.1. The v-topology for perfect schemes. In this section, set-theoretic carefullness is necessary. We advise the reader to review the definition and basic properties of cut-off cardinals [[Sch17](#), §4].

Denote by $\mathrm{PCAlg}_{\mathbb{F}_p}^{\mathrm{op}}$ the category of perfect affine schemes over \mathbb{F}_p . If κ is a cut-off cardinal we let $\mathrm{PCAlg}_{\mathbb{F}_p, \kappa}^{\mathrm{op}}$ be the category of perfect affine schemes over \mathbb{F}_p whose underlying topological space and whose ring of global sections have cardinality bounded by κ . Given $S = \mathrm{Spec}(A) \in \mathrm{PCAlg}_{\mathbb{F}_p}^{\mathrm{op}}$ we associate to it a v-sheaf in Perf given by: $S^\diamond((R, R^+)) = \{f : A \rightarrow R^+ \mid f \text{ is a morphism of rings}\}$.

Remark 3.1. Notice that $\mathrm{Spec}(A)^\diamond = \mathrm{Spd}(A)$ when A is given the discrete topology.

Proposition 3.2. *If κ is a cut-off cardinal and $S \in \mathrm{PCAlg}_{\mathbb{F}_p, \kappa}^{\mathrm{op}}$ then S^\diamond is a κ -small v-sheaf.*

[Proposition 3.2](#) gives rise to functors $\diamond_\kappa : \mathrm{PCAlg}_{\mathbb{F}_p, \kappa}^{\mathrm{op}} \rightarrow \widetilde{\mathrm{Perf}}_\kappa$ that are compatible when we vary κ and give rise to a functor $\diamond : \mathrm{PCAlg}_{\mathbb{F}_p}^{\mathrm{op}} \rightarrow \widetilde{\mathrm{Perf}}$

Proposition 3.3. *The functors $\diamond : \mathrm{PCAlg}_{\mathbb{F}_p}^{\mathrm{op}} \rightarrow \widetilde{\mathrm{Perf}}$ and $\diamond_\kappa : \mathrm{PCAlg}_{\mathbb{F}_p, \kappa}^{\mathrm{op}} \rightarrow \widetilde{\mathrm{Perf}}_\kappa$ are fully-faithful and commute with finite limits.*

Proof. This is a direct consequence of [Theorem 2.29](#). \square

After embedding $\mathrm{PCAlg}_{\mathbb{F}_p}^{\mathrm{op}}$ in $\widetilde{\mathrm{Perf}}$ one can define a Grothendieck topology on $\mathrm{PCAlg}_{\mathbb{F}_p}^{\mathrm{op}}$ by considering a small family of maps of affine schemes, $(S_i \rightarrow T)_{i \in \mathcal{F}}$, to be a cover if the map $\coprod_{i \in \mathcal{F}} S_i^\diamond \rightarrow T^\diamond$ is a surjective map of v-sheaves. However, there is an intrinsic way of defining this topology which we now discuss.

Definition 3.4. ([[BS17](#), Definition 2.1])

- (1) A morphism of qcqs schemes $S \rightarrow T$, is said to be universally subtrusive (or a v-cover) if for any valuation ring V and a map $\mathrm{Spec}(V) \rightarrow T$ there is an extension of valuation rings $V \subseteq W$ ([[Sta18](#), Tag 0ASG]) and a map $\mathrm{Spec}(W) \rightarrow S$ making the following diagram commutative:

$$\begin{array}{ccc} \mathrm{Spec}(W) & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathrm{Spec}(V) & \longrightarrow & T \end{array}$$

- (2) A small family of morphisms in $\mathrm{PCAlg}_{\mathbb{F}_p}^{\mathrm{op}}$, $(S_i \rightarrow T)_{i \in \mathcal{F}}$, is said to be universally subtrusive (or a v-cover) if there is a finite subset $\mathcal{F}' \subseteq \mathcal{F}$ for which $\coprod_{i \in \mathcal{F}'} S_i \rightarrow T$ is universally subtrusive.

Lemma 3.5. ([[BS17](#), Remark 2.2]) *A morphism $f : \mathrm{Spec}(B) \rightarrow \mathrm{Spec}(A)$ of affine schemes (not necessarily over \mathbb{F}_p) is universally subtrusive if and only if the map of topological spaces $|f^{\mathrm{ad}}| : |\mathrm{Spa}(B)| \rightarrow |\mathrm{Spa}(A)|$ is surjective.*

Lemma 3.6. *Let $f : S \rightarrow T$ be a morphism of perfect affine schemes over \mathbb{F}_p . The map $f^\diamond : S^\diamond \rightarrow T^\diamond$ is a quasicompact map of v-sheaves.*

Proof. Observe that for a perfect discrete ring A we have the identity $\mathrm{Spd}(A)^\dagger = \mathrm{Spd}(A)$. We can apply [Lemma 2.23](#). \square

Proposition 3.7. (1) *Let $f : S \rightarrow T$ be a morphism of perfect affine schemes over \mathbb{F}_p . The map f is universally subtrusive if and only if $f^\diamond : S^\diamond \rightarrow T^\diamond$ is a surjective map of v-sheaves.*

(2) *A family of morphisms $(S_i \rightarrow T)_{i \in \mathcal{F}}$ is universally subtrusive if and only if $(\prod_{i \in \mathcal{F}} S_i^\diamond) \rightarrow T^\diamond$ is a surjective map of v-sheaves.*

Proof. Since $f^\diamond : S^\diamond \rightarrow T^\diamond$ is quasicompact, by [[Sch17](#), Lemma 12.11] it is surjective if and only if $|f^\diamond|$ is surjective. By [Proposition 2.11](#) and [Lemma 3.5](#), it suffices to prove that $\mathrm{Spo}(B, B) \rightarrow \mathrm{Spo}(A, A)$ is surjective if and only if the map $\mathrm{Spa}(B) \rightarrow \mathrm{Spa}(A)$ is. Surjectivity of h proves one direction, the converse is a consequence of [Proposition 2.31](#). The second claim, follows easily from the first. \square

Remark 3.8. One can discuss the analogue of [Example 1.1](#). Given an index set I and $\{V_i\}_{i \in I}$ a family of perfect valuation rings over \mathbb{F}_p , we let $R = \prod_{i \in I} V_i$. We call the affine schemes constructed in this way a scheme-theoretic product of points. They form a basis for the v-topology on $\mathrm{PCAlg}_{\mathbb{F}_p}^{\mathrm{op}}$ [[BS17](#), Lemma 6.2].

Given a cut-off cardinal κ we let $\widetilde{\mathrm{SchPerf}}_\kappa$ be the topos associated to the site $\mathrm{PCAlg}_{\mathbb{F}_p, \kappa}^{\mathrm{op}}$ with the v-topology, and we will refer to an object in this topos as a κ -small scheme-theoretic v-sheaf. For any pair of cut-off cardinals $\kappa < \lambda$ we have a continuous fully-faithful embedding of sites $\iota_{\kappa, \lambda}^* : \mathrm{PCAlg}_{\mathbb{F}_p, \kappa}^{\mathrm{op}} \rightarrow \mathrm{PCAlg}_{\mathbb{F}_p, \lambda}^{\mathrm{op}}$, which induces a morphism of topoi $\iota_{\kappa, \lambda} : \widetilde{\mathrm{SchPerf}}_\lambda \rightarrow \widetilde{\mathrm{SchPerf}}_\kappa$.

Proposition 3.9. *The functor $\iota_{\kappa, \lambda}^* : \widetilde{\mathrm{SchPerf}}_\kappa \rightarrow \widetilde{\mathrm{SchPerf}}_\lambda$ is fully-faithful [[Sch17](#), Proposition 8.2].*

Proof. It is enough to prove that the adjunction $\mathcal{F} \rightarrow \iota_{\kappa, \lambda, *}\iota_{\kappa, \lambda}^*\mathcal{F}$ is an isomorphism. Define a presheaf $S \mapsto \mathcal{G}(S)$ constructed as follows. Let \mathcal{C}_S^κ denote the category of maps of affine schemes $S \rightarrow T$ with $T \in \mathrm{PCAlg}_{\mathbb{F}_p, \kappa}^{\mathrm{op}}$. This category is cofiltered and there is a λ -small set of objects $I_S^\kappa \subseteq \mathcal{C}_S^\kappa$, that is cofinal in \mathcal{C}_S^κ . We let $\mathcal{G}(S) = \varinjlim_{T \in I_S^\kappa} \mathcal{F}(T)$, for any choice of I_S^κ . Unraveling the definitions we see that $\iota_{\kappa, \lambda}^*\mathcal{F}$ is the sheafification of \mathcal{G} .

We claim that \mathcal{G} is already a sheaf. Indeed, since filtered colimits are exact it suffices to prove that v-covers $S' \rightarrow S$ in $\mathrm{PCAlg}_{\mathbb{F}_p, \lambda}^{\mathrm{op}}$ are filtered colimits of v-covers in $\mathrm{PCAlg}_{\mathbb{F}_p, \kappa}^{\mathrm{op}}$. Let $S = \mathrm{Spec}(A)$ and let $S' = \mathrm{Spec}(B)$, write $A = \varinjlim_{i \in I_S^\kappa} A_i$ and $B = \varinjlim_{j \in I_{S'}^\kappa} B_j$ with A_i and B_j κ -small rings, we may assume that the transition maps are injective. By [Lemma 3.10](#) below we may assume that the $\mathrm{Spec}(A) \rightarrow \mathrm{Spec}(A_i)$ are v-covers. Consequently, $S' \rightarrow S \rightarrow \mathrm{Spec}(A_i)$ are v-covers and when $S' \rightarrow \mathrm{Spec}(A_i)$ factors through $\mathrm{Spec}(B_j) \rightarrow \mathrm{Spec}(A_i)$ this later one is also a v-cover. Replacing our index sets I_S^κ and $I_{S'}^\kappa$ by a common index set I and replacing B_j by the smallest subring of B containing B_j and A_i for some $i \in I_S^\kappa$ we can ensure $(\mathrm{Spec}(B_i) \rightarrow \mathrm{Spec}(A_i))_{i \in I}$ is defined for all $i \in I$ and is a v-cover. We get our desired expression

$$(S' \rightarrow S) = \varinjlim_{i \in I} (\mathrm{Spec}(B_i) \rightarrow \mathrm{Spec}(A_i))_{i \in I}.$$

Once we know $\iota_{\kappa, \lambda}^*\mathcal{F} = \mathcal{G}$, we compute $\iota_{\kappa, \lambda, *}\iota_{\kappa, \lambda}^*\mathcal{F}(S) = \mathcal{F}(S)$ since the identity is cofinal in \mathcal{C}_S^κ . \square

Lemma 3.10. *Let κ be a cut-off cardinal, $S \in \mathrm{PCAlg}_{\mathbb{F}_p}^{\mathrm{op}}$ and $T \in \mathrm{PCAlg}_{\mathbb{F}_p, \kappa}^{\mathrm{op}}$. Given a morphism $g : S \rightarrow T$, there is $T' \in \mathrm{PCAlg}_{\mathbb{F}_p, \kappa}^{\mathrm{op}}$ together with morphisms $f : S \rightarrow T'$ and $h : T' \rightarrow T$ such that f is a v-cover and $g = h \circ f$.*

Proof. Let $S = \mathrm{Spec}(B)$ and $T = \mathrm{Spec}(A)$. By replacing A by its image in B we may assume $g^* : A \rightarrow B$ to be injective. We construct recursively a countable sequence of κ -small subrings

$$A = A_0 \subseteq \dots \subseteq A_n \subseteq A_{n+1} \subseteq \dots \subseteq B$$

such that the image of $\mathrm{Spa}(B) \rightarrow \mathrm{Spa}(A_n)$ coincides with that of $\mathrm{Spa}(A_{n+1}) \rightarrow \mathrm{Spa}(A_n)$. Assume A_n is defined and let $Z_n \subseteq \mathrm{Spa}(A_n)$ be the image of $\mathrm{Spa}(B)$ in $\mathrm{Spa}(A_n)$. If $x \in \mathrm{Spa}(A_n) \setminus Z_n$ the valuation $|\cdot|_x : A_n \rightarrow \Gamma_x$ can't be extended to a valuation $|\cdot| : B \rightarrow \Gamma$. A compactness argument proves there are finitely many elements $\{a_1, \dots, a_m\}$ such that $|\cdot|_x$ does not extend to $A_n[a_1, \dots, a_m] \subseteq B$. Since $\mathrm{Spa}(A_n) \setminus Z_n$ is κ -small, there is $\lambda < \kappa$ and a set $\{a_i\}_{i \in \lambda} \subseteq B$ such that $A_n[a_i]_{i \in \lambda}$ does not extend any $x \in \mathrm{Spa}(A_n) \setminus Z_n$. We let $A_{n+1} = A_n[a_i^{\frac{1}{p^\kappa}}]_{i \in \lambda}$.

We let $A_\infty = \varinjlim_{i \in \mathbb{N}} A_i$, it is κ -small and we claim that the map $\mathrm{Spec}(B) \rightarrow \mathrm{Spec}(A_\infty)$ is a v -cover. We use [Lemma 3.5](#) to prove instead that $\mathrm{Spa}(B) \rightarrow \mathrm{Spa}(A_\infty)$ is surjective. One verifies that $\mathrm{Spa}(A_\infty) = \varprojlim_{i \in \mathbb{N}} \mathrm{Spa}(A_i)$. Given a compatible sequence $x_i \in \mathrm{Spa}(A_i)$ let M_i be the preimage of x_i in $\mathrm{Spa}(B)$. This gives a sequence $\mathrm{Spa}(B) \supseteq M_0 \supseteq M_1 \dots$. Since the maps $\mathrm{Spa}(B) \rightarrow \mathrm{Spa}(A_i)$ are spectral, each M_i is compact in the patch topology. Any element in this intersection maps to x_∞ . \square

We define $\widetilde{\mathrm{SchPerf}}$ as the big colimit $\bigcup_\kappa \widetilde{\mathrm{SchPerf}}_\kappa$ along all cut-off cardinals and the fully-faithful embeddings $\iota_{\kappa, \lambda}^*$. Objects in $\widetilde{\mathrm{SchPerf}}$ are called small scheme-theoretic v -sheaves.

The general formalism of topoi, specifically ([\[AGV71, IV 4.9.4\]](#)), allows us to promote $\diamond_\kappa : \mathrm{PCAlg}_{\mathbb{F}_p, \kappa}^{\mathrm{op}} \rightarrow \widetilde{\mathrm{Perf}}_\kappa$ to a morphism of topoi $f_\kappa : \widetilde{\mathrm{Perf}}_\kappa \rightarrow \widetilde{\mathrm{SchPerf}}_\kappa$ for which $f_\kappa^*|_{\mathrm{PCAlg}_{\mathbb{F}_p, \kappa}^{\mathrm{op}}} = \diamond_\kappa$.

Proposition 3.11. *Given two cut-off cardinals $\kappa < \lambda$ we have a commutative diagram of morphism of topoi:*

$$\begin{array}{ccc} \widetilde{\mathrm{Perf}}_\lambda & \xrightarrow{f_\lambda} & \widetilde{\mathrm{SchPerf}}_\lambda \\ \downarrow \iota_{\kappa, \lambda} & & \downarrow \iota_{\kappa, \lambda} \\ \widetilde{\mathrm{Perf}}_\kappa & \xrightarrow{f_\kappa} & \widetilde{\mathrm{SchPerf}}_\kappa \end{array}$$

Moreover, the natural morphism $\iota_{\kappa, \lambda}^* \circ f_{\kappa, *}\mathcal{F} \rightarrow f_{\lambda, *}\mathcal{F} \circ \iota_{\kappa, \lambda}^*$ is an isomorphism.

Proof. The commutativity of morphism of topoi follows formally from the similar commutativity of continuous functors. For the second claim, given an element $S \in \mathrm{PCAlg}_{\mathbb{F}_p, \lambda}^{\mathrm{op}}$ we let I_S^κ be an index set category as in the proof of [Proposition 3.9](#). If $S = \mathrm{Spec}(A)$ we let $X = \mathrm{Spa}(A((t^{\frac{1}{p^\infty}})), A[[t^{\frac{1}{p^\infty}}]])$ and $Y = X \times_{S^\diamond} X$. In a similar way, for $T \in I_S^\kappa$ with $T = \mathrm{Spec}(B)$ we let $X_T = \mathrm{Spa}(B((t^{\frac{1}{p^\infty}})), B[[t^{\frac{1}{p^\infty}}]])$ and $Y_T = X_T \times_{T^\diamond} X_T$. The family of perfectoid spaces $(X_T)_{T \in I_S^\kappa}$ ($(Y_T)_{T \in I_S^\kappa}$ respectively) is cofinal in the category \mathcal{C}_X^κ of maps $X \rightarrow X'$ with X' a κ -small perfectoid space (\mathcal{C}_Y^κ respectively). We get the following chain of isomorphisms:

$$\iota_{\kappa, \lambda}^* f_{\kappa, *} \mathcal{F}(S) = \varinjlim_{T \in I_S^\kappa} \mathrm{Hom}_{\widetilde{\mathrm{SchPerf}}_\kappa}(h_T, f_{\kappa, *} \mathcal{F}) \quad (1)$$

$$= \varinjlim_{T \in I_S^\kappa} \mathrm{Hom}_{\widetilde{\mathrm{Perf}}_\kappa}(f_\kappa^* h_T, \mathcal{F}) \quad (2)$$

$$= \varinjlim_{T \in I_S^\kappa} \mathrm{Hom}_{\widetilde{\mathrm{Perf}}_\kappa}(T^{\diamond \kappa}, \mathcal{F}) \quad (3)$$

$$= \varinjlim_{T \in I_S^\kappa} \mathrm{Eq}_{\widetilde{\mathrm{Perf}}_\kappa}(\mathrm{Hom}(X_T, \mathcal{F}) \rightrightarrows \mathrm{Hom}(Y_T, \mathcal{F})) \quad (4)$$

$$= \mathrm{Eq}_{\widetilde{\mathrm{Perf}}_\lambda}(\varinjlim_{T \in I_S^\kappa} \mathrm{Hom}(X_T, \mathcal{F}) \rightrightarrows \varinjlim_{T \in I_S^\kappa} \mathrm{Hom}(Y_T, \mathcal{F})) \quad (5)$$

$$= \mathrm{Eq}_{\widetilde{\mathrm{Perf}}_\lambda}(\mathrm{Hom}(X_S, \iota_{\kappa, \lambda}^* \mathcal{F}) \rightrightarrows \mathrm{Hom}(Y_S, \iota_{\kappa, \lambda}^* \mathcal{F})) \quad (6)$$

$$= \mathrm{Hom}_{\widetilde{\mathrm{Perf}}_\lambda}(S^{\diamond \lambda}, \iota_{\kappa, \lambda}^* \mathcal{F}) \quad (7)$$

$$= \mathrm{Hom}_{\widetilde{\mathrm{SchPerf}}_\lambda}(h_S, f_{\lambda, *} \iota_{\kappa, \lambda}^* \mathcal{F}) \quad (8)$$

$$= f_{\lambda, *} \iota_{\kappa, \lambda}^* \mathcal{F}(S) \quad (9)$$

\square

Recall that a morphism of topoi consists of a pair of adjoint functors (f^*, f_*) such that f^* commutes with finite limits. By [Proposition 3.11](#) above we can gather all of the morphisms of topoi $f_\kappa : \widetilde{\mathrm{Perf}}_\kappa \rightarrow \widetilde{\mathrm{SchPerf}}_\kappa$ into a pair of adjoint functors $(f^*, f_*) : \widetilde{\mathrm{Perf}} \rightarrow \widetilde{\mathrm{SchPerf}}$ such that f^* commutes with finite limits. This is not a morphism of topoi because $\widetilde{\mathrm{Perf}}$ and $\widetilde{\mathrm{SchPerf}}$ are not topoi, but they behave as such.

Definition 3.12. Let (f^*, f_*) be the pair of adjoint functors described above, given $\mathcal{F} \in \widetilde{\mathrm{SchPerf}}$ we will denote $f^* \mathcal{F}$ by \mathcal{F}^\diamond and given $\mathcal{G} \in \widetilde{\mathrm{Perf}}$ we will denote $f_* \mathcal{G}$ by $(\mathcal{G})^{\mathrm{red}}$. We refer to $(-)^{\mathrm{red}}$ as the reduction functor.

Remark 3.13. By adjunction $\mathcal{F}^{\mathrm{red}}(S) = \mathrm{Hom}_{\widetilde{\mathrm{Perf}}}(S^\diamond, \mathcal{F})$. We could have simply defined it in this way, but it is useful to know that “reduction” preserves smallness.

We can endow small $\widetilde{\text{SchPerf}}$ v-sheaf with a topological space in a similar fashion to [Definition 1.7](#). Given $\mathfrak{S} \in \widetilde{\text{SchPerf}}$ we let $|\mathfrak{S}|$ denote the set of equivalence classes of maps $\text{Spec}(k) \rightarrow \mathfrak{S}$, where k is a perfect field over \mathbb{F}_p . Two maps p_1, p_2 are equivalent if we can complete a commutative diagram as below:

$$\begin{array}{ccc}
 & \text{Spec}(k_1) & \\
 q_1 \nearrow & & \searrow p_1 \\
 \text{Spec}(k_3) & \xrightarrow{p_3} & \mathfrak{S} \\
 q_2 \searrow & & \nearrow p_2 \\
 & \text{Spec}(k_2) &
 \end{array}$$

Proposition 3.14. *Let $\mathfrak{S} \in \widetilde{\text{SchPerf}}$ the following hold:*

- (1) *There is a pair of cut-off cardinals $\kappa < \lambda$ and a λ -small family $\{S_i\}_{i \in I}$ of objects in $\text{PCAlg}_{\mathbb{F}_p, \kappa}^{\text{op}}$ together with a surjective map $X = (\coprod_{i \in I} S_i) \rightarrow \mathfrak{S}$.*
- (2) *The small scheme-theoretic v-sheaf $R = X \times_{\mathfrak{S}} X$ has a similar cover $Y = (\coprod_{j \in J} T_j) \rightarrow R$, there is a natural map $|X| \rightarrow |\mathfrak{S}|$ which induces a bijection $|\mathfrak{S}| \cong |X|/|Y|$. We endow $|\mathfrak{S}|$ with the quotient topology induced by this bijection.*
- (3) *The topology on $|\mathfrak{S}|$ does not depend on the choices of X or Y .*
- (4) *Any map of small v-sheaves $\mathfrak{S}_1 \rightarrow \mathfrak{S}_2$ induces a continuous map of topological spaces $|\mathfrak{S}_1| \rightarrow |\mathfrak{S}_2|$.*

3.2. Reduction functor and formal adicness.

Definition 3.15. Let $\mathcal{F} \in \widetilde{\text{SchPerf}}$, we say it is *reduced* if $\mathcal{F} \rightarrow (\mathcal{F}^\diamond)^{\text{red}}$ is an isomorphism.

Proposition 3.16. ([\[SW20, Proposition 18.3.1\]](#))

- (1) *If S is a perfect scheme over \mathbb{F}_p then the Yoneda functor h_S is reduced.*
- (2) *The functor $\diamond : \widetilde{\text{SchPerf}} \rightarrow \widetilde{\text{Perf}}$ is fully-faithful when restricted to small reduced v-sheaves.*

Proof. The first claim follows from [Theorem 2.29](#). The second claim follows from adjunction. Indeed, $\text{Hom}_{\widetilde{\text{Perf}}}(\mathcal{G}^\diamond, \mathcal{F}^\diamond) = \text{Hom}_{\widetilde{\text{SchPerf}}}(\mathcal{G}, (\mathcal{F}^\diamond)^{\text{red}}) = \text{Hom}_{\widetilde{\text{SchPerf}}}(\mathcal{G}, \mathcal{F})$. \square

Intuitively, the reduction functor kills all topological nilpotent elements and removes analytic points. One can think of reduction functor as taking the underlying reduced subscheme of a formal scheme.

Lemma 3.17. *The scheme-theoretic v-sheaf $\text{Spd}(\mathbb{Z}_p)^{\text{red}}$ is represented by $\text{Spec}(\mathbb{F}_p)$.*

Proof. This is a direct consequence of [Lemma 2.27](#). \square

For an f-adic ring A over \mathbb{Z}_p , we let $A_{\text{red}} = (A/(A \cdot A^\circ))^{\text{perf}}$ where $A \cdot A^\circ$ is the ideal generated by the topological nilpotent elements. The following statement generalizes [Lemma 3.17](#)

Proposition 3.18. *Let X be a pre-adic space over \mathbb{Z}_p and let X^{na} be the reduced adic space associated to the non-analytic locus of [Proposition 1.19](#). The following hold:*

- (1) *The map $(X^{\text{na}, \diamond})^{\text{red}} \rightarrow (X^\diamond)^{\text{red}}$ is an isomorphism.*
- (2) *If $X = \text{Spa}(A, A^+)$ for (A, A^+) a Huber pair over \mathbb{Z}_p , then $\text{Spd}(A, A^+)^{\text{red}}$ is represented by $\text{Spec}(A_{\text{red}})$.*

Proof. By [Theorem 2.29](#) if $S = \text{Spec}(R) \in \text{PCAlg}_{\mathbb{F}_p}^{\text{op}}$ then morphisms $S^\diamond \rightarrow X$ are given by maps of pre-adic spaces $f : \text{Spa}(R) \rightarrow X$. These factor through the non-analytic locus. The non-analytic locus of $\text{Spa}(A, A^+)$ is represented by the Huber pair $(A/A^\circ \cdot A, A^\circ \cdot A^+)$. Since R is perfect the map $f^* : A/A^\circ \cdot A \rightarrow R$ factors uniquely through its perfection. \square

Proposition 3.19. *If Y is a quasiseparated diamond, then $Y^{\text{red}} = \emptyset$.*

Proof. It suffices to prove that there are no maps $f : S^\diamond \rightarrow Y$ for $S = \text{Spec}(k)$ and k an algebraically closed field. Suppose f exists and let $y \in |Y|$ be the unique point in the image of $|f|$. Consider Y_y the sub-v-sheaf of points that factors through y . By [\[Sch17, Proposition 11.10\]](#) it is a quasiseparated diamond and $|Y_y|$ consists of one point. Using [\[Sch17, Proposition 21.9\]](#) we write $Y_y = \text{Spa}(C, O_C)/\underline{G}$

with C a nonarchimedean algebraically closed field over \mathbb{F}_p and \underline{G} a profinite group acting continuously and faithfully on C .

Consider the v-cover $S' = \mathrm{Spa}(K_1, O_{K_1}) \rightarrow \mathrm{Spec}(k)^\diamond$ where K_1 is an algebraic closure of $k((t^{\frac{1}{p^\infty}}))$. Similarly, let $T = \mathrm{Spa}(K_2, O_{K_2})$ where K_2 is an algebraically closed nonarchimedean field containing k discretely and whose value group $\Gamma_{K_2} \subseteq \mathbb{R}^{>0}$ has at least two \mathbb{Q} -linearly independent elements. By hypothesis on K_2 , we can find two embeddings $\iota_i^* : K_1 \rightarrow K_2$ with $\iota_1^*(K_1) \cap \iota_2^*(K_1) = k$.

The composition of $[g] : \mathrm{Spa}(K_1, K_1^+) \rightarrow S^\diamond \rightarrow Y_y$ satisfies $[g] \circ \iota_1 = [g] \circ \iota_2$. Since $\mathrm{Spa}(K_1, K_1^+)$ and $\mathrm{Spa}(K_2, K_2^+)$ are algebraically closed the maps to Y_y are given by G -orbits of maps to $\mathrm{Spa}(C, O_C)$. Let $g^* : (C, O_C) \rightarrow (K_1, O_{K_1})$ represent $[g]$ in $\mathrm{Hom}(\mathrm{Spa}(K_1, K_1^+), Y_y)$, we get maps $\iota_i^* \circ g^* : (C, O_C) \rightarrow (K_2, O_{K_2})$ and since $[g] \circ \iota_1 = [g] \circ \iota_2$ we have $\iota_1^* \circ g^*(C) = \iota_2^* \circ g^*(C) \subseteq k$. The incompatibility of topology between k and C gives the contradiction. \square

Recall that a morphism of adic spaces $X \rightarrow Y$ is said to be adic if the image of an analytic point is again an analytic point. For v-sheaves we can define a related notion.

Definition 3.20. A morphism $\mathcal{F} \rightarrow \mathcal{G}$ is *formally adic* if the following diagram is Cartesian:

$$\begin{array}{ccc} (\mathcal{F}^{\mathrm{red}})^\diamond & \longrightarrow & (\mathcal{G}^{\mathrm{red}})^\diamond \\ \downarrow & & \downarrow \\ \mathcal{F} & \longrightarrow & \mathcal{G} \end{array}$$

Although the notion of a morphism of adic spaces being adic is related to the morphism of v-sheaves being formally adic neither of these notions implies the other.

Example 3.21. Endow $\mathbb{F}_p((t))$ with the discrete topology, then $\mathrm{Spa}(\mathbb{F}_p((t)), \mathbb{F}_p[[t]]) \rightarrow \mathrm{Spa}(\mathbb{F}_p, \mathbb{F}_p)$ is adic. Nevertheless, $\mathrm{Spd}(\mathbb{F}_p((t)), \mathbb{F}_p[[t]]) \rightarrow \mathrm{Spd}(\mathbb{F}_p, \mathbb{F}_p)$ is not formally adic. Observe that $\mathrm{Spo}(\mathbb{F}_p((t)), \mathbb{F}_p[[t]])$ has an unbounded meromorphic point.

Example 3.22. Let K be a perfect nonarchimedean field and consider $\mathrm{Id} : \mathrm{Spa}(K_1, O_{K_1}) \rightarrow \mathrm{Spa}(K_2, O_{K_2})$ where $K_2 = K$ given the discrete topology and $K_1 = K$ given the norm topology. This morphism is not adic, but the reduction diagram is Cartesian. Indeed, it looks like this:

$$\begin{array}{ccc} \emptyset & \longrightarrow & \mathrm{Spec}(K_2)^\diamond \\ \downarrow & & \downarrow \\ \mathrm{Spd}(K_1, O_{K_1}) & \longrightarrow & \mathrm{Spd}(K_2, O_{K_2}) \end{array}$$

Although formal adicness does not capture adicness in general, it does in important situations:

Proposition 3.23. Let (A, A) and (B, B) be formal Huber pairs over \mathbb{Z}_p with ideals of definition I_A and I_B respectively. Then $\mathrm{Spa}(A) \rightarrow \mathrm{Spa}(B)$ is adic if and only if $\mathrm{Spd}(A) \rightarrow \mathrm{Spd}(B)$ is formally adic.

Proof. The reduction diagram looks as follows:

$$\begin{array}{ccccc} \mathrm{Spec}(A_{\mathrm{red}})^\diamond & & & & \\ & \searrow & & \searrow & \\ & & (\mathrm{Spec}(A/I_B)^{\mathrm{perf}})^\diamond & \longrightarrow & \mathrm{Spec}(B_{\mathrm{red}})^\diamond \\ & \searrow & \downarrow & & \downarrow \\ & & \mathrm{Spd}(A) & \longrightarrow & \mathrm{Spd}(B) \end{array}$$

Continuity of the morphism $B \rightarrow A$ ensures that $I_B^n \subseteq I_A$ for some n . The morphism is adic if and only if $I_A^m \subseteq A \cdot I_B$ for some m . If the morphism is adic, then A/I_A and $(A/A \cdot I_B)$ become isomorphic after taking perfection which gives formal adicness. Conversely, if the morphism is formally adic, by hypothesis the rings $(A/I_B)^{\mathrm{perf}}$, and A_{red} are isomorphic with the isomorphism being induced by the natural surjective ring map with source $(A/p)^{\mathrm{perf}}$. This implies that the ideals I_A and I_B define the same Zariski closed subset in $\mathrm{Spec}(A)$. In particular, the elements of I_A are nilpotent in A/I_B , and since I_A is finitely generated $I_A^m \subseteq I_B$ for some m . \square

Proposition 3.24. Let \mathcal{F} , \mathcal{G} and \mathcal{H} be small v-sheaves.

- (1) If $\mathcal{F} \rightarrow \mathcal{H}$ and $\mathcal{H} \rightarrow \mathcal{G}$ are formally adic, the composition $\mathcal{F} \rightarrow \mathcal{G}$ is formally adic.
- (2) If $\mathcal{F} \rightarrow \mathcal{H}$ is formally adic, the basechange $\mathcal{G} \times_{\mathcal{H}} \mathcal{F} \rightarrow \mathcal{G}$ is formally adic.

Proof. The first claim is clear. The second follows from the commutativity of $(-)^{\text{red}}$ and $(-)^{\diamond}$ with finite limits. \square

Definition 3.25. We say that a v-sheaf \mathcal{F} over $\text{Spd}(\mathbb{Z}_p)$ is *formally p -adic* (or just *p -adic* when the context is clear) if the morphism $\mathcal{F} \rightarrow \text{Spd}(\mathbb{Z}_p)$ is formally adic.

Over \mathbb{Z}_p the situation of [Example 3.22](#) does not happen.

Proposition 3.26. *Suppose we have a Huber pair (A, A^+) and a map $f : \text{Spa}(A, A^+) \rightarrow \text{Spa}(\mathbb{Z}_p)$, if f^{\diamond} is formally adic then f is adic (as a morphism of adic spaces).*

Proof. Let $U \subseteq \text{Spa}(A, A^+)$ the open subset of analytic points. It follows from [Proposition 3.19](#) and [Proposition 3.18](#) that $U^{\diamond} \rightarrow \text{Spd}(A, A^+)$ is formally adic. By [Proposition 3.24](#), $U^{\diamond} \rightarrow \text{Spd}(\mathbb{Z}_p)$ is formally adic and the map must factor through $\text{Spd}(\mathbb{Q}_p)$. This proves that f is adic. \square

Recall that a v-sheaf \mathcal{F} is said to be separated if the diagonal $\mathcal{F} \rightarrow \mathcal{F} \times \mathcal{F}$ is a closed immersion [[Sch17](#), Definition 10.7]. We need the following related notion:

Definition 3.27. Let \mathcal{F} and \mathcal{G} be small v-sheaves.

- (1) We say $\mathcal{F} \rightarrow \mathcal{G}$ is *formally closed* if it is a formally adic closed immersion.
- (2) We say that a v-sheaf is *formally separated* if the diagonal map $\mathcal{F} \rightarrow \mathcal{F} \times \mathcal{F}$ is formally closed.

Lemma 3.28. *The v-sheaf $\text{Spd}(\mathbb{Z}_p)$ is formally separated.*

Proof. To prove that the diagonal $\text{Spd}(\mathbb{Z}_p) \rightarrow \text{Spd}(\mathbb{Z}_p) \times \text{Spd}(\mathbb{Z}_p)$ is a closed immersion observe that the basechanges by maps $\text{Spa}(R, R^+) \rightarrow \text{Spd}(\mathbb{Z}_p) \times \text{Spd}(\mathbb{Z}_p)$, with $\text{Spa}(R, R^+) \in \text{Perf}$ define the locus on which two untilts agree in $|\text{Spa}(R, R^+)|$. Each untilt is individually cut out of $\text{Spa}(W(R^+)) \setminus \{V([\varpi])\}$ as a closed Cartier divisor [[SW20](#), Proposition 11.3.1]. The intersection defines a Zariski closed subset in each of the untilts and these are represented by a perfectoid space.

We compute directly $(\text{Spd}(\mathbb{Z}_p) \times \text{Spd}(\mathbb{Z}_p))^{\text{red}} = \mathbb{F}_p$ since $(-)^{\text{red}}$ commutes with limits. On the other hand, $\text{Spd}(\mathbb{F}_p) \times_{\text{Spd}(\mathbb{Z}_p)^2} \text{Spd}(\mathbb{Z}_p) = \text{Spd}(\mathbb{F}_p)$, which proves that the diagonal is formally adic. \square

Proposition 3.29. *If \mathcal{F} is formally p -adic, then the diagonal $\mathcal{F} \rightarrow \mathcal{F} \times \mathcal{F}$ is formally adic.*

Proof. We have a formally adic map $\mathcal{F} \rightarrow \text{Spd}(\mathbb{Z}_p)$, and since formal adicness is preserved by basechange and composition we get a formally adic map $\mathcal{F} \times_{\text{Spd}(\mathbb{Z}_p)} \mathcal{F} \rightarrow \text{Spd}(\mathbb{Z}_p)$. By a general property of Cartesian diagrams, the diagonal map $\mathcal{F} \rightarrow \mathcal{F} \times_{\text{Spd}(\mathbb{Z}_p)} \mathcal{F}$ is also formally adic. Now, $\mathcal{F} \times_{\text{Spd}(\mathbb{Z}_p)} \mathcal{F}$ is the basechange of the diagonal $\text{Spd}(\mathbb{Z}_p) \rightarrow \text{Spd}(\mathbb{Z}_p) \times \text{Spd}(\mathbb{Z}_p)$ by the projection $\mathcal{F} \times \mathcal{F} \rightarrow \text{Spd}(\mathbb{Z}_p) \times \text{Spd}(\mathbb{Z}_p)$. This gives that $\mathcal{F} \times_{\text{Spd}(\mathbb{Z}_p)} \mathcal{F} \rightarrow \mathcal{F} \times \mathcal{F}$ and by composition that $\mathcal{F} \rightarrow \mathcal{F} \times \mathcal{F}$ are also formally adic. \square

Lemma 3.30. *The diagonal $\mathcal{F} \rightarrow \mathcal{F} \times \mathcal{F}$ is formally adic if and only if the adjunction morphism $(\mathcal{F}^{\text{red}})^{\diamond} \rightarrow \mathcal{F}$ is injective. Let $\text{Spa}(A, A^+) \in \text{Perf}$ and $m \in \mathcal{F}(A, A^+)$. Then $m \in (\mathcal{F}^{\text{red}})^{\diamond}(A, A^+)$ if and only if $\text{Spa}(A, A^+)$ admits a v-cover $\text{Spa}(R, R^+) \rightarrow \text{Spa}(A, A^+)$ and a map $\text{Spec}(R^+)^{\diamond} \rightarrow \mathcal{F}$ making the following diagram commutative:*

$$\begin{array}{ccc} \text{Spa}(R, R^+) & \longrightarrow & \text{Spec}(R^+)^{\diamond} \\ \downarrow & & \downarrow \\ \text{Spa}(A, A^+) & \xrightarrow{m} & \mathcal{F} \end{array}$$

Proof. In general, a map of sheaves $\mathcal{G} \rightarrow \mathcal{F}$ is injective if and only if $(\mathcal{G} \times \mathcal{G}) \times_{\mathcal{F} \times \mathcal{F}} \mathcal{F} = \mathcal{G}$. Now, $(\mathcal{F}^{\text{red}})^{\diamond}$ is the sheafification of $(R, R^+) \mapsto \text{Hom}(\text{Spec}(R^+)^{\diamond}, \mathcal{F})$. The description given in the statement above is what one gets from taking sheafification and assuming injectivity of $(\mathcal{F}^{\text{red}})^{\diamond} \rightarrow \mathcal{F}$. \square

The following lemma will be key for our theory of specialization, it roughly says that formally adic closed immersions behave as expected:

Lemma 3.31. *Let (A, A) be a formal Huber pair and let $\mathcal{F} \rightarrow \text{Spd}(A)$ be formally adic closed immersion. Then $(\mathcal{F}^{\text{red}})^{\diamond} = \text{Spec}(A/J)^{\diamond}$ for some open ideal $J \subseteq A$.*

Proof. $|\mathcal{F}| \subseteq \mathrm{Spo}(A, A)$ is closed and we get an expression $\mathcal{F} = \mathrm{Spd}(A) \times_{|\mathrm{Spd}(A, A^+)|} |\mathcal{F}|$. By [Proposition 3.18](#), $(\mathrm{Spd}(A)^{\mathrm{red}})^{\diamond} = \mathrm{Spec}(A_{\mathrm{red}})^{\diamond}$ which is closed in $\mathrm{Spd}(A)$. By formal adicness $(\mathcal{F}^{\mathrm{red}})^{\diamond}$ is closed in $\mathrm{Spd}(A)$ determined by $|\mathcal{F}| \cap |\mathrm{Spec}(A_{\mathrm{red}})^{\diamond}|$. By [Lemma 3.30](#), a map $\mathrm{Spa}(R, R^+) \rightarrow \mathcal{F}$ factors through $(\mathcal{F}^{\mathrm{red}})^{\diamond}$ if after possibly replacing R by a v-cover it factors through $\mathrm{Spec}(R^+) \rightarrow \mathcal{F} \cap \mathrm{Spec}(A_{\mathrm{red}})^{\diamond}$. This proves that $|\mathcal{F}^{\mathrm{red}})^{\diamond}|$ is a schematic closed subset of $\mathrm{Spo}(A, A)$ as in [Definition 2.32](#). By [Proposition 2.33](#), it is a Zariski closed subset corresponding to an open ideal $J \subseteq A$. \square

We will often use implicitly the following easy result.

Lemma 3.32. *Let \mathcal{F} and \mathcal{G} be two small v-sheaves, and $f : \mathcal{F} \rightarrow \mathcal{G}$ a map between them. Suppose that the adjunction map $(\mathcal{G}^{\mathrm{red}})^{\diamond} \rightarrow \mathcal{G}$ is injective and that $\mathcal{F} \times_{\mathcal{G}} (\mathcal{G}^{\mathrm{red}})^{\diamond}$ is representable by a reduced scheme-theoretic v-sheaf, then f is formally adic.*

Proof. Let $T \in \widetilde{\mathrm{SchPerf}}$ be reduced and such that $T^{\diamond} = \mathcal{F} \times_{\mathcal{G}} (\mathcal{G}^{\mathrm{red}})^{\diamond}$. By hypothesis $(\mathcal{G}^{\mathrm{red}})^{\diamond} \rightarrow \mathcal{G}$ is a monomorphism and since $(-)^{\mathrm{red}}$ is a right adjoint $((\mathcal{G}^{\mathrm{red}})^{\diamond})^{\mathrm{red}} \rightarrow \mathcal{G}^{\mathrm{red}}$ is also a monomorphism. Recall that for any pair of adjoint functors (L, R) the compositions $R \rightarrow R \circ L \circ R \rightarrow R$ and $L \rightarrow L \circ R \circ L \rightarrow L$ are the identity. This implies that $((\mathcal{G}^{\mathrm{red}})^{\diamond})^{\mathrm{red}} \rightarrow \mathcal{G}^{\mathrm{red}}$ is an isomorphism. We compute directly:

$$(T^{\diamond})^{\mathrm{red}} = (\mathcal{F} \times_{\mathcal{G}} (\mathcal{G}^{\mathrm{red}})^{\diamond})^{\mathrm{red}} = \mathcal{F}^{\mathrm{red}} \times_{\mathcal{G}^{\mathrm{red}}} ((\mathcal{G}^{\mathrm{red}})^{\diamond})^{\mathrm{red}} = \mathcal{F}^{\mathrm{red}} \times_{\mathcal{G}^{\mathrm{red}}} \mathcal{G}^{\mathrm{red}} = \mathcal{F}^{\mathrm{red}} \text{ and } (\mathcal{F}^{\mathrm{red}})^{\diamond} = ((T^{\diamond})^{\mathrm{red}})^{\diamond} = T^{\diamond}. \quad \square$$

4. SPECIALIZATION

4.1. Specialization for Tate Huber pairs.

Definition 4.1. Given a Tate Huber pair (A, A^+) over \mathbb{Z}_p and a pseudo-uniformizer $\varpi \in A$, we define the specialization map $\mathrm{sp}_A : |\mathrm{Spa}(A, A^+)| \rightarrow |\mathrm{Spec}(A_{\mathrm{red}}^+)|$ by sending a valuation $|\cdot|_x \in |\mathrm{Spa}(A, A^+)|$ to the ideal $\mathfrak{p} \subseteq A^+$ given by $\mathfrak{p} = \{a \in A^+ \mid |a|_x < 1\}$.

These maps of sets are functorial in the category of Tate Huber pairs. We thank David Hansen for providing reference and an explanation of the following statement.

Proposition 4.2. ([\[Bha17, Theorem 8.1.2\]](#)) *The specialization map $\mathrm{sp}_A : |\mathrm{Spa}(A, A^+)| \rightarrow |\mathrm{Spec}(A_{\mathrm{red}}^+)|$ is a continuous, surjective, spectral and closed map of spectral topological spaces.*

Proposition 4.3. *For a strictly totally disconnected space $\mathrm{Spa}(R, R^+)$, the specialization map sp_R is a homeomorphism.*

Proof. By [Proposition 4.2](#) the map is surjective and a quotient map so it suffices to prove injectivity. One first proves that if $\mathrm{sp}_R(x) = \mathrm{sp}_R(y)$, then x and y are in the same connected component of $|\mathrm{Spa}(R, R^+)|$. In this way one reduces to prove injectivity component by component. Using [Proposition 1.5](#) we can assume $R = C$, and this case follows from generalities of valuation rings. \square

Remark 4.4. One can also prove [Proposition 4.2](#) if we knew already that [Proposition 4.3](#) holds.

4.2. Specializing v-sheaves. We now discuss the specialization map for v-sheaves. The idea is to descend the specialization map from the case of formal Huber pairs.

Definition 4.5. We say that a small v-sheaf \mathcal{F} is *v-locally formal* if there is a set I , a family $(B_i, B_i)_{i \in I}$ of formal Huber pairs over \mathbb{Z}_p and a surjective map of v-sheaves $\coprod_{i \in I} \mathrm{Spd}(B_i) \rightarrow \mathcal{F}$.

Definition 4.6. Let $\mathcal{F} \in \widetilde{\mathrm{Perf}}$, (A, A^+) be a Tate Huber pair and $f : \mathrm{Spd}(A, A^+) \rightarrow \mathcal{F}$ a map.

- (1) We say that \mathcal{F} *formalizes* f (or that f is *formalizable*) if there is $t : \mathrm{Spd}(A^+) \rightarrow \mathcal{F}$ factoring f .
- (2) We say that \mathcal{F} *v-formalizes* f if for a v-cover $g : \mathrm{Spa}(B, B^+) \rightarrow \mathrm{Spa}(A, A^+)$, \mathcal{F} formalizes $f \circ g$.
- (3) We say that \mathcal{F} is *formalizing* if it formalizes maps with source an affinoid perfectoid space.
- (4) We say that \mathcal{F} is *v-formalizing* if it v-formalizes any f as above.

We use this extensively because it gives an abstract way to verify that a v-sheaf is v-locally formal.

Lemma 4.7. *The following statements hold:*

- (1) *The v-sheaf $\mathrm{Spd}(\mathbb{Z}_p)$ is formalizing.*
- (2) *$\mathrm{Spd}(B)$ is formalizing for any formal Huber pair over \mathbb{Z}_p .*
- (3) *A small v-sheaf \mathcal{F} is v-formalizing if and only if it is v-locally formal.*

Proof. Let $\mathrm{Spa}(R, R^+) \in \mathrm{Perf}$ in characteristic p and an untilt $\iota : (R^\sharp)^\flat \rightarrow R$. Let $\xi = p + [\varpi]\alpha$ be a generator of the kernel of $W(R^+) \rightarrow (R^\sharp)^+$. The image of ξ under $W(R^+) \rightarrow W(A^+)$ defines an untilt of $\mathrm{Spa}(A, A^+)$ for every map $\mathrm{Spa}(A, A^+) \rightarrow \mathrm{Spd}(R^+)$, and gives a map $\mathrm{Spd}(R^+) \rightarrow \mathrm{Spd}(\mathbb{Z}_p)$. Now, given $\mathrm{Spa}(R^\sharp, R^{\sharp,+}) \rightarrow \mathrm{Spa}(B)$ we get $B \rightarrow R^{\sharp,+}$ and $\mathrm{Spd}(R^{\sharp,+}) \rightarrow \mathrm{Spd}(B)$. For the third claim, assume that \mathcal{F} is v -formalizing. Since it is small there is a set I and a surjective map by a union of affinoid perfectoid spaces $\coprod_{i \in I} \mathrm{Spa}(R_i, R_i^+) \rightarrow \mathcal{F}$. After refining this cover we may assume that each $\mathrm{Spa}(R_i, R_i^+) \rightarrow \mathcal{F}$ formalizes to $\mathrm{Spd}(R_i^+) \rightarrow \mathcal{F}$. Then $\coprod_{i \in I} \mathrm{Spd}(R_i^+) \rightarrow \mathcal{F}$ is surjective, so \mathcal{F} is v -locally formal. If \mathcal{F} is v -locally formal a map $\mathrm{Spa}(R, R^+) \rightarrow \mathcal{F}$ will v -locally factor through a map $\mathrm{Spa}(R, R^+) \rightarrow \mathrm{Spd}(B_i)$. By the second claim, this map formalizes $\mathrm{Spd}(R^+) \rightarrow \mathrm{Spd}(B_i)$. \square

Proposition 4.8. *The following properties are easy to verify.*

- (1) *If $f : \mathcal{F} \rightarrow \mathcal{G}$ is a surjective map of small v -sheaves and \mathcal{F} is v -formalizing then \mathcal{G} is v -formalizing.*
- (2) *If $\mathrm{Spec}(R) \in \mathrm{PCAlg}_{\mathbb{F}_p}^{\mathrm{op}}$ then $\mathrm{Spec}(R)^\diamond$ is formalizing.*
- (3) *If $X \in \widetilde{\mathrm{SchPerf}}$ then X^\diamond is v -formalizing by [Lemma 3.30](#).*
- (4) *Non-empty v -formalizing v -sheaves have non-empty reduction. Quasi-separated diamonds are not v -formalizing.*
- (5) *If \mathcal{F} formalizes $f : \mathrm{Spa}(A, A^+) \rightarrow \mathcal{F}$ then \mathcal{F} formalizes $f \circ g$ for any map $g : \mathrm{Spa}(B, B^+) \rightarrow \mathrm{Spa}(A, A^+)$.*

Proposition 4.9. *Let \mathcal{F} be a small v -sheaf, and $f : \mathrm{Spa}(R, R^+) \rightarrow \mathcal{F}$ a map with $\mathrm{Spa}(R, R^+)$ affinoid perfectoid in characteristic p . If \mathcal{F} is formally separated then f admits at most one formalization.*

Proof. Pick two maps $g_i : \mathrm{Spd}(R^+) \rightarrow \mathcal{F}$ that agree on $\mathrm{Spa}(R, R^+)$. Consider $(g_1, g_2) : \mathrm{Spd}(R^+) \rightarrow \mathcal{F} \times \mathcal{F}$, and the pullback along $\Delta_{\mathcal{F}} : \mathcal{F} \rightarrow \mathcal{F} \times \mathcal{F}$ to get $\mathcal{G} \subseteq \mathrm{Spd}(R^+)$ a formally closed subsheaf. We prove $\mathcal{G} = \mathrm{Spd}(R^+)$, it suffices to show $|\mathcal{G}| = |\mathrm{Spd}(R^+)|$. Moreover, since $|\mathrm{Spa}(R, R^+)| \subseteq |\mathcal{G}|$ and $|\mathrm{Spd}(R^+)| = |\mathrm{Spa}(R, R^+)| \cup |\mathrm{Spec}(R_{\mathrm{red}}^+)^\diamond|$ it suffices to prove $|\mathcal{G}^{\mathrm{red}}|^\diamond = |\mathrm{Spec}(R_{\mathrm{red}}^+)^\diamond|$. We first assume $(R, R^+) = (C, C^+)$ for C is a nonarchimedean field and $C^+ \subseteq C$ an open and bounded valuation subring. Let $k^+ = C_{\mathrm{red}}^+$ and $k = \mathrm{Frac}(k^+)$, then $\mathrm{Spec}(k^+) = \mathrm{Spd}(C^+)_{\mathrm{red}}$ and by [Lemma 3.31](#) $(\mathcal{G}^{\mathrm{red}})^\diamond = \mathrm{Spec}(k^+/I)^\diamond$ for some ideal I . Since $\mathrm{Spa}(C, C^+) \subseteq \mathcal{G}$ and $|\mathcal{G}|$ is closed, $|\mathcal{G}|$ contains the formal specialization of $\mathrm{Spa}(C, O_C)$, which is the image of $\mathrm{Spec}(k)^\diamond$. By formal adicness $|\mathcal{G}^{\mathrm{red}}|^\diamond = |\mathcal{G}| \cap |\mathrm{Spec}(k^+)^\diamond|$ and we can conclude that $\mathrm{Spec}(k)^\diamond \subseteq (\mathcal{G}^{\mathrm{red}})^\diamond$. This proves $I = \{0\}$ and $(\mathcal{G}^{\mathrm{red}})^\diamond = \mathrm{Spec}(k^+)^\diamond$ in this case.

In the general case, for every map $\mathrm{Spa}(C, C^+) \rightarrow \mathrm{Spa}(R, R^+)$ the canonical formalization $\mathrm{Spd}(C^+) \rightarrow \mathrm{Spd}(R^+)$ factors through \mathcal{G} . In particular, after taking reduction, the map $\mathrm{Spec}(k^+) \rightarrow \mathrm{Spec}(R_{\mathrm{red}}^+)$ factors through $\mathcal{G}^{\mathrm{red}}$. This says that $|\mathcal{G}^{\mathrm{red}}|$ contains every point of $|\mathrm{Spec}(R_{\mathrm{red}}^+)|$ in the image of the specialization map. By [Lemma 3.31](#) $\mathcal{G}^{\mathrm{red}} \rightarrow \mathrm{Spec}(R_{\mathrm{red}}^+)$ is a closed immersion and by [Proposition 4.2](#) the specialization map is surjective, these two imply that $\mathcal{G}^{\mathrm{red}} = \mathrm{Spec}(R_{\mathrm{red}}^+)$. \square

Proposition 4.10. *The following statements hold:*

- (1) *Given two maps of v -sheaves $\mathcal{F} \rightarrow \mathcal{H}$, $\mathcal{G} \rightarrow \mathcal{H}$ if \mathcal{F} and \mathcal{G} are v -formalizing and \mathcal{H} is formally separated then $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ is v -formalizing.*
- (2) *The subcategory of v -sheaves that are v -formalizing and formally separated is stable under fiber product and contains $\mathrm{Spd}(\mathbb{Z}_p)$.*

Proof. Given a map $\mathrm{Spa}(A, A^+) \rightarrow \mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ we can find a cover $\mathrm{Spa}(B, B^+) \rightarrow \mathrm{Spa}(A, A^+)$ for which the compositions with the projections to \mathcal{F} and \mathcal{G} are both formalizable. By formal separatedness any pair of choices of formalizations $\mathrm{Spd}(B^+) \rightarrow \mathcal{G}$ and to $\mathrm{Spd}(B^+) \rightarrow \mathcal{F}$ define the same formalization to \mathcal{H} and a map to $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$. The second claim follows from the stability of separatedness by basechange and composition, from [Lemma 3.28](#) and from [Lemma 3.30](#). Indeed, we need to prove that $(\mathcal{F}^{\mathrm{red}})^\diamond \times_{(\mathcal{H}^{\mathrm{red}})^\diamond} (\mathcal{G}^{\mathrm{red}})^\diamond$ is a subsheaf of $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$, but this follows from knowing that $\mathcal{F}^{\mathrm{red}}$ (respectively \mathcal{H} , \mathcal{G}) is a subsheaf of \mathcal{F} (respectively \mathcal{H} , \mathcal{G}). \square

Definition 4.11. Let $\mathcal{F} \in \widetilde{\mathrm{Perf}}$, we say it is *specializing* if it is formally separated and v -locally formal.

Definition 4.12. Let \mathcal{F} be a specializing v -sheaf and let $f : \coprod_{i \in I} \mathrm{Spd}(B_i) \rightarrow \mathcal{F}$ be a surjective map. The specialization map for \mathcal{F} , denoted $\mathrm{sp}_{\mathcal{F}}$, is the unique map $\mathrm{sp}_{\mathcal{F}} : |\mathcal{F}| \rightarrow |\mathcal{F}^{\mathrm{red}}|$ making the following diagram commutative:

$$\begin{array}{ccc}
\coprod_{i \in I} |\mathrm{Spd}(B_i)| & \xrightarrow{f} & |\mathcal{F}| \\
\downarrow \mathrm{sp}_{B_i} & & \downarrow \\
\coprod_{i \in I} |\mathrm{Spec}((B_i)_{\mathrm{red}})| & \xrightarrow{|f^{\mathrm{red}}|} & |\mathcal{F}^{\mathrm{red}}|
\end{array}$$

Remark 4.13. We use [Proposition 4.9](#) to prove that this map of sets is well defined and does not depend on the choices taken. Indeed, given $[x] \in |\mathcal{F}|$ we take a formalizable representative $x : \mathrm{Spa}(K_x, K_x^+) \rightarrow \mathcal{F}$. Consider, its unique formalization $\mathrm{Spd}(K_x^+) \rightarrow \mathcal{F}$ and apply reduction to obtain $\mathrm{Spec}((K_x^+)_{\mathrm{red}}) \rightarrow \mathcal{F}^{\mathrm{red}}$. The maximal ideal of $(K_x^+)_{\mathrm{red}}$ maps $\mathrm{sp}_{\mathcal{F}}([x])$.

Proposition 4.14. *For any specializing v-sheaf \mathcal{F} the specialization map $\mathrm{sp}_{\mathcal{F}} : |\mathcal{F}| \rightarrow |\mathcal{F}^{\mathrm{red}}|$ is continuous. Moreover, this construction is functorial in the category of specializing v-sheaves.*

Proof. Functoriality follows from uniqueness of formalizations, functoriality of the reduction functor and [Remark 4.13](#). For continuity, take a cover $f : \coprod_{i \in I} \mathrm{Spd}(R_i^+) \rightarrow \mathcal{F}$. We get the following commutative diagram:

$$\begin{array}{ccc}
|\coprod_{i \in I} \mathrm{Spd}(R_i^+)| & \xrightarrow{f} & |\mathcal{F}| \\
\downarrow \mathrm{sp}_{R_i^+} & & \downarrow \mathrm{sp}_{\mathcal{F}} \\
|\mathrm{Spec}((R_i^+)_{\mathrm{red}})| & \xrightarrow{|f^{\mathrm{red}}|} & |\mathcal{F}^{\mathrm{red}}|
\end{array}$$

Now, f^{red} is continuous by [Proposition 3.14](#), f is continuous and a quotient map, and the maps $\mathrm{sp}_{R_i^+}$ are continuous by [Proposition 4.2](#). Since the diagram is commutative, the map $\mathrm{sp}_{\mathcal{F}}$ is also continuous. \square

4.3. Pre-kimberlites, formal schemes and formal neighborhoods.

Definition 4.15. Let \mathcal{F} be a specializing v-sheaf. We say \mathcal{F} is a *prekimberlite* if:

- (1) $\mathcal{F}^{\mathrm{red}}$ is represented by a scheme.
- (2) The map $(\mathcal{F}^{\mathrm{red}})^{\diamond} \rightarrow \mathcal{F}$ coming from adjunction is a closed immersion.

If \mathcal{F} is a prekimberlite, we let the *analytic locus* be $\mathcal{F}^{\mathrm{an}} = \mathcal{F} \setminus (\mathcal{F}^{\mathrm{red}})^{\diamond}$.

In what follows, we prove that the v-sheaf associated to separated formal schemes are prekimberlites. For this we fix a convention of what we mean by a ‘‘formal scheme’’. We follow [\[SW13, §2.2\]](#).

Convention 1. Denote by $\mathrm{Nilp}_{\mathbb{Z}_p}$ the category of algebras in which p is nilpotent, and endow $\mathrm{Nilp}_{\mathbb{Z}_p}^{\mathrm{op}}$ with the structure of a site by giving it the Zariski topology. By a formal scheme \mathfrak{X} over \mathbb{Z}_p we mean a Zariski sheaf on $\mathrm{Nilp}_{\mathbb{Z}_p}^{\mathrm{op}}$ which is Zariski locally of the form $\mathrm{Spf}(A)$. Here A is a topological ring given the I -adic topology for a finitely generated ideal of A containing p , and $\mathrm{Spf}(A)$ denotes the functor $\mathrm{Spec}(B) \mapsto \varinjlim_n \mathrm{Hom}(A/I^n, B)$.

For a formal scheme \mathfrak{X} over \mathbb{Z}_p we let $\mathfrak{X}_{\mathrm{red}}$ denote its reduction in the sense of formal schemes ([\[Sta18, Tag 0AIN\]](#)). Recall that this is a sheaf in $\mathrm{Nilp}_{\mathbb{Z}_p}^{\mathrm{op}}$ which is representable by a scheme. Moreover, the map $\mathfrak{X}_{\mathrm{red}} \rightarrow \mathfrak{X}$ is relatively representable in schemes, it is a closed immersion and for any open $\mathrm{Spf}(A) \subseteq \mathfrak{X}$ the pullback to $\mathfrak{X}_{\mathrm{red}}$ is given by the reduced subscheme of $\mathrm{Spec}(A/I)$ (for an ideal of definition $I \subseteq A$).

We say that \mathfrak{X} is separated if $\mathfrak{X}_{\mathrm{red}}$ is a separated scheme ([\[Sta18, Tag 0AJ7\]](#)).

Recall the following result of Scholze and Weinstein.⁶

Proposition 4.16. ([\[SW13, Proposition 2.2.1\]](#)) *The functor $\mathrm{Spf}(A) \mapsto \mathrm{Spa}(A, A)$ extends to a fully faithful functor $\mathfrak{X} \mapsto \mathfrak{X}^{\mathrm{ad}}$ from formal schemes over \mathbb{Z}_p as in [Convention 1](#) to the category of pre-adic spaces.*

Proposition 4.17. *If \mathfrak{X} is a separated formal scheme over \mathbb{Z}_p , then $(\mathfrak{X}^{\mathrm{ad}})^{\diamond}$ is a prekimberlite.*

Proof. Let $X = \mathfrak{X}^{\mathrm{ad}}$ and let $W = X^{\mathrm{na}}$, then $W = (\mathfrak{X}_{\mathrm{red}})^{\mathrm{ad}}$. Clearly X^{\diamond} is v-locally formal. By [Proposition 3.18](#) we have $(W^{\diamond})^{\mathrm{red}} = (X^{\diamond})^{\mathrm{red}}$ which is the perfection of $\mathfrak{X}_{\mathrm{red}}$. The adjunction morphism agrees with the map $W^{\diamond} \rightarrow X^{\diamond}$ which by [Proposition 1.21](#) is a closed immersion.

The only thing left to prove is that $X^{\diamond} \rightarrow \mathrm{Spd}(\mathbb{Z}_p)$ is separated, we first prove that X^{\diamond} is quasi-separated. Let $Z = \mathrm{Spa}(R, R^+)$ be a strictly totally disconnected space and take a map $f : Z \rightarrow$

⁶What is called adic spaces in [\[SW13\]](#) is what we call pre-adic spaces here and in [\[SW20\]](#).

$X^\diamond \times_{\mathrm{Spd}(\mathbb{Z}_p)} X^\diamond$. Since Z splits any open cover we may assume that f factors through an open neighborhood of the form $\mathrm{Spd}(B_1) \times_{\mathrm{Spd}(\mathbb{Z}_p)} \mathrm{Spd}(B_2)$ for an open subset $\mathrm{Spf}(B_1) \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathrm{Spf}(B_2) \subseteq \mathfrak{X} \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathfrak{X}$. Consider the following basechange diagrams, where $Y = \mathfrak{Y}^{\mathrm{ad}}$

$$\begin{array}{ccc} \mathfrak{Y} & \longrightarrow & \mathrm{Spf}(B_1) \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathrm{Spf}(B_2) \\ \downarrow & & \downarrow \\ \mathfrak{X} & \longrightarrow & \mathfrak{X} \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathfrak{X} \end{array} \quad \begin{array}{ccc} Y & \longrightarrow & \mathrm{Spa}(B_1) \times_{\mathbb{Z}_p} \mathrm{Spa}(B_2) \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times_{\mathbb{Z}_p} X \end{array}$$

Since \mathfrak{X} is separated \mathfrak{Y} is quasicompact. This implies that Y admits a finite open cover of the form $\coprod_{i=1}^n \mathrm{Spa}(A_i) \rightarrow Y$. Moreover, the diagonal map $X \rightarrow X \times_{\mathbb{Z}_p} X$ is adic. By [Lemma 2.23](#) the maps $\mathrm{Spd}(A_i) \rightarrow \mathrm{Spd}(B_1) \times_{\mathbb{Z}_p} \mathrm{Spd}(B_2)$ are quasicompact, which proves that $Y^\diamond \rightarrow \mathrm{Spd}(B_1) \times_{\mathbb{Z}_p} \mathrm{Spd}(B_2)$ and any basechange of it is also quasicompact. Now we may use the valuative criterion of separatedness [[Sch17](#), Proposition 10.9]. Given $\mathrm{Spa}(K, O_K) \rightarrow X^\diamond$ we must show there is at most one extension to $\mathrm{Spa}(K, K^+) \rightarrow X^\diamond$ where $K^+ \subseteq O_K$ is an open and bounded valuation subring. Maps $\mathrm{Spa}(K, K^+) \rightarrow X^\diamond$ are in bijection with maps $\mathrm{Spf}(K^+) \rightarrow \mathfrak{X}$. On the other hand, maps $g : \mathrm{Spf}(K^+) \rightarrow \mathfrak{X}$ are in bijection with pairs (g_η, g_s) where $g_\eta : \mathrm{Spf}(O_K) \rightarrow \mathfrak{X}$, $g_s : \mathrm{Spec}(K^+/K^{\circ\circ}) \rightarrow \mathfrak{X}_{\mathrm{red}}$ and such that $g_\eta = g_s$ when we restrict the maps to $\mathrm{Spec}(O_K/K^{\circ\circ})$. At this point we may use the valuative criterion of separatedness of $\mathfrak{X}_{\mathrm{red}}$. \square

Definition 4.18. Let \mathcal{F} be a prekimberlite and let $S \subseteq \mathcal{F}^{\mathrm{red}}$ be a locally closed immersion of schemes. We let $\widehat{\mathcal{F}}_{/S}$, the *formal neighborhood* of S on \mathcal{F} , be the subsheaf given by the Cartesian diagram:

$$\begin{array}{ccc} \widehat{\mathcal{F}}_{/S} & \longrightarrow & |S| \\ \downarrow & & \downarrow \\ \mathcal{F} & \longrightarrow & |\mathcal{F}| \xrightarrow{\mathrm{sp}_{\mathcal{F}}} |\mathcal{F}^{\mathrm{red}}| \end{array}$$

If $S \subseteq \mathcal{F}^{\mathrm{red}}$ is open we call it *open formal neighborhood*.

Proposition 4.19. Suppose (A, A) is a formal Huber pair over \mathbb{Z}_p with ideal of definition I . Let $J \subseteq A$ be a finitely generated ideal containing I and B the completion of A with respect to J . The closed immersion of schemes, $S = \mathrm{Spec}(B_{\mathrm{red}}) \rightarrow \mathrm{Spec}(A_{\mathrm{red}})$, induces an identification $\widehat{\mathrm{Spd}(A)}_{/S} = \mathrm{Spd}(B)$.

Proof. Let $S = \mathrm{Spec}(B_{\mathrm{red}})$ and $T = \mathrm{Spec}(A_{\mathrm{red}})$. The reduction of $\mathrm{Spd}(B) \rightarrow \mathrm{Spd}(A)$ induces $S \rightarrow T$. Since specialization is functorial, points coming from $\mathrm{Spd}(B)$ specialize to S . Consequently, the map factors as $\mathrm{Spd}(B) \rightarrow \widehat{\mathrm{Spd}(A)}_{/S} \rightarrow \mathrm{Spd}(A)$. Since A is dense in B , this map is an injection. To prove surjectivity onto $\widehat{\mathrm{Spd}(A)}_{/S}$, let $f : A \rightarrow R^+$ be a map such that $f : \mathrm{Spec}(R^+) \rightarrow \mathrm{Spec}(A_{\mathrm{red}})$ factors through $|S|$. If $a \in J$, then $f(a)$ is nilpotent in $\mathrm{Spec}(R^+/\varpi^n)$. Since J is finitely generated there is an m for which $J^m \subseteq (\varpi^n)$ in R^+ . This proves that the map $f : A \rightarrow R^+$ is continuous for the J -adic topology on A . Since R^+ is complete the map $f : A \rightarrow R^+$ factors through B . \square

Proposition 4.20. Let $f : \mathcal{G} \rightarrow \mathcal{F}$ be a map of prekimberlites and $S \subseteq |\mathcal{F}^{\mathrm{red}}|$ a locally closed subscheme. Let $T = S \times_{\mathcal{F}^{\mathrm{red}}} \mathcal{G}^{\mathrm{red}}$, then $\widehat{\mathcal{F}}_{/S} \times_{\mathcal{F}} \mathcal{G} = \widehat{\mathcal{G}}_{/T}$. In particular, $\mathcal{G} \rightarrow \mathcal{F}$ factors through $\widehat{\mathcal{F}}_{/S}$ if and only if $\mathcal{G}^{\mathrm{red}} \rightarrow \mathcal{F}^{\mathrm{red}}$ factors through S .

Proof. Since S is a locally closed $|T| = |S| \times_{|\mathcal{F}^{\mathrm{red}}|} |\mathcal{G}^{\mathrm{red}}|$. The rest is a standard diagram chase. \square

Proposition 4.21. Let \mathcal{F} be a prekimberlite and let $S \subseteq |\mathcal{F}^{\mathrm{red}}|$ a locally closed subset, then $\widehat{\mathcal{F}}_{/S}$ is a prekimberlite and $(\widehat{\mathcal{F}}_{/S})^{\mathrm{red}} = S$.

Proof. The formula $(\widehat{\mathcal{F}}_{/S})^{\mathrm{red}} = S$ follows from observing that by [Proposition 4.20](#) a map $\mathrm{Spec}(A)^\diamond \rightarrow \mathcal{F}$ factors through $\widehat{\mathcal{F}}_{/S}$ if and only if the adjunction map $\mathrm{Spec}(A) \rightarrow \mathcal{F}^{\mathrm{red}}$ factors through S . Since $\widehat{\mathcal{F}}_{/S}$ is a subsheaf of a separated v-sheaf it is separated as well. The map $S^\diamond \rightarrow \widehat{\mathcal{F}}_{/S}$ is injective, so $\widehat{\mathcal{F}}_{/S}$ is formally separated. Take a map $\mathrm{Spa}(R, R^+) \rightarrow \widehat{\mathcal{F}}_{/S} \subseteq \mathcal{F}$. After replacing $\mathrm{Spa}(R, R^+)$ by a v-cover we get a formalization $\mathrm{Spd}(R^+) \rightarrow \mathcal{F}$. By [Proposition 4.20](#) this factors through $\widehat{\mathcal{F}}_{/S}$ since $\mathrm{Spec}(R^+) \rightarrow \mathcal{F}^{\mathrm{red}}$ factors through S . We have proved $\widehat{\mathcal{F}}_{/S}$ is specializing, we prove $S^\diamond \rightarrow \widehat{\mathcal{F}}_{/S}$ is a closed immersion.

Consider $\mathcal{G} = \widehat{\mathcal{F}}_{/S} \times_{\mathcal{F}} (\mathcal{F}^{\text{red}})^{\diamond}$. Now, $\mathcal{G} \rightarrow \widehat{\mathcal{F}}_{/S}$ is a closed immersion, and by [Proposition 4.20](#) \mathcal{G} is $((\widehat{\mathcal{F}^{\text{red}}})^{\diamond})_{/S}$. If S is a closed subscheme of \mathcal{F}^{red} , then $S^{\diamond} \rightarrow (\mathcal{F}^{\text{red}})^{\diamond}$ is proper, so $S^{\diamond} \rightarrow ((\widehat{\mathcal{F}^{\text{red}}})^{\diamond})_{/S}$ is a closed immersion. If S is an open in \mathcal{F}^{red} , then $((\widehat{\mathcal{F}^{\text{red}}})^{\diamond})_{/S} = S^{\diamond}$. One can reduce to these cases. \square

Proposition 4.22. *Let \mathcal{F} be a prekimberlite, $S \subseteq |\mathcal{F}^{\text{red}}|$ a locally closed constructible subset, then the map $\widehat{\mathcal{F}}_{/S} \rightarrow \mathcal{F}$ is an open immersion.*

Proof. The question is Zariski local in \mathcal{F}^{red} . Indeed, an open cover $\coprod_{i \in I} U_i \rightarrow \mathcal{F}^{\text{red}}$ induces an open cover $\coprod_{i \in I} \widehat{\mathcal{F}}_{/U_i} \rightarrow \mathcal{F}$. We may assume that $\mathcal{F}^{\text{red}} = \text{Spec}(A)$ and that S is closed and constructible in $\text{Spec}(A)$. Write $S = \text{Spec}(A/I)$ for $I \subseteq A$ an ideal, by constructibility we may assume that I is finitely generated. Pick $\{i_1, \dots, i_n\}$ a list of generators for I , $(R, R^+) \in \text{Perf}$ and a map $\text{Spd}(R^+) \rightarrow \mathcal{F}$. Let $X := \text{Spd}(R^+) \times_{\mathcal{F}} \widehat{\mathcal{F}}_{/S}$, and let $\varpi \in R^+$ be a pseudo-uniformizer. Let $\{j_1, \dots, j_n\}$ be a list of lifts of $\{i_1, \dots, i_n\}$ to R^+ . Then X is the open subsheaf of $\text{Spd}(R^+)$ defined by $\bigcap_{k=1}^n N_{j_k \ll 1}$. Indeed, this follows from [Proposition 4.20](#), [Proposition 4.19](#) and [Lemma 2.21](#). Since \mathcal{F} is v-formalizing every map $\text{Spa}(R, R^+) \rightarrow \mathcal{F}$ factors through $\text{Spd}(R^+)$ after replacing $\text{Spa}(R, R^+)$ by a v-cover. By [[Sch17](#), Proposition 10.11] $\widehat{\mathcal{F}}_{/S} \rightarrow \mathcal{F}$ is open. \square

4.4. Heuer's specialization map and étale formal neighborhoods. In [[Heu21](#)], Heuer considers certain specialization maps. These are maps of v-sheaves rather than a map of topological spaces. We discuss his construction and use it to enhance our theory.

Definition 4.23. ([[Heu21](#), Definition 5.1]) Let X be a scheme over \mathbb{F}_p . We attach a presheaf $X^{\diamond/\circ}$ defined by the (analytic sheafification of the) formula $(R, R^+) \mapsto X(\text{Spec}(R_{\text{red}}^+))$, where $R_{\text{red}}^+ = R^+/R^{\circ\circ}$.

In [[Heu21](#), Lemma 5.2], Heuer proves that $X^{\diamond/\circ}$ is a v-sheaf and that when X is affine the sheafification is not necessary. There is an evident map $X^{\diamond} \rightarrow X^{\diamond/\circ}$.

Proposition 4.24. *If X is a scheme over \mathbb{F}_p , then $X = (X^{\diamond/\circ})^{\text{red}}$. Moreover, if $\text{Spa}(R, R^+)$ is a totally disconnected perfectoid space then $X^{\diamond/\circ}(\text{Spa}(R, R^+)) = X^{\diamond/\circ}(\text{Spd}(R^+))$.*

Proof. Let $X = \text{Spec}(A)$. The inclusion $X \subseteq (X^{\diamond/\circ})^{\text{red}}$ of scheme-theoretic v-sheaves is easy to verify. Now, $X^{\diamond/\circ}(\text{Spec}(R)^{\diamond}) \subseteq X^{\diamond/\circ}(\text{Spd}(R[[t^{\frac{1}{p^{\infty}}}]])$, $R[[t^{\frac{1}{p^{\infty}}}]])$ and this latter is by [[Heu21](#), Lemma 5.2] the set of maps $A \rightarrow R$. So $(X^{\diamond/\circ})^{\text{red}} = X$. Moreover, let $\text{Spa}(R, R^+) \in \text{Perf}$, with pseudo-uniformizer ϖ . Consider $U = (\text{Spd}(R^+[[t^{\frac{1}{p^{\infty}}}]])^{\text{an}}$ with its cover $\text{Spa}(R_1, R_1^+) = U(\frac{\varpi}{t})$ and $\text{Spa}(R_2, R_2^+) = U(\frac{t}{\varpi})$. Let $\text{Spa}(R_3, R_3^+) = U(\frac{t}{\varpi}) \cap U(\frac{\varpi}{t})$. One computes explicitly that $X^{\diamond/\circ}(\text{Spa}(R_i, R_i^+)) = X^{\diamond/\circ}(\text{Spa}(R, R^+))$ for $i \in \{1, 2, 3\}$, so $X^{\diamond/\circ}(U) = X^{\diamond/\circ}(\text{Spa}(R, R^+))$. This proves $X^{\diamond/\circ}(\text{Spd}(R^+)) = X^{\diamond/\circ}(\text{Spa}(R, R^+))$ when X is affine. Using the techniques of [Theorem 2.29](#) we can glue and prove the general case. Indeed, open subschemes $f : U \subseteq X$ induce open subsheaves $f^{\diamond/\circ} : U^{\diamond/\circ} \subseteq X^{\diamond/\circ}$ and the delicate part of the glueing happens on $|\text{Spec}(R)^{\diamond}|$ for test objects R , which is easily reduced to $R = V$ a valuation ring. Following the proof loc. cit. we get to a similar diagram:

$$\begin{array}{ccccc}
\text{Spa}(V_b[\frac{1}{b}], V_b) & \longrightarrow & \text{Spd}(V[\frac{1}{b}], V) & & \\
\downarrow & \searrow & \downarrow & & \\
& & \text{Spec}(B_3)^{\diamond/\circ} & \longrightarrow & \text{Spec}(B_2)^{\diamond/\circ} \\
& & \downarrow & & \downarrow \\
\text{Spd}(V_b, V_b) & \longrightarrow & \text{Spec}(B_1)^{\diamond/\circ} & \longrightarrow & X^{\diamond}
\end{array}$$

Now, $\text{Spec}(B_1)^{\diamond/\circ}(\text{Spa}(V_b[\frac{1}{b}], V_b)) = \text{Spec}(B_1)^{\diamond/\circ}(\text{Spa}(V_b, V_b))$ proves $\text{Spd}(V_b)$ factors through $\text{Spec}(B_3)^{\diamond/\circ}$ and $\text{Spec}(V)^{\diamond}$ factors through $\text{Spec}(B_2)^{\diamond/\circ}$. This proves $(X^{\diamond/\circ})^{\text{red}} = X$. It also proves that for perfectoid fields (K, K^+) and a map $\text{Spd}(K^+) \rightarrow X^{\diamond/\circ}$ if $\text{Spa}(K, K^+) \rightarrow X^{\diamond/\circ}$ factors through an affine then $\text{Spd}(K^+)$ also does. Since totally disconnected perfectoid spaces split open covers we can conclude $X^{\diamond/\circ}(\text{Spa}(R, R^+)) = X^{\diamond/\circ}(\text{Spd}(R^+))$. \square

Suppose X is a prekimberlite and $\text{Spa}(R, R^+) \in \text{Perf}$. Let $f : \text{Spa}(R, R^+) \rightarrow X$ be a formalizable map, applying reduction to the formalization we obtain a map $\text{Spec}(R_{\text{red}}^+) \rightarrow X^{\text{red}}$, or in other words a map $\text{Spa}(R, R^+) \rightarrow (X^{\text{red}})^{\diamond/\circ}$. Overall, we obtain a natural transformation $\text{SP}_X : X \rightarrow (X^{\text{red}})^{\diamond/\circ}$. This is the type of specialization map that Heuer considers. With this switch of perspective we can

reinterpret formal neighborhoods: if $S \subseteq |X^{\text{red}}|$ is a locally closed subset we get a map $S^{\diamond/\diamond} \rightarrow (X^{\text{red}})^{\diamond/\diamond}$ and $\widehat{\mathcal{F}}/S = X \times_{(X^{\text{red}})^{\diamond/\diamond}} S^{\diamond/\diamond}$. Under this light, [Proposition 4.22](#) is simply proving that $S^{\diamond/\diamond} \rightarrow (X^{\text{red}})^{\diamond/\diamond}$ is an open immersion when S is constructible. Moreover, this leads to a good notion of “étale formal neighborhoods” of a prekimberlite.

Lemma 4.25. *If $V \rightarrow X$ is a quasicompact, separated étale map of perfect schemes over \mathbb{F}_p , then $V^{\diamond/\diamond} \rightarrow X^{\diamond/\diamond}$ is separated, quasicompact, formally adic and étale.*

Proof. The formation of $X^{\diamond/\diamond}$ commutes with finite limits and preserves open immersions. If $V \rightarrow X$ is separated then $V \rightarrow V \times_X V$ is open and closed, the same holds for $V^{\diamond/\diamond} \rightarrow V^{\diamond/\diamond} \times_{X^{\diamond/\diamond}} V^{\diamond/\diamond}$ which proves separatedness. We prove formal adicness. Let $Y = V^{\diamond/\diamond} \times_{X^{\diamond/\diamond}} X^{\diamond}$, we get a map $V^{\diamond} \rightarrow Y$. We prove this map is an isomorphism after basechange by any map $\text{Spa}(R, R^+) \rightarrow X^{\diamond}$. Let $Y_R := Y \times_{X^{\diamond}} \text{Spa}(R, R^+)$, we may assume m factors through $\text{Spec}(R^+)^{\diamond} \rightarrow X^{\diamond}$ since this happens v-locally. Now, before sheafification $Y_R(S, S^+)$ parametrizes lifts of $\text{Spec}(S^+_{\text{red}}) \rightarrow \text{Spec}(R^+_{\text{red}}) \rightarrow X$ to V . By invariance of the étale site under perfection (respectively nilpotent thickenings), this is the same as parametrizing lifts of $\text{Spec}(S^+/\varpi) \rightarrow X$ to V (respectively of $\text{Spf}(S^+) \rightarrow X$ to V). In other words, Y_R fits in the following Cartesian diagram:

$$\begin{array}{ccc} Y_R & \longrightarrow & \text{Spa}(R, R^+) \\ \downarrow & & \downarrow \\ \text{Spec}(R^+)^{\diamond} \times_{X^{\diamond}} V^{\diamond} & \longrightarrow & \text{Spec}(R^+)^{\diamond} \end{array}$$

But this is precisely $V^{\diamond} \times_{X^{\diamond}} \text{Spa}(R, R^+)$. For quasicompactness and étaleness we can argue locally and assume $X = \text{Spec}(A)$ and $V = \text{Spec}(B)$. Indeed, this follows from the quasicompactness of the specialization map for Tate Huber pairs. Arguing as above, for a map $\text{Spa}(R, R^+) \rightarrow X^{\diamond/\diamond}$ the basechange is represented by $\text{Spa}(S, S^+)$ where S^+ is the unique étale R^+ -algebra lifting $R^+_{\text{red}} \otimes_A B$. \square

Definition 4.26. Suppose X is a prekimberlite, we let $(X)_{\text{qc,for-ét}}$ be the category that has as objects maps $f : T \rightarrow X$ where T is a prekimberlite and f is formally adic, étale and quasicompact. Morphisms are maps of v-sheaves commuting with the structure map. We call objects in this category the *étale formal neighborhoods* of X .

Morphisms in $X_{\text{for-ét}}$ are automatically quasicompact, formally adic, étale and separated. Given a perfect scheme S we let $(S)_{\text{ét,qc,sep}}$ denote the category of schemes étale, quasicompact and separated over S .

Theorem 4.27. *For X a prekimberlite, reduction $(-)^{\text{red}} : (X)_{\text{qc,for-ét}} \cong (X^{\text{red}})_{\text{qc,ét,sep}}$ is an equivalence.*

Step 1: If a morphism $f : Y \rightarrow W$ in $(X)_{\text{qc,for-ét}}$ induces an isomorphism $f^{\text{red}} : Y^{\text{red}} \rightarrow W^{\text{red}}$ then f is an isomorphism.

Proof. The sheaf-theoretic image of Y in W is an open subsheaf of W containing W^{red} so it must be W . This proves surjectivity. Since the map is qcqs we may prove injectivity on geometric points. We reduce to the case where $W = \text{Spd}(C^+, C^+)$, and the geometric point is given by the inclusion $\text{Spa}(C, C^+) \subseteq \text{Spd}(C^+)$. In this case, Y^{an} has the form $\prod_{i=1}^n \text{Spa}(C, C'^+)$ with $C^+ \subseteq C'^+ \subseteq O_C$. To count connected components we may restrict to $W = \text{Spd}(O_C, O_C)$ so that $Y^{\text{an}} = \prod_{i=1}^n \text{Spa}(C, O_C)$ and $Y^{\text{red}} = \text{Spec}(O_C/C^{\circ\circ}) = W^{\text{red}}$. Replacing C by a v-cover we may assume that $\text{Spa}(C, O_C) \rightarrow Y$ is formalizable. The map $\text{Spd}(O_C, O_C) \rightarrow Y$ is étale, and since its image is open $n = 1$. \square

Step 2: $(-)^{\text{red}} : (X)_{\text{qc,for-ét}} \rightarrow (X^{\text{red}})_{\text{qc,ét,sep}}$ is fully-faithful.

Proof. Maps from Z to Y over X are in bijection with open and closed subsheaves $W \subseteq Z \times_X Y$ whose projection to Z is an isomorphism. Identically, maps $Z^{\text{red}} \rightarrow Y^{\text{red}}$ are in bijection with open and closed subschemes of $Z^{\text{red}} \times_{X^{\text{red}}} Y^{\text{red}}$ whose projection to Z^{red} is an isomorphism. Since we can check isomorphisms by passing to reduction we get a bijection $W \mapsto W^{\text{red}}$ with inverse $V \mapsto (\widehat{Z \times_X Y})_{/V^{\text{red}}}$. \square

Step 3: $(-)^{\text{red}} : (X)_{\text{qc,for-ét}} \rightarrow (X^{\text{red}})_{\text{qc,ét,sep}}$ is essentially surjective.

Proof. If $V \in (X^{\text{red}})_{\text{qc,ét,sep}}$ we let $\widehat{X}/V := X \times_{(X^{\text{red}})^{\diamond/\diamond}} V^{\diamond/\diamond}$, where the map $X \rightarrow (X^{\text{red}})^{\diamond/\diamond}$ is SP_X . By [Lemma 4.25](#), $\widehat{X}/V \rightarrow X$ is quasicompact, formally adic, étale and separated. Since X is formally separated \widehat{X}/V also is. Although they are not formally separated, by [Proposition 4.24](#), $(X^{\text{red}})^{\diamond/\diamond}$ and $V^{\diamond/\diamond}$ still formalize uniquely \widehat{X}/V totally disconnected spaces, this proves \widehat{X}/V is also

v-formalizing and a specializing v-sheaf. It is a prekimberlite since $(\widehat{X}/V)^{\text{red}} = V$, and the map $V^\diamond \rightarrow \widehat{X}/V$ is closed. \square

If X is a prekimberlite and $V \rightarrow X^{\text{red}}$ is an étale map, [Theorem 4.27](#) associates to V the étale formal neighborhood $\widehat{X}/V := X \times_{X^{\diamond/\circ}} V^{\diamond/\circ}$.

Corollary 4.28. *Let X be a prekimberlite and let $Y \in (X)_{\text{qc,for-ét}}$. If $X = \mathfrak{X}^\diamond$ for a formal scheme \mathfrak{X} , then there is a formal scheme \mathfrak{Y} with $Y = \mathfrak{Y}^\diamond$.*

Remark 4.29. If $\mathcal{K} = (\mathcal{F}, \mathcal{D})$ is a smelted kimberlite, by [Theorem 4.27](#), we obtain a morphism of sites $\Psi' : \mathcal{D}_{\text{ét}} \rightarrow (\mathcal{F}^{\text{red}})_{\text{ét}}$. This allows us to form a “naive nearby cycles functor” $\text{R}\Psi'$. If $j : \mathcal{D} \rightarrow \mathcal{F} \leftarrow (\mathcal{F}^{\text{red}})^\diamond : i$ denote the inclusions, Scholze’s 6-functor formalism give us already a nearby cycles functor $i^* \text{R}j_*$. It is an interesting question to understand the relation between these two functors. We have partial progress in answering this question. We will report our findings on a future work.

Observe that $\text{SP}_X : X \rightarrow (X^{\text{red}})^{\diamond/\circ}$ is separated. This allows us to make the following definition.

Definition 4.30. A prekimberlite is *valuative* if $\text{SP}_X : X \rightarrow (X^{\text{red}})^{\diamond/\circ}$ is partially proper.

Proposition 4.31. *If \mathfrak{X} is a separated formal scheme over \mathbb{Z}_p as in [Convention 1](#), then \mathfrak{X}^\diamond is valuative.*

Proof. By [[Sch17](#), Proposition 18.6], one can verify partial properness open locally on the target, this reduces to the case $\mathfrak{X} = \text{Spd}(B)$. The valuative criterion asks if given a map $B \rightarrow R^{\sharp,\circ}$ such that $B_{\text{red}} \rightarrow R_{\text{red}}^{\sharp,\circ}$ factors through $R_{\text{red}}^{\sharp,+}$ then $B \rightarrow R^{\sharp,\circ}$ factors through $R^{\sharp,+}$. This follows from $R^{\sharp,+} = R^{\sharp,\circ} \times_{R_{\text{red}}^{\sharp,\circ}} R_{\text{red}}^{\sharp,+}$. \square

Proposition 4.32. *Let $f : X \rightarrow Y$ be a map of prekimberlites. If f is partially proper and Y is valuative then X is valuative.*

Proof. It follows from two facts: $X \rightarrow (Y^{\text{red}})^{\diamond/\circ}$ is partially proper and $X \rightarrow (X^{\text{red}})^{\diamond/\circ}$ is separated. \square

Proposition 4.33. *If \mathcal{F} is a valuative prekimberlite then $\text{sp}_{\mathcal{F}} : |\mathcal{F}| \rightarrow |\mathcal{F}^{\text{red}}|$ is specialising as a map of topological spaces.*

Proof. Let $r \in |\mathcal{F}|$, $x = \text{sp}_{\mathcal{F}}(r)$ and $y \in |\mathcal{F}^{\text{red}}|$ specializing from x . We construct q specializing from r that maps to y . Pick a formalizable representative $f_r : \text{Spa}(C, C^+) \rightarrow \mathcal{F}$. Let $K = O_C/C^{\circ\circ}$ and $K^+ = C^+/C^{\circ\circ}$, then x is the image of closed point under $f_x : \text{Spec}(K^+) \rightarrow \mathcal{F}^{\text{red}}$. Let R be the local ring obtained by intersecting the closure of x and the localization at y . Let $k = K^+/\mathfrak{m}_{K^+}$, so that $R \subseteq k$. By [[Sta18](#), Tag 00IA], we have a valuation subring $R \subseteq V \subseteq k$ such that $\text{Frac}(V) = k$ and V dominates R . This induces a valuation subring $K'^+ \subseteq K^+$ and a map $f_y : \text{Spec}(K'^+) \rightarrow \mathcal{F}^{\text{red}}$ whose closed point maps to y . In turn, this induces a valuation subring $C'^+ \subseteq C^+$ with $C'^+/C^{\circ\circ} = K'^+$. Now, f_y induces a map $\text{Spa}(C, C'^+) \rightarrow (\mathcal{F}^{\text{red}})^{\diamond/\circ}$ extending $\text{SP}_{\mathcal{F}} \circ f_r$. By partial properness this lifts to a map $f_q : \text{Spa}(C, C'^+) \rightarrow \mathcal{F}$ and clearly $\text{sp}_{\mathcal{F}}(q) = y$. \square

Proposition 4.34. *Formal neighborhoods and étale formal neighborhoods of valuative prekimberlites are valuative prekimberlites.*

Proof. This is immediate from their expression as a basechange. \square

4.5. Kimberlites and smelted kimberlites.

Definition 4.35. Let \mathcal{F} be a valuative prekimberlite.

- (1) A *smelted kimberlite* \mathcal{K} is a pair $\mathcal{K} := (\mathcal{F}, \mathcal{D})$, where $\mathcal{D} \subseteq \mathcal{F}^{\text{an}}$ is partially proper open subsheaf and \mathcal{D} is a quasiseparated locally spatial diamond.
- (2) We define the specialization map $\text{sp}_{\mathcal{K}} : |\mathcal{D}| \rightarrow |\mathcal{F}^{\text{red}}|$ as the composition $|\mathcal{D}| \rightarrow |\mathcal{F}| \xrightarrow{\text{sp}_{\mathcal{F}}} |\mathcal{F}^{\text{red}}|$. If the context is clear, we write $\text{sp}_{\mathcal{D}}$ instead of $\text{sp}_{\mathcal{K}}$.
- (3) We say \mathcal{G} is a *kimberlite* if $\mathcal{K}_{\mathcal{G}} := (\mathcal{G}, \mathcal{G}^{\text{an}})$ is a smelted-kimberlite and $\text{sp}_{\mathcal{K}_{\mathcal{G}}}$ is quasicompact.

Remark 4.36. Given a valuative prekimberlite \mathcal{F} one is mostly interested in smelted kimberlites $(\mathcal{F}, \mathcal{D})$ where $\mathcal{D} = \mathcal{F}^{\text{an}}$ or where $\mathcal{D} = \mathcal{F} \times_{\text{Spd}(O_K)} \text{Spd}(K, O_K)$ when \mathcal{F} comes with a map $\mathcal{F} \rightarrow \text{Spd}(O_K)$ for O_K a complete rank 1 valuation ring. Notice that $\mathcal{F}^{\text{an}} \rightarrow \mathcal{F}$ is always partially proper if \mathcal{F} is a prekimberlite.

Remark 4.37. Let \mathcal{F} be a kimberlite. The quasicompactness hypothesis of $\text{sp}_{\mathcal{F}^{\text{an}}}$ is equivalent to asking that $\text{sp}_{\mathcal{F}^{\text{an}}}^{-1}(U)$ is a spatial diamond for all $U \subseteq \mathcal{F}^{\text{red}}$ with U affine.

Definition 4.38. Let $\mathcal{K} = (\mathcal{F}, \mathcal{D})$ be a smelted kimberlite. Given a constructible locally closed subset $S \subseteq \mathcal{F}^{\text{red}}$ we let $\mathcal{D}_{/S}^{\circledast} = \widehat{\mathcal{F}}_{/S} \times_{\mathcal{F}} \mathcal{D}$. We call this subsheaf the *tubular neighborhood* of \mathcal{D} around S . If \mathcal{G} is a kimberlite we write $\mathcal{G}_{/S}^{\circledast}$ for $\widehat{\mathcal{G}}_{/S} \times_{\mathcal{G}} \mathcal{G}^{\text{an}}$.

Proposition 4.39. *If $\mathcal{K} = (\mathcal{F}, \mathcal{D})$ is a smelted kimberlite, then $(\widehat{\mathcal{F}}_{/S}, \mathcal{D}_{/S}^{\circledast})$ is a smelted kimberlite. Moreover, $\mathcal{D}_{/S}^{\circledast}$ is the open subdiamond corresponding to the interior of $\text{sp}_{\mathcal{K}}^{-1}(S)$ in $|\mathcal{D}|$.*

Proof. By [Proposition 4.34](#) $\widehat{\mathcal{F}}_{/S}$ is a valuative prekimberlite, and by basechange $\mathcal{D}_{/S}^{\circledast} \rightarrow \widehat{\mathcal{F}}_{/S}$ is partially proper open. Moreover, [Proposition 4.22](#) $\mathcal{D}_{/S}^{\circledast} \rightarrow \mathcal{D}$ is an open immersion, so $\mathcal{D}_{/S}^{\circledast}$ is a quasiseparated locally spatial diamond. For the second claim, let $T \subseteq \text{sp}_{\mathcal{K}}^{-1}(S)$ be the largest subset stable under generization. It suffices to prove $T \subseteq \mathcal{D}_{/S}^{\circledast}$ since by [Proposition 4.22](#) we already have $\mathcal{D}_{/S}^{\circledast} \subseteq (\text{sp}_{\mathcal{K}}^{-1}(S))^{\text{int}} \subseteq T$. Take $x \in T$ and a formalizable geometric point $\iota_x : \text{Spa}(C_x, C_x^+) \rightarrow \mathcal{F}$ over x . Since every generization of x is in $\text{sp}_{\mathcal{K}}^{-1}(S)$ the map $\text{Spec}((C_x^+)_{\text{red}}) \rightarrow \mathcal{F}^{\text{red}}$ factors through S , so ι_x factors through $\mathcal{D}_{/S}^{\circledast} = \widehat{\mathcal{F}}_{/S} \cap \mathcal{D}$ by [Proposition 4.20](#). \square

Theorem 4.40. *Let $\mathcal{K} = (\mathcal{F}, \mathcal{D})$ be a smelted kimberlite and \mathcal{G} be a kimberlite, the following hold:*

- (1) $\text{sp}_{\mathcal{D}} : |\mathcal{D}| \rightarrow |\mathcal{F}^{\text{red}}|$ is a specializing, spectral map of locally spectral spaces.
- (2) $\text{sp}_{\mathcal{G}^{\text{an}}} : |\mathcal{G}^{\text{an}}| \rightarrow |\mathcal{G}^{\text{red}}|$ is a closed map.

Proof. Being specializing follows from [Proposition 4.33](#) and the hypothesis that $\mathcal{D} \rightarrow \mathcal{F}$ is partially proper. The second claim follows easily from [Corollary 1.15](#), from the first claim and from quasicompactness of $\text{sp}_{\mathcal{G}^{\text{an}}}$. We need to prove that $\text{sp}_{\mathcal{D}}$ is continuous for the constructible topology. We can argue on an open cover of $|\mathcal{D}|$, so we may assume that \mathcal{D} is spatial. Find a formalizable cover $X \rightarrow \mathcal{D}$ with $X = \text{Spa}(A, A^+) \in \text{Perf}$, and consider the diagram:

$$\begin{array}{ccc} |\text{Spa}(A, A^+) |^{\text{cons}} & \xrightarrow{g} & |\mathcal{D} |^{\text{cons}} \\ \downarrow \text{sp}_X & & \downarrow \text{sp}_{\mathcal{D}} \\ |\text{Spec}(A_{\text{red}}^+) |^{\text{cons}} & \xrightarrow{g^{\text{red}}} & |\mathcal{F}^{\text{red}} |^{\text{cons}} \end{array}$$

Since \mathcal{F}^{red} is represented by a scheme, by [Proposition 3.16](#) g^{red} is continuous for the patch topology. Similarly, sp_X is continuous and since \mathcal{D} is spatial by [Proposition 1.18](#) g is also continuous. Moreover, g is a surjective map of compact Hausdorff spaces and consequently a quotient map. Since the diagram commutes, $\text{sp}_{\mathcal{K}}$ is continuous for the patch topology. \square

Proposition 4.41. *Let $f : \mathcal{G} \rightarrow \mathcal{F}$ be a formally closed immersion. The following hold:*

- (1) *If \mathcal{F} is a specializing v-sheaf, then \mathcal{G} is a specializing v-sheaf.*
- (2) *If \mathcal{F} is a prekimberlite, then \mathcal{G} is a prekimberlite.*
- (3) *If $(\mathcal{F}, \mathcal{D})$ forms a smelted kimberlite then $(\mathcal{G}, \mathcal{G} \cap \mathcal{D})$ forms a smelted kimberlite.*
- (4) *If \mathcal{F} is a kimberlite, then \mathcal{G} is a kimberlite.*

Proof. Suppose \mathcal{F} is specializing, since \mathcal{G} is a subsheaf of \mathcal{F} it is separated, and by formal adicness $(\mathcal{G}^{\text{red}})^{\circ} \rightarrow \mathcal{G}$ is injective. Pick $\text{Spa}(R, R^+) \in \text{Perf}$ and a map $\text{Spd}(R^+) \rightarrow \mathcal{F}$, the basechange $X := \mathcal{G} \times_{\mathcal{F}} \text{Spd}(R^+)$ is a formally closed subsheaf of $\text{Spd}(R^+)$. Reasoning as in [Proposition 4.9](#) we conclude $X = \text{Spd}(R^+)$ when $\text{Spa}(R, R^+) \rightarrow \mathcal{F}$ factors through \mathcal{G} . This proves that \mathcal{G} is v-formalizing and a specializing sheaf. Suppose \mathcal{F} is a prekimberlite. By formal adicness $(\mathcal{G}^{\text{red}})^{\circ} \rightarrow \mathcal{G}$ is a closed immersion. Now, $(\mathcal{G}^{\text{red}})^{\circ} \rightarrow (\mathcal{F}^{\text{red}})^{\circ}$ is also a formally closed immersion and by [Lemma 3.31](#) \mathcal{G}^{red} is represented by a closed subscheme of \mathcal{F} . This proves that \mathcal{G} is a prekimberlite. Since closed immersions are partially proper the map $\mathcal{F} \rightarrow (\mathcal{G}^{\text{red}})^{\circ/\circ}$ is partially proper. Consequently, the same holds for $\mathcal{F} \rightarrow (\mathcal{F}^{\text{red}})^{\circ/\circ}$. That $(\mathcal{G}, \mathcal{G} \cap \mathcal{D})$ is a smelted kimberlite follows from [[Sch17](#), Proposition 11.20]. If \mathcal{F} is a kimberlite, then $\mathcal{G}^{\text{an}} = \mathcal{F}^{\text{an}} \times_{\mathcal{F}} \mathcal{G}$ so $(\mathcal{G}, \mathcal{G}^{\text{an}})$ is a smelted kimberlite. The quasicompactness hypothesis on $\text{sp}_{\mathcal{G}^{\text{an}}}$ follows from that of $\text{sp}_{\mathcal{F}^{\text{an}}}$ and that closed immersions are quasicompact. \square

Proposition 4.42. *Let \mathcal{F} be a prekimberlite and $V \rightarrow \mathcal{F}^{\text{red}}$ a quasicompact, separated and étale map. The following hold:*

- (1) *If $(\mathcal{F}, \mathcal{D})$ is a smelted kimberlite, then $(\widehat{\mathcal{F}}_{/V}, \mathcal{D} \times_{\mathcal{F}} \widehat{\mathcal{F}}_{/V})$ is a smelted kimberlite.*
- (2) *If \mathcal{F} is a kimberlite $\widehat{\mathcal{F}}_{/V}$ is a kimberlite.*

Proof. This follows from [Lemma 4.25](#), from [Proposition 4.34](#), and [[Sch17](#), Corollary 11.28]. \square

4.6. Finiteness and normality. In this section, we discuss a finiteness and a normality hypothesis imposed on kimberlites to tame them. These notions are ad hoc, but they turned out to be useful for applications.

Let us give some motivation. Suppose \mathcal{X} is a formal scheme topologically of finite type over \mathbb{Z}_p , let X_η denote the generic fiber considered as an adic space and suppose let X^{red} denote the reduced special fiber. In this situation, we have a specialization map $\text{sp}_{X_\eta} : |X_\eta| \rightarrow |X^{\text{red}}|$, and for a fixed closed point $x \in |X^{\text{red}}|$ we have the following chain of inclusions $|\mathcal{X}/_x| \subseteq \text{sp}_{X_\eta}^{-1}(x) \subseteq |X_\eta|$. These inclusions satisfy:

- (1) $\text{sp}_{X_\eta}^{-1}(x)$ is a closed subset.
- (2) $|\mathcal{X}/_x|$ is the interior of $\text{sp}_{X_\eta}^{-1}(x)$ in $|X_\eta|$.
- (3) $|\mathcal{X}/_x|$ is dense in $\text{sp}_{X_\eta}^{-1}(x)$

The first two conditions generalize, by [Proposition 4.39](#), to the case of kimberlites whose closed points are constructible. Our finiteness condition is sufficient to make a kimberlite have the third property as well. As the next example shows finiteness hypothesis are necessary for this third property to hold.

Example 4.43. Let C be a nonarchimedean field and $C^+ \subseteq C$ an open and bounded valuation subring whose rank is strictly larger than 1. Then sp_C is a homeomorphism between $\text{Spa}(C, C^+)$ and $\text{Spec}(C^+/C^{\circ\circ})$. If x denotes the closed point of $\text{Spec}(C^+/C^{\circ\circ})$ then $y = \text{sp}_C^{-1}(x)$ is the closed point in $\text{Spa}(C, C^+)$. The interior of $\{y\}$ is empty.

Definition 4.44. We say that a locally spatial diamond X is *constructibly Jacobson* if the subset of rank 1 points are dense for the constructible topology of $|X|$. We refer to them as *cJ-diamonds*.

Proposition 4.45. *Suppose that $\mathcal{K} = (\mathcal{F}, \mathcal{D})$ is a smelted kimberlite with \mathcal{D} a cJ-diamond, let $S \subseteq |\mathcal{F}|$ a constructible subset. Then $|\mathcal{D} \cap \widehat{\mathcal{F}}/_S|$ is dense in $\text{sp}_{\mathcal{K}}^{-1}(S)$.*

Proof. By the proof of [Proposition 4.22](#), $|\mathcal{D} \cap \widehat{\mathcal{F}}/_S|$ is the largest subset of $\text{sp}_{\mathcal{K}}^{-1}(S)$ stable under generization. Now, $\text{sp}_{\mathcal{K}}^{-1}(S)$ is open in the patch topology of $|\mathcal{D}|$ and by assumption rank 1 points are dense in it. Since rank 1 points are stable under generization, they belong to $|\mathcal{D} \cap \widehat{\mathcal{F}}/_S|$. This proves that $|\mathcal{D} \cap \widehat{\mathcal{F}}/_S|$ is dense in $\text{sp}_{\mathcal{K}}^{-1}(S)$. \square

Proposition 4.46. *Let $f : X \rightarrow Y$ be a morphism of locally spatial diamonds the following hold:*

- (1) *If $|f|$ is surjective and X is a cJ-diamond, then Y is a cJ-diamond.*
- (2) *If f is an open immersion and Y is a cJ-diamond, then X is a cJ-diamond.*
- (3) *If f realizes X as a quasi-pro-étale \underline{J} -torsor over Y for a profinite group J and X is a cJ-diamond, then Y is a cJ-diamond.*
- (4) *If f is étale and Y is a cJ-diamond, then X is a cJ-diamond.*

Proof. Maps of locally spatial diamonds induce spectral and generalizing maps of locally spectral spaces. The first claim follows easily from this. Suppose that Y is a cJ-diamond. If f is an open immersion, any open in the patch topology of X is also open in the patch topology of Y . This proves the second claim. Moreover, this allow us to argue locally, so we can assume X and Y are spatial. If f is étale, by [[Sch17](#), Lemma 11.31], locally we can write f as the composition of an open immersion and a finite étale map. Spaces that are finite étale over a fixed spatial diamond form a Galois category and using the first claim we may reduce to the case in which f is Galois with finite Galois group G . In this way, the fourth claim follows from the third. For the third claim, we prove that f is an open map for the patch topology. This would finish the proof since the fibers of rank 1 points of a quasi-pro-étale map are also rank 1.

Let $J = \varprojlim_i J_i$ with J_i a cofiltered family of finite groups and denote by $f_i : X_i \rightarrow Y$ the induced J_i -torsors. We get continuous action maps $J_i \times |X_i|^{\text{cons}} \rightarrow |X_i|^{\text{cons}}$. Moreover, if $S \subseteq |X_i|$ then $f_i^{-1}(f_i(S)) = J_i \cdot S$. Now, the formation of the patch topology on a spectral space commutes with limits along spectral maps. This gives a continuous action map $J \times |X|^{\text{cons}} \rightarrow |X|^{\text{cons}}$. Let $U \subseteq X$ be open in the patch topology, then $f^{-1}(f(U)) = J \cdot U$ which is also open. We can conclude since $|f|^{\text{cons}} : |X|^{\text{cons}} \rightarrow |Y|^{\text{cons}}$ is a quotient map. \square

We recall a theorem of Huber. His statement is stronger, but we only need this weaker form.

Theorem 4.47. ([[Hub93](#), Theorem 4.1]) *Let K be a complete nonarchimedean field, and let A be a topologically of finite type K -algebra. Then, $\text{Max}(A) \subseteq \text{Spa}(A, A^\circ)$ is dense for the patch topology.*

Corollary 4.48. *If X is an adic space topologically of finite type over $\text{Spa}(K, O_K)$, with K a complete nonarchimedean field over \mathbb{Z}_p . Then X^\diamond is a cJ-diamond.*

Example 4.49. The perfectoid unit ball $\mathbb{B}_n = \mathrm{Spa}(C\langle T_1^{\frac{1}{p^\infty}} \dots T_n^{\frac{1}{p^\infty}} \rangle, O_C\langle T_1^{\frac{1}{p^\infty}} \dots T_n^{\frac{1}{p^\infty}} \rangle)$ over a perfectoid field C over \mathbb{F}_p , is a cJ-diamond. Indeed, we have an equality $\mathrm{Spa}(C\langle T_1 \dots T_n \rangle, O_C\langle T_1 \dots T_n \rangle)^\diamond = \mathbb{B}_n$.

Definition 4.50. Let C be a characteristic p perfectoid field and X a locally spatial diamond over $\mathrm{Spa}(C, O_C)$. We say that X has *enough facets* over C if it admits a v -cover of the form $\coprod_{i \in I} \mathrm{Spd}(A_i, A_i^\circ) \rightarrow X$ where each A_i is an algebra topologically of finite type over C .

Proposition 4.51. *Let X and Y be two locally spatial diamonds with enough facets over C , let C^\sharp be an untilt of C , and $C \rightarrow C'$ a map of perfectoid fields. The following hold:*

- (1) *The base change $X \times_{\mathrm{Spa}(C, O_C)} \mathrm{Spa}(C', O_{C'})$ has enough facets over C' .*
- (2) *The fiber product $X \times_{\mathrm{Spa}(C, O_C)} Y$ has enough facets over C .*
- (3) *X is a cJ-diamond.*
- (4) *If $X = \mathrm{Spd}(A, A^\circ)$ for a smooth and topologically of finite type C^\sharp -algebra A , then X has enough facets.*

Proof. Since being topologically of finite type is stable under products and change of the ground field the first two claims follows. The third claim follows from [Theorem 4.47](#) and [Proposition 4.46](#). For the last claim, let $\mathbb{T}_{C^\sharp}^n$ denote $\mathrm{Spa}(C^\sharp\langle T_1^\pm, \dots, T_n^\pm \rangle, O_{C^\sharp}\langle T_1^\pm, \dots, T_n^\pm \rangle)$, and let $\tilde{\mathbb{T}}_{C^\sharp}^n = \varprojlim_{T_i \rightarrow T_i^p} \mathbb{T}_{C^\sharp}^n$ analogously for \mathbb{T}_C^n and $\tilde{\mathbb{T}}_C^n$. For $x \in \mathrm{Spa}(A, A^\circ)$ there is U a neighborhood of x and an étale map $\eta : U \rightarrow \mathbb{T}_{C^\sharp}^n$. Let \tilde{U} be the pullback of η along $\tilde{\mathbb{T}}_{C^\sharp}^n \rightarrow \mathbb{T}_{C^\sharp}^n$, we get an étale map $\tilde{U}^b \rightarrow \tilde{\mathbb{T}}_{C^\sharp}^n$. By the invariance of the étale site under perfection ([\[Sch17, Lemma 15.6\]](#)) $\tilde{U}^b = U'^\diamond$ for an adic space U' étale over \mathbb{T}_C^n . Now, U' admits an open cover of the form $\coprod_{i \in I} \mathrm{Spa}(A_i, A_i^\circ) \rightarrow U'$ with each A_i topologically of finite type over C . This gives a cover, $\coprod_{i \in I} \mathrm{Spd}(A_i, A_i^\circ) \rightarrow \tilde{U}^b \rightarrow U^\diamond$. \square

Definition 4.52. Let \mathcal{G} be a kimberlite and $\mathcal{K} = (\mathcal{F}, \mathcal{D})$ a smelted kimberlite.

- (1) We say that \mathcal{K} is *rich* if: \mathcal{F} is valuative, \mathcal{D} is a cJ-diamond, $|\mathcal{F}^{\mathrm{red}}|$ is locally Noetherian and $\mathrm{sp}_{\mathcal{D}} : |\mathcal{D}| \rightarrow |\mathcal{F}^{\mathrm{red}}|$ is surjective.
- (2) We say that \mathcal{G} is *rich* if: $(\mathcal{G}, \mathcal{G}^{\mathrm{an}})$ is rich.
- (3) If \mathcal{K} is rich we say it is *topologically normal* if for every closed point $x \in |\mathcal{F}^{\mathrm{red}}|$ the tubular neighborhood $\mathcal{D}_{/x}^\circ$ is connected.⁷

Lemma 4.53. *If $\mathcal{K} = (\mathcal{F}, \mathcal{D})$ is a rich smelted kimberlite, then $\mathrm{sp}_{\mathcal{D}}$ is a quotient map.*

Proof. We can argue locally on $|\mathcal{F}^{\mathrm{red}}|$, so we may assume $\mathcal{F}^{\mathrm{red}} = \mathrm{Spec}(A)^\circ$. We first prove the case that $|\mathcal{F}^{\mathrm{red}}|$ is irreducible. Let g be the generic point of $|\mathcal{F}^{\mathrm{red}}|$, let $r \in |\mathcal{D}|$ with $\mathrm{sp}_{\mathcal{D}}(r) = g$ represented by formalizable map $f_r : \mathrm{Spa}(C, O_C) \rightarrow \mathcal{D}$. Let C_g^{min} be the minimal ring of integral elements of C such that $C_g^{\mathrm{min}}/C^\circ$ contains A . This defines a map $\mathrm{Spa}(C, C_g^{\mathrm{min}}) \rightarrow \mathrm{Spec}(A)^\circ$. By valuativity, we get a lift $\mathrm{Spa}(C, C_g^{\mathrm{min}}) \rightarrow \mathcal{D}$. The map $f^{\mathrm{min}} : |\mathrm{Spa}(C, C^{\mathrm{min}})| \rightarrow |\mathcal{F}^{\mathrm{red}}|$ is specializing, surjective and a spectral map of spectral spaces. By [Corollary 1.15](#), f^{min} is a closed map and consequently a quotient map of topological spaces. The case in which $|\mathcal{F}^{\mathrm{red}}|$ has a finite number of irreducible components is analogous. Since we are allowed to work locally and we assumed $|\mathcal{F}^{\mathrm{red}}|$ is locally Noetherian, every point has an affine neighborhood with finitely many irreducible components. \square

Proposition 4.54. *Constructible formal neighborhoods and étale formal neighborhoods preserve rich smelted kimberlites.*

Proof. This follows from [Proposition 4.46](#), [Proposition 4.22](#) and [Proposition 4.24](#). \square

The following lemma was the starting point of our theory of specialization and a key input for our work on connected components of moduli spaces of p -adic shtukas [\[Gle22\]](#).

Lemma 4.55. *Let $\mathcal{K} = (\mathcal{F}, \mathcal{D})$ be a topologically normal rich smelted kimberlite. Then $\pi_0(\mathrm{sp}_{\mathcal{D}}) : \pi_0(|\mathcal{D}|) \rightarrow \pi_0(|\mathcal{F}^{\mathrm{red}}|)$ is bijective.*

Proof. Let $U, V \subseteq |\mathcal{D}|$ be two non-empty closed-open subsets with $V \cup U = |\mathcal{D}|$. Clearly, $\pi_0(\mathrm{sp}_{\mathcal{D}})$ is surjective. Suppose that $\emptyset \neq \mathrm{sp}_{\mathcal{F}}(U) \cap \mathrm{sp}_{\mathcal{F}}(V)$, it suffices to show that $U \cap V \neq \emptyset$. Since $\mathrm{sp}_{\mathcal{D}}$ is specializing we can assume there is a closed point $x \in \mathrm{sp}_{\mathcal{D}}(U) \cap \mathrm{sp}_{\mathcal{D}}(V)$. Since $|\mathcal{F}^{\mathrm{red}}|$ is locally Noetherian, closed

⁷This definition is motivated by [\[AGLR22, Proposition 2.38\]](#), where the authors observed that normality “could be captured” in terms of tubular neighborhoods.

points are open in the constructible topology. By [Proposition 4.45](#), $\mathcal{D}_{/x}^\circledast$ is dense in $\mathrm{sp}_{\mathcal{D}}^{-1}(x)$, which implies that $\mathrm{sp}_{\mathcal{D}}^{-1}(x)$ is connected. Connectedness gives that $(\mathrm{sp}_{\mathcal{D}}^{-1}(x) \cap U) \cap (\mathrm{sp}_{\mathcal{D}}^{-1}(x) \cap V) \neq \emptyset$ and in particular $U \cap V \neq \emptyset$ which is what we wanted to show. \square

5. THE SPECIALIZATION MAP FOR UNRAMIFIED p -ADIC BEILINSON–DRINFELD GRASSMANNIANS

So far our discussion has been purely theoretical. In this section, we apply the theory to construct and study the specialization map for some p -adic Beilinson–Drinfeld Grassmannians [[SW20](#), Definition 20.3.1]. For the rest of the section G denotes a reductive over \mathbb{Z}_p and we let $T \subseteq B \subseteq G$ denote integrally defined maximal torus and Borel subgroups respectively. Let $\mu \in X_*^+(T_{\overline{\mathbb{Q}}_p})$ be a dominant cocharacter with reflex field $E \subseteq \overline{\mathbb{Q}}_p$. Let O_E denote the ring of integers of E and let k_E denote the residue field. Let $\mathrm{Gr}_{O_E}^{G, \leq \mu}$ denote the v -sheaf parametrizing B_{dR}^+ -lattices with G -structure whose relative position is bounded by μ as in [[SW20](#), Definition 20.5.3] and let $\mathrm{Gr}_{\mathcal{W}, k_E}^{G, \leq \mu}$ denote the Witt vector affine Grassmannian. Let F be a nonarchimedean field extension of E . Let O_F denote the ring of integers of F and the residue field k_F , assume that F is complete for the p -adic topology and that k_F is perfect. Let $\mathrm{Gr}_{O_F}^{G, \leq \mu} := \mathrm{Gr}_{O_E}^{G, \leq \mu} \times_{\mathrm{Spd}(O_E)} \mathrm{Spd}(O_F)$ and let $\mathrm{Gr}_{\mathcal{W}, k_F}^{G, \leq \mu} = \mathrm{Gr}_{\mathcal{W}, k_E}^{G, \leq \mu} \times_{\mathrm{Spec}(k_E)} \mathrm{Spec}(k_F)$. Here is our result:

Theorem 5.1. $\mathrm{Gr}_{O_F}^{G, \leq \mu}$ is a topologically normal rich p -adic kimberlite with $(\mathrm{Gr}_{O_F}^{G, \leq \mu})^{\mathrm{red}} = \mathrm{Gr}_{\mathcal{W}, k_F}^{G, \leq \mu}$.

Remark 5.2. This result has partially been generalized in our collaboration [[AGLR22](#)]. There, we prove that the local models for parahoric groups are rich p -adic kimberlites. Nevertheless, we only improve the “normality” part of the result if we assume that μ is minuscule and outside certain cases in small characteristic.

5.1. Twisted loop sheaves. Fix a perfect field k in characteristic p . Let $X = \mathrm{Spec}(A)$ be a scheme of finite type over $W(k)$. In [[Sch17](#)], Scholze associates to X two v -sheaves over $\mathrm{Spd}(W(k))$, which we denote X^\diamond and X° following [[AGLR22](#), Definition 2.10]. Here $X^\diamond : \mathrm{Perf}_k \rightarrow \mathrm{Sets}$ assigns to $\mathrm{Spa}(R, R^+)$ triples (R^\sharp, ι, f) with (R^\sharp, ι) an untilt and $f \in \mathrm{Hom}_{W(k)}(A, R^\sharp)$ is a $W(k)$ -algebra homomorphism. On the other hand, X assigns triples (R^\sharp, ι, f) with $f \in \mathrm{Hom}_{W(k)}(A, R^{\sharp,+})$. Now, $X^\circ \subseteq X^\diamond$ is open. Both of these functors glue to a construction defined for schemes X locally of finite type over $\mathrm{Spec}(W(k))$. Visibly, these constructions are related to $\diamond : \mathrm{PreAd}_{W(k)} \rightarrow \mathrm{Perf}$. Let us elaborate.

Let X_p denote the p -adic completion of X . Now, X_p is a p -adic Noetherian formal scheme that we may regard as an adic space. We have an identification $X_p^\diamond = X^\circ$. If $X = \mathrm{Spec}(A)$ and $X_f \rightarrow X$ is the open $X_f = \mathrm{Spec}(A[\frac{1}{f}])$ with $f \in A$, then $(X_f)_p \rightarrow X_p$ is the locus in X_p where $1 \leq |f|$.

The construction of X^\diamond is more elaborate. Given an adic space S (thought of as a triple $(|S|, \mathcal{O}_S, \{v_s : s \in |S|\})$ in Huber’s category \mathcal{Y} see [[Hub94](#)]), we let S^H denote the topologically ringed space $(|S|, \mathcal{O}_S)$ that is obtained from S by forgetting the last entry of data. Suppose we are given a morphism of schemes $f : X \rightarrow Y$ that is locally of finite type and a morphism $g : S^H \rightarrow Y$ of locally ringed spaces where S is an adic space for which every point $s \in S$ has an affinoid open neighborhood with Noetherian ring of definition. In [[Hub94](#), Proposition 3.8]), Huber constructs an adic space ${}^{\circ}S \times_Y X$ together with a map of adic spaces $p_1 : {}^{\circ}S \times_Y X \rightarrow S$ and a map of locally ringed spaces $p_2 : ({}^{\circ}S \times_Y X)^H \rightarrow X$ with the following universal property. If T is an adic space, $\pi_1 : T \rightarrow S$ is a map of adic spaces and $\pi_2 : T^H \rightarrow X$ is a map of locally ringed spaces such that $f \circ \pi_1 = g \circ \pi_2^H$, then there is a unique map $\pi : T \rightarrow {}^{\circ}S \times_Y X$ such that $p_1 \circ \pi = \pi_1$ and $p_2 \circ \pi^H = \pi_2$. Letting $Y = \mathrm{Spec}(W(k))$ and $S = \mathrm{Spa}(W(k))$ we define X^{ad} as $({}^{\circ}S \times_Y X)$. Then, $X^\diamond = (X^{\mathrm{ad}})^\diamond$. Moreover, if $X = \mathrm{Spec}(A)$ and $X_f = \mathrm{Spec}(A[\frac{1}{f}])$ for $f \in A$ we can see from the universal property that X_f^{ad} is the open locus of X^{ad} where $f \neq 0$.

Proposition 5.3. If $X \rightarrow \mathrm{Spec}(W(k))$ is a proper map of schemes, then the natural map $X^\circ \rightarrow X^\diamond$ is an isomorphism.

Proof. It follows directly from [[Hub94](#), Remark 4.6.(iv).d]. \square

Proposition 5.4. If X and Y are qcqs finite type schemes over $\mathrm{Spec}(W(k))$ and that $X \rightarrow Y$ is universally subtrusive as in [Definition 3.4](#), then $X^\diamond \rightarrow Y^\diamond$ and $X^\circ \rightarrow Y^\circ$ are surjective.

Proof. Replacing Y by an open cover we may assume $Y = \mathrm{Spec}(A)$. By [[Ryd10](#), Theorem 3.12] we may assume that $X \rightarrow Y$ factors as $X \rightarrow Y' \rightarrow Y$ with $Y' \rightarrow Y$ a proper surjection and $X \rightarrow Y'$ a quasicompact open cover. Open covers give surjective maps so we can reduce to the proper case. By Chow’s lemma [[Sta18](#), Tag 0200], we may assume $Y' \rightarrow Y$ is projective. Now, $Y'^\diamond \rightarrow Y^\diamond$ and

$Y'^{\diamond} \rightarrow Y^{\diamond}$ are quasicompact. Indeed, they are the composition of a closed immersion and projection from $(\mathbb{P}_{W(k)}^n)^{\diamond} \times_{\mathrm{Spd}(W(k))} Y^{\diamond}$ (respectively $(\mathbb{P}_{W(k)}^n)^{\diamond} \times_{\mathrm{Spd}(W(k))} Y^{\diamond}$). By [Sch17, Lemma 12.11], we may check surjectivity at a topological level. Take geometric points $r : \mathrm{Spa}(C, C^+) \rightarrow Y^{\diamond}$ and $s : \mathrm{Spa}(C, C^+) \rightarrow Y^{\diamond}$ given by ring maps $r^* : A \rightarrow C$ and $s^* : A \rightarrow C^+$. Since $Y' \rightarrow Y$ is proper and surjective $\mathrm{Spec}(C) \times_Y Y' \rightarrow \mathrm{Spec}(C)$ admits a section inducing a lift to Y'^{\diamond} . Analogously, $\mathrm{Spec}(C^+) \times_Y Y' \rightarrow \mathrm{Spec}(C^+)$ admits a section (by the valuative criterion of properness). \square

Proposition 5.5. *Let X be locally of finite type over $W(k)$ with special fiber X_s . Then $(X^{\diamond})^{\mathrm{red}} = X_s^{\mathrm{perf}} = (X^{\circ})^{\mathrm{red}}$.*

Proof. Both identifications follow from Proposition 3.18. By the construction of X_p as a p -adic completion in the case of X° , and by the universal property of X^{ad} in the case of X^{\diamond} . \square

For the rest of the section let C be an algebraically closed nonarchimedean field over k with ring of integers O_C and residue field k_C . Fix a characteristic 0 untilt C^{\sharp} and fix $\xi \in W(O_C)$ a generator for the kernel of $W(O_C) \rightarrow O_{C^{\sharp}}$. The choice of untilt determines a unique map $\mathrm{Spd}(O_C) \rightarrow \mathrm{Spd}(O_{C^{\sharp}})$.

Definition 5.6. We denote ring sheaves $W^+(\mathcal{O}), B_{\mathrm{dR}}^+(\mathcal{O}^{\sharp}), W(\mathcal{O}), B_{\mathrm{dR}}(\mathcal{O}^{\sharp}) : \mathrm{Perf}_{\mathrm{Spd}(O_C)} \rightarrow \mathrm{Sets}$

- (1) Where $W^+(\mathcal{O})$ assigns to $\mathrm{Spa}(R, R^+) \rightarrow \mathrm{Spd}(O_C)$ the ring $W(R^+)$.
- (2) Where $B_{\mathrm{dR}}^+(\mathcal{O}^{\sharp})$ assigns to $\mathrm{Spa}(R, R^+) \rightarrow \mathrm{Spd}(O_C)$ the ring $B_{\mathrm{dR}}^+(R^{\sharp})$.
- (3) Where $W(\mathcal{O})$ assigns to $\mathrm{Spa}(R, R^+) \rightarrow \mathrm{Spd}(O_C)$ the ring $W(R^+)_{[\frac{1}{\xi}]}$.
- (4) Where $B_{\mathrm{dR}}(\mathcal{O}^{\sharp})$ assigns to $\mathrm{Spa}(R, R^+) \rightarrow \mathrm{Spd}(O_C)$ the ring $B_{\mathrm{dR}}(R^{\sharp}) := B_{\mathrm{dR}}^+(R^{\sharp})_{[\frac{1}{\xi}]}$.

Note that we have reduction maps $\mathrm{red} : W^+(\mathcal{O}) \rightarrow \mathcal{O}^{\sharp,+}$ and $\mathrm{red} : B_{\mathrm{dR}}^+(\mathcal{O}^{\sharp}) \rightarrow \mathcal{O}^{\sharp}$.

Definition 5.7. Let H be a finite type affine scheme over $\mathbb{G}_{m,W(k)}$, and (\mathcal{H}, ρ) a finite type affine scheme over $\mathbb{A}_{W(k)}^1$ with an isomorphism $\rho : \mathcal{H} \times_{\mathbb{A}_{W(k)}^1} \mathbb{G}_m \rightarrow H$. We associate v-sheaves over $\mathrm{Spd}(O_C)$.

- (1) $W^+\mathcal{H}$ assigns to $\mathrm{Spa}(R, R^+)$ the set of sections to $\mathcal{H}_{W(R^+)} \rightarrow \mathrm{Spec}(W(R^+))$.
- (2) WH assigns to $\mathrm{Spa}(R, R^+)$ the set of sections to $H_{W(R^+)_{[\frac{1}{\xi}]}} \rightarrow \mathrm{Spec}(W(R^+)_{[\frac{1}{\xi}]})$.
- (3) $L^+\mathcal{H}$ assigns to $\mathrm{Spa}(R, R^+)$ the set of sections to $\mathcal{H}_{B_{\mathrm{dR}}^+(R^{\sharp})} \rightarrow \mathrm{Spec}(B_{\mathrm{dR}}^+(R^{\sharp}))$.
- (4) LH assigns to $\mathrm{Spa}(R, R^+)$ the set of sections to $\mathcal{H}_{B_{\mathrm{dR}}(R^{\sharp})} \rightarrow \mathrm{Spec}(B_{\mathrm{dR}}(R^{\sharp}))$.

Here the base change of \mathcal{H} and H are given by maps to $\mathbb{A}_{W(k)}^1$ and $\mathbb{G}_{m,W(k)}$ defined by $t \mapsto \xi$.

ρ induces maps $L^+\mathcal{H} \xrightarrow{\rho} LH$ and $W^+\mathcal{H} \xrightarrow{\rho} WH$. We get the following diagrams of inclusions:

$$\begin{array}{ccc}
 & W(\mathcal{O}) & \\
 \swarrow & & \searrow \\
 W^+(\mathcal{O}) & & B_{\mathrm{dR}}(\mathcal{O}^{\sharp}) \\
 \searrow & & \swarrow \\
 & B_{\mathrm{dR}}^+(\mathcal{O}^{\sharp}) &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & WH & \\
 \swarrow & & \searrow \\
 W^+\mathcal{H} & & LH \\
 \searrow & & \swarrow \\
 & L^+\mathcal{H} &
 \end{array}$$

Moreover, if $\overline{\mathcal{H}}$ denotes the basechange $\mathcal{H} \times_{\mathbb{A}_{W(k)}^1} \mathrm{Spec}(W(k))$ at $t = 0$ we get reduction maps $W^+\mathcal{H} \rightarrow (\overline{\mathcal{H}})_{\mathrm{Spd}(O_C)}^{\diamond}$ and $L^+\mathcal{H} \rightarrow (\overline{\mathcal{H}})_{\mathrm{Spd}(O_C)}^{\diamond}$.

Proposition 5.8. *If \mathcal{H} is smooth over $\mathrm{Spec}(W(k)[t])$ the reduction maps are surjective.*

Proof. We claim that the map is surjective even at the level of presheaves. The (R, R^+) -valued points of $(\overline{\mathcal{H}})^{\diamond}$ and $(\overline{\mathcal{H}})^{\circ}$ can be seen as maps $\mathrm{Spec}(R^{\sharp}) \rightarrow \mathcal{H}_{B_{\mathrm{dR}}(R^{\sharp})}$ and $\mathrm{Spec}(R^{\sharp,+}) \rightarrow \mathcal{H}_{W(R^+)}$ whose composition with the projections to $\mathrm{Spec}(B_{\mathrm{dR}}(R^{\sharp}))$ and $\mathrm{Spec}(W(R^+))$ are the usual closed embeddings. By smoothness of \mathcal{H} , for any $n \in \mathbb{N}$ the maps can be lifted to maps $\mathrm{Spec}(B_{\mathrm{dR}}(R^{\sharp})/\xi^n) \rightarrow \mathcal{H}_{B_{\mathrm{dR}}(R^{\sharp})}$ and $\mathrm{Spec}(W(R^+)/\xi^n) \rightarrow \mathcal{H}_{W(R^+)}$ respectively. Since \mathcal{H} is an affine scheme and since both $B_{\mathrm{dR}}(R^{\sharp})$ and $W(R^+)$ are (ξ) -adically complete we may pass to the inverse limit by choosing compatible lifts. \square

Definition 5.9. We define scheme-theoretic v-sheaves $\mathcal{W}_{\mathrm{red}}^+(\mathcal{O}), \mathcal{W}_{\mathrm{red}}^+\mathcal{H} : \mathrm{PCAlg}_{k_C}^{\mathrm{op}} \rightarrow \mathrm{Sets}$.

- (1) Let $\mathcal{W}_{\mathrm{red}}^+(\mathcal{O})$ attach to $\mathrm{Spec}(R)$ the ring $W(R)$.
- (2) Let $\mathcal{W}_{\mathrm{red}}^+\mathcal{H}$ attach to $\mathrm{Spec}(R)$ the sections to $\mathcal{H} \times_{W(R)} \rightarrow \mathrm{Spec}(W(R))$.

Here the base change of \mathcal{H} is given by the map to $\mathbb{A}_{W(k)}^1$ defined by $t \mapsto p$.

Remark 5.10. These v-sheaves are Zhu’s p -adic jet spaces in [Zhu17, §1.1.1].

Proposition 5.11. *With the notation as above, $W^+\mathcal{H}$ is a p -adic kimberlite and $(W^+\mathcal{H})^{\text{red}} = (\mathcal{W}_{\text{red}}^+\mathcal{H})$.*

Proof. $W^+(\mathcal{O})$ is represented by $\text{Spd}(O_C\langle T_n \rangle_{n \in \mathbb{N}})$, by Proposition 4.17, Proposition 3.23 and Proposition 3.18 $W^+(\mathcal{O})$ is a p -adic kimberlite with $W^+(\mathcal{O})^{\text{red}} = \text{Spec}(k_C[T_n]_{n \in \mathbb{N}})$ which is $\mathcal{W}_{\text{red}}^+(\mathcal{O})$. If $\mathcal{H} = \text{Spec}(A)$ is presented as $A = W(k)[t][\bar{x}]/I$ with $I = (f_1(t, \bar{x}), \dots, f_m(t, \bar{x}))$. Then $W^+\mathcal{H}$ fits in the commutative diagram with Cartesian square:

$$\begin{array}{ccc} W^+\mathcal{H} & \longrightarrow & \text{Spd}(O_C) \\ \downarrow & & \downarrow 0 \searrow \text{id} \\ W^+(\mathcal{O})^n & \xrightarrow{F} & W^+(\mathcal{O})^m \longrightarrow \text{Spd}(O_C) \end{array}$$

Here $F(\bar{r}) = (f_1(\xi, \bar{r}), \dots, f_m(\xi, \bar{r}))$. All of these maps are formally adic, and $\text{Spd}(O_C) \xrightarrow{0} W^+(\mathcal{O})^m$ is formally closed. By Proposition 4.41 $W^+\mathcal{H}$ is a p -adic kimberlite. Finally, we can basechange by $\text{Spec}(k_C)^\circ \rightarrow \text{Spd}(O_C)$ to compute reductions. \square

5.2. Demazure resolution. We assume the reader has some familiarity with Bruhat–Tits theory and parahoric group schemes [BT84]. We use twisted loop sheaves to consider integrally defined Demazure resolutions. Our main observation is that the Demazure resolution can be constructed using either parahoric loop groups or what we call below “formal parahoric loop” groups. The difference is whether one uses B_{dR} or A_{inf} coefficients. We keep the notation as above.

- (1) Let H be a split $W(k)$ -reductive group, and $T \subseteq B \subseteq H$ maximal split torus and Borel subgroups.
- (2) Let $(X^*, \Phi, X_*, \Phi^\vee)$ be the root datum associated to (H, T) .
- (3) We let $\langle \cdot, \cdot \rangle : X^* \times X_* \rightarrow \mathbb{Z}$ denote the perfect pairing between roots and coroots.
- (4) Let Φ^+ be the set of positive roots associated to B .
- (5) Let N be the normalizer of T in H .
- (6) Let $W = N/T$ be the Weyl group of H .
- (7) We let $\mathcal{A} = \mathcal{A}(H, T)$ denote $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$.
- (8) We let $\Psi = \{\alpha + n \mid \alpha \in \Phi, n \in \mathbb{Z}\}$ denote the set of affine roots on \mathcal{A} .
- (9) Given a point $q \in \mathcal{A}$ we let $\Phi_q = \{\alpha \in \Phi \mid \alpha(q) \in \mathbb{Z}\}$ this is a closed sub-root system.
- (10) Let M_q be the Levi subgroup containing T with root datum given by $(X^*, \Phi_q, X_*, \Phi_q^\vee)$.
- (11) If $q \in \mathcal{A}$ we associate $F_q \subseteq \mathcal{A}$ containing q and bounded by the hyperplanes defined by Ψ .
- (12) We let $o \in \mathcal{A}$ be the origin and $o \in \mathcal{C}$ be the alcove contained in the Bruhat chamber of B .
- (13) Let \mathbb{S} be the reflections along the walls of \mathcal{C} , we let W^{aff} the affine Weyl group generated by \mathbb{S} .
- (14) For facets $\mathcal{F} \subseteq \mathcal{C}$ let $\mathbb{S}_{\mathcal{F}}$ be elements of \mathbb{S} fixing \mathcal{F} and let $W_{\mathcal{F}}$ be the subgroup generated by $\mathbb{S}_{\mathcal{F}}$.
- (15) Let \tilde{W} be the Iwahori–Weyl. Recall that if $\Omega_H = \pi_1(H^{\text{der}})$ then $\tilde{W} = W^{\text{aff}} \rtimes \Omega_H$.

Fix a point $q \in \mathcal{A}$. Using Bruhat–Tits theory and dilatation techniques ([PZ13, §3]), one can construct smooth affine algebraic groups \mathcal{H}_q over $\text{Spec}(W(k)[t])$ and isomorphism $\rho : \mathcal{H}_q \times_{W(k)[t]} \text{Spec}(W(k)[t, t^{-1}]) \cong H \times_{W(k)} \text{Spec}(W(k)[t, t^{-1}])$ satisfying the following:

- a) For a DVR V with uniformizer $\pi \in V$ the basechange $\mathcal{H}_q \times_{\mathbb{A}_{W(k)}^1} \text{Spec}(V)$ along $t \mapsto \pi$ is the parahoric group scheme of $q \in \mathcal{A}(H, V[\frac{1}{\pi}]) = \mathcal{A}$.
- b) For $\alpha \in \Phi$ there are closed subgroups $\mathcal{U}_\alpha^q \subseteq \mathcal{H}_q$ and $\mathcal{T} \subseteq \mathcal{H}_q$ with $\mathbb{G}_a \cong \mathcal{U}_\alpha^q$ and $\mathbb{G}_m^n \cong \mathcal{T}$. These extend the root groups $U_\alpha \subseteq H$ and the torus $T \subseteq H$.
- c) There is an open cell decomposition: $\mathcal{V}_q := \prod_{\alpha \in \Phi^-} \mathcal{U}_\alpha^q \times \mathcal{T} \times \prod_{\alpha \in \Phi^+} \mathcal{U}_\alpha^q \subseteq \mathcal{H}_q$. It is an open immersion surjecting onto a fiberwise Zariski-dense neighborhood of the identity section.
- d) The group multiplication map $\mathcal{V}_q \times \mathcal{V}_q \xrightarrow{\mu} \mathcal{H}_q$ is smooth and surjective.
- e) The basechange $\overline{\mathcal{H}}_q := \mathcal{H}_q \times_{\mathbb{A}_{W(k)}^1} \text{Spec}(W(k))$ along $t = 0$ supports a split reductive quotient $(\overline{\mathcal{H}}_q)^{\text{Red}}$ over $W(k)$ with root datum identified with $(X^*, \Phi_q, X_*, \Phi_q^\vee)$ so that $M_q = (\overline{\mathcal{H}}_q)^{\text{Red}}$.
- f) If $\alpha \in \Phi_q$, then $\overline{\mathcal{U}}_{\alpha, t=0}^q \rightarrow (\overline{\mathcal{H}}_q)^{\text{Red}}$ identifies with the root group of $(\overline{\mathcal{H}}_q)^{\text{Red}}$ corresponding to α .
- g) If $\alpha \in \Phi \setminus \Phi_q$ then the composition $\overline{\mathcal{U}}_{\alpha, t=0}^q \rightarrow (\overline{\mathcal{H}}_q)^{\text{Red}}$ factors through the identity section.
- h) We have a commutative diagram of open cell decompositions:

$$\begin{array}{ccccc}
\prod_{\alpha \in \Phi \setminus \Phi_q} \overline{U}_\alpha & \longrightarrow & \prod_{\alpha \in \Phi^-} \overline{U}_\alpha \times \overline{\mathcal{T}} \times \prod_{\alpha \in \Phi^+} \overline{U}_\alpha & \xrightarrow{\pi} & \prod_{\alpha \in \Phi_q^-} \overline{U}_\alpha \times \overline{\mathcal{T}} \times \prod_{\alpha \in \Phi_q^+} \overline{U}_\alpha \\
\downarrow \cong & & \downarrow \mu & & \downarrow \mu \\
\text{Ker}(m) & \longrightarrow & \overline{\mathcal{H}}_q & \xrightarrow{m} & \overline{\mathcal{H}}_q^{\text{Red}}
\end{array}$$

Also, given $q_1, q_2 \in \mathcal{A}$ with $F_{q_1} \subseteq F_{q_2}$ we get a map groups $f : \mathcal{H}_{q_2} \rightarrow \mathcal{H}_{q_1}$ satisfying:

- i) $\rho_1 \circ f = \rho_2$ over $\text{Spec}(W(k)[t, t^{-1}])$.
- j) $\overline{\mathcal{H}}_{q_2} \rightarrow (\overline{\mathcal{H}}_{q_1})^{\text{Red}}$ surjects onto the parabolic subgroup associated to the sub-root system $\Phi_{q_1, q_2} := \{\alpha \in \Phi_{q_1} \mid [\alpha(q_2)] = \alpha(q_1)\}$.
- k) The kernel of $\overline{\mathcal{H}}_{q_2} \rightarrow (\overline{\mathcal{H}}_{q_1})^{\text{Red}}$ is fiberwise a vector group.

Definition 5.12. Let $q \in \mathcal{A}$. Since \mathcal{H}_q and H are defined over $\text{Spec}(W(k)[t])$ and $\text{Spec}(W(k)[t, t^{-1}])$ we can use the construction of [Definition 5.7](#). We call LH the loop group, we call $L^+\mathcal{H}_q$ the parahoric loop group and we call $W^+\mathcal{H}_q$ the ‘‘formal parahoric loop group’’.

Notice that we have injective maps of v-sheaves $W^+\mathcal{H}_q \subseteq L^+\mathcal{H}_q \subseteq LH$.

Proposition 5.13. *With the notation as above, for any point $q \in \mathcal{A}$ we have surjective morphisms of v-sheaves in groups: $L^+\mathcal{H}_q \rightarrow [(\overline{\mathcal{H}}_q)^{\text{Red}}]^\diamond = M_q^\diamond$ and $W^+\mathcal{H}_q \rightarrow [(\overline{\mathcal{H}}_q)^{\text{Red}}]^\diamond = M_q^\diamond$.*

Proof. This is a direct consequence of [Proposition 5.8](#) and [Proposition 5.4](#) since $\overline{\mathcal{H}} \rightarrow \overline{\mathcal{H}}^{\text{Red}}$ is smooth. \square

We let $L^u\mathcal{H}_q$ and $W^u\mathcal{H}_q$ denote respectively the kernels of the morphisms of [Proposition 5.13](#) above.

Proposition 5.14. *If $q_1, q_2 \in \mathcal{A}$ are such that $F_{q_1} \subseteq F_{q_2}$, then we get inclusions of v-sheaves in groups: $L^u\mathcal{H}_{q_1} \subseteq L^u\mathcal{H}_{q_2} \subseteq L^+\mathcal{H}_{q_2} \subseteq L^+\mathcal{H}_{q_1}$ and $W^u\mathcal{H}_{q_1} \subseteq W^u\mathcal{H}_{q_2} \subseteq W^+\mathcal{H}_{q_2} \subseteq W^+\mathcal{H}_{q_1}$. Moreover, the map from $L^+\mathcal{H}_{q_2}$ to $M_{q_1}^\diamond$ surjects onto $P_{\Phi_{q_1, q_2}}^\diamond \subseteq M_{q_1}^\diamond$. Analogously, $W^+\mathcal{H}_{q_2}$ surjects onto $P_{\Phi_{q_1, q_2}}^\diamond \subseteq M_{q_1}^\diamond$.*

Proof. We deal with the parahoric loop group case, the other is analogous. Recall that $f : \mathcal{H}_{q_2} \rightarrow \mathcal{H}_{q_1}$ satisfies $\rho_1 \circ f = \rho_2$. Functoriality of L^+ , gives $L^+\mathcal{H}_{q_2} \rightarrow L^+\mathcal{H}_{q_1} \rightarrow LH$. Since $L^+\mathcal{H}_{q_2} \rightarrow LH$ is injective, then $L^+\mathcal{H}_{q_2} \rightarrow L^+\mathcal{H}_{q_1}$ also is. The map of affine schemes $\overline{\mathcal{H}}_{q_2} \rightarrow P_{\Phi_{q_1, q_2}}$ is faithfully flat of finite presentation. By [Proposition 5.8](#) and [Proposition 5.4](#), the map $L^+\mathcal{H}_{q_2} \rightarrow (\overline{\mathcal{H}}_{q_2})^\diamond \rightarrow P_{\Phi_{q_1, q_2}}^\diamond$ is surjective. Finally, any map $g : \text{Spec}(B_{\text{dR}}^+(R^\sharp)) \rightarrow \mathcal{H}_{q_1, B_{\text{dR}}^+(R^\sharp)}$ whose induced map $\text{Spec}(R^\sharp) \rightarrow M_q$ factors through the identity lifts to a map $\text{Spec}(B_{\text{dR}}^+(R^\sharp)) \rightarrow \mathcal{H}_{q_2, B_{\text{dR}}^+(R^\sharp)}$. Indeed, $\text{Spec}(R^\sharp) \rightarrow \overline{\mathcal{H}}_{q_1}$ is in the open cell $\overline{\mathcal{V}}_{q_1}$, so g has the form $g = (\prod_{\alpha \in \Phi^-} u_\alpha(g)) \cdot t(g) \cdot (\prod_{\alpha \in \Phi^+} u_\alpha(g))$ with $t(g)$ and $\{u_\alpha(g)\}_{\alpha \in \Phi_{q_1}}$ reducing to the identity modulo ξ . By inspection, each of this elements lifts uniquely to \mathcal{V}_{q_2} . For \mathcal{T} and U_α with $\alpha \in (\Phi \setminus \Phi_{q_1}) \cup \Phi_{q_1, q_2}$, f is an isomorphism. For $\alpha \in \Phi_{q_1} \setminus \Phi_{q_1, q_2}$ we may, after making some choices, write $U_\alpha^{q_1}$ as $\text{Spec}(W(k)[t, u])$ and $U_\alpha^{q_2}$ as $\text{Spec}(W(k)[t, \frac{u}{t}])$ such that f is the natural map. Now, $u_\alpha(g)^* : W(k)[t, u] \rightarrow B_{\text{dR}}^+(R^\sharp)$ with $t \mapsto \xi$ extends (uniquely) to $W(k)[t, \frac{u}{t}] \rightarrow B_{\text{dR}}^+(R^\sharp)$ if ξ divides the image of u , but this happens whenever $u_\alpha(g)$ reduces to identity. \square

Proposition 5.15. *Let $q_1, q_2 \in \mathcal{A}$ such that $F_{q_1} \subseteq F_{q_2}$, then we have identifications $L^+\mathcal{H}_{q_1}/L^+\mathcal{H}_{q_2} = W^+\mathcal{H}_{q_1}/W^+\mathcal{H}_{q_2} = (\text{Fl}_{q_1, q_2, O_C})^\diamond$, where Fl_{q_1, q_2} denotes the flag variety.*

Proof. We compute: $L^+\mathcal{H}_{p_1}/L^+\mathcal{H}_{p_2} = (L^+\mathcal{H}_{p_1}/L^u\mathcal{H}_{p_1})/(L^+\mathcal{H}_{p_2}/L^u\mathcal{H}_{p_1}) = M_{p_1}^\diamond/P_{\Phi_{p_1, p_2}}^\diamond = \text{Fl}_{p_1, p_2}^\diamond$ and $W^+\mathcal{H}_{p_1}/W^+\mathcal{H}_{p_2} = (W^+\mathcal{H}_{p_1}/W^u\mathcal{H}_{p_1})/(W^+\mathcal{H}_{p_2}/W^u\mathcal{H}_{p_1}) = M_{p_1}^\diamond/P_{\Phi_{p_1, p_2}}^\diamond = \text{Fl}_{p_1, p_2}^\diamond$. Finally, [Proposition 5.3](#) gives $\text{Fl}_{p_1, p_2}^\diamond = \text{Fl}_{p_1, p_2}^\diamond$. \square

Lemma 5.16. *Fix $q_1, q_2 \in \mathcal{A}$ with $F_{q_1} \subseteq F_{q_2}$. Let \mathcal{F} be a locally spatial diamond with a map $\mathcal{F} \rightarrow \text{Spd}(O_C)$ and let $\text{Spa}(R, R^+)$ be an affinoid perfectoid with a map $\text{Spa}(R, R^+) \rightarrow \text{Fl}_{q_1, q_2}^\diamond$.*

- (1) *The map $L^+\mathcal{H}_{q_1} \times_{\text{Spd}(O_C)} \mathcal{F} \rightarrow \text{Fl}_{q_1, q_2}^\diamond \times_{\text{Spd}(O_C)} \mathcal{F}$ admits pro-étale locally a section.*
- (2) *The map $L^+\mathcal{H}_{q_1} \times_{\text{Fl}_{q_1, q_2}^\diamond} \text{Spa}(R, R^+) \rightarrow \text{Spa}(R, R^+)$ admits analytic locally a section.*

Proof. We may reduce the first claim to the second one by [\[Sch17, Proposition 11.24\]](#). Indeed, by [Proposition 5.15](#) the map is a $L^+\mathcal{H}_{q_2}$ -torsor. Let us prove the second claim. Let $\mathbf{obs} \in H_v^1(\text{Spa}(R, R^+), L^+\mathcal{H}_{q_2})$ be the obstruction to triviality. We prove $\mathbf{obs} = e$ after analytic localization. Consider the sequences of maps: $(L^+\mathcal{H}_{q_1}) \rightarrow M_{q_1}^\diamond \rightarrow \text{Fl}_{q_1, q_2}^\diamond$ and $e \rightarrow L^u\mathcal{H}_{q_1} \rightarrow L^+\mathcal{H}_{q_2} \rightarrow P_{\Phi_{q_1, q_2}}^\diamond \rightarrow e$. Now, $M_{q_1}^\diamond \rightarrow \text{Fl}_{q_1, q_2}^\diamond$ is a $(P_{\Phi_{q_1, q_2}}^\diamond)$ -torsor with obstruction lying in $H_v^1(\text{Fl}_{q_1, q_2}^\diamond, P_{\Phi_{q_1, q_2}}^\diamond)$. Since $M_{q_1} \rightarrow \text{Fl}_{q_1, q_2}$ admits Zariski locally

a section, replacing $\mathrm{Spa}(R, R^+)$ by an analytic cover, we may assume $\mathbf{obs} = e$ in $H_v^1(\mathrm{Spa}(R, R^+), P_{\Phi_{q_1, q_2}}^\diamond)$. We claim $H_v^1(\mathrm{Spa}(R, R^+), L^u \mathcal{H}_{q_1}) = \{e\}$. Recall the exact sequence $e \rightarrow \mathrm{Ker}(L^+ \mathcal{H}_{q_1} \rightarrow (\overline{\mathcal{H}}_{q_1})^\diamond) \rightarrow L^u \mathcal{H}_{q_1} \rightarrow \mathrm{Ker}(\overline{\mathcal{H}}_{q_1}^\diamond \rightarrow [(\overline{\mathcal{H}}_{q_1})^{\mathrm{Red}}]^\diamond) \rightarrow e$, we prove vanishing of $H^1(\mathrm{Spa}(R, R^+), -)$ on the other two groups.

For the left group consider the family of groups $\{L^{u, n}\}_{n=1}^\infty$ filtering $L^{u, 1} := \mathrm{Ker}(L^+ \mathcal{H}_{q_1} \rightarrow \overline{\mathcal{H}}_{q_1}^\diamond)$. Define them as: $L^{u, n}(R, R^+) := \mathrm{Ker}(\mathcal{H}_{q_1, B_{\mathrm{dR}}^+(R^\sharp)}(B_{\mathrm{dR}}^+(R^\sharp)) \rightarrow \mathcal{H}_{q_1, B_{\mathrm{dR}}^+(R^\sharp)}(B_{\mathrm{dR}}^+(R^\sharp)/\xi^n))$. Now, after sheafification $L^{u, n}/L^{u, n+1} = \mathrm{Ker}(\mathcal{H}_{q_1, B_{\mathrm{dR}}^+(R^\sharp)}(B_{\mathrm{dR}}^+(R^\sharp)/\xi^{n+1}) \rightarrow \mathcal{H}_{q_1, B_{\mathrm{dR}}^+(R^\sharp)}(B_{\mathrm{dR}}^+(R^\sharp)/\xi^n))$. Since $\mathrm{Spec}(B_{\mathrm{dR}}^+(R^\sharp)/\xi^n) \rightarrow \mathrm{Spec}(B_{\mathrm{dR}}^+(R^\sharp)/\xi^{n+1})$ is a first order nilpotent thickening, deformation theory gives:

$$L^{u, n}/L^{u, n+1} = \mathrm{Hom}(e^* \Omega_{\mathcal{H}_{q_1}}^1 \otimes_{W(k)[t]} B_{\mathrm{dR}}^+(R^\sharp), (\xi^n \cdot B_{\mathrm{dR}}^+(R^\sharp)/\xi^{n+1})) = \mathrm{Hom}(e^* \Omega_{\mathcal{H}_{q_1}}^1 \otimes R^\sharp, R^\sharp).$$

Now, $e^* \Omega_{\mathcal{H}_{q_1}/W(k)[t]}^1$ is a finite free $W(k)[t]$ -module so $L^{u, n}/L^{u, n+1} \cong (\mathcal{O}^\sharp)^k$ for some k . By [Sch17, Proposition 8.8], $H_v^1(\mathrm{Spa}(R, R^+), \mathcal{O}^\sharp) = 0$. This shows that $H_v^1(\mathrm{Spa}(R, R^+), L^{u, 1}/L^{u, n}) = \{e\}$ for all n . Since $L^{u, 1} = \varprojlim L^{u, 1}/L^{u, n}$ and the transition maps are surjective at the level of presheaves we conclude that $H_v^1(\mathrm{Spa}(R, R^+), L^{u, 1}) = \{e\}$.

For the right group, we may use [Sch17, Proposition 8.8] again since $\mathrm{Ker}(\overline{\mathcal{H}}_{q_1} \rightarrow (\overline{\mathcal{H}}_{q_1})^{\mathrm{Red}})$ is a vector group over $W(k)$. \square

We can now consider families of Demazure varieties, see [FS21, Definition VI.5.6].

Definition 5.17. Let $\sigma_r := \{r_i\}_{1 \leq i \leq n}$ and $\sigma_q := \{q_i\}_{1 \leq i \leq n}$ be a pair of sequences of points in \mathcal{A} such that $F_{r_i}, F_{r_{i+1}} \subseteq F_{q_i}$, and let $\sigma := (\sigma_r, \sigma_q)$. To such σ we associate a v-sheaf given by the contracted

$$\text{group product: } D(\sigma) = L^+ \mathcal{H}_{r_1} \times_{\mathrm{Spd}(O_C)}^{L^+ \mathcal{H}_{q_1}} L^+ \mathcal{H}_{r_2} \times_{\mathrm{Spd}(O_C)}^{L^+ \mathcal{H}_{q_2}} \cdots \times_{\mathrm{Spd}(O_C)}^{L^+ \mathcal{H}_{q_{n-1}}} L^+ \mathcal{H}_{r_n} / L^+ \mathcal{H}_{q_n}.$$

Proposition 5.18. *The map of v-sheaves $D(\sigma) \rightarrow \mathrm{Spd}(O_C)$ is representable in spatial diamonds, proper and ℓ -cohomologically smooth for any $\ell \neq p$.*

Proof. Let σ be as above and let $\sigma' = (\{r_i\}_{1 \leq i \leq n-1}, \{q_i\}_{1 \leq i \leq n-1})$ be the subsequence of the first $n-1$ points of σ . We have a projection morphism of v-sheaves $f : D(\sigma) \rightarrow D(\sigma')$ given by forgetting the last entry corresponding to r_n . One can inductively show that this map satisfies all of the properties in the hypothesis, given that it is a $(\mathrm{Fl}_{r_n, q_n})^\diamond$ -fibration. \square

Proposition 5.19. *The map $\pi : W^+ \mathcal{H}_{r_1} \times_{\mathrm{Spd}(O_C)} \cdots \times_{\mathrm{Spd}(O_C)} W^+ \mathcal{H}_{r_n} \rightarrow D(\sigma)$ coming from $W^+ \mathcal{H}_{r_i} \subseteq L^+ \mathcal{H}_{r_i}$ is surjective, it induces an isomorphism $\iota : D(\sigma) \cong W^+ \mathcal{H}_{r_1} \times_{\mathrm{Spd}(O_C)}^{W^+ \mathcal{H}_{q_1}} \cdots \times_{\mathrm{Spd}(O_C)}^{W^+ \mathcal{H}_{q_{n-1}}} W^+ \mathcal{H}_{r_n} / W^+ \mathcal{H}_{q_n}$. Consequently, $D(\sigma)$ is v-formalizing.*

Proof. Consider the following basechange diagram:

$$\begin{array}{ccccc} W^+ \mathcal{H}_{r_1} \times_{\mathrm{Spd}(O_C)} \cdots \times_{\mathrm{Spd}(O_C)} (L^+ \mathcal{H}_{r_n} / L^+ \mathcal{H}_{q_n}) & \longrightarrow & L^+ \mathcal{H}_{r_1} \times_{\mathrm{Spd}(O_C)} \cdots \times_{\mathrm{Spd}(O_C)} (L^+ \mathcal{H}_{r_n} / L^+ \mathcal{H}_{q_n}) & \longrightarrow & D(\sigma) \\ \downarrow & & \downarrow & & \downarrow \\ W^+ \mathcal{H}_{r_1} \times_{\mathrm{Spd}(O_C)} \cdots \times_{\mathrm{Spd}(O_C)} W^+ \mathcal{H}_{r_{n-1}} & \longrightarrow & L^+ \mathcal{H}_{r_1} \times_{\mathrm{Spd}(O_C)} \cdots \times_{\mathrm{Spd}(O_C)} L^+ \mathcal{H}_{r_{n-1}} & \longrightarrow & D(\sigma') \end{array} \quad (10)$$

Proposition 5.15 gives $W^+ \mathcal{H}_{r_n} / W^+ \mathcal{H}_{q_n} = L^+ \mathcal{H}_{r_n} / L^+ \mathcal{H}_{q_n}$ and the surjectivity by induction. Assume

that we have an identification: $\iota' : D(\sigma') \cong W^+ \mathcal{H}_{r_1} \times_{\mathrm{Spd}(O_C)}^{W^+ \mathcal{H}_{q_1}} \cdots \times_{\mathrm{Spd}(O_C)}^{W^+ \mathcal{H}_{q_{n-2}}} W^+ \mathcal{H}_{r_{n-1}} / W^+ \mathcal{H}_{q_{n-1}}$. Since $W^+ \mathcal{H}_{q_k} \subseteq L^+ \mathcal{H}_{q_k}$, the map ι is defined and surjective, we prove that it is also injective. Let $[g_1]$

and $[g_2]$ be two maps $[g_1], [g_2] : \mathrm{Spa}(R, R^+) \rightarrow W^+ \mathcal{H}_{r_1} \times_{\mathrm{Spd}(O_C)}^{W^+ \mathcal{H}_{q_1}} \cdots \times_{\mathrm{Spd}(O_C)}^{W^+ \mathcal{H}_{q_{n-1}}} W^+ \mathcal{H}_{r_n} / W^+ \mathcal{H}_{q_n}$ with $\iota([g_1]) = \iota([g_2])$. By inductive hypothesis, we may v-locally find representatives g_1 and g_2 of $[g_1]$ and $[g_2]$ of the form $g_i = (g_i^1, \dots, g_i^n)$ such that $g_i^j = g_j^j$ for $j \in \{1, \dots, n-1\}$. Since $[g_1]$ and $[g_2]$ get identified in $D(\sigma)$, v-locally g_1 and g_2 are on the same $L^+ \mathcal{H}_{q_1} \times_{\mathrm{Spd}(O_C)} \cdots \times_{\mathrm{Spd}(O_C)} L^+ \mathcal{H}_{q_n}$ -orbit. Since g_1 and g_2 share all of their entries except possibly the last, g_1^n and g_2^n are in the same $L^+ \mathcal{H}_{q_n}$ -orbit. Since $g_1^n, g_2^n \in W^+ \mathcal{H}_{r_n}$ and $W^+ \mathcal{H}_{q_n} = W^+ \mathcal{H}_{r_n} \cap L^+ \mathcal{H}_{q_n}$ they are in the same $W^+ \mathcal{H}_{q_n}$ -orbit, so $[g_1] = [g_2]$.

Finally, by **Proposition 5.11** each $W^+ \mathcal{H}_{r_i}$ is formalizing, **Proposition 4.10** implies the same for the product, and since $D(\sigma)$ is the quotient of a v-formalizing sheaf it is also v-formalizing. \square

Proposition 5.20. *The map $D(\sigma) \rightarrow \mathrm{Spd}(O_C)$ is formally adic. Moreover, $D(\sigma)^{\mathrm{red}}$ is represented by a qcqs scheme that is perfectly finitely presented and proper over $\mathrm{Spec}(k_C)$ [BS17, Proposition 3.11, Definition 3.14].*

Proof. In any Grothendieck topos, pullback commutes with finite limits and colimits. **Proposition 5.11**

gives: $D(\sigma) \times_{\mathrm{Spd}(O_C)} \mathrm{Spec}(k_C)^\diamond = (\mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{r_1})^\diamond \times_{\mathrm{Spec}(k_C)^\diamond}^{\mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{q_1}^\diamond} \cdots \times_{\mathrm{Spec}(k_C)^\diamond}^{\mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{q_{n-1}}^\diamond} (\mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{r_n})^\diamond / (\mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{q_n})^\diamond$. Now,

this is $\left(\mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{p_1} \times_{k_C}^{\mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{q_1}} \cdots \times_{k_C}^{\mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{q_{n-1}}} \mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{p_n} / \mathcal{W}_{\mathrm{red}}^+ \mathcal{H}_{q_n} \right)^\diamond$ since $(\cdot)^\diamond$ is a left adjoint and commutes

with colimits. **Lemma 3.32** proves that $D(\sigma) \rightarrow \mathrm{Spd}(O_C)$ is formally adic and that $D(\sigma)^{\mathrm{red}} = D(\sigma) \times_{\mathrm{Spd}(O_C)} \mathrm{Spec}(k_C)^\diamond$. The structural properties of $D(\sigma)^{\mathrm{red}}$ are well known, see [Zhu17], [BS17]. \square

Proposition 5.21. *$D(\sigma) \times_{\mathrm{Spd}(O_C)} \mathrm{Spa}(C, O_C)$ has enough facets over C .*

Proof. We prove this by induction. Let $\sigma = (\{r_i\}_{1 \leq i \leq n}, \{q_i\}_{1 \leq i \leq n})$ and $\sigma' = (\{r_i\}_{1 \leq i \leq n-1}, \{q_i\}_{1 \leq i \leq n-1})$. Suppose that $D(\sigma')_C$ has enough facets, let $S := \prod_{i \in I} \mathrm{Spd}(B_i, B_i^\diamond)$ and let $f : S \rightarrow D(\sigma')_C$ be a cover as in **Definition 4.50**. Let $\mathcal{F} = D(\sigma)_C \times_{D(\sigma')_C} S$, we prove that \mathcal{F} has a enough facets. By **Lemma 5.16**, $\mathcal{F} \rightarrow S$ is analytically locally a trivial $(\mathrm{Fl}_{r_n, q_n, C^\#})^\diamond$ -fibration. We may conclude by **Proposition 4.51**. \square

Proposition 5.22. *For any σ as in **Definition 5.17**, $D(\sigma)$ is a topologically normal rich p -adic kimberlite.*

Proof. By **Proposition 5.19**, **Proposition 3.29** and **Proposition 5.20** it is a p -adic prekimberlite, and by **Proposition 5.18** together with **Proposition 4.32** it is a valuative kimberlite whose analytic locus is qcqs. Since $D(\sigma)^{\mathrm{red}}$ is a proper perfectly finitely presented scheme over k_C , $|D(\sigma)^{\mathrm{red}}|$ is Noetherian. Also, $D(\sigma)^{\mathrm{an}}$ coincides with $D(\sigma) \times_{\mathrm{Spd}(O_C)} \mathrm{Spa}(C, O_C)$ and by **Proposition 5.21** and this is a qcqs cJ-diamond. By **Lemma 5.23**, we may check surjectivity of $\mathrm{sp}_{D(\sigma)^{\mathrm{an}}}$ on closed points. At this point, it suffices to prove that if $x \in |D(\sigma)^{\mathrm{red}}|$ is a closed point, then $D(\sigma)_{/x}^\circ$ is non-empty and connected. This follows inductively from **Lemma 5.24**. Indeed, **Item 2** holds by induction over the maps $D(\sigma) \rightarrow D(\sigma')$ and **Item 1** follows from the diagram 10 since each of the $W^+ \mathcal{H}_{r_i}$ is formalizing and basechanges along maps that factor through $W^+ \mathcal{H}_{r_1} \times_{\mathrm{Spd}(O_C)} \cdots \times_{\mathrm{Spd}(O_C)} W^+ \mathcal{H}_{r_{n-1}}$ will give a trivial bundle. \square

Lemma 5.23. *Let C be a characteristic zero nonarchimedean algebraically closed field, and let $k = O_C/\mathfrak{m}_C$. Let $(\mathcal{F}, \mathcal{F}_C)$ be a smelted kimberlite over $\mathrm{Spd}(O_C)$ with $\mathcal{F}_C = \mathcal{F} \times_{\mathrm{Spd}(O_C)} \mathrm{Spa}(C, O_C)$. Consider $\mathcal{F}_{O_{C'}} := \mathcal{F} \times_{\mathrm{Spd}(O_C)} \mathrm{Spd}(O_{C'})$ ranging over algebraically closed nonarchimedean field extension C'/C . Suppose that for every C' and every closed point $x \in |\mathcal{F}_{O_{C'}}^{\mathrm{red}}|$ the tubular neighborhood $(\mathcal{F}_{O_{C'}})_{/x}^\circ$ is non-empty. Then $\mathrm{sp}_{\mathcal{F}_n}$ is a surjection.*

Proof. Given a point $x \in |\mathcal{F}^{\mathrm{red}}|$ we can find a field extension of perfect fields K/k for which $\mathcal{F}^{\mathrm{red}} \times_k \mathrm{Spec}(K)$ has a section $y : \mathrm{Spec}(K) \rightarrow \mathcal{F}^{\mathrm{red}} \times_k \mathrm{Spec}(K)$ mapping to x under $\mathcal{F}^{\mathrm{red}} \times_k \mathrm{Spec}(K) \rightarrow \mathcal{F}^{\mathrm{red}}$. Since \mathcal{F} is formally separated, $\mathcal{F}^{\mathrm{red}} \times_k \mathrm{Spec}(K)$ is also separated and sections to the structure map define closed points. We can construct a nonarchimedean field C' with $C \subseteq C'$ and $W(k)[\frac{1}{p}] \subseteq W(K)[\frac{1}{p}] \subseteq C'$. We get a map $\mathcal{F}_{O_{C'}} \rightarrow \mathcal{F}$, and in $|\mathcal{F}_{O_{C'}}^{\mathrm{red}}|$ there is a closed point y mapping to x . Any point $r \in |\mathcal{F}_{C'}|$ with $\mathrm{sp}_{\mathcal{F}_{C'}}(r) = y$ maps to a point whose image under the specialization map is x . \square

Lemma 5.24. *Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be a proper, ℓ -cohomologically smooth map of p -adic kimberlites over $\mathrm{Spd}(O_C)$. Suppose $\mathcal{G}^{\mathrm{red}}$ and $\mathcal{F}^{\mathrm{red}}$ are perfectly finitely presented over $\mathrm{Spec}(k_C)$ [BS17, Definition 3.10]. Let $X \rightarrow \mathrm{Spec}(O_{C^\#})$ be a smooth projective scheme and suppose that f is X^\diamond -bundle. Suppose that:*

- (1) *For any nonarchimedean field C' extension C and a map $t : \mathrm{Spd}(C', O_{C'}) \rightarrow \mathcal{G}$ there is an extension C'' of C' such that $\mathcal{F} \times_{\mathcal{G}} \mathrm{Spd}(O_{C''})$ is isomorphic to $X^\diamond \times_{\mathrm{Spd}(O_C)} \mathrm{Spd}(O_{C''})$.*
- (2) *For any closed point $x \in |\mathcal{G}^{\mathrm{red}}|$ the tubular neighborhood $\mathcal{G}_{/x}^\circ$ is non-empty and connected.*

Then, for any closed point $y \in |\mathcal{F}^{\mathrm{red}}|$ the tubular neighborhood $\mathcal{F}_{/y}^\circ$ is also non-empty and connected.

Proof. Take a closed point $y \in |\mathcal{F}^{\mathrm{red}}|$ with $x = f(y)$ and consider the map $f : \mathcal{F}_{/y}^\circ \rightarrow \mathcal{G}_{/x}^\circ$. Assume for now that for all maps $\mathrm{Spa}(C', O_{C'}) \rightarrow \mathcal{G}_{/x}^\circ$ the base change $\mathcal{F}_{/y}^\circ \times_{\mathcal{G}_{/x}^\circ} \mathrm{Spa}(C', O_{C'})$ is non-empty and connected, we finish the proof under this assumption. The map $|\mathcal{F}_{/y}^\circ| \rightarrow |\mathcal{G}_{/x}^\circ|$ is specializing, and by assumption surjective on rank 1 points. Let U and V non-empty open and closed subsets with $U \cup V = |\mathcal{F}_{/y}^\circ|$. Since f is open and closed [Sch17, Proposition 23.11], $f(U) \cup f(V) = |\mathcal{G}_{/x}^\circ|$ and $f(U)$ and $f(V)$ meet at a rank 1 point.

Let us prove our assumption holds, take a map $t : \mathrm{Spa}(C', O_{C'}) \rightarrow \mathcal{G}_{/x}^{\circledast}$. After, replacing $\mathrm{Spa}(C', O_{C'})$ by a v -cover we can assume \mathcal{G} formalizes t and that t has the base change property of [Item 1](#). We get a pair of Cartesian diagrams, the right being obtained by taking the reduction of the left:

$$\begin{array}{ccccc} \widehat{\mathcal{F}}_{/y} \times_{\mathcal{G}} \mathrm{Spd}(O_{C'}) & \longrightarrow & X^{\circ} \times_{\mathrm{Spd}(O_C)} \mathrm{Spd}(O_{C'}) & \longrightarrow & \mathrm{Spd}(O_{C'}) \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{\mathcal{F}}_{/y} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} \end{array} \quad \begin{array}{ccccc} Z & \longrightarrow & X \times \mathrm{Spec}(k') & \longrightarrow & \mathrm{Spec}(k') \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(k(y)) & \xrightarrow{y} & \mathcal{F}^{\mathrm{red}} & \longrightarrow & \mathcal{G}^{\mathrm{red}} \end{array}$$

Since $\mathcal{F}^{\mathrm{red}} \rightarrow \mathcal{G}^{\mathrm{red}}$ is perfectly finitely presented and k is algebraically closed, $k = k(y) = k(x)$ and the composition $y : \mathrm{Spec}(k) \rightarrow \mathcal{G}^{\mathrm{red}}$ is a closed immersion. Consequently $Z \rightarrow \mathrm{Spec}(k')$ is an isomorphism. This gives $\widehat{\mathcal{F}}_{/y} \times_{\mathcal{G}} \mathrm{Spd}(O_{C'}) = X^{\circ}_{O_{C'}} \times_{\mathcal{F}} \widehat{\mathcal{F}}_{/y} = \widehat{(X^{\circ}_{O_{C'}})}_{/Z}$ by [Proposition 4.20](#). But $Z \rightarrow X \times \mathrm{Spec}(k')$ is a closed point, so $(X^{\circ}_{O_{C'}})_{/Z}^{\circledast}$ is isomorphic to an open unit ball $\mathbb{B}_{<1}^n$ over C'^{\sharp} , proving the assumption. \square

We keep the notation from the beginning of the previous subsection and we restrict our attention to parahoric loop groups associated to points contained in our chosen alcove \mathcal{C} . Given $s_j \in \mathbb{S}$ we denote by $L^+ \mathcal{H}_{s_j}$ the parahoric loop group associated to the wall F_{s_j} in \mathcal{C} corresponding to the reflection s_j . For a point $r \in \mathcal{C}$ we let $J_r \subseteq \mathbb{S}$ denote the set $\{s_j \mid r \in F_{s_j}\}$. We will denote by $L^+ B$ the parahoric loop group associated to \mathcal{C} .

By functoriality of $L(-)$ we have loop group versions of the Weyl and Iwahori–Weyl groups by the formula $LW := LN/LT$ and $L\tilde{W} := LN/L^+ \mathcal{T}$. They fit in an exact sequence: $e \rightarrow LT/L^+ \mathcal{T} \rightarrow L\tilde{W} \rightarrow LW \rightarrow e$. A direct computation shows that $LW = L(N/T) = \underline{W} \times \mathrm{Spd}(O_C)$, that $LT/L^+ \mathcal{T} = X_*(T) \times \mathrm{Spd}(O_C)$ and that $L\tilde{W} = \tilde{W} \times \mathrm{Spd}(O_C)$. Since H is a split reductive group over $W(k)[t, t^{-1}]$, for any element $w \in W$ we can find a section $n_w : \mathrm{Spec}(W(k)[t, t^{-1}]) \rightarrow N$ whose projection to W is w [[Con14](#), Corollary 5.1.11]. This allow us to define a similar section $n_w : \mathrm{Spd}(O_C) \rightarrow LN \subseteq LH$. Also for any $\mu \in X_*(T)$ and any $\mathrm{Spa}(R, R^+) \rightarrow \mathrm{Spd}(O_C)$ we can consider the element $\xi^\mu \in T(B_{\mathrm{dR}}(R^\sharp))$. This is functorial and defines a section $\mathrm{Spd}(O_C) \rightarrow LT$ mapping to $\mu \in X_*(T) \times \mathrm{Spd}(O_C)$. In particular, for any element $\tilde{w} \in \tilde{W}$ there is a section $n_{\tilde{w}} : \mathrm{Spd}(O_C) \rightarrow LN$ projecting to \tilde{w} in $L\tilde{W}$. We can use $n_{\tilde{w}}$ to construct an automorphism $n_{\tilde{w}} : \mathrm{Gr}_{O_C}^H \rightarrow \mathrm{Gr}_{O_C}^H$ with $n_{\tilde{w}}(x \cdot L^+ H) := n_{\tilde{w}} \cdot x \cdot L^+ H$. We will use this discussion in the proof of [Theorem 5.1](#).

Lemma 5.25. *Let $\sigma = (\sigma_r, \sigma_q)$ with σ as in the previous subsection except that we require $\sigma_r, \sigma_q \subseteq \mathcal{C}$. Suppose that $L^+ \mathcal{H}_{q_n} = L^+ \mathcal{H}_{r_n} = L^+ H$ then the multiplication map $\mu : D(\sigma) \rightarrow \mathrm{Gr}_{O_C}^H = LH/L^+ H$ has geometrically connected fibers.*

Proof. The proof is combinatorial and follows the classical case. The key geometric inputs are as follows, the basechange of $D(\sigma) \rightarrow \mathrm{Spd}(O_C)$ by geometric points are proper spatial diamonds, rank 1 points are always dense for any spatial diamond and the group of rank 1 geometric points of a parahoric loop group coincide with the ‘‘parabolic subgroups’’ of a Tits-systems (or BN -pair). These two observations together with [[Sch17](#), Lemma 12.11] reduces the proof to the classical combinatorial case. We omit the details. \square

We can now prove [Theorem 5.1](#).

Proof (of Theorem 5.1). Observe that $\mathrm{Gr}_{O_F}^{G, \leq \mu} \rightarrow \mathrm{Spd}(O_F)$ is formally adic. Indeed, $\mathrm{Gr}_{O_F}^{G, \leq \mu} \times_{\mathrm{Spd}(O_F)} \mathrm{Spec}(k_F)^\diamond$ is $(\mathrm{Gr}_{\mathcal{W}, k_F}^{G, \leq \mu})^\diamond$ and since $\mathrm{Gr}_{\mathcal{W}, k_F}^{G, \leq \mu}$ is proper this is $(\mathrm{Gr}_{\mathcal{W}, k_F}^{G, \leq \mu})^\diamond$ we may conclude by [Lemma 3.32](#). This gives that $(\mathrm{Gr}_{O_F}^{G, \leq \mu})^{\mathrm{red}} = \mathrm{Gr}_{\mathcal{W}, k_F}^{G, \leq \mu}$, which is represented by a scheme [[BS17](#), Theorem 8.3], that the adjunction map $((\mathrm{Gr}_{O_F}^{G, \leq \mu})^{\mathrm{red}})^\diamond \rightarrow \mathrm{Gr}_{O_F}^{G, \leq \mu}$ is a closed immersion and that $(\mathrm{Gr}_{O_F}^{G, \leq \mu})^{\mathrm{an}} = \mathrm{Gr}_{O_F}^{G, \leq \mu} \times_{\mathrm{Spd}(O_F)} \mathrm{Spd}(F)$, which is represented by a spatial diamond by [[SW20](#), Proposition 20.4.5]. Also, $\mathrm{Gr}_{O_F}^{G, \leq \mu}$ is separated by [[SW20](#), Proposition 20.5.4], and by [Proposition 3.29](#) it is formally separated. To prove that $\mathrm{Gr}_{O_F}^{G, \leq \mu}$ is a p -adic kimberlite we need to prove it is v -formalizing. To prove that it is rich it suffices to prove $\mathrm{Gr}_{O_F}^{G, \leq \mu} \times_{\mathrm{Spd}(O_F)} \mathrm{Spd}(F)$ has enough facets and that the tubular neighborhoods at closed points are non-empty. These can be checked after basechange to $\mathrm{Spd}(O_C)$ for C/F a completed algebraic closure.

Suppose for now that $F = C$. In this case $G \times_{\mathbb{Z}_p} W(k_F)$ is a split reductive group, and since $\mathrm{Gr}_{O_C}^{G, \leq \mu}$ only depends on $G_{W(k_F)}$, we may assume $G = H$ with H split reductive. Furthermore, using the discussion of the beginning of the section one can reduce to the case in which H is semisimple and simply connected. In this case $\tilde{W} = W^{\mathrm{aff}}$. Recall that we have inclusions $X_*^+(T) \subseteq X_*(T) \subseteq \tilde{W}$ so we may think of μ

as an element of the Iwahori–Weyl group. By definition, $\mathrm{Gr}_{O_C}^{H, \leq \mu}(R, R^+)$ consists of those elements in $\mathrm{Gr}_{O_C}^H(R, R^+)$ satisfying that for any geometric point $q : \mathrm{Spa}(C', C'^+) \rightarrow \mathrm{Spa}(R, R^+)$ the type of q , μ_q , is in the double coset $H(B_{\mathrm{dR}}^+(C'^{\sharp})) \backslash H(B_{\mathrm{dR}}(C'^{\sharp})) / H(B_{\mathrm{dR}}^+(C'^{\sharp})) = X_*^+(T) = W_o \backslash W^{\mathrm{aff}} / W_o$ satisfies $\mu_q \leq \mu$ in the Bruhat order. Now, given $w \in \tilde{W}$ we can consider the subsheaf $\mathrm{Gr}_{O_C}^{G, \leq w} \subseteq \mathrm{Gr}_{O_C}^G$ with the similar property on a geometric point using instead the double coset $B(B_{\mathrm{dR}}^+(C'^{\sharp})) \backslash H(B_{\mathrm{dR}}(C'^{\sharp})) / H(B_{\mathrm{dR}}^+(C'^{\sharp})) = W^{\mathrm{aff}} / W_o$. The projection map $\pi : W^{\mathrm{aff}} / W_o \rightarrow W_o \backslash W^{\mathrm{aff}} / W_o$ is order preserving and $\pi^{-1}(\mu)$ has a unique element $[w_\mu]$ of largest length, it has the property that $v \leq w_\mu$ if and only if $\pi(v) \leq \mu$. In particular, we have an equality of sheaves $\mathrm{Gr}_{O_C}^{H, \leq w_\mu} = \mathrm{Gr}_{O_C}^{H, \leq \mu}$. We prove that for $w \in W^{\mathrm{aff}}$ the v-sheaf $\mathrm{Gr}_{O_C}^{H, \leq w}$ satisfies the conclusions of the theorem.

Find a reduced expression for $w = s_{j_1} \dots s_{j_n}$ and consider $D(w) := L^+ H_{s_{j_1}}^{L^+ B} \times_{\mathrm{Spd}(O_C)} \dots \times_{\mathrm{Spd}(O_C)}^{L^+ B} L^+ H_{s_{j_n}} / L^+ H$. The multiplication map $m : D(w) \rightarrow \mathrm{Gr}_{O_C}^H$ factors through $\mathrm{Gr}_{O_C}^{H, \leq w}$ and surjects onto it. This implies $\mathrm{Gr}_{O_C}^H$ is a kimberlite. [Proposition 4.32](#), [Proposition 4.46](#) and [Theorem 4.40](#) imply it is rich. [Proposition 5.22](#) and [Lemma 5.25](#) combined with [Lemma 5.26](#) allow us to conclude in this case.

Finally, let us deal with the case $F \neq C$. Let F' the completed maximal unramified subextension of F in C . We have surjective maps of v-sheaves: $\mathrm{Gr}_{O_C}^{G, \leq \mu} \rightarrow \mathrm{Gr}_{O_{F'}}^{G, \leq \mu} \rightarrow \mathrm{Gr}_{O_F}^{G, \leq \mu}$. We may argue as above to prove $\mathrm{Gr}_{O_F}^{G, \leq \mu}$ and $\mathrm{Gr}_{O_{F'}}^{G, \leq \mu}$ are rich kimberlites. By [Proposition 4.20](#), $\mathrm{Gr}_{O_{F'}}^{G, \leq \mu}$ has connected tubular neighborhoods since $(\mathrm{Gr}_{O_C}^{G, \leq \mu})^{\mathrm{red}} = (\mathrm{Gr}_{O_{F'}}^{G, \leq \mu})^{\mathrm{red}}$. On the other hand, $\mathrm{Gr}_{O_{F'}}^{G, \leq \mu} \rightarrow \mathrm{Gr}_{O_F}^{G, \leq \mu}$ is a $\pi_1^{\mathrm{fét}}(\mathrm{Spec}(O_F))$ -torsor and for any closed point $x \in |(\mathrm{Gr}_{O_F}^{G, \leq \mu})^{\mathrm{red}}|$ the action of $\pi_1^{\mathrm{fét}}(\mathrm{Spec}(O_F))$ permutes transitively the closed points $y \in |(\mathrm{Gr}_{O_{F'}}^{G, \leq \mu})^{\mathrm{red}}|$ over x . The action permutes transitively the tubular neighborhoods associated to y which proves that the tubular neighborhood over x is also connected. \square

Lemma 5.26. *Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be a surjective map of rich p -adic kimberlites over $\mathrm{Spd}(O_F)$, such that $\mathcal{G}^{\mathrm{red}}$ and $\mathcal{F}^{\mathrm{red}}$ are perfectly of finite type over $\mathrm{Spec}(k_F)$. Suppose f^{red} has geometrically connected fibers and that \mathcal{F} is topologically normal. Then \mathcal{G} is topologically normal.*

Proof. Pick a closed point $x \in |\mathcal{G}^{\mathrm{red}}|$, by [Proposition 4.20](#) $\mathcal{G}_{/x}^{\otimes} \times_{\mathcal{G}} \mathcal{F} = \mathcal{F}_{/S}^{\otimes}$ with $S = (f^{\mathrm{red}})^{-1}(x)$. By [Lemma 4.55](#) and [Proposition 4.54](#), $\mathcal{F}_{/S}^{\otimes}$ is connected. Since f is surjective $\mathcal{G}_{/x}^{\otimes}$ is also connected. \square

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