ON THE p-ADIC THEORY OF LOCAL MODELS

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Abstract. We prove the Scholze–Weinstein conjecture on the existence and uniqueness of local models of local Shimura varieties and the test function conjecture of Haines–Kottwitz in this setting. In order to achieve this, we establish the specialization principle for well-behaved p-adic kimberlites, show that these include the v-sheaf local models, determine their special fibers using hyperbolic localization for the étale cohomology of small v-stacks and analyze the resulting specialization morphism using convolution.

Contents

1. Introduction

The general theory of Shimura varieties has been developed by Deligne [Del71, Del79] in the seventies. It generalizes classical objects such as modular curves, moduli spaces of principally polarized abelian varieties or Hilbert modular varieties. Shimura varieties naturally occur in the search for higher reciprocity laws within the Langlands program [Lan79]. Their arithmetic properties are encoded in the reduction to positive characteristic $p > 0$ and have contributed to spectacular developments in number theory and arithmetic geometry in the past decades.

Local models are flat projective schemes over complete discrete valuation rings of characteristic $(0, p)$ that are designed to model the singularities of Shimura varieties arising in the completion at p. Starting from the pioneering works [DR73, Rap90, CN90, dJ93, DP94], the theory is formalized to some extent in the book of Rapoport–Zink [RZ96] for those Shimura varieties arising as moduli problems of abelian varieties with extra structures. The geometric properties of the corresponding local models are studied in a series of works notably by Faltings, Görtz, Pappas and Rapoport [Fal97, Fal01, Pap00, Gör01, Gör03, Fal03, PR03, PR05, PR08, PR09], see the survey article [PRS13] for details and further references. A breakthrough is due to Pappas and Zhu [PZ13, Zhu14], who gave a purely group-theoretic construction of (flat) local models, later refined by Levin [Lev16] and the third named author [Lou19]. Roughly, the local models in this approach are constructed as flat, closed subschemes in a power series affine Grassmannian, which depends on certain auxiliary choices, see Section 7.3. A more functorial approach (without ad hoc choices) was initiated by Scholze–Weinstein [SW20], who constructed small v-sheaves inside mixed characteristic Beilinson–Drinfeld Grassmannians by taking closures of partial flag varieties. Unfortunately, the approach has the drawback of not producing schemes, at least a priori.

The aim of the present manuscript is to connect the scheme-theoretic local models to Scholze–Weinstein's p-adic approach. More precisely, we prove the Scholze–Weinstein conjecture [SW20, Conjecture 21.4.1] on the existence and uniqueness of (weakly normal) local models, representing minuscule portions of the parahoric Beilinson–Drinfeld Grassmannian. Our methods allow us to prove, furthermore, the test function conjecture of Haines–Kottwitz [Hai14, Conjecture 6.1.1] for these local models in full generality, expressing the trace of Frobenius function on the nearby cycles sheaf in terms of spectral data.

These local models are intimately related with moduli spaces of p-adic shtukas by [SW20, Lecture XXV]. Recently, progress in their study was made by Pappas–Rapoport [PR21] in the Hodge type case, partially relying on a positive solution of the Scholze–Weinstein conjecture as given here. Let us now discuss our results in more detail.

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1.1. **Main results.** Fix a prime number p. Let G be a connected reductive $\mathbb{Q}_p$-group and $\mathcal{G}$ a parahoric $\mathbb{Z}_p$-model in the sense of Bruhat–Tits [BT84].[1] Let μ be a conjugacy class of geometric cocharacters in G. Denote by $E/\mathbb{Q}_p$ its reflex field with ring of integers O_E and finite residue field k.

Scholze–Weinstein [SW20, Section 20.3] introduced the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Spd}\,\mathbb{Z}_p$ which is representable in ind-(proper, spatial diamonds). Its generic fiber is the B_{dR}^+-affine Grassmannian Gr_G and its special fiber is the v-sheaf $\mathcal{F}\ell_{\mathcal{G}}^{\diamondsuit}$ associated to the Witt vector partial affine flag variety. Attached to the pair $(\mathcal{G},\mu)$ is the v-sheaf local model

$$\mathcal{M}_{\mathcal{G},\mu} \subset \mathrm{Gr}_{\mathcal{G}}|_{\mathrm{Spd}\,O_E} \tag{1.1}$$

defined as the v-closure of the affine Schubert variety $\mathrm{Gr}_{G,\mu}$, in analogy to the local models from [PZ13]. If μ is minuscule[2], then the generic fiber $\mathcal{M}_{\mathcal{G},\mu}|_{\mathrm{Spd}\,E} = \mathrm{Gr}_{G,\mu}$ is canonically isomorphic to $\mathcal{F}_{G,\mu}^{\diamondsuit}$, the v-sheaf associated with the homogenous space $\mathcal{F}_{G,\mu}$ of parabolic subgroups of type μ. The main result of the present work is the representability of local models as given by the Scholze–Weinstein conjecture [SW20, Conjecture 21.4.1]:

Theorem 1.1 (Theorem 7.21, Theorem 7.23)**.** *Let μ be minuscule. Then, there is a unique (up to unique isomorphism) flat, projective and weakly normal O_E-model $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{sch}}$ of the E-scheme $\mathcal{F}_{G,\mu}$ endowable with an isomorphism*

$$\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{sch},\diamondsuit} \cong \mathcal{M}_{\mathcal{G},\mu} \tag{1.2}$$

of v-sheaves over $\mathrm{Spd}\,O_E$, prolonging $\mathcal{F}_{G,\mu}^{\diamondsuit} \cong \mathcal{M}_{\mathcal{G},\mu}|_{\mathrm{Spd}\,E}$. Furthermore, the special fiber of $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{sch}}$ is reduced, in virtually all cases (see below), and then equal to $\mathcal{A}_{\mathcal{G},\mu}^{\mathrm{can}}$, the canonical deperfection of the μ-admissible locus inside $\mathcal{F}\ell_{\mathcal{G}}$.

Our result provides a functorial and group-theoretic framework for the theory of local models that goes beyond the previous results [SW20, HPR20, Lou20] for certain pairs (G,μ) of abelian type using Hodge embeddings. The scheme $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{sch}}$ agrees with the local models defined in [PZ13, Lev16, Lou19] whenever $p \nmid |\pi_1(G_{\mathrm{der}})|$ and otherwise with their weak normalization[3]. The works [PZ13, Lev16, Lou19] are complemented by the results of Fakhruddin–Haines–L.–R. [FHLR22] which handles new cases for wildly ramified groups. The upshot is that we are able to show reducedness of the special fiber of $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{sch}}$ (and other geometric properties such as Cohen-Macaulayness) for all pairs $(\mathcal{G},\mu)$ except possibly if the adjoint group G_{ad} contains one of the following non-split $\breve{\mathbb{Q}}_p$-factors: for $p=2$ an odd unitary group defined by a ramified, quadratic root-of-unit extension; for $p=3$ the triality defined by a ramified, cubic non-(root-of-a-prime) extension. In particular, our result is complete for all primes $p \geq 5$ and, up to the two non-split examples, also for $p=2,3$. In addition, we conjecture that $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{sch}}$ always has reduced special fiber equal to $\mathcal{A}_{\mathcal{G},\mu}^{\mathrm{can}}$ and, furthermore, that its singularities are pseudo-rational, see Conjecture 7.25 and Conjecture 7.27. We remark that the schemes $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{sch}}$ seem not to admit moduli-theoretic interpretations in general, see, however, [Pap00, Gör01, Gör03, PR03, PR05, PR09] for interesting special cases.

As a cohomological application, we prove the test function conjecture for $\mathcal{M}_{\mathcal{G},\mu}$, all primes p and all pairs $(\mathcal{G},\mu)$, see Section 8. Namely, fix an auxiliary prime $\ell \neq p$, a square root $\sqrt{q}$ of the residue cardinality of E and an embedding $E \hookrightarrow \bar{\mathbb{Q}}_p$. Put $\Lambda = \mathbb{Q}_\ell(\sqrt{q})$ which we will use as sheaf coefficients. Let Γ be the absolute Galois group of E with inertia subgroup I, and fix a lift $\Phi \in \Gamma$ of geometric Frobenius. Let $E_0 = W(k)[\frac{1}{p}]$ be the maximal unramified subextension of $E/\mathbb{Q}_p$. Let IC_μ be the intersection complex on $\mathrm{Gr}_{G,\mu}$ with Λ-coefficients as in [FS21, Chapter VI] normalized to be of weight zero as in (8.1). The function $\tau_{\mathcal{G},\mu}^{\Phi}\colon G(E_0)/\mathcal{G}(O_{E_0}) \to \Lambda$ is defined, up to sign, by the alternating trace of Φ on the nearby cycle stalks

$$\tau_{\mathcal{G},\mu}^{\Phi}(x) = (-1)^{\langle 2\rho,\mu\rangle} \sum_{i\in\mathbb{Z}} (-1)^i \,\mathrm{trace}\left(\Phi \mid \mathrm{R}^i\Psi_{\mathcal{M}_{\mathcal{G},\mu}}(\mathrm{IC}_\mu)_{\bar{x}}\right), \tag{1.3}$$

if $x \in \mathcal{M}_{\mathcal{G},\mu}(\mathrm{Spd}\,k)$ and 0 else. Here $\mathcal{M}_{\mathcal{G},\mu}(\mathrm{Spd}\,k)$ is viewed as a subset of $\mathcal{F}\ell_{\mathcal{G}}(k) = G(E_0)/\mathcal{G}(O_{E_0})$, and the trace is well-defined by Theorem 1.8, proving that the nearby cycles are constructible in this setting. The function $\tau_{\mathcal{G},\mu}^{\Phi}$ is left-$\mathcal{G}(O_{E_0})$-invariant and supported on finitely many orbits, hence $\tau_{\mathcal{G},\mu}^{\Phi} \in \mathcal{H}(G(E_0),\mathcal{G}(O_{E_0}))_\Lambda$ naturally lies in the parahoric Hecke algebra of Λ-valued functions.

Theorem 1.2 (Lemma 8.4)**.** *The function $\tau_{\mathcal{G},\mu}^{\Phi}$ lies in the center of $\mathcal{H}(G(E_0),\mathcal{G}(O_{E_0}))_\Lambda$. It is characterized as the unique function in the center that acts on any $\mathcal{G}(\mathcal{O}_{E_0})$-spherical smooth irreducible Λ-representation π by the scalar*

$$\mathrm{trace}\Big(s^{\Phi}(\pi) \mid V_\mu\Big), \tag{1.4}$$

where $s^{\Phi}(\pi) \in [\widehat{G}^I \rtimes \Phi]_{\mathrm{ss}}/\widehat{G}^I$ is the Satake parameter for π and V_μ the representation of the L-group $\widehat{G} \rtimes \Gamma$ of highest weight μ. Moreover, $(\sqrt{q})^{\langle 2\rho,\mu\rangle}\tau_{\mathcal{G},\mu}^{\Phi}$ takes values in $\mathbb{Z}$ and is independent of $\ell \neq p$, $\sqrt{q}$ and $E \hookrightarrow \bar{\mathbb{Q}}_p$.

[1] For simplicity of exposition, we assume the base field to be $\mathbb{Q}_p$ in the introduction. In the main body of the text, we allow more general p-adic base fields.

[2] This means $\langle \mu, a\rangle \in \{0,\pm 1\}$ for every root a of $G_{\mathbb{C}_p}$. In particular, we include central cocharacters.

[3] Passing to the weak normalization is necessary, in general, due to the existence of non-normal Schubert varieties, see [HLR18].

The theorem is a solution to the test function conjecture of Haines–Kottwitz [Hai14, Conjecture 6.1.1] for v-sheaf local models. This is new when μ is non-minuscule: then $\mathcal{M}_{\mathcal{G},\mu}$ is not representable by a scheme due to the theory of Banach–Colmez spaces and, hence, not related to their schematic counterparts defined in [PZ13, Lev16, Lou19, FHLR22]. If μ is minuscule, then the analogue of Theorem 1.2 holds for $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ as well, using Theorem 1.1. In this case, we can work purely algebraically and replace IC_μ by the constant sheaf $\mathbb{Q}_\ell$ on the smooth E-scheme $\mathcal{F}_{G,\mu}$. Here, our result is new for the wildly ramified groups that were excluded in previous work [PZ13, HR20, HR21]. With a view towards applications, say, to point counting formulas in the context of the Langlands–Kottwitz method, we have an analogous result where E is replaced by a finite unramified extension, and correspondingly E_0 by its unramified subextension. Also, Theorem 1.2 easily implies the version for the semi-simple trace by averaging over the Frobenius Φ, see [HR20, Appendix].

1.2. **Strategy of proof.** The proofs of Theorem 1.1 and Theorem 1.2 are purely group-theoretic and do not rely on the classification of reductive groups over local fields. Our method is inspired by a conjecture of He–Pappas–Rapoport [HPR20, Conjecture 2.13] of which we prove a variant, see Theorem 7.12. Following a suggestion in [Lou20, Introduction], we approach $\mathcal{M}_{\mathcal{G},\mu}$ by studying its generic fiber, its special fiber and the specialization morphisms between them. The basic idea is that the data should characterize $\mathcal{M}_{\mathcal{G},\mu}$, making the comparison with the to-be-constructed $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ possible. Once there is enough progress towards Theorem 1.1, the proof of Theorem 1.2 roughly follows the method from [HR21], and the reader is referred to Section 8 for details. Along the way, we address further questions and conjectures in the field, notably Zhu's conjecture [Zhu17a, Appendix B, Conjecture III] on the geometry of deperfections of affine Schubert varieties in the Witt vector affine flag variety.

To make the above ideas precise, there are several technical difficulties to overcome. The most basic is the determination of the underlying topological space $|\mathcal{M}_{\mathcal{G},\mu}|$, that is, to show the density of $|\mathrm{Gr}_{G,\mu}|$ inside $|\mathcal{M}_{\mathcal{G},\mu}|$. In the following, all v-stacks will be on the category $\mathrm{Perf}_{\mathbb{F}_p}$ of perfectoid spaces over $\mathbb{F}_p$:

Proposition 1.3 (Lemma 2.7, Proposition 2.8)**.** *Let X be a small v-stack with underlying topological space $|X|$. Then $Z \mapsto |Z|$ defines a bijection between the set of closed sub-v-stacks of X and the set of weakly generalizing closed subspaces of $|X|$, with inverse $S \mapsto \underline{S} \times_{\underline{|X|}} X$.*

Recall that $\mathcal{M}_{\mathcal{G},\mu} \subset \mathrm{Gr}_{\mathcal{G}}|_{\mathrm{Spd}\, O_E}$ is defined as the v-closure of $\mathrm{Gr}_{G,\mu}$. So Proposition 1.3 implies that $|\mathcal{M}_{\mathcal{G},\mu}|$ agrees with the weakly generalizing closure of $|\mathrm{Gr}_{G,\mu}|$, a first step towards showing the density. We warn the reader that density of v-closures fails in easy examples (see Example 2.5) and are a feature of the situation at hand. More precisely, we show in Proposition 4.14 that $\mathcal{M}_{\mathcal{G},\mu}$ satisfies the assumptions of the following theorem which is one of our main tools towards Theorem 1.1:

Theorem 1.4 (Theorem 2.36)**.** *The functor sending a well-behaved p-adic kimberlite over $\mathrm{Spd}\, O_E$ (see Definition 2.35) to its specialization triple*

$$X \mapsto \big(X_\eta, X_s, \mathrm{sp}_X\big) \tag{1.5}$$

is fully faithful.

Kimberlites are introduced by the second named author in [Gle20] and are a certain class of v-sheaves (containing the v-sheaves associated to flat, projective O_E-schemes) that admit a well-behaved theory of specialization maps, see Section 2.3 for details. A similar result for the functor $X \mapsto X^\diamondsuit$ from weakly normal flat projective O_E-schemes to v-sheaves over $\mathrm{Spd}\, O_E$ is shown by the third named author in [Lou20]. Theorem 1.4 is key for us because $\mathcal{M}_{\mathcal{G},\mu}$ are only v-sheaves, a priori. In order to apply the theorem, we need to establish an isomorphism between the specialization triples for $\mathcal{M}_{\mathcal{G},\mu}$ and for $\mathcal{M}^{\mathrm{sch},\diamondsuit}_{\mathcal{G},\mu}$ when μ is minuscule. The generic fiber of $\mathcal{M}_{\mathcal{G},\mu}$ is $\mathrm{Gr}_{G,\mu}$ by definition, and likewise for $\mathcal{M}^{\mathrm{sch},\diamondsuit}_{\mathcal{G},\mu}$. In Section 1.3 and Section 1.4 below, we explain how to determine the special fibers in both cases, and in Section 1.5, how to control the associated specialization maps. The final Section 1.6 concludes with a more detailed version of Theorem 1.1.

1.3. **Special fibers of v-sheaf local models.** The Witt vector partial affine flag variety $\mathcal{F}\ell_{\mathcal{G}}$ is the increasing union of perfect projective varieties $\mathcal{F}\ell_{\mathcal{G},w}$ indexed by the double coset of the Iwahori–Weyl group, see [Zhu17a, BS17]. The μ-admissible locus $\mathcal{A}_{\mathcal{G},\mu}$ in the sense of Kottwitz–Rapoport is defined as the k-descent of the $\bar{\mathbb{F}}_p$-union

$$\mathcal{A}_{\mathcal{G},\mu,\bar{\mathbb{F}}_p} = \bigcup_{\lambda} \mathcal{F}\ell_{\mathcal{G},\bar{\mathbb{F}}_p,\lambda_I(p)}, \tag{1.6}$$

where λ runs over all absolute Weyl conjugates of μ and $\lambda_I(p)$ denotes the associated translation in the Iwahori–Weyl group of $G_{\breve{\mathbb{Q}}_p}$, see Definition 3.11.

Theorem 1.5 (Theorem 6.16)**.** *The special fiber of $\mathcal{M}_{\mathcal{G},\mu}$ is the v-sheaf $\mathcal{A}^\diamondsuit_{\mathcal{G},\mu}$ associated with the μ-admissible locus inside $\mathcal{F}\ell^\diamondsuit_{\mathcal{G}}$.*

The difficulty in determining the special fiber lies in the rather abstract definition of $\mathcal{M}_{\mathcal{G},\mu}$ via closure operations inside $\mathrm{Gr}_{\mathcal{G}}$. In a different context, [HR21, Theorem 6.12] determines such special fibers based on cohomological considerations by calculating the support of the nearby cycles $\Psi_{\mathcal{M}_{\mathcal{G},\mu}}\mathrm{IC}_\mu$ appearing in (1.3). A key input is the study of $\mathbb{G}_m$-actions on $\mathrm{Gr}_{\mathcal{G}}$ coming from the choice of a cocharacter $\lambda\colon \mathbb{G}_m \to \mathcal{G}$. Let $\mathcal{M}$, respectively $\mathcal{P}$ the closed subgroup scheme of $\mathcal{G}$ with Lie algebra the subspace of $\mathrm{Lie}\,\mathcal{G}$ with weights $\lambda = 0$, respectively $\lambda \geq 0$. This induces maps $\mathcal{M} \leftarrow \mathcal{P} \to \mathcal{G}$ of $\mathbb{Z}_p$-group schemes and, by functoriality, also of the associated Beilinson–Drinfeld Grassmannians. The geometric input towards Theorem 1.5 is the following result which extends [FS21, Chapter VI.3] beyond the case where $\mathcal{G}$ is reductive:

Theorem 1.6 (Theorem 5.2)**.** *The $\mathbb{G}_m^{\diamondsuit}$-action on $\mathrm{Gr}_{\mathcal{G}}$ induced by λ admits a commutative diagram of ind-(spatial $\mathbb{Z}_p^{\diamondsuit}$-diamonds)*

$$\begin{tikzcd} \mathrm{Gr}_{\mathcal{M}} \arrow[d] & \mathrm{Gr}_{\mathcal{P}} \arrow[l] \arrow[r] \arrow[d] & \mathrm{Gr}_{\mathcal{G}} \arrow[d, "\mathrm{id}"'] \\ (\mathrm{Gr}_{\mathcal{G}})^0 & (\mathrm{Gr}_{\mathcal{G}})^+ \arrow[l] \arrow[r] & \mathrm{Gr}_{\mathcal{G}}, \end{tikzcd} \tag{1.7}$$

of attractors and fixers such that the vertical arrows are open and closed immersions that induce isomorphisms over $\mathrm{Spd}\,\mathbb{Q}_p$, and if $\mathcal{G}$ is special parahoric over $\breve{\mathbb{Z}}_p$ (for example, reductive), then also over $\mathrm{Spd}\,\mathbb{F}_p$.

If λ is regular, then the special fiber of $(\mathrm{Gr}_{\mathcal{G}})^+$ consists of the v-sheaves associated to the semi-infinite orbits $\mathcal{S}_w$ inside $\mathcal{F}\ell_{\mathcal{G}}$ indexed by certain cosets of the Iwahori–Weyl group, see (5.12). By comparison, $\mathrm{Gr}_{\mathcal{P}}$ corresponds to those semi-infinite orbits attached to translation elements. The following proposition generalizes the closure relation of semi-infinite orbits and the equi-dimensionality of Mirković–Vilonen cycles [MV07, Theorem 3.2] from split groups to twisted groups, questions left open in the context of the ramified Satake equivalence [Zhu15, Ric16]:

Proposition 1.7 (Proposition 5.4, Lemma 5.5)**.** *For a regular coweight λ and the induced semi-infinite orbits, there is an equality inside $\mathcal{F}\ell_{\mathcal{G}}$:*

$$\overline{\mathcal{S}_w} = \bigcup_v \mathcal{S}_v, \tag{1.8}$$

where v runs through all elements less than or equal to w in Lusztig's semi-infinite Bruhat order. If $\mathcal{G}$ is very special (for example, reductive), then the non-empty intersections $\mathcal{S}_{\nu_I} \cap \mathcal{F}\ell_{\mathcal{G},\mu_I}$ are equi-dimensional of dimension $\langle \rho, \nu + \mu \rangle$.

Let us explain the cohomological results going into Theorem 1.5. As there is no general theory of nearby cycles for v-sheaves, we develop the foundational results for the Hecke stack. Fix a prime number $\ell \neq p$ and let Λ be one of the rings $\mathbb{Z}/\ell^n$, $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$ for some $n \geq 0$. For a small v-stack X, let $\mathrm{D}(X, \Lambda)$ be the derived category of étale Λ-sheaves on X as defined in [Sch17], where for $\Lambda = \mathbb{Q}_\ell$ we additionally invert ℓ in the adic formalism developed in [Sch17, Section 26]. We consider the inclusion of the geometric generic, respectively geometric special fiber into the integral Hecke stack

$$\mathrm{Hk}_G|_{\mathrm{Spd}\,\mathbb{C}_p} \xrightarrow{j} \mathrm{Hk}_{\mathcal{G}}|_{\mathrm{Spd}\,O_{\mathbb{C}_p}} \xleftarrow{i} \mathrm{Hk}_{\mathcal{G}}|_{\mathrm{Spd}\,\bar{\mathbb{F}}_p}. \tag{1.9}$$

The following result is inspired by work of Hansen–Scholze [HS21] for schemes and proves constructibility of nearby cycles in our context:

Theorem 1.8 (Proposition 6.7, Proposition 6.12)**.** *The pullback functor*

$$j^*\colon \mathrm{D}\big(\mathrm{Hk}_{\mathcal{G}}|_{\mathrm{Spd}\,O_{\mathbb{C}_p}}, \Lambda\big) \to \mathrm{D}\big(\mathrm{Hk}_G|_{\mathrm{Spd}\,\mathbb{C}_p}, \Lambda\big) \tag{1.10}$$

induces an equivalence on the full subcategories of universally locally acyclic sheaves with bounded support, with inverse given by the derived push forward Rj_*. Consequently, the nearby cycles functor $\Psi_{\mathcal{G}} := i^* \circ Rj_*$ restricts to a functor

$$\mathrm{D}\big(\mathrm{Hk}_G|_{\mathrm{Spd}\,\mathbb{C}_p}, \Lambda\big)^{\mathrm{bd,ula}} \to \mathrm{D}\big(\mathrm{Hk}_{\mathcal{G}}|_{\mathrm{Spd}\,\bar{\mathbb{F}}_p}, \Lambda\big)^{\mathrm{bd,ula}}. \tag{1.11}$$

Furthermore, the target category is equivalent to the derived category on the schematic Witt vector Hecke stack $\mathrm{D}_{\mathrm{cons}}\big(\mathrm{Hk}_{\mathcal{G}}^{\mathrm{sch}}|_{\mathrm{Spec}\,\bar{\mathbb{F}}_p}, \Lambda\big)^{\mathrm{bd}}$ of perfect-constructible sheaves with bounded support (see Section A).

The intersection complex IC_μ descends along $\mathrm{Gr}_G \to \mathrm{Hk}_G$ and defines an object in $\mathrm{D}(\mathrm{Hk}_G|_{\mathrm{Spd}\,\mathbb{C}_p}, \Lambda)^{\mathrm{bd,ula}}$. So Theorem 1.8 implies constructibility of $\Psi_{\mathcal{M}_{\mathcal{G},\mu}}\mathrm{IC}_\mu$. To compute the support, we use that nearby cycles commute with the constant term functors defined by the generic and special fibers of diagram (1.7):

$$\mathrm{CT}_{\mathcal{P}} \circ \Psi_{\mathcal{G}} \cong \Psi_{\mathcal{M}} \circ \mathrm{CT}_P, \tag{1.12}$$

see Proposition 6.13 for details. If λ is regular, then $\mathcal{M}$ is the connected Néron model of a torus so that the right hand side of (1.12) can be computed in representation theoretic terms using the geometric Satake equivalence for the B_{dR}^+-affine Grassmannian [FS21, Chapter VI]. In down to earth terms, we are able to determine the compactly supported cohomology of $\Psi_{\mathcal{M}_{\mathcal{G},\mu}}\mathrm{IC}_\mu$ on the stratification given by the semi-infinite orbits. For carefully chosen λ (see Lemma 5.3), this is enough to deduce Theorem 1.5, see Theorem 6.16.

1.4. **Special fibers of schematic local models.** In this subsection, we assume that μ is minuscule. The construction of schematic local models [PZ13, Lev16, Lou20, FHLR22] relies on Breuil–Kisin type lifts of $\mathcal{G}$ to a group scheme $\underline{\mathcal{G}}$ over $\breve{\mathbb{Z}}_p[\![t]\!]$ equipped with isomorphisms

$$\underline{\mathcal{G}} \otimes_{\breve{\mathbb{Z}}_p[\![t]\!], t\mapsto p} \breve{\mathbb{Z}}_p \simeq \mathcal{G} \otimes \breve{\mathbb{Z}}_p, \quad \underline{\mathcal{G}} \otimes \breve{\mathbb{Q}}_p[\![t-p]\!] \simeq G \otimes \breve{\mathbb{Q}}_p[\![t-p]\!]. \tag{1.13}$$

Let us temporarily denote by $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{G}},\mu}$ the weak normalization of the closure of $\mathcal{F}_{G,\mu}|_{\operatorname{Spec} \breve{E}}$ inside the schematic Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{\underline{\mathcal{G}}}|_{\operatorname{Spec} O_{\breve{E}}}$ attached to $\underline{\mathcal{G}}$, a weakly normal, flat, projective $O_{\breve{E}}$-scheme. As we explain in Section 1.5, this is isomorphic to the base changed local model $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}|_{\operatorname{Spec} O_{\breve{E}}}$ appearing in Theorem 1.1, in virtually all cases in the above sense. The most general group lifts $\underline{\mathcal{G}}$ are constructed in [FHLR22], based on [Lou20], under the following assumption:

Assumption 1.9. If $p = 2$, then G_{ad} has no odd unitary $\breve{\mathbb{Q}}_2$-factors.

The reason for its appearance is the difficult structure of the integral root groups inside $\mathcal{G} \otimes \breve{\mathbb{Z}}_p$ in the wildly ramified, odd unitary case. More precisely, quadratic field extensions of $\breve{\mathbb{Q}}_2$ fall into two classes: square roots of uniformizers and of units. The first class is handled in [Lou20], leading to the milder assumption for $p = 2$ in Theorem 1.1, and it is the second class that appears most difficult.

The determination of the special fiber of $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{G}},\mu}$ relies on the coherence conjecture of Pappas–Rapoport proved by Zhu [Zhu14]. The following result is the version in our context, and moreover, confirms [Zhu17a, Appendix B, Conjecture III] for the Schubert varieties in the μ-admissible locus:

Theorem 1.10 (Theorem 3.16)**.** *Under Assumption 1.9, the canonical deperfection $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}$ of the μ-admissible locus is Cohen–Macaulay and its components are compatibly Frobenius-split. Moreover, for every ample line bundle $\mathcal{L}$ on $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}$, there is an equality*

$$\dim_k \mathrm{H}^0(\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}, \mathcal{L}) = \dim_E \mathrm{H}^0(\mathcal{F}_{G,\mu}, \mathcal{O}(c)), \tag{1.14}$$

where c denotes the central charge of $\mathcal{L}$.

The canonical deperfection of $\mathcal{A}_{\mathcal{G},\mu}$ is induced from the Greenberg realization of the Witt vector loop groups, see Section 3.3. The proof of Theorem 1.10 proceeds by comparing the p-adic admissible loci to their analogues in equicharacteristic, and ultimately relies on the normality of Schubert varieties [Fal03, PR08] where we use [FHLR22] for wildly ramified groups. We first compare the perfect(ed) Demazure resolutions and then apply Bhatt–Scholze's h-descent results [BS17] to the ample line bundles on the resolutions. A key ingredient is He–Zhou's calculation [HZ20] of the Picard group of $\mathcal{F}\ell_{\mathcal{G}}$ as the free $\mathbb{Z}[p^{-1}]$-module dual to the lines stable under a fixed Iwahori dilated from $\mathcal{G}$.

Theorem 1.11 (Lemma 3.15, [FHLR22])**.** *Under Assumption 1.9, there is an isomorphism of $\bar{\mathbb{F}}_p$-schemes*

$$\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{G}},\mu}|_{\operatorname{Spec} \bar{\mathbb{F}}_p} \simeq \mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}|_{\operatorname{Spec} \bar{\mathbb{F}}_p}. \tag{1.15}$$

Hence, $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{G}},\mu}$ is normal, Cohen–Macaulay and has reduced special fiber.

The theorem holds, more generally, under the milder assumption explained above when $p = 2$ by [Lou20]. The reader is referred to [FHLR22] for a finer study of the singularities of the local models.

1.5. **Specialization maps.** We continue to assume that μ is minuscule and focus on the v-sheaf local models $\mathcal{M}_{\mathcal{G},\mu}$. The study of specialization maps for $\mathcal{M}_{\mathcal{G},\mu}$ is challenging. A basic problem is that, beyond rare exceptions, the set of $\mathcal{G}(O_{\mathbb{C}_p})$-orbits in $\mathcal{F}_{G,\mu}(\mathbb{C}_p)$ is infinite. However, we understand relatively well the reduction of $\operatorname{Spd} O_{\mathbb{C}_p}$-valued points lying in a certain cohomologically smooth sub-v-sheaf

$$\mathcal{M}^{\circ}_{\mathcal{G},\mu} \subset \mathcal{M}_{\mathcal{G},\mu}, \tag{1.16}$$

given as the $\operatorname{Spd} O_E$-descent of the $\mathcal{G}^{\diamondsuit}$-semi-orbit of the $\operatorname{Spd} O_{\breve{E}}$-sections rationally conjugate to μ. Unfortunately, $\mathcal{M}^{\circ}_{\mathcal{G},\mu}$ alone does not afford sufficiently many integral points. Even varying $(\mathcal{G}, \mu)$ barely improves the situation. Here we resort to variants of the splitting models of Pappas–Rapoport [PR05] in our situation, that is, we use convolutions to partially desingularize the local models. For a sequence $\mu_\bullet = (\mu_1, \dots, \mu_n)$ of minuscule coweights with pairwise disjoint supports, we consider the sub-v-sheaf

$$\mathcal{M}_{\mathcal{G},\mu_\bullet} \subset \mathrm{Gr}_{\mathcal{G}} \tilde{\times} \dots \tilde{\times} \mathrm{Gr}_{\mathcal{G}}, \tag{1.17}$$

defined as the v-closure of $\mathcal{F}^{\diamondsuit}_{G,\mu_\bullet} = \mathcal{F}^{\diamondsuit}_{G,\mu_1} \tilde{\times} \dots \tilde{\times} \mathcal{F}^{\diamondsuit}_{G,\mu_n}$ inside the convolution Beilinson–Drinfeld Grassmannian, see Section 7.1. Most of the notions discussed before have their convolution counterparts. It then becomes true that functoriality in $(\mathcal{G}, \mu_\bullet)$ is enough to control the specialization map:

Theorem 1.12 (Theorem 7.12)**.** *The specialization maps*

$$\mathrm{sp}_{\mathcal{G},\mu_\bullet} \colon \mathcal{F}_{G,\mu_\bullet}(\mathbb{C}_p) \to \mathcal{A}_{\mathcal{G},\mu_\bullet}(\bar{\mathbb{F}}_p) \tag{1.18}$$

for all pairs $(\mathcal{G}, \mu_\bullet)$ as above are the only functorial collection of continuous maps whose restrictions to the sets $\mathcal{M}^{\circ}_{\mathcal{G},\mu_\bullet}(\operatorname{Spd} O_{\mathbb{C}_p})$ agree with the natural reduction maps.

The theorem is a result of our reflections on the He–Pappas–Rapoport conjecture [HPR20, Conjecture 2.12], which roughly states that the local model $\mathcal{M}_{\mathcal{G},\mu}$ should be uniquely recovered from its fibers equipped with the $\mathcal{G}$-action. The key calculation concerns the case where all non-zero components of $\mu_\bullet$ have irreducible support and G is a restriction of scalars along $E/\mathbb{Q}_p$ of a split group. Applying the Iwasawa decomposition and an induction on the number of non-zero components, we see $\mathcal{F}_{G,\mu_\bullet}(E) = \mathcal{M}^\circ_{\mathcal{G},\mu_\bullet}(\operatorname{Spd} O_E)$, that is, all rational points of the flag variety extend to integral points of the semi-orbit. Then functoriality forces uniqueness for all remaining cases.

1.6. **Conclusion.** The formulation of Theorem 1.12 requires functoriality of local models in $(\mathcal{G},\mu)$. This is clear for $\mathcal{M}_{\mathcal{G},\mu}$ but, a priori, problematic for its schematic version $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{G}},\mu}$. We need to impose the following assumption which relates to functoriality problems with the association $\mathcal{G} \mapsto \underline{\mathcal{G}}$ of Breuil–Kisin lifts:

Assumption 1.13. If $p = 3$, then G_{ad} has no triality $\breve{\mathbb{Q}}_3$-factors.

The following result confirms [SW20, Conjecture 21.4.1], except for very few cases when $p \leq 3$. More precisely, It shows that [Lou20, Conjecture IV.4.18] holds in full generality, whereas for [Lou20, Conjecture IV.4.19] to hold, we only have to impose the extremely mild restrictions[4] of Assumption 1.9 and Assumption 1.13:

Theorem 1.14. *There is a unique flat, projective and weakly normal O_E-model $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ of the E-scheme $\mathcal{F}_{G,\mu}$ endowable with an isomorphism of v-sheaves*

$$(\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu})^\diamondsuit \cong \mathcal{M}_{\mathcal{G},\mu}, \tag{1.19}$$

prolonging $\mathcal{F}^\diamondsuit_{G,\mu} \cong \mathcal{M}_{\mathcal{G},\mu}|_{\operatorname{Spd} E}$. Under Assumption 1.9 and Assumption 1.13, there is a unique isomorphism

$$\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}|_{\operatorname{Spec} O_{\breve{E}}} \cong \mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{G}},\mu}|_{\operatorname{Spec} O_{\breve{E}}} \tag{1.20}$$

inducing the identity on generic fibers. So (Theorem 1.11), $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is normal, Cohen–Macaulay and has reduced special fiber equal to $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}$. Furthermore, the isomorphisms (1.19) and (1.20) are equivariant for $\mathcal{G}^\diamondsuit_{O_E}$, respectively for $\mathcal{G}_{O_{\breve{E}}}$.

The unique scheme $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ satisfying the Scholze–Weinstein conjecture is the weak normalization of the closure of $\mathcal{F}_{G,\mu}$ inside the Beilinson–Drinfeld Grassmannian attached to a Breuil–Kisin lift of $\mathrm{Res}_{O_K/\mathbb{Z}_p}\mathcal{H}$, where K is the splitting field of G and $\mathcal{H}$ the parahoric O_K-model of the split Chevalley form. The required functoriality can be enforced by choosing further Breuil–Kisin lifts accordingly, so that the analogue of Theorem 1.12 holds true for $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$. By a careful analysis, we get an isomorphism between the relevant specialization triples, which is enough to conclude by Theorem 1.4.

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2. v-sheaf theory

Our main reference for the theory of diamonds, v-sheaves and v-stacks is [Sch17]. Here we gather some complementary results in the geometry of these objects that we will need later on.

2.1. **Closures.** In this subsection, we discuss closures of small v-stacks. Let $\mathrm{Perf}_{\mathbb{F}_p}$ be the v-site of perfectoid spaces of characteristic p. All the small v-stacks in the following will be stacks on $\mathrm{Perf}_{\mathbb{F}_p}$.

Let X be a small v-stack and let $Y \subset X$ be a sub-v-stack, by which we mean a monomorphism of small v-stacks $Y \to X$, that is, a morphism whose diagonal

$$Y \xrightarrow{\Delta} Y \times_X Y \tag{2.1}$$

is an isomorphism. Just as in [Sch17, Definition 10.7], we say that Y is a locally closed sub-v-stack if, for every totally disconnected perfectoid space $S \to X$, the pullback $Y \times_X S \to S$ is representable by an immersion of perfectoid spaces, see [Sch17, Definition 5.6]. It is called a closed, respectively open immersion, if so are the respective pullbacks.

This admits a simpler description for closed sub-v-stacks.

[4] The assumptions in Theorem 1.14 can be weakened as in Theorem 1.1 by resorting to [Lou20].

Lemma 2.1. *A morphism $Y \to X$ of small v-stacks is a closed immersion if and only if $Y \to X$ is a quasi-compact monomorphism and the induced map $|Y| \subset |X|$ is a closed embedding.*

Proof. Assume $Y \to X$ is a closed immersion and let $f\colon S \to X$ be a surjection from a disjoint union of totally disconnected perfectoid spaces. By assumption $Z := Y \times_X S$ is representable and $Z \to S$ is a closed immersion, in particular it is quasi-compact. By [Sch17, Proposition 10.11 (o)], the map $Y \to X$ is then quasi-compact as well. We may check that $Y \xrightarrow{\Delta} Y \times_X Y$ is an isomorphism after base change to S. This amounts to verifying that $Z \xrightarrow{\Delta} Z \times_S Z$ is an isomorphism which follows from the fact that closed immersions are monomorphisms [Sch17, Definition 5.6]. The inclusion $|Z| \subset |S|$ is a closed subset and equal to $|f|^{-1}(|Y|)$. Indeed, if $s \in |S| \setminus |Z|$, $y \in |Y|$ and we let $\tilde{s}\colon \mathrm{Spa}(C_s, C_s^+) \to S$ and $\tilde{y}\colon \mathrm{Spa}(C_y, C_y^+) \to Y$ represent s and y respectively, then $\tilde{s} \times_X \tilde{y} = \emptyset$. So s and y map to different points in $|X|$. As $|f|$ is a quotient map [Sch17, Proposition 12.9], this implies that $|Y| \subset |X|$ is a closed embedding.

Conversely, assume that $Y \subset X$ is a quasi-compact monomorphism and induces a closed embedding of underlying topological spaces. Let $f\colon S \to X$ be a map from a totally disconnected space S. The base change $Y \times_X S \to S$ is still a quasi-compact monomorphism of v-sheaves. By [Sch17, Corollary 10.6, Lemma 7.6] the v-sheaf $Y \times_X S$ is representable by a pro-constructible generalizing affinoid subset of S, and $|Y \times_X S|$ carries the subspace topology of $|S|$. Arguing as above, the image of $|Y \times_X S|$ in $|S|$ is $|f|^{-1}(|Y|)$ which is closed by assumption. This implies that the morphism $Y \times_X S \to S$ is a closed immersion in the sense of [Sch17, Definition 5.6]. □

We now define the v-sheaf closure, or v-closure, of a sub-v-sheaf.

Definition 2.2. Let X be a small v-stack and $Y \subset X$ a sub-v-stack $Y \subset X$. We define the v-closure Y^{cl} in X as the limit (in the 2-category of v-stacks) of all closed sub-v-stacks of X containing Y.

The sub-v-stack $Y^{\mathrm{cl}} \subset X$ is a closed sub-v-stack. Indeed, we can verify this after base change by a totally disconnected perfectoid space and use [Sch17, Proposition 6.4 (o)] to conclude.

Next, we discuss the relation between the topological space $|Y^{\mathrm{cl}}|$ of the v-closure Y^{cl} and the topological closure $|Y|^{\mathrm{cl}}$ of the image of $|Y|$ in $|X|$.

Definition 2.3. Let $S \subset |X|$ be a subset.

(1) We call S weakly generalizing if, for any perfectoid field C with open and bounded valuation subring $C^+ \subset C$, and every morphism $\mathrm{Spa}(C, C^+) \to X$, the induced morphism $|\mathrm{Spa}(C, C^+)| \to |X|$ factors over S if and only if the closed point of $|\mathrm{Spa}(C, C^+)|$ maps into S.

(2) The weakly generalizing closure S^{wgc} of S is defined as the intersection of all closed, weakly generalizing subsets $S' \subset |X|$ containing S.

We note that if X is (the v-sheaf associated to) a perfectoid space, then a subset $S \subset X$ is weakly generalizing if and only if it is generalizing. Indeed, for each analytic adic space specializations happen only at the same residue field.

The images of morphisms of small v-stacks are weakly generalizing as the next lemma shows.

Lemma 2.4. *For every morphism $f\colon X \to X'$ of small v-stacks, the image of $|f|\colon |X| \to |X'|$ is weakly generalizing in $|X'|$.*

Proof. Let C be a perfectoid field with open and bounded valuation subring $C^+ \subset C$. Assume that the morphism
$$\mathrm{Spa}(C, C^+) \to X' \tag{2.2}$$
sends the closed point of $\mathrm{Spa}(C, C^+)$ into $f(|X|)$. This means that the above morphism factors through X after possibly enlarging (C, C^+), see [Sch17, Proposition 12.7]. But then the full image of $|\mathrm{Spa}(C, C^+)|$ in $|X'|$ will factor through $f(|X|)$ and this shows that $f(|X|)$ is weakly generalizing. □

In particular, the topological space of the v-closure Y^{cl} of some sub-v-stack $Y \subset X$ is always weakly generalizing. Thus, the topological space of the v-closure does not coincide, in general, with the topological closure.

Example 2.5. As a concrete example, consider the inclusion
$$\mathbb{D}_C^\diamondsuit \to \mathbb{B}_C^\diamondsuit := \mathrm{Spd}(C\langle T\rangle, \mathcal{O}_C\langle T\rangle) \tag{2.3}$$
of the open unit ball into the closed unit ball over a perfectoid base field C. Then $|\mathbb{D}_C^\diamondsuit|^{\mathrm{cl}}$ is the complement of the torus $|\mathbb{T}_C^\diamondsuit| = \mathrm{Spd}(C\langle T^{\pm 1}\rangle, \mathcal{O}_C\langle T^{\pm 1}\rangle)$, hence not weakly generalizing, as it misses the Gaußpoint but contains a rank 2 specialization thereof.

The weakly generalizing closure $|\mathbb{D}_C^\diamondsuit|^{\mathrm{wgc}}$ is given in turn by the complement of every open unit ball $\mathbb{D}_{x,C}^\diamondsuit$ centered around $x \in \mathbb{T}_C(C)$. This will give rise to the v-closure of $\mathbb{D}_C^\diamondsuit$ inside $\mathbb{B}_C^\diamondsuit$, see Proposition 2.8.

Let us recall that, for every small v-stack X, there is the canonical morphism
$$X \to \underline{|X|},\ (f\colon T \to X) \mapsto (|f|\colon |T| \to |X|) \tag{2.4}$$
of v-stacks where $\underline{|X|}$ denotes the v-sheaf represented by the topological space $|X|$, that is, it is given by $\underline{|X|}(T) = \mathrm{Hom}_{\mathrm{cts}}(|T|, |X|)$ for each $T \in \mathrm{Perf}_{\mathbb{F}_p}$.

Remark 2.6. We warn the reader that $\underline{|X|}$ is not small whenever $|X|$ fails to satisfy the separation axiom $T1$. Indeed, if $|X|$ is a trait then $\underline{|X|}(R,R^+)$ is the set of closed subsets of $|\mathrm{Spa}(R,R^+)|$, and for each fixed κ, there is $|\mathrm{Spa}(R,R^+)|$ large enough so that not all closed subsets come from pullback of κ-small ones.

Lemma 2.7. *Let X be a small v-stack, and $S \subset |X|$ be a weakly generalizing closed subset. Then $Y := \underline{S} \times_{\underline{|X|}} X$ is a small closed sub-v-stack satisfying $|Y| = S$ and, moreover, every closed sub-v-stack is of this form.*

Proof. By [Sch17, Proposition 10.11], we may check that $Y \subset X$ is a closed sub-v-stack after pullback along a v-cover $f\colon Z \to X$ with Z a disjoint union of totally disconnected perfectoid spaces. Then $Y \times_X Z = \underline{|f|^{-1}(S)} \times_{\underline{|Z|}} Z$ and note that $|f|^{-1}(S) \subset |Z|$ is closed as $|f|$ is continuous. Moreover, $|f|^{-1}(S)$ is weakly generalizing, and thus generalizing because Z is a perfectoid space. Consequently, $|f|^{-1}(S)$ is representable by a perfectoid space by [Sch17, Lemma 7.6]. Its v-sheaf coincides with $Y \times_X Z$ by [Sch17, Lemma 12.5], so $Y \subset X$ is a closed immersion by Lemma 2.1. Clearly, we also have $|Y| = S$ as $|f|$ is surjective.

Now assume that $Y \subset X$ is a closed sub-v-stack and let $Y' = X \times_{\underline{|X|}} \underline{|Y|}$. The identity $Y = Y \times_X Y'$ is easy to verify (by base change to totally disconnected S and [Sch17, Proposition 5.3.(iv)]), so the map $Y \to Y'$ is a closed sub-v-stack with the same underlying topological space. By [Sch17, Lemma 12.11], $Y \to Y'$ is a surjective map of v-stacks and consequently an isomorphism. □

Lemma 2.4 and Lemma 2.7 characterize closed weakly generalizing subsets $S \subset |X|$ as exactly those closed subsets $S \subset |X|$ for which the inclusion

$$\left|\underline{S} \times_{\underline{|X|}} X\right| \subset S \tag{2.5}$$

is an equality. Note that the v-sheaf $Y := \underline{S} \times_{\underline{|X|}} X$ may even be empty if S is not weakly generalizing. For example, this happens if $S = \{s\}$ for $s \in \mathrm{Spa}(C, C^+)$ the closed point of a perfectoid field C with $C^+ \subsetneq O_C \subset C$ an open and bounded valuation subring of rank > 1.

Proposition 2.8. *Let X be a small v-stack, and let $Y \subset X$ be a sub-v-stack. Let $Y^{\mathrm{cl}} \subset X$ be the v-closure of Y in X. Then*

$$Y^{\mathrm{cl}} = \underline{|Y|^{\mathrm{wgc}}} \times_{\underline{|X|}} X \tag{2.6}$$

as sub-v-stacks of X. Hence, $|Y^{\mathrm{cl}}| \subset |X|$ is the weakly generalizing closure $|Y|^{\mathrm{wgc}}$ of $|Y|$ in $|X|$.

Proof. Set $Y' := \underline{|Y|^{\mathrm{wgc}}} \times_{\underline{|X|}} X$. Then Y' is a closed sub-v-stack of X containing Y and $|Y'| = |Y|^{\mathrm{wgc}}$ by Lemma 2.7. Therefore, the v-closure Y^{cl} is contained in Y'. But conversely, the topological space $|Y^{\mathrm{cl}}|$ must contain $|Y|^{\mathrm{wgc}}$ by Lemma 2.4. Since $Y^{\mathrm{cl}} = \underline{|Y^{\mathrm{cl}}|} \times_{\underline{|X|}} X$ again by Lemma 2.7, we conclude that $Y' \subset Y^{\mathrm{cl}}$ and thus they coincide as desired. □

The next result will turn out to be a useful tool later on when computing v-closures.

Corollary 2.9. *The formation of v-closures commutes with base change by partially proper morphisms that are also open maps.*

Proof. In the following, we identify open substacks of small v-stacks with open subsets of their topological space, see [Sch17, Proposition 12.9]. By Proposition 2.8, we need to verify the corresponding assertion at the topological level. Let $f\colon Z \to X$ be an open and partially proper morphism between small v-stacks and set $g := |f|$. Let $S \subset |X|$ be a subset, clearly $g^{-1}(S)^{\mathrm{wgc}} \subset g^{-1}(S^{\mathrm{wgc}})$. Let $T := g^{-1}(S)^{\mathrm{wgc}} \subset |Z|$. Its complement V is an open subset of $|Z|$, and the map $V \to Z$ is partially proper because T is weakly generalizing. Since the map $Z \to X$ is open, the subset $U := g(V)$ is also open. Since $V \to Z$ is partially proper, the map $U \to X$ is partially proper as well. The complement $F \subset |X|$ of U is closed and $g^{-1}(F) \subset T$. Also, F is weakly generalizing since U is partially proper. This implies $S^{\mathrm{wgc}} \subset F$ and consequently $g^{-1}(S^{\mathrm{wgc}}) \subset T$, as we wanted to show. □

2.2. **The two different diamond functors.** Let O be a complete discrete valuation ring with perfect residue field k of characteristic p, and assume that O is flat over $\mathbb{Z}_p$, that is, p-torsion free. Let $\pi \in O$ be a uniformizer.

If X is a pre-adic space over O, we can attach to it a v-sheaf $X^{\diamondsuit}$ over $\mathrm{Spd}\, O$[5] as in [SW20, Section 18]. Namely, if $S \in \mathrm{Perf}_{\mathbb{F}_p}$, then

$$X^{\diamondsuit}(S) = \{(S^\sharp, \iota, f)\}/\mathrm{isom}. \tag{2.7}$$

with $(S^\sharp, \iota)$ an untilt of S and $f\colon S^\sharp \to X$ a morphism of pre-adic spaces (and the obvious notion of isomorphism between these triples). On the other hand, given an algebra A over O there are two different ways to associate a v-sheaf to $\mathrm{Spec}(A)$.

Definition 2.10. Let A be an O-algebra.

(1) We let $\mathrm{Spec}(A)^\diamond$ denote the functor

$$(R, R^+) \mapsto \{(R^\sharp, \iota, f)\}/\mathrm{isom}. \tag{2.8}$$

where $(R^\sharp, \iota)$ is an untilt over O and $f\colon A \to R^{\sharp,+}$ is a ring homomorphism.

[5]Whenever $R^+ = R$, we simply write $\mathrm{Spd}(R)$ for $\mathrm{Spd}(R, R)$ following [SW20].

(2) We let $\mathrm{Spec}(A)^{\diamondsuit}$ denote the functor

$$(R, R^+) \mapsto \{(R^\sharp, \iota, f)\}/\mathrm{isom}. \tag{2.9}$$

where $(R^\sharp, \iota)$ is an untilt over O and $f\colon A \to R^\sharp$ is a ring homomorphism.

Both of these constructions are compatible with localization and glue to functors from the category of schemes over O to the category of v-sheaves over $\mathrm{Spd}\, O$. Indeed, given $g \in A$, the open subscheme $\mathrm{Spec}(A[1/f])$ is sent to the open subfunctors of $\mathrm{Spec}(A)^\diamond$, respectively $\mathrm{Spec}(A)^{\diamondsuit}$ defined by the conditions $f(g) \in (R^\sharp)^\times$, respectively $f(g) \in (R^{\sharp,+})^\times$, that is, by the open loci $\{|g| \neq 0\} \subset \mathrm{Spa}(R^\sharp, R^{\sharp,+})$, respectively $\{|g| = 1\} \subset \mathrm{Spa}(R^\sharp, R^{\sharp,+})$. We still denote these functors on Sch_O by $\diamond$ and $\diamondsuit$.

Remark 2.11. For schemes locally of finite type over O both functors admit a two step construction.

We can first associate to a scheme X adic spaces $\widehat{X}$ and X^{an} with an adic structure map to $\mathrm{Spa}(O)$ that satisfy $\widehat{X}^{\diamondsuit} = X^\diamond$ and $(X^{\mathrm{an}})^{\diamondsuit} = X^{\diamondsuit}$ (with $(-)^{\diamondsuit}$ the diamond functor on pre-adic spaces). This is the approach taken in [Gle20, Section 2.2.2]. The space $\widehat{X}$ is easy to describe, for $X = \mathrm{Spec}(A)$ we have $\widehat{X} = \mathrm{Spa}(\widehat{A}, \widehat{A})$ where we let $\widehat{A}$ denote the π-adic completion of A. On the other hand, the construction of X^{an} is more subtle and goes back to work of Huber [Hub94, Proposition 3.8]. In general, even if X is affinoid X^{an} might not be affinoid and very often the structure map $X^{\diamondsuit} \to \mathrm{Spd}\, O$ is not quasi-compact. Now, for $X = \mathrm{Spec}(A)$, not necessarily of finite type, and $\pi = 0$ in A we have $X^{\diamondsuit} = \mathrm{Spd}(A^{\mathrm{perf}}, \overline{k})$, where $\overline{k}$ denotes the integral closure of k in the perfection A^{perf} of A and both rings are given the discrete topology.

Put differently, there is an adjunction

$$\mathrm{Sch}_O \underset{(\text{-})^{\mathrm{ad}/O}}{\overset{(\text{-})^{\mathrm{ad}}}{\rightleftarrows}} \mathrm{DiscAdSp}_{O_{\mathrm{disc}}}$$

from the category of schemes over O to the category of discrete adic spaces over O_{disc} characterized by the formulas on affinoids, respectively affines:

$$\mathrm{Spa}(A, A^+) \mapsto \mathrm{Spec}(A), \quad \mathrm{Spec}(A)^{\mathrm{ad}} = \mathrm{Spa}(A, A), \quad \mathrm{Spec}(A)^{\mathrm{ad}/O} = \mathrm{Spa}(A, \widetilde{O}),$$

where $\widetilde{O}$ is the integral closure of O in A. Then, for a scheme X locally of finite type over O, the pre-adic space $\widehat{X}$, respectively X^{an} is the base change of X^{ad}, respectively $X^{\mathrm{ad}/O}$ along $O_{\mathrm{disc}} \to O$.

Let X be an O-scheme. There is an evident natural transformation $X^\diamond \to X^{\diamondsuit}$. If X is separated over O, this map is a monomorphism of small v-sheaves, and an open immersion if X is also locally of finite type over O. If X is proper over O, then the open immersion $X^\diamond \to X^{\diamondsuit}$ is an isomorphism because it is surjective on points by the valuative criterion for properness. Therefore we will abuse notation and let $X^{\diamondsuit}$ denote the common value $X^\diamond = X^{\diamondsuit}$ whenever X is proper over O.

The two diamond functors play different roles in the following sections: the "analytic" functor $(\text{-})^{\diamondsuit}$ is the most natural to study $\mathbb{G}_m$-actions and nearby cycles while the "formal" functor $(\text{-})^\diamond$ carries a specialization map which we will exploit.

A natural question is to what extent the associated v-sheaves $X^\diamond$ and $X^{\diamondsuit}$ reflect the geometry of X. In general, neither of the functors is full or faithful. For example, if A is any O-algebra and $\hat{A}$ its π-adic completion, then the natural morphism

$$\mathrm{Spec}(\hat{A})^\diamond \to \mathrm{Spec}(A)^\diamond \tag{2.10}$$

is an isomorphism (because $R^{\sharp,+}$ is π-complete by uniformity of affinoid perfectoid spaces). In particular, if F is the fraction field of O, and A an F-algebra, then $\mathrm{Spec}(A)^\diamond = \emptyset$. If $A = F[t]$, then

$$\mathrm{Spec}(F[t])^{\diamondsuit} = (\mathbb{A}_F^{1,\mathrm{ad}})^{\diamondsuit} \tag{2.11}$$

is the rigid-analytic affine line over F, which has many non-algebraic automorphisms.

When we restrict to schemes over O for which $\pi = 0$, the situation is more clear. Both $\diamond$ and $\diamondsuit$ are fully faithful on perfect schemes and if we let Y denote the perfection of X, then $X^\diamond = Y^\diamond$ and $X^{\diamondsuit} = Y^{\diamondsuit}$ [SW20, Proposition 18.3.1], [Gle20, Theorem 1.2.32]. That is, up to a fully faithful embedding, both functors are the perfection functor. Nevertheless, we stress again that the essential images of the functors $(\text{-})^\diamond, (\text{-})^{\diamondsuit}$ on (perfect) schemes over k are different.

To prove the Scholze-Weinstein conjecture we need to work with schemes that are proper and flat over O, and their associated small v-sheaves. Therefore, we have to relate these two notions.

The functor $X \mapsto X^\diamond (= X^{\diamondsuit}$ if X is proper) from the category schemes over O to small v-sheaves over $\mathrm{Spd}\, O$ factors as the composition of the functor

$$\widehat{(\text{-})}_\pi \colon \mathrm{Sch}_O \to \mathrm{fSch}_O,\ Y \mapsto \widehat{Y}_\pi \tag{2.12}$$

of π-adic completion, the functor sending (locally) a formal scheme $\mathrm{Spf}(A)$ over $\mathrm{Spf}(O)$ (with locally finitely generated ideal of definition) to the (pre-)adic space $\mathrm{Spa}(A)$, and then the functor $(\text{-})^{\diamondsuit}$ on pre-adic spaces over $\mathrm{Spa}(O)$. Let us abuse of notation and denote for a formal scheme Y (admitting locally a finite ideal of definition) by $Y^{\diamondsuit}$ the v-sheaf for the pre-adic space associated with Y.

On the category of schemes which are proper over O, the functor $\widehat{(\text{-})}_\pi$ of π-adic completion is fully faithful by Grothendieck's existence theorem [Sta21, Tag 08BF] or [Gro61, Théorème 5.4.1]. Let us note that π-adic completion maps schemes, which are flat over O, to formal schemes, which are flat over O.

The first thing to note is that for any formal scheme Y over O (locally admitting a finite ideal of definition), the small v-sheaf $Y^\diamondsuit$ only depends on the absolute weak normalization[6] of Y, see Lemma 2.13.

Let us review the following terms.

Definition 2.12 ([Sta21, Tag 0EUL])**.** A ring A is called semi-normal if for all $a, b \in A$ with $a^3 = b^2$ there exists a unique $c \in A$ with $a = c^2$ and $b = c^3$. Similarly, A is called absolutely weakly normal if it is semi-normal and if, for any prime ℓ and elements $a, b \in A$ with $\ell^\ell a = b^\ell$, there exists a unique $c \in A$ with $a = c^\ell$ and $b = \ell c$.

Note that the last property is automatic for any prime ℓ, which is invertible in A, and that each semi-normal ring is reduced.

Since ring localizations preserve semi-normality or absolute weak normality, they can be generalized to schemes, see [Sta21, Tag 0EUN]. Moreover, given any scheme X, there exists an initial morphism $X^{\mathrm{sn}} \to X$ (respectively, $X^{\mathrm{awn}} \to X$) from a semi-normal (respectively, absolutely weakly normal) scheme, which is called the semi-normalization (respectively, absolutely weak normalization) of X, and which is also the initial morphism $Y \to X$, which is a universal homeomorphism inducing isomorphisms on each residue field (respectively, universal homeomorphism), see [Sta21, Tag 0EUS]. If A is an $\mathbb{F}_p$-algebra, then A is absolutely weakly normal if and only if A is perfect [Sta21, 0EVV], and thus the absolute weak normalization agrees with the perfection of schemes over $\mathbb{F}_p$. From the universal property of X^{awn} and the fact that universal homeomorphisms are integral, radicial and surjective [Sta21, Tag 04DF], it is clear that normal, integral schemes X with perfect function field are absolutely weakly normal.

Lemma 2.13. *Let $Y = \mathrm{Spf}(A)$ be an affine formal scheme over O, and let $I \subset A$ be a finitely generated ideal of definition. Let $B := \widehat{A^{\mathrm{awn}}}_I$ be the I-adic completion of the absolute weak normalization of A. Then the natural map*

$$\mathrm{Spf}(B)^\diamondsuit \to \mathrm{Spf}(A)^\diamondsuit \tag{2.13}$$

is an isomorphism.

Proof. Let $\mathrm{Spa}(R, R^+) \to \mathrm{Spa}(A)$ be a morphism from some affinoid perfectoid space over O. Then R^+ is automatically I-adically complete. By the universal property of the absolute weak normalization and the fact that $\mathrm{Spf}(B)^\diamondsuit, \mathrm{Spf}(A)^\diamondsuit$ are v-sheaves it suffices to see that every affinoid perfectoid space admits a v-cover by one of the form $\mathrm{Spa}(R, R^+)$ with R^+ absolutely weakly normal. We can always choose (R, R^+) so that R^+ equals a product of perfectoid valuation rings with algebraically closed fraction fields. In this case, R^+ is absolutely weakly normal because the conditions in Definition 2.12 can be checked in each factor. Indeed, it is clear that any such factor is a normal, integral domain with algebraically closed, in particular perfect, fraction field. □

Definition 2.14. We will work with formal schemes X that are "weakly normal" flat and "topologically of finite type"[7] over O. By this we mean formal schemes that are locally of the form $\mathrm{Spf}(A)$ where A is a weakly normal, flat and π-adically complete topological algebra of the form $O\langle T_1, \dots, T_n\rangle/I$ for some ideal $I \subset O\langle T_1, \dots, T_n\rangle$.

To justify Definition 2.14, we need to prove that "weak normality" glues and localizes for the formal schemes that we work with. This is the content of the next statement.

Proposition 2.15. *Assume that A is flat and topologically of finite type over O. Let $\emptyset \neq U_{f_i} \subset \mathrm{Spf}(A)$ with $i \in \{1 \dots, n\}$ be an open cover by distinguished open subsets with $U_i = \mathrm{Spf}(B_i)$. Then A is weakly normal if and only if all of the B_i are weakly normal.*

Proof. Since weak normality is compatible with localization A is weakly normal if and only if all of the $A[f_i^{-1}]$ are weakly normal. Now, B_i is the π-adic completion of $A[f_i^{-1}]$, in particular flat over it. We claim that $A[f_i^{-1}] \to B_i$ is a regular map [Sta21, Tag 07BZ] and that $A \to \prod_i B_i$ is regular and faithfully flat. Given these, the statement follows directly from [Man80, Proposition III.3] since a regular map is a reduced and normal map. Let us prove the claim. Observe that all of the rings are Noetherian and excellent because they are obtained from O by adding variables, taking quotient by ideals, completing or localizing. By [Sta21, Tag 07C0], we may check regularity after localizing at a maximal ideal $\mathfrak{m} \subset B_i$. Consider the following maps of rings:

$$A_{(\mathfrak{m})} \to (B_i)_{(\mathfrak{m})} \to \widehat{(A_{(\mathfrak{m})})_\pi} \to \widehat{(A)_\mathfrak{m}} \tag{2.14}$$

They are all faithfully flat. Since A is excellent, the map $A_{(\mathfrak{m})} \to \widehat{(A)_\mathfrak{m}}$ is regular. By [Sta21, Tag 07QI], we conclude $A_{(\mathfrak{m})} \to (B_i)_{(\mathfrak{m})}$ is so as well. □

[6]We do not define absolute weak normalizations of formal schemes though and stick to the more concrete statement in Lemma 2.13.

[7]Recall that in the context of adic spaces there are two related notions for Huber rings over O: namely, "topologically of finite type" O-algebras and "strictly topologically of finite type" O-algebras [Wed19, Section 6.6]. We only need to work with O-algebras that, in the adic space lingo, are called strictly topologically of finite type. Nevertheless, we adopt the formal schemes convention in which these same algebras are called topologically of finite type instead.

We prove that the diamond functor is fully faithful on absolute weakly normal formal schemes that are locally topologically of finite type over O. Variants of this statement already appear in [SW20, Proposition 18.4.1] and [Lou20, IV, Theorem 4.6]. The precise form is as follows:

Theorem 2.16. *The functor $X \mapsto X^\diamondsuit$ from the category of absolute weakly normal formal schemes flat, separated and topologically of finite type over O, to v-sheaves over $\operatorname{Spd} O$ is fully faithful.*

Proof. We begin by proving the case in which X and Y are affine formal schemes. Confusing a formal scheme with its associated adic space we may assume that $X = \operatorname{Spa}(A), Y = \operatorname{Spa}(B)$ with A, B absolutely weakly normal flat and topologically of finite type over O. Faithfulness follows from the fact that B admits an injection (as it is reduced) into a product of perfectoid valuation rings. For fullness, let $f\colon X^\diamondsuit \to Y^\diamondsuit$ be a morphism of small v-sheaves. We are seeking a morphism $\psi\colon X \to Y$ such that $\psi^\diamondsuit = f$. Let $K = O[\pi^{-1}]$ be the fraction field of O. As A, B are π-adic the generic fibers X_η, Y_η are given by $\operatorname{Spa}(A[\pi^{-1}], A')$, $\operatorname{Spa}(B[\pi^{-1}], B')$ with A', B' the integral closure of A, B in $A[\pi^{-1}]$, $B[\pi^{-1}]$. The localizations $A[\pi^{-1}]$ and $B[\pi^{-1}]$ are absolutely weakly normal by [Sta21, Tag 0EUM], and thus semi-normal. By [SW20, Proposition 10.2.3], we get a morphism $\psi_\eta\colon Y_\eta \to X_\eta$, so that $\psi_\eta^\diamondsuit = f_\eta$. Because A, B are topologically of finite type over O and reduced, the rings A', B' are finite over A, B, and thus in particular the subspace topology coming from $A[\pi^{-1}]$, $B[\pi^{-1}]$ is π-adic on A', B'. In particular, A', B' are Huber. By definition the map $\psi_\eta\colon Y_\eta \to X_\eta$ induces a morphism $\psi'\colon Y' := \operatorname{Spa}(B') \to X$ over O, so that $\psi'_\eta = \psi_\eta$. Denoting by $B'' \subset B'$ the (automatically closed as B is noetherian and B' finite over B) image of $A \widehat{\otimes}_O B$ in $B[\pi^{-1}]$, we even get $\psi''\colon Y'' := \operatorname{Spa}(B'') \to X$ such that the morphism $Y'' \to X \times_O Y$ is a closed embedding of formal schemes. It is easy to see that $(Y'')^\diamondsuit \to (X \times_{\operatorname{Spa}(O)} Y)^\diamondsuit \cong X^\diamondsuit \times_{\operatorname{Spd} O} Y^\diamondsuit$ is a closed immersion of v-sheaves. Inside $X^\diamondsuit \times_{\operatorname{Spd} O} Y^\diamondsuit$, we then have two closed sub-v-sheaves, namely $Y''^\diamondsuit$ induced by $\psi''^\diamondsuit$ and $Y^\diamondsuit \simeq \Gamma_f$ induced by the graph of f. In both of these closed sub-v-sheaves, the generic fiber is dense by Lemma 2.17 below (applied to B and B''), and they carry the same generic fiber. Therefore, the finite birational morphism $Y'' \to Y$ induced by the inclusion $B \subset B''$ becomes an isomorphism in the category of v-sheaves. Passing to special fibers, this implies that $\operatorname{Spec}(B''/\pi)^{\mathrm{perf}} \to \operatorname{Spec}(B/\pi)^{\mathrm{perf}}$ is an isomorphism [SW20, Proposition 18.3.1]. As $B''[\pi^{-1}] \cong B[\pi^{-1}]$ we can conclude that $\operatorname{Spec}(B'') \to \operatorname{Spec}(B)$ is a universal homeomorphism. Indeed, $B \to B''$ is integral, radicial (as can be checked on each fiber over $\operatorname{Spec}(O)$) and surjective. Since B is absolutely weakly normal, we get $B'' = B$ and thus $(\psi'')^\diamondsuit = f$.

We now extend the argument to the general case. To verify faithfulness one can easily argue locally on X and Y because if $X = \cup_{i\in I} X_i$ is an open cover by formal schemes, then $\cup_{i \in I} X_i^\diamondsuit$ is an open cover of $X^\diamondsuit$. Proving fullness is more subtle since one has to justify that for a map $f\colon X^\diamondsuit \to Y^\diamondsuit$ and an open subset $U \subset Y$ with $U = \operatorname{Spf}(A)$, the pullback $f^{-1}(U^\diamondsuit) \subset X^\diamondsuit$ is "classical". In other words,

$$f^{-1}(U^\diamondsuit) = V^\diamondsuit \tag{2.15}$$

for some open immersion of formal schemes $V \subset X$. Now, by [SW20, Proposition 18.3.1] the special fiber map $f \times_{\operatorname{Spd} O} \operatorname{Spd} k$ is induced by a map of perfect schemes $f_{\mathrm{red}}\colon X_{\mathrm{red}}^{\mathrm{perf}} \to Y_{\mathrm{red}}^{\mathrm{perf}}$. Identifying $|X|$, $|Y|$ with $|X_{\mathrm{red}}^{\mathrm{perf}}|$ and $|Y_{\mathrm{red}}^{\mathrm{perf}}|$, we can construct V as $f_{\mathrm{red}}^{-1}(U_{\mathrm{red}})$. That the identity in Equation (2.15) holds will follow from functoriality of the specialization map considered in [Gle20]. Indeed, $U^\diamondsuit = \mathrm{sp}_{Y^\diamondsuit}^{-1}(U_{\mathrm{red}})$. □

We used the following lemma. Here, for a Huber pair (A, A^+) over O the notation $\operatorname{Spd}(A, A^+)$ is a shorthand for $\operatorname{Spa}(A, A^+)^\diamondsuit$.

Lemma 2.17. *Suppose that B is a π-adically complete flat and topologically of finite type O-algebra, let B' denote the integral closure of B in $B[\pi^{-1}]$. Then the generic fiber $\operatorname{Spd}(B[\pi^{-1}], B')$ is a dense open subset of $\operatorname{Spd}(B)$.*

Proof. Let $X = \operatorname{Spa}(B)$ with B given the π-adic topology. Let Y be the punctured open unit ball over X. That is, $Y = \{y \in \operatorname{Spa}(B[\![t]\!]) \mid |t|_y \neq 0\}$, where $B[\![t]\!]$ is endowed with the (π, t)-adic topology. The map $Y^\diamondsuit \to X^\diamondsuit$ is a v-cover so it is enough to prove $|Y_\eta^\diamondsuit|$ is dense in $|Y^\diamondsuit|$. Now, Y is the diamond associated to an analytic adic space so $|Y| = |Y^\diamondsuit|$ by [SW20, Proposition 10.3.7]. Let $\operatorname{Spa}(R, R^+) \subset Y$ be a non-empty affinoid rational subset (with (R, R^+) a complete Huber pair). Since $B[\![t]\!]$ is noetherian, flat over B, and rational localizations are flat for Huber pairs admitting a noetherian ring of definition, we can conclude that R is flat over O. Now $\operatorname{Spa}(R, R^+)$ is a pseudorigid space over $\operatorname{Spa}(O)$ in the sense of [Lou17], and thus in particular R is a Jacobson ring [Lou17, Proposition 3.3.(3), 4.6]. By flatness of R over O we get that π is not nilpotent in R. There is a maximal ideal $\mathfrak{m} \subset R$ with $\pi \notin \mathfrak{m}$ as R is a Jacobson ring. By [Hub94, Lemma 1.4] there is an element $x \in \operatorname{Spa}(R, R^+)$ whose support ideal is $\mathfrak{m}$. In particular, this point lies in $\operatorname{Spa}(R, R^+) \cap Y_\eta \neq \emptyset$, which finishes the proof. □

The following consequence is the main statement we need from this chapter.

Proposition 2.18. (1) *Let X be a proper, flat scheme over O. Then the absolute weak normalization $X^{\mathrm{awn}} \to \operatorname{Spec}(O)$ is proper and flat, and the canonical morphism*

$$(X^{\mathrm{awn}})^\diamondsuit \to X^\diamondsuit \tag{2.16}$$

is an isomorphism

(2) *The functor $X \mapsto X^{\diamondsuit}$ is fully faithful when restricted to proper, flat and absolutely weakly normal schemes over O.*

Proof. Using Theorem 2.16 and Grothendieck's existence theorem as explained before there remain two statements to check: firstly that $X^{\mathrm{awn}} \to \mathrm{Spec}(O)$ is locally of finite type, and secondly that π-adic completion preserves absolute weak normality of O-algebras of finite type. The first follows from the fact that X is excellent (implying finiteness of the normalization of the reduction of X), and that the absolute weak normalization of an integral domain with field of fraction of characteristic 0 embeds into its normalization. The second follows by stability of absolute weak normality under regular ring homomorphisms, see [GT80, Proposition 5.1] and [Man80, Proposition III.3]. □

2.3. **π-adic kimberlites.** As in Section 2.2, we let O be a complete discrete valuation ring, which is flat over $\mathbb{Z}_p$, with perfect residue field k (of characteristic p) and uniformizer $\pi \in O$. We let F denote its fraction field and C a completed algebraic closure of F.

In [Gle20], the second named author introduced a set of axioms for a v-sheaf to have a well behaved specialization map to its reduced locus. The v-sheaves satisfying these axioms are called kimberlites [Gle20, Definition 1.4.16] and they mimic the behavior of formal schemes. Actually (under the very mild conditions of being separated and locally admitting a finitely generated ideal of definition), the v-sheaves associated to a formal scheme are always kimberlites [Gle20, Proposition 1.4.23] and the specialization map of the kimberlite attached to the formal scheme agrees with the traditional one.

On the other hand, in [Lou17] the third named author considers the functor from the category of formal schemes X over O to the category $\mathcal{C}$ of specialization triples $(X_\eta, X_s, \mathrm{sp})$ where X_η is a rigid analytic space over F, X_s is a scheme over k and $\mathrm{sp}\colon |X_\eta| \to |X_s|$ is a continuous map. This functor turns out to be fully faithful when one restricts to X locally formally of finite type[8], normal and flat over O, see [SW20, 18.4.2].

In this section we take this approach to study π-adic kimberlites. That is, to a π-adic kimberlite X over $\mathrm{Spd}\, O$ we attach a specialization triple $(X_\eta, X_s, \mathrm{sp})$ where now X_η a diamond over $\mathrm{Spd}(F)$, X_s a perfect scheme over $\mathrm{Spec}(k)$ and $\mathrm{sp}\colon |X_\eta| \to |X_s|$ a continuous map. More importantly, we discuss some conditions on X that make this functor fully faithful.

We start by giving a small review of the theory of specialization for kimberlites. Set $\mathrm{SchPerf}_k$ as the v-site of perfect schemes over k[9], and $\widetilde{\mathrm{SchPerf}}_k$ the associated topos.

Definition 2.19 ([Gle20, Definition 1.3.13])**.** Given a v-sheaf X on $\mathrm{Perf}_{\mathbb{F}_p}$ over $\mathrm{Spd}\, O$, we define X^{red} as the functor on $\mathrm{SchPerf}_k$ given by $Y \mapsto \mathrm{Hom}(Y^{\diamond}, X)$.

Thus, if $Y = \mathrm{Spec}(A)$ is an affine perfect scheme, then $X^{\mathrm{red}}(\mathrm{Spec}(A)) = X(\mathrm{Spd}(A))$. By [Gle20, Proposition 1.3.8], X^{red} is in fact a small v-sheaf on $\mathrm{SchPerf}_k$. The functor $(\text{-})^{\diamond}\colon \mathrm{Spec}(A) \mapsto \mathrm{Spd}(A)$ extends to small scheme-theoretic v-sheaves and the pair $(\diamond, (\text{-})^{\mathrm{red}})$ forms an adjunction, see [Gle20, Definition 1.3.13].

For formal schemes over O, the reduction functor is simply the functor that assigns the perfection of the reduced locus [Gle20, Proposition 1.3.20]. More precisely, if (B, B) is a formal Huber pair over O, that is B is a complete I-adic O-algebra (with I finitely generated), then $\mathrm{Spd}(B)^{\mathrm{red}} = \mathrm{Spec}(B/I)^{\mathrm{perf}}$.

Definition 2.20. (1) A map of v-sheaves $X \to Y$ is said to be formally adic if the following diagram is Cartesian:

$$\begin{tikzcd} (X^{\mathrm{red}})^{\diamond} \arrow[r] \arrow[d] & X \arrow[d] \\ (Y^{\mathrm{red}})^{\diamond} \arrow[r] & Y \end{tikzcd}$$

(2) We say that a v-sheaf over $\mathrm{Spd}\, O$ is π-adic if the structure morphism $X \to \mathrm{Spd}\, O$ is formally adic.

If $\mathrm{Spa}(A, A^+)$ is an affinoid adic space, we let $\mathrm{Spd}(A, A^+)$ denote the associated v-sheaf given by homomorphisms to untilts, see [SW20, Subsection 18.1]. If $A = A^+$, we abbreviate this by $\mathrm{Spd}(A)$.

Definition 2.21. Given a v-sheaf X, we say that a map $f\colon \mathrm{Spa}(R, R^+) \to X$ from an affinoid perfectoid space formalizes if it factors through a map $g\colon \mathrm{Spd}(R^+) \to X$. Any such g is called a formalization of f. We say that f v-formalizes if there is a v-cover $h\colon \mathrm{Spa}(S, S^+) \to \mathrm{Spa}(R, R^+)$ such that $f \circ h$ formalizes.

Proposition 2.22. *For a small v-sheaf X, the following are equivalent:*

(1) *There is a set I, a family of formal Huber pairs $(B_i, B_i)_{\{i\in I\}}$ and a v-cover*
$$\coprod_{i\in I} \mathrm{Spd}(B_i) \to X. \tag{2.17}$$

(2) *There is a set J, a family of perfectoid Huber pairs $(R_j, R_j^+)_{\{j\in J\}}$ and a v-cover*
$$\coprod_{j\in J} \mathrm{Spd}(R_j^+) \to X. \tag{2.18}$$

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[8] That is, locally of the form $O[\![T_1, \dots, T_n]\!]\langle X_1, \dots, X_m\rangle/I$ for some ideal I.

[9] Subject to the usual set-theoretic constraints of fixing some cut-off cardinal.

(3) *For any perfectoid Huber pair (R, R^+) all the maps $f\colon \mathrm{Spa}(R, R^+) \to X$ v-formalize.*

Proof. This is [Gle20, Lemma 1.4.8]. □

Any v-sheaf satisfying the conditions in Proposition 2.22 is said to be *v-locally formal* or alternatively *v-formalizing.*

Definition 2.23. A v-locally formal v-sheaf X separated over $\mathrm{Spd}\, O$ is a *π-adic kimberlite* if the structure map $X \to \mathrm{Spd}\, O$ is formally adic, the generic fiber X_η is a locally spatial diamond and if X^{red} is represented on $\mathrm{SchPerf}_k$ by a perfect scheme.

The more general definition of a kimberlite is given in [Gle20, Definition 1.4.18], and we justify below why π-adic kimberlites are a special type of kimberlite. For this reason, in our context, we can take Definition 2.23 as our definition.

Proposition 2.24. *A v-sheaf X equipped with a separated morphism $X \to \mathrm{Spd}\, O$ is a π-adic kimberlite if and only if X is a kimberlite and the map $X \to \mathrm{Spd}\, O$ is formally adic.*

Proof. Formal adicness implies that $X^{\mathrm{an}} = X_\eta$ and $(X^{\mathrm{red}})^\diamond = X \times_{\mathrm{Spd}\, O} \mathrm{Spd}\, k$. From this it is clear how one definition translates to the other except that to prove X is a kimberlite we need to justify why it is formally separated. Now, the argument given in [Gle20, Proposition 1.3.31] applies with the role of $\mathbb{Z}_p$ exchanged for O. □

If $f\colon S \to T$ is a map of locally spectral spaces, then we call f spectral if for any quasi-compact open $U \subset S, V \subset T$ with $f(U) \subset V$ the induced map $f\colon U \to V$ of spectral spaces is spectral, that is, quasi-compact.

Proposition 2.25. *If X is a π-adic kimberlite, then there is a continuous and spectral specialization map* $\mathrm{sp}_X\colon |X_\eta| \to |X^{\mathrm{red}}|$. *Moreover,*

$$X \mapsto (X_\eta, X^{\mathrm{red}}, \mathrm{sp}_X) \tag{2.19}$$

is functorial when X varies along π-adic kimberlites.

By construction, the specialization map is the restriction to $|X_\eta|$ of a continuous map $|X| \to |X^{\mathrm{red}}|$.

Proof. This is [Gle20, Proposition 1.4.20] specialized to the π-adic case considered here. □

One of the main features of kimberlites is that, as with formal schemes, they come with a notion of tubular neighborhoods (or completion at a point).

Definition 2.26 ([Gle20, 1.4.24])**.** Given a π-adic kimberlite X and a locally closed subset $S \subset |X^{\mathrm{red}}|$, we define $\widehat{X}_{/S}$ as the v-sheaf making the following diagram Cartesian

$$\begin{array}{ccc} \widehat{X}_{/S} & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ \underline{|S|} & \longrightarrow & \underline{|X^{\mathrm{red}}|} \end{array}$$

Here, the right vertical arrow is the composition of the natural map $X \to \underline{|X|}$ and the map $\underline{|X|} \to \underline{|X^{\mathrm{red}}|}$ mentioned in Proposition 2.25. We will mostly use tubular neighborhoods when $S = \{x\}$ is a closed (and constructible) point in X^{red}.

Remark 2.27. In general,

$$|S^\diamond| \subset |\widehat{X}_{/S}| \subset \mathrm{sp}_X^{-1}(S), \tag{2.20}$$

but more often than not neither of the equalities hold.

Example 2.28. Let $X = \mathrm{Spd}(A)$ with A a perfect k-algebra and let $S \subset \mathrm{Spec}(A) = X^{\mathrm{red}}$ the Zariski closed subset defined by a finitely generated ideal $I \subset A$ with generators $a_1, \dots, a_n$. Then $S^\diamond$ is the locus in $\mathrm{Spd}(A)$ where $a_1 = \cdots = a_n = 0$, $\widehat{X}_{/S}$ is the (open) locus in $\mathrm{Spd}(A, A)$ where $a_1, \dots, a_n$ are all topologically nilpotent and $\mathrm{sp}_X^{-1}(S)$ is the closed subset of points for which $|a_i| < 1$. With this description it is immediate to verify the containment of eq. (2.20). Now, the complement $\mathrm{sp}_X^{-1}(S) \setminus |\widehat{X}_{/S}|$ consists of those higher rank points $(A, A) \to (C, C^+)$, for which at least one of $a_i^{-1} \in C^\circ \setminus C^+$. Note the associated point $(A, A) \to (C, C^\circ)$ is not in $\mathrm{sp}_X^{-1}(S)$. In particular, $\mathrm{sp}_X^{-1}(S)$ is usually not weakly generalizing and does not define a closed subsheaf.

Proposition 2.29. *If $S \subset |X^{\mathrm{red}}|$ is locally closed and constructible then $\widehat{X}_{/S} \to X$ is an open immersion.*

Proof. This is proved in [Gle20, Proposition 1.4.29]. □

We now introduce a weak form of flatness over O for π-adic kimberlites.

Definition 2.30. A π-adic kimberlite X over $\operatorname{Spd} O$ is said to be *flat* if there is a set I, a family of F-perfectoid Huber pairs $\{(R_i^\sharp, R_i^{\sharp+})\}_{i\in I}$ and a v-cover over $\operatorname{Spd} O$

$$\coprod_{i\in I} \operatorname{Spd}(R_i^{\sharp+}) \to X. \tag{2.21}$$

We now construct our first examples of flat π-adic kimberlites.

Proposition 2.31. *Let $f\colon A \to B$ be a map of complete π-adic algebras that are flat over O. Suppose that A is integrally closed in $A[\pi^{-1}]$ and that $\operatorname{Spd}(B[\pi^{-1}], B) \to \operatorname{Spd}(A[\pi^{-1}], A)$ is a v-cover. Then $\operatorname{Spd}(B) \to \operatorname{Spd}(A)$ is also a v-cover. In particular, for any such A the v-sheaf $\operatorname{Spd}(A)$ is a flat π-adic kimberlite.*

Proof. By [Gle20, Lemma 1.2.26], the map $\operatorname{Spd}(B) \to \operatorname{Spd}(A)$ is quasi-compact, so it is enough to prove $|\operatorname{Spd}(B)| \to |\operatorname{Spd}(A)|$ is surjective by [Sch17, Lemma 12.11]. Surjectivity on the generic fiber follows from the hypothesis. On the special fiber, we use [Gle20, Lemma 1.3.5, Proposition 1.3.8] to prove instead that the map $\operatorname{Spa}(B/\pi) \to \operatorname{Spa}(A/\pi)$ is surjective.

Let $x \in \operatorname{Spa}(A/\pi)$ and let $\operatorname{Spa}(k(x), k(x)^+) \to \operatorname{Spa}(A/\pi)$ the affinoid residue field map. Let $\mathfrak{p}_x \in \operatorname{Spec}(A/\pi)$ denote the support ideal of x. Since A is integrally closed in $A[\pi^{-1}]$, the pair $(A[\pi^{-1}], A)$ is a complete Tate Huber pair and we have a surjective specialization map $\operatorname{sp}_A\colon \operatorname{Spa}(A[\pi^{-1}], A) \to \operatorname{Spec}(A/\pi)$ by [Gle20, Proposition 1.4.2], [Bha17, Remark 7.4.12]. Let $y \in \operatorname{Spa}(A[\pi^{-1}], A)$ with $\operatorname{sp}_A(y) = \mathfrak{p}_x$. We obtain a map $\operatorname{Spa}(R[p^{-1}], R) \to \operatorname{Spa}(A[\pi^{-1}], A)$ with $R := k(y)^+$. The residue field of R is $k(x)$ and we can consider $R^+ \subset R$ defined as $R \times_{k(x)} k(x)^+$. This promotes to a map $\operatorname{Spa}(R^+) \to \operatorname{Spa}(A)$. As $\operatorname{Spd}(B[\pi^{-1}], B) \to \operatorname{Spd}(A[\pi^{-1}], A)$ is a v-cover we can find a v-cover of $\operatorname{Spa}(C, C^+) \to \operatorname{Spa}(R[\pi^{-1}], R^+)$ with (C, C^+) a perfectoid field and a commutative diagram

$$\begin{tikzcd} \operatorname{Spa}(C^+) \arrow[r] \arrow[d] & \operatorname{Spa}(R^+) \arrow[d] \\ \operatorname{Spa}(B) \arrow[r] & \operatorname{Spa}(A). \end{tikzcd}$$

The map $\operatorname{Spa}(C^+) \to \operatorname{Spa}(R^+)$ is easily seen to be surjective since it is an extension of valuation rings. So x lies in the image of $\operatorname{Spa}(B/\pi)$ as we needed to show.

That $\operatorname{Spd}(A)$ is a kimberlite for A as above follows from [Gle20, Proposition 1.4.23]. Now, we may always find a cover by an affinoid perfectoid space $\operatorname{Spd}(P, P^+) \to \operatorname{Spd}(A[p^{-1}], A)$ by [Sch17, Lemma 15.3]. What we have shown so far implies that $\operatorname{Spd}(P^+) \to \operatorname{Spd}(A)$ is also a v-cover. This finishes the proof. □

Proposition 2.32. *If A is the π-adic completion of a flat and finite type algebra over O, then $\operatorname{Spd}(A)$ is a flat π-adic kimberlite.*

Proof. We may assume that A is reduced as passing to the absolute weak normalization does not change $\operatorname{Spd}(A)$ by Theorem 2.16 and Proposition 2.18. As A is noetherian and quasi-excellent, the integral closure of A in its total ring of fractions is therefore a finite A-module. In particular, the integral closure A' of A in $A[p^{-1}]$ is finite over A. Thus, we can conclude that $\operatorname{Spd}(A')$ (with A' given the π-adic topology) is flat by Proposition 2.31 and the map $\operatorname{Spd}(A') \to \operatorname{Spd}(A)$ is a v-cover since it is isomorphism over $\operatorname{Spd}(F)$ (this uses that the π-adic topology on A' agrees with the subspace topology on $A[\pi^{-1}]$) and the map $\operatorname{Spec}(A'/\pi) \to \operatorname{Spec}(A/\pi)$ is proper and surjective (here we use again [Gle20, Lemma 1.3.5, Proposition 1.3.8] as in Proposition 2.31). □

Remark 2.33. A careful inspection of the proof of Proposition 2.31 above allows us to conclude that a π-adic formal Huber pair (A, A) will give rise to a flat π-adic kimberlite $\operatorname{Spd}(A)$ if and only if the specialization map

$$\operatorname{sp}_A\colon \{x \in \operatorname{Spa}(A) \mid |\pi|_x \neq 0\} \subset \operatorname{Spd}(A) \to \operatorname{Spec}(A/\pi) \tag{2.22}$$

is surjective. The hypothesis taken in Proposition 2.31 are easy to verify assumptions that ensure this happens. Without assuming flatness of A, this might not hold since for a discrete and perfect O-algebra A in characteristic p, the v-sheaf $\operatorname{Spd}(A)$ is a π-adic kimberlite that is not flat.

We can relate flatness for π-adic kimberlites to surjectivity of the specialization map.

Proposition 2.34. *Let X be a π-adic kimberlite over $\operatorname{Spd} O$.*

(1) *If X is flat, then the specialization map $\operatorname{sp}\colon |X_\eta| \to |X^{\mathrm{red}}|$ is surjective.*
(2) *Conversely, if $X \to \operatorname{Spd} O$ is proper and $\operatorname{sp}\colon |X_\eta| \to |X^{\mathrm{red}}|$ surjective, then X is flat over $\operatorname{Spd} O$.*

Proof. The first statement reduces to the case $X = \operatorname{Spd}(R_i^{\sharp+})$ for $(R_i^\sharp, R_i^{\sharp+})$ a perfectoid Huber pair over F, where it follows from [Gle20, Proposition 1.4.2]. Let us prove the second. It follows from the hypothesis that X_η is quasi-compact over $\operatorname{Spd} F$, and thus we may find a v-cover $\operatorname{Spa}(R, R^+) \to X_\eta$ by affinoid perfectoid. Refining the cover if necessary we may assume this map factors through a map $\operatorname{Spd}(R^+) \to X$ because X is v-formalizing. Since X is quasi-separated over $\operatorname{Spd} O$ and $\operatorname{Spd}(R^+)$ is quasi-compact over $\operatorname{Spd} O$ (see [Gle20, Lemma 1.2.26]), we may conclude that $\operatorname{Spd}(R^+)$ is quasi-compact over X. To prove it is a v-cover, it is therefore enough to prove that the map of topological spaces is surjective. On the generic fiber this is clear. Using [Gle20, Lemma 1.3.5, Proposition 1.3.8], we need to show $\mathrm{Spec}((R^+/\pi)^{\mathrm{perf}}) \to X^{\mathrm{red}}$ is a scheme-theoretic v-cover, or equivalently that the map of the associated adic spectra induced by the morphism of schemes is surjective.

The proof now follows a similar argument to the one given in Proposition 2.31. Given a point $x \in |(X^{\mathrm{red}})^{\mathrm{ad}}|$ in the adic spectrum of X with affinoid residue field $\mathrm{Spa}(k(x), k(x)^+)$ we consider the point in $\mathfrak{p}_x \in |X^{\mathrm{red}}|$ corresponding to the support of x. By surjectivity of the specialization map there is a point $y \in |X_\eta|$ with $\mathrm{sp}_X(y) = \mathfrak{p}_x$. Represent y by a map $\mathrm{Spa}(C, C^+) \to X_\eta$ with (C, C^+) a perfectoid affinoid field over F. Replacing $\mathrm{Spd}(C, C^+)$ by a v-cover we may assume this map factors over a map $\mathrm{Spd}(C^+, C^+) \to X$. In particular, it promotes to a map $\mathrm{Spd}(C^+) \to X$. The closed point of $\mathrm{Spd}(C, C^+)$ specializes to a point with the same support as x. Let $\kappa(y)$ be the residue field of C^+. Then $\kappa(y)$ is a field extension of $k(x)$, and we can find a valuation ring $\kappa(y)^+ \subset \kappa(y)$ making $\kappa(y)^+/k(x)^+$ an extension of valuation rings. By pullback along the surjection $C^+ \twoheadrightarrow \kappa(y)$ we may construct from $\kappa(y)^+$ an open and bounded valuation $C_1^+ \subset C^+$. Since X_η is partially proper we may extend $\mathrm{Spd}(C, C^+)$ to a map $\mathrm{Spd}(C^+, C_1^+) \to X_\eta$. After possibly replacing $\mathrm{Spa}(C, C_1^+)$ by a v-cover, we may assume it factors through $\mathrm{Spa}(R, R^+)$. Then the map extends to $\mathrm{Spd}(C_1^+) \to \mathrm{Spd}(R^+) \to X$. The map of adic spectra $\mathrm{Spec}((C_1^+/\pi)^{\mathrm{perf}})^{\mathrm{ad}} = \mathrm{Spa}((C_1^+/\pi)^{\mathrm{perf}}, (C_1^+/\pi)^{\mathrm{perf}}) \to (X^{\mathrm{red}})^{\mathrm{ad}}$ has x in its image as we wanted to show. $\square$

We now discuss some ad hoc hypothesis on π-adic kimberlites that allow us to recover them from their specialization triple.

Definition 2.35. We let $\mathcal{K}$ denote the category of flat π-adic kimberlites X that are quasi-compact and separated over $\mathrm{Spd}\, O$ and satisfy the following properties:

1. The $\mathrm{Spd}\, C$-valued points of X define a dense subset of $|X_C|$.
2. The reduction X^{red} is a perfect k-scheme perfectly of finite type.
3. Every section $\mathrm{Spd}\, C \to X_C$ formalizes to a map $\mathrm{Spd}\, O_C \to X_{O_C}$.

Our main theorem about the category $\mathcal{K}$ is the following.

Theorem 2.36. *When restricted to the category $\mathcal{K}$ of Definition 2.35, the functor sending a π-adic kimberlite to its generic fiber is faithful and the functor that sends it to its specialization triple*

$$X \mapsto (X_\eta, X^{\mathrm{red}}, \mathrm{sp}_X) \tag{2.23}$$

is fully faithful.

Proof. Let us prove faithfulness. Let $f, g \colon X \to Y$ be two maps such that $f_\eta = g_\eta$. Since X is flat and quasi-compact we may replace it by a cover of the form $\mathrm{Spd}(R^+)$. Since Y is separated and π-adic the map $\Delta \colon Y \to Y \times_{\mathrm{Spd}\, O} Y$ is formally adic and a closed immersion. The pullback of Δ by (f, g) is closed and formally adic subsheaf of $\mathrm{Spd}(R^+)$ with the same generic fiber. We may finish by arguing as in the proof of [Gle20, Proposition 1.4.10].

Let us prove the map is full. Fix a map $f := (f_\eta, f^{\mathrm{red}})$ of triples

$$f \colon (X_\eta, X^{\mathrm{red}}, \mathrm{sp}_X) \to (Y_\eta, Y^{\mathrm{red}}, \mathrm{sp}_Y) \tag{2.24}$$

and let $W = X \times_{\mathrm{Spd}\, O} Y$. Let $g \colon \mathrm{Spa}(R, R^+) \to X_\eta$ be a formalizable v-cover which extends to a surjection $\mathrm{Spd}(R^+) \to X$ and for which $f \circ g$ is also formalizable (this is possible using Proposition 2.31). Let $(g, f \circ g) \colon \mathrm{Spd}(R^+) \to W$ be the induced map and define Z as the sheaf-theoretic image of $(g, f \circ g)$ in W. We have a projection map $Z \to X$ and we wish to prove that it is an isomorphism. Observe that the graph morphism $(\mathrm{id}, f_\eta) \colon X_\eta \to W_\eta$ already identifies X_η with Z_η. In particular, $Z(C)$ is dense inside $|Z_C|$ by our assumption on X.

By construction Z is v-locally formal since $\mathrm{Spd}(R^+, R^+)$ surjects onto it. Moreover, since $Z \subset W$ and W is separated over O, we see that Z is also separated over O. Let us prove that Z if formally π-adic and that Z^{red} is isomorphic to X^{red}.

We claim that $Z_s \subset W_s = (W^{\mathrm{red}})^\diamond$ factors through the graph of $(f^{\mathrm{red}})^\diamond$. Indeed, since X_{O_C} and Y_{O_C} formalize C-sections, for any map $q \colon \mathrm{Spd}\, C \to Z_\eta$ we obtain maps $q_x^{\mathrm{red}} \colon \mathrm{Spec}(\bar{k}) \to X^{\mathrm{red}}$ and $q_y^{\mathrm{red}} \colon \mathrm{Spec}(\bar{k}) \to Y^{\mathrm{red}}$ intertwinned under f^{red} (because $|f^{\mathrm{red}}| \circ \mathrm{sp}_X = \mathrm{sp}_Y \circ |f_\eta|$ by assumption), in other words $\mathrm{sp}_W(q) \in \Gamma(f^{\mathrm{red}})$. In particular, $\mathrm{sp}_W(|Z_\eta|) \subset \overline{\mathrm{sp}_W(Z(C))} \subset |W^{\mathrm{red}}|$ is contained in $\Gamma(f^{\mathrm{red}})$. By [Gle20, Proposition 1.4.2], we know that the specialization map $\mathrm{Spd}(R, R^+) \to \mathrm{Spec}((R^+/\pi)^{\mathrm{perf}})$ is surjective. Because $(g, f_\eta \circ g) \colon \mathrm{Spd}(R, R^+) \to W$ has image $|Z_\eta|$ (on topological spaces), this implies by naturality of the specialization map that the morphism $g^{\mathrm{red}} \colon \mathrm{Spec}(R^+/\pi)^{\mathrm{perf}} \to W^{\mathrm{red}}$ factors through $\Gamma(f^{\mathrm{red}})$ as well. Consequently, $Z_s \to W_s$ factors through $\Gamma(f^{\mathrm{red}})^\diamond$. On the other hand, since $\mathrm{Spd}(R^+) \to X$ is surjective the projection map

$$(\mathrm{Spec}(R^+/p)^{\mathrm{perf}})^\diamond \to (X^{\mathrm{red}})^\diamond \tag{2.25}$$

is a surjection. This implies that the morphism $\mathrm{Spec}((R^+/\pi)^{\mathrm{perf}}) \to W_s$ surjects onto $\Gamma(f^{\mathrm{red}})$, and this in turn implies that $Z_s \to \Gamma(f^{\mathrm{red}})^\diamond$ is an isomorphism, as it is a monomorphism and surjective. In particular, we get that $Z_s \cong (Z^{\mathrm{red}})^\diamond$, that is, Z is formally π-adic.

As we have seen the map $Z \to X$ is an isomorphism on the generic fiber and on the special fiber. Since $\mathrm{Spd}(R^+) \to Z$ is surjective Z is quasi-compact over $\mathrm{Spd}\, O$, which is enough to conclude $Z \to X$ is an isomorphism (by [Sch17, Lemma 12.5], note that $Z \to X$ is quasi-compact, as X is qcqs over $\mathrm{Spd}\, O$). $\square$

It is also relevant to relate this to a notion of topological flatness that appears in [PR21].

Lemma 2.37. *Let X be a proper π-adic kimberlite over* $\mathrm{Spd}\, O$ *satisfying conditions (1)-(3) of Definition 2.35. If $|X_\eta|$ is a dense open*[10] *of $|X|$, then X is flat, thus lies in $\mathcal{K}$.*

Proof. By [Gle20, Proposition 1.4.20], the specialization map is a spectral map of spectral spaces, and by [Gle20, Lemma 1.4.43], the map is specializing. In particular, it sends closed subsets to closed subsets. Since X^{red} is perfectly of finite type, it suffices to prove surjectivity of the map $X(\mathrm{Spd}\, C) \to X^{\mathrm{red}}(\bar{k})$ induced by sp.

For this, take the associated tubular neighborhood $\widehat{X}_{/x}$ over a closed point x of the reduction, which can be represented by $\mathrm{Spec}\bar{k} \to X^{\mathrm{red}}$ uniquely up to Galois automorphisms. It is a non-empty open by [Gle20, Proposition 1.4.29]. Hence, it must have topologically dense generic fiber, which is in particular non-empty. By hypothesis, we can find a C-valued point mapping to x. □

The following statement gives a v-sheaf theoretic criterion to determine when a weakly normal scheme is already normal.

Proposition 2.38. *Let A be a flat, weakly normal and topologically of finite type π-adically complete domain over O. Suppose that $A[\pi^{-1}]$ is normal and that, for every closed point $x \in \mathrm{Spec}(A/\pi)$, the diamond $(\widehat{\mathrm{Spd}(A)}_{/x})_\eta$ is connected* [11]*. Then A is normal.*

Proof. Let B denote the integral closure of A in $A[\pi^{-1}]$. Since $A[\pi^{-1}]$ is normal, B is also normal and B is a finite A-algebra. We claim that $f\colon \mathrm{Spd}(B) \to \mathrm{Spd}(A)$ is an isomorphism, so that $A = B$ by Theorem 2.16. By quasi-compactness, it is enough to check this on the generic and special fibers. The generic case follows from the definition of B. We need to prove $\mathrm{Spec}(B/\pi)^{\mathrm{perf}} \cong \mathrm{Spec}(A/\pi)^{\mathrm{perf}}$ which amounts to proving that the fibers at closed points consists of singletons. Let $x \in \mathrm{Spec}(A/\pi)$ denote a closed point. By [Gle20, Proposition 1.4.26], we have an identification

$$(\widehat{\mathrm{Spd}(B)}_{/f^{-1}(x)})_\eta \cong (\widehat{\mathrm{Spd}(A)}_{/x})_\eta \tag{2.26}$$

In turn we also have $\coprod_{y\in f^{-1}(x)} (\widehat{\mathrm{Spd}(B)}_{/y})_\eta \cong (\widehat{\mathrm{Spd}(B)}_{/f^{-1}(x)})_\eta$. By Proposition 2.32 and Proposition 2.34 for all $y \in f^{-1}(x)$ the tubular neighborhood $(\widehat{\mathrm{Spd}(B)}_{/y})_\eta$ is a non-empty open subset of $(\widehat{\mathrm{Spd}(A)}_{/x})_\eta$. Since we assumed this to be connected we can conclude $f^{-1}(x)$ contains a unique element. □

3. The affine flag variety

In this section, we discuss some relevant material on perfect schemes and Witt vector affine flag varieties. Namely, we review the calculation of the Picard group by He–Zhou [HZ20], the definition of canonical finite type deperfections of Schubert perfect schemes and apply a Stein factorization argument to construct a comparison isomorphism between the p-adic canonical deperfections of depth 0 Schubert perfect schemes with the corresponding weakly normal Schubert schemes in equicharacteristic. In particular, we prove [Zhu17b, Conjecture III] on their singularities in this case.

3.1. **Perfect schemes.** Here, we present some facts on perfect schemes that we will need later. Let p be a prime number. All our schemes in this subsection will be assumed to lie over $\mathbb{F}_p$.

The basic theory of perfect schemes is discussed in [Zhu17a, A.] and [BS17, Section 3]. In particular, we will use the notions of a perfectly finitely presented map between qcqs perfect schemes [BS17, Proposition 3.11] of a perfectly proper morphism [BS17, Defintion 3.14], [Zhu17a, Appendix A.18] and, if k is a perfect field, we occasionally call a separated, perfectly finitely presented scheme X over k a perfect k-variety [Zhu17a, Remark A.14].

A morphism $Y \to X$ of perfect schemes is called perfectly smooth if, étale locally on Y, there exists étale morphisms to the perfection of some relative affine space over X, see [Zhu17a, Definition A.25].

Given any normal finite type k-scheme Y, its perfection Y_{perf} is normal as it is a filtered colimit of normal schemes along affine transition maps. Conversely, if X is a qcqs normal perfect scheme perfectly of finite type, then using [Sta21, Tag 01ZA] (and finiteness for integral closures of schemes of finite type over a field), then we can write X as the filtered colimit of perfections of normal schemes $Y_i, i \in I$, which are of finite type over k.

The following result gives a topological criterion for normality of perfect schemes. We stress that perfectness is crucial as one sees, for example, by looking at the normalization morphisms of the cuspidal curve.

Lemma 3.1. *Let $f\colon Y \to X$ be a surjective, perfectly proper morphism between qcqs integral perfect schemes. Assume that Y is normal and f birational. Then X is normal if and only if the geometric fibers of f are connected.*

[10]The converse, however, fails. Indeed, let $O\langle t\rangle \subset V$ be a higher rank valuation ring endowed with its π-adic topology. Then $\mathrm{Spd}(V)$ is a flat π-adic kimberlite. Using the olivine spectrum [Gle20, Definition 1.2.1, Proposition 1.2.17], one can prove that the locus $N_{t\ll 1}$ where t is topologically nilpotent is an open of $|\mathrm{Spd}(V)|$ that does not meet the generic fiber.

[11]Instead of tubular neighborhoods, one could use formal completions at closed points to get a more classical formulation.

Proof. If all geometric fibers of f are connected, then the natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is an isomorphism, see [BS17, Proposition 6.1], [Zhu17a, Lemma A.21]. Thus,

$$\mathcal{O}_X(U) \cong \mathcal{O}_Y(f^{-1}(U)),$$

for any open affine $U \subset X$. As $\mathcal{O}_Y(V)$ is a normal ring for any open subset $V \subset Y$, the claim follows (here we use that Y is integral [Sta21, Tag 0358]).

Conversely, we can write f as the perfection of a proper, finitely presented morphism $f_0 \colon Y_0 \to X$ by [BS17, Proposition 3.13, Corollary 3.15]. Let $g_0 \colon Y_0 \to Z_0 = \underline{\mathrm{Spec}}((f_0)_*(\mathcal{O}_{Y_0}))$ be the Stein factorization of f_0, see [Sta21, Tag 03H2]. Perfecting again, we get a factorization $f = h \circ g$ with $g \colon Y \to Z := (Z_0)_{\mathrm{perf}}$ having connected geometric fibers, and $h \colon Z \to X$ an integral, dominant morphism of integral schemes inducing an isomorphism at generic points (because f is birational). As X is normal we obtain that $X \cong Z$, which implies the claim. □

We now turn to Picard groups of perfect schemes. Given any qcqs perfect k-scheme X, we have $\mathrm{Pic}(X) \cong \mathrm{Pic}(X_0)[1/p]$ for any preferred choice of finite type deperfection X_0, cf. [BS17, Lemma 3.5]. In particular, the Picard groups of perfect schemes are always uniquely p-divisible.

If X is perfectly finitely presented over some perfect field k and $X_0 \to \mathrm{Spec}(k)$ a finitely presented model for X, then the localized Weil divisor class group $\mathrm{Cl}(X_0)[1/p]$ only depends on X and not on X_0, and we set

$$\mathrm{Cl}(X) := \mathrm{Cl}(X_0)[1/p]. \tag{3.1}$$

If X is normal, then by [Sta21, 0BE8] (and passage to the limit over Frobenius for some normal model) there exists a natural, injective map

$$\mathrm{Pic}(X) \hookrightarrow \mathrm{Cl}(X). \tag{3.2}$$

Let us recall that a line bundle on a (qcqs) scheme is semi-ample if some positive power of it is globally generated.

Proposition 3.2. *Let X be a perfectly proper perfect k-scheme and $\mathcal{L}$ be a semi-ample line bundle on X. There is a unique perfectly proper surjection $X \to Y$ of perfect k-schemes with connected geometric fibers such that all sufficiently divisible powers of $\mathcal{L}$ descend uniquely to ample line bundles on Y.*

Proof. By semi-ampleness of $\mathcal{L}$, we can take X_0 to be a finite type deperfection of X over k, and let $\mathcal{L}_0$ be a base point free line bundle on X_0 whose pullback to X is a power of $\mathcal{L}$. Let Y_0 be the Stein factorization of the canonical morphism

$$X_0 \to Z_0 \subset \mathbb{P}(\Gamma(X_0, \mathcal{L}_0)), \tag{3.3}$$

where Z_0 is the (scheme-theoretic) image of X_0. Clearly, $\mathcal{L}_0$ descends by construction to an ample line bundle on Y_0, pulling back $\mathcal{O}(1)$ on the right side of (3.3). After taking perfections, we get $X \to Y$ with the desired properties (see [BS17, Proposition 6.1] for unique descent of line bundles).

In order to prove uniqueness of Y, we proceed as in [BS17, Proof of Theorem 8.3]. The morphism $X \to Y$ is a v-cover (by properness), hence Y is determined by the closed subscheme $X \times_Y X \subset X \times_{\mathrm{Spec}(k)} X$. To identify this closed (and necessarily reduced) subscheme it suffices to identify the geometric fibers of the map $X \to Y$ in terms of $\mathcal{L}$, and we only have to argue on k-valued points as these are dense inside $X \times_Y X$. We claim that two k-rational points of X lie in the same fiber over Y if and only if both points can be linked by a chain of closed integral perfect k-curves C, such that the restriction $\mathcal{L}|_C$ is torsion in $\mathrm{Pic}(C)$.

By connectedness of the fibers and the definition of $X \to Y$, every two points in it can be linked by such a chain of integral perfect k-curves C on which $\mathcal{L}$ is torsion. Conversely, given an integral perfect k-curve $C \subset X$ whose image in Y is not a point, all sufficiently large powers of $\mathcal{L}$ restrict to an ample line bundle on C. Indeed, after passing to a finitely presented deperfection of C over k the morphism $C_0 \to Y$ is finite and pullback of ample line bundles along affine morphisms are ample. □

Next, we discuss finite type deperfections. Let k be a perfect field and let X be a qcqs perfect k-scheme of perfectly finite presentation. For each of the finitely many generic points $\eta \in X$, fix a subfield $k(\eta_0) \subset k(\eta)$, which is finitely generated over k and has perfection $k(\eta)$. Then there exists a unique (up to unique isomorphism) weakly normal[12] finite type k-scheme X_0 such that $(X_0)_{\mathrm{perf}} \cong X$ and for each generic point $\eta_0 \in |X_0| \cong |X|$ the function field of X_0 at η_0 identifies with $k(\eta_0)$, see [Zhu17a, Proposition A.15]. Note that X_0 also reflects normality of X, that is, X normal implies X_0 normal.

For group actions we can draw the following consequence.

Proposition 3.3. *Let G be an affine perfect k-group of perfectly finite presentation and X a qcqs perfect k-scheme of perfectly finite presentation equipped with a G-action with finitely many orbits.*

(1) *Any reduced deperfection G_0 of G is a smooth affine k-group.*
(2) *For such G_0, there are unique weakly normal deperfections X_0 with G_0-action, whose generic fixers are also smooth.*

[12] A finite type reduced k-scheme Y is called weakly normal if every finite birational universal homeomorphism $Z \to Y$ with Z reduced is an isomorphism.

Proof. The first item is [Zhu17a, Lemma A.26]. For the second item, we notice that X has a dense open subset U consisting of the disjoint union of its maximal orbits, cf. [Zhu17a, Proposition A.32]. Having constructed the unique deperfection U_0 with the desired properties, it has a unique extension to a deperfection X_0 of X by [Zhu17a, Lemma A.15]. Furthermore, the action map $G \times X \to X$ also deperfects, because it does so over a dense open (and X_0 is weakly normal).

Therefore, we may and do assume that $X = G/H$ is a single orbit around a certain k-valued point x. But then taking $H_0 \subset G_0$ to be the unique reduced closed subscheme whose perfection recovers $H \subset G$, we get a G_0-orbit $X_0 = G_0/H_0$ deperfecting X with smooth fixers. Uniqueness is clear. □

Proposition 3.3 will be useful for constructing finite type deperfections for Schubert varieties in Witt vector affine Grassmannians, see Section 3.3.

3.2. **Affine flag varieties.** We now study the geometry of Witt vector affine flag varieties. Assume that k is a perfect field of characteristic $p > 0$ and that F is a complete discretely valued field with residue field k and ring of integers O. Exceptionally, we allow $F \cong k(\!(\pi)\!)$ to be a Laurent series field, since it is needed in Section 3.3.

We denote by $\mathrm{Alg}_k^{\mathrm{perf}}$ the category of perfect k-algebras. For $R \in \mathrm{Alg}_k^{\mathrm{perf}}$, we denote by $W_O(R)$ the associated ring of O-Witt vectors, see [FF, Section 1.2.1]: if O is p-adic, then $W_O(R) = W(R) \otimes_{W(k)} O$; if $O \cong k[\![\pi]\!]$, then $W_O(R) = R\widehat{\otimes}_k O \cong R[\![\pi]\!]$.

Moreover, we fix a (connected) reductive F-group G and a parahoric O-model $\mathcal{G}$ in the sense of Bruhat–Tits. We note that, over the completion $\breve{F}$ of the maximal unramified extension of F, the group $G_{\breve{F}}$ is automatically quasi-split by Steinberg's theorem, see [Ser94, Chapitre III.2.3]. We let $\mathcal{G}_k = \mathcal{G} \otimes_O k$ be the special fiber of $\mathcal{G}$.

Recall the definition of the Witt vector affine flag variety associated to $\mathcal{G}$.

Definition 3.4. (1) The loop group of G is the functor

$$L_k G \colon \mathrm{Alg}_k^{\mathrm{perf}} \to (\mathrm{Sets}),\ R \mapsto G(W_O(R) \otimes_O F). \tag{3.4}$$

(2) The positive loop group of $\mathcal{G}$ is the functor

$$L_k^+ \mathcal{G} \colon \mathrm{Alg}_k^{\mathrm{perf}} \to (\mathrm{Sets}),\ R \mapsto \mathcal{G}(W_O(R)) \tag{3.5}$$

(3) The affine flag variety for $\mathcal{G}$ is the quotient (for the étale topology)

$$\mathcal{F}\ell_{\mathcal{G}} := L_k G / L_k^+ \mathcal{G}. \tag{3.6}$$

Because any $\mathcal{G}$-torsor on $W_O(R)$ can be trivialized over $W_O(R')$ for some with $R \to R'$ étale, the affine flag variety $\mathcal{F}\ell_{\mathcal{G}}$ is equivalently the functor on perfect k-algebras R that classifies $\mathcal{G}$-torsors $\mathcal{P}$ on $\mathrm{Spec}(W_O(R))$ together with a trivialization over $\mathrm{Spec}(W_O(R) \otimes_O F)$.

We have the following crucial representability result, see [BS17, Corollary 9.6].

Theorem 3.5 (Bhatt-Scholze)**.** *The functor $\mathcal{F}\ell_{\mathcal{G}}$ is representable by an ind-(perfectly projective) ind-(perfect k-scheme).*

Representability as an ind-(perfect algebraic space) was previously proved by Zhu, [Zhu17a], but is not sufficient for our purpose.

Fix an auxiliary maximal split F-torus A, a maximal $\breve{F}$-split F-torus $A \subset S \subset G$ whose connected Néron O-model $\mathcal{S}$ is contained in $\mathcal{G}$, see [BT84, Proposition 5.1.10]. Let $T \subset G$ be the centralizer of S, and let $\mathcal{T}$ be the connected Néron O-model of T. This yields the Iwahori–Weyl group

$$\tilde{W} := N_G(T)(\breve{F})/\mathcal{T}(\breve{O}) \tag{3.7}$$

associated with S, see [HR08, Definition 7]. By [HR08, Lemma 14], there exists a short exact sequence

$$1 \to W_{\mathrm{af}} \to \tilde{W} \to \pi_1(G)_I \to 1 \tag{3.8}$$

with $W_{\mathrm{af}} \subset \tilde{W}$ the affine Weyl group, I the absolute Galois group of $\breve{F}$, and $\pi_1(G)$ Borovoi's algebraic fundamental group of G. The choice of an alcove in the apartment for S yields a splitting $W_{\mathrm{af}} \rtimes \pi_1(G)_I$ of the sequence. By declaring the elements of $\pi_1(G)_I$ to have length 0 and to be pairwise incomparable, we can further extend the length function and the Bruhat partial order on the Coxeter group W_{af} to $\tilde{W}$.

By the Cartan decomposition, we may identify the double coclasses

$$\mathcal{G}(\breve{O})\backslash G(\breve{F})/\mathcal{G}(\breve{O}) \cong W_{\mathcal{G}}\backslash \tilde{W}/W_{\mathcal{G}} \tag{3.9}$$

where $W_{\mathcal{G}} := (N_G(T)(\breve{F}) \cap \mathcal{G}(\breve{O}))/\mathcal{T}(\breve{O})$ is the Weyl O-group of $\mathcal{G}$ relative to its maximal O-torus $\mathcal{S}$, see also [HR08, Proposition 8]. This double coset carries a natural action of the Galois group $\mathrm{Gal}(\breve{F}/F)$.

Definition 3.6. Given a finite subset $W \subset W_{\mathcal{G}}\backslash \tilde{W}/W_{\mathcal{G}}$ with reflex field[13] k_W, we define the associated Schubert perfect k_W-scheme $\mathcal{F}\ell_{\mathcal{G},W} \subset \mathcal{F}\ell_{\mathcal{G}}$ as the closure of the Schubert perfect orbit $\mathcal{F}\ell^{\circ}_{\mathcal{G},W}$, the étale descent to k_W of the union of the $L_k^+\mathcal{G}$-orbits of the maximal elements $w \in W$.

[13]Concretely, the residue field defined by the $\mathrm{Gal}(\breve{F}/F)$-stabilizer of W.

If $W = \{w\}$, then these are perfect k_w-varieties denoted by $\mathcal{F}\ell_{\mathcal{G},w}$, respectively $\mathcal{F}\ell^\circ_{\mathcal{G},w}$, which are usually called the Schubert perfect variety, respectively Schubert perfect orbit associated with w. More generally, if we fix an Iwahori $\mathcal{I}$ dilating $\mathcal{G}$ and containing $\mathcal{S}$, then its $L^+_{\bar{k}}\mathcal{I}$-orbits are enumerated by $\tilde{W}/W_{\mathcal{G}}$. Given some finite subset $W \subset \tilde{W}/W_{\mathcal{G}}$, we can define in the same manner

$$\mathcal{F}\ell^\circ_{(\mathcal{I},\mathcal{G}),W} \subset \mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),W} \tag{3.10}$$

the finite disjoint union of orbits and their closure inside $\mathcal{F}\ell_{\mathcal{G}}$. The latter is called an Iwahori–Schubert perfect scheme. We observe that Schubert perfect schemes are always Iwahori–Schubert (but the converse is false). Indeed, given $w \in W_{\mathcal{G}}\backslash\tilde{W}/W_{\mathcal{G}}$ with lift $\dot{w} \in \tilde{W}/W_{\mathcal{G}}$ of maximal length, we have

$$\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\dot{w}} = \mathcal{F}\ell_{\mathcal{G},w}. \tag{3.11}$$

Here we recall that the length function and Bruhat partial order on $\tilde{W}$ induces one on the cosets $W_{\mathcal{G}}\backslash\tilde{W}/W_{\mathcal{G}}$, respectively $\tilde{W}/W_{\mathcal{G}}$ compatibly with the dimensions and closure relations of Schubert varieties, respectively Iwahori–Schubert varieties, see [Ric13, Section 1, Proposition 2.8] for details and proofs in equicharacteristic (the arguments translate literally).

Proposition 3.7. *For each $w \in W_{\mathcal{G}}\backslash\tilde{W}/W_{\mathcal{G}}$, the Schubert perfect variety $\mathcal{F}\ell_{\mathcal{G},w}$ is normal and $\mathcal{F}\ell^\circ_{\mathcal{G},w}$ is a perfectly smooth dense open with connected fixers.*

Proof. Let $\mathscr{B}(G,F)$ be the Bruhat-Tits building of G, and let $\mathbf{f} \subset \mathscr{B}(G,F)$ be the facet associated to $\mathcal{G}$, see [BT84]. Given $w \in \tilde{W}/W_{\mathcal{G}}$, the stabilizer of $wL^+_{\bar{k}}\mathcal{G} \in \mathcal{F}\ell_{\mathcal{G}}$ is $L^+_{\bar{k}}\mathcal{G} \cap wL^+_{\bar{k}}\mathcal{G}w^{-1}$, which is the positive loop group associated to the parahoric group scheme, which is the connected fixer of $\mathbf{f} \cup w(\mathbf{f})$. In particular, this stabilizer is pro-(perfectly smooth and connected). We deduce that $\mathcal{F}\ell^\circ_{\mathcal{G},w}$ is perfectly smooth.

Fix an auxiliary Iwahori $\mathcal{I}$ dilating $\mathcal{G}$ and containing $\mathcal{S}$. This yields the subgroup functor $L^+_{\bar{k}}\mathcal{I} \subset L^+_{\bar{k}}\mathcal{G}$ and, as explained before, we know that $\mathcal{F}\ell_{\mathcal{G},w} = \mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),w^{\mathbf{f}}}$ where $w^{\mathbf{f}}$ is the maximal lift of w to $\tilde{W}/W_{\mathcal{G}}$. Let ${}_{\mathbf{f}}w^{\mathbf{f}}$ be the minimal lift to $\tilde{W}$, write it as $w_{\mathrm{af}}\tau$ with $w_{\mathrm{af}} \in W_{\mathrm{af}}, \tau \in \pi_1(G)_I$, and fix some reduced word $\dot{w}$ in simple reflections (along the alcove defined by $\mathcal{I}$) for w_{af}.

After all those combinatorics, we may consider the Demazure variety

$$\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}} := L^+_{\bar{k}}\mathcal{P}_{i_1} \times^{L^+_{\bar{k}}\mathcal{I}} \cdots \times^{L^+_{\bar{k}}\mathcal{I}} L^+_{\bar{k}}\mathcal{P}_{i_n}/L^+_{\bar{k}}\mathcal{I}, \tag{3.12}$$

where $L^+_{\bar{k}}\mathcal{I} \subset L^+_{\bar{k}}\mathcal{P}_{i_j}$ are the minimal parahoric overgroups attached to the simple reflections. It follows easily by induction that the geometric fibers of the birational resolution (induced by multiplication)

$$\pi_{\dot{w}} \colon \mathcal{D}_{\mathcal{I},\bar{k},\dot{w}} \to \mathcal{F}\ell_{\mathcal{G},w} \tag{3.13}$$

are perfected projective line fibrations, hence connected. As $\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}}$ is perfectly smooth over $\bar{k}$, normality becomes a consequence of Lemma 3.1. □

The Picard group of Schubert perfect schemes over $\bar{k}$ can be explicitly determined, see [HZ20, Theorem 3.1] for the case when $\mathcal{G} = \mathcal{I}$ is Iwahori and $W = \{w\}$.

Theorem 3.8 (He–Zhou)**.** *The homomorphism*

$$\mathrm{Pic}(\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W}) \to \mathrm{Pic}(\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},\mathbb{S}_W}) \cong \mathbb{Z}[p^{-1}]^{|\mathbb{S}_W|} \tag{3.14}$$

is a bijection where $\mathbb{S}_W$ is the set of all length 1 elements in $W \subset \tilde{W}/W_{\mathcal{G}}$. (Note that $\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k}\mathbb{S}_W} \cong \mathbb{P}^{1,\mathrm{perf}}_{\bar{k}}$ if $\mathbb{S}_W$ is a singleton.)

Proof. To reduce the question to Iwahori–Schubert perfect varieties, we contemplate the Mayer–Vietoris sequence

$$1 \to \mathcal{O}^\times_{\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W_0}} \to \mathcal{O}^\times_{\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W_1}} \oplus \mathcal{O}^\times_{\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W_2}} \to \mathcal{O}^\times_{\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W_3}} \to 1 \tag{3.15}$$

where the subsets W_i are closed for the Bruhat order $W_0 = W_1 \cup W_2$ and $W_3 = W_1 \cap W_2$. Since we may and do assume all these Schubert perfect schemes to be contained in a single connected component of $\mathcal{F}\ell_{\mathcal{G},\bar{k}}$ (which implies $H^0(\mathcal{O}^\times) \cong \bar{k}^\times$ by perfectly properness), we get a natural isomorphism

$$\mathrm{Pic}(\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W_0}) \cong \mathrm{Pic}(\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W_1}) \times_{\mathrm{Pic}(\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W_3})} \mathrm{Pic}(\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W_2}). \tag{3.16}$$

By definition $S_0 = S_1 \cup_{S_3} S_2$, which implies that it suffices to show the claim for $X = \mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},w}$ an Iwahori–Schubert perfect variety.

Injectivity can be reduced to Demazure varieties, see [BS17, Theorem 6.1]. The Demazure varieties $\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}}$ are $\mathbb{P}^{1,\mathrm{perf}}_{\bar{k}}$-fibrations and can be handled directly, see [HZ20, Proposition 3.4]. To treat surjectivity, it suffices to descend certain line bundles on $\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}}$ back to the Iwahori–Schubert varieties. By [BS17, Theorem 6.13], it remains to check that restriction of $\mathcal{L}$ to geometric fibers is trivial. For this, see [HZ20, Proposition 3.9]. □

The choice of a $\mathbb{Z}[p^{-1}]$-basis in $\mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G},\bar{k},W})$ seems arbitrary, due to p-divisibility. However, using the deperfection $\mathcal{I} \otimes_O \bar{k}$ for the quotient $R \in \mathrm{Alg}^{\mathrm{perf}}_{\bar{k}} \mapsto \mathcal{I}(R)$ of $L^+_{\bar{k}}\mathcal{I}$, the perfect curve $\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},\mathbb{S}_W}$ has a canonical equivariant deperfection, see Proposition 3.3, yielding a natural $\mathbb{Z}$-lattice.

Remark 3.9. During the proof, we have also determined the Picard group of the Demazure varieties $\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}}$, or more generally those of the convolutions

$$\mathcal{F}\ell_{\mathcal{I},\bar{k},W_1}\tilde{\times}\dots\tilde{\times}\mathcal{F}\ell_{\mathcal{I},\bar{k},W_{n-1}}\tilde{\times}\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W_n} \tag{3.17}$$

of Iwahori–Schubert perfect schemes, where at most the last one is not at full level.

Together with Proposition 3.2, this tells us how to recover, for instance, the perfect Schubert variety $\mathcal{F}\ell_{\mathcal{G},\bar{k},w}$ just from its Demazure resolution and the sub-$\mathbb{Z}[p^{-1}]$-module $\mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G},\bar{k},w})\subset \mathrm{Pic}(\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}})$: take any $\mathcal{L}$ on $\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}}$ which is the pullback of a line bundle on $\mathcal{F}\ell_{\mathcal{G},\bar{k},w}$ whose restriction to $\mathcal{F}\ell_{\mathcal{G},\bar{k},s}$ has positive degree for each $s\in\mathbb{S}_W$.

We now turn to equivariant automorphisms of (connected) Schubert schemes.

Proposition 3.10. *The group of $L^+_{\bar{k}}\mathcal{G}$-equivariant automorphisms of a connected Schubert perfect scheme $\mathcal{F}\ell_{\mathcal{G},\bar{k},W}$ is trivial. In particular, the stabilizers are self-normalizing* [14] *subgroups of $L^+_{\bar{k}}\mathcal{G}$.*

Proof. We prove the more general statement for Iwahori–Schubert perfect schemes. Consider the disjoint irreducible components in the dense open $\mathcal{F}\ell^{\circ}_{(\mathcal{I},\mathcal{G}),\bar{k},W}\subset\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W}$. These will be permuted under any equivariant automorphism σ. Moreover, σ preserves the $\bar{k}$-valued points of $\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W}$ fixed under $\mathcal{S}(\breve{O})$. For the entire flag variety, we claim that the $\mathcal{S}(\breve{O})$-fixed points in $G(\breve{F})/\mathcal{G}(\breve{O})$ lie in the image of $N(\breve{F})$. Indeed, let $[g]\in G(\breve{F})/\mathcal{G}(\breve{O})$ be a fixed point. Then $g\mathbf{f}$ is a $\mathcal{S}(\breve{O})$-stable facet (with $\mathbf{f}$ the facet determined by $\mathcal{G}$), hence contained in $\mathscr{A}(G,S,\breve{F})$ by [BT84, Proposition 5.1.37]. Multiplying on the left by a suitable element of $N(\breve{F})$, we can trivialize $[g]$, that is, $[g]\in N(\breve{F})\mathcal{G}(\breve{O})/\mathcal{G}(\breve{O})$.

Now, observe that the $L^+_{\bar{k}}\mathcal{I}$-fixer of some $w\in\tilde{W}/W_{\mathcal{G}}$ equals $L^+_{\bar{k}}\mathcal{I}\cap wL^+_{\bar{k}}\mathcal{G}w^{-1}$, and it suffices to recover w from this subgroup alone. Indeed, then σ must preserve w, and then $\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W}$ pointwise by $L^+_{\bar{k}}\mathcal{I}$-equivariance as $w\in\tilde{W}/W_{\mathcal{G}}\cap\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W}$ was arbitrary. If $\mathbf{a}$ is the alcove fixed by $\mathcal{I}(\breve{O})$ and $\tilde{w}\in\tilde{W}$ the minimal lift of w, then birationality of the Demazure resolution $\pi_{\dot{w}}$ implies

$$L^+_{\bar{k}}\mathcal{I}\cap wL^+_{\bar{k}}\mathcal{G}w^{-1}=L^+_{\bar{k}}\mathcal{I}\cap\tilde{w}L^+_{\bar{k}}\mathcal{I}\tilde{w}^{-1}. \tag{3.18}$$

Note that the right side is the Bruhat–Tits group attached to $\mathbf{a}\cup\tilde{w}(\mathbf{a})$. We need to recover $\tilde{w}$. Moreover, because $\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W}$ was assumed to be connected, all $\tilde{w}$ considered here project to the same constant $\tau\in\pi_1(G)$, so it is enough to get $w_{\mathrm{af}}\in W_{\mathrm{af}}$ if $\tilde{w}=w_{\mathrm{af}}\tau$.

By [BT84, Corollaire 5.1.39], the fixed point set of $L^+_{\bar{k}}\mathcal{I}\cap\tilde{w}L^+_{\bar{k}}\mathcal{I}\tilde{w}^{-1}$ inside $\mathscr{B}(G,F)$ equals the closed convex hull of $\mathbf{a}\cup\tilde{w}(\mathbf{a})$. In turn, every alcove inside this closed convex hull lies in some minimal gallery connecting $\mathbf{a}$ to $\tilde{w}(\mathbf{a})$ by [BT72, Lemme 2.4.4]. Since a minimal gallery describes a unique word of simple reflections necessary to move from one alcove to another, this gives back the affine transformation w_{af}. ☐

Among Schubert schemes, we are especially interested in the μ-admissible locus. Recall that C is a completed algebraic closure of F and that I denotes the inertia group of F. Moreover, let $B\subset G_{\breve{F}}$ be a Borel containing $T_{\breve{F}}$. Recall that the inverse of the Kottwitz morphism [Kot97, Equation (7.2.1)] induces an isomorphism of the coinvariants

$$X_*(T)_I\cong T(\breve{F})/\mathcal{T}(\breve{O}),\ \nu_I\mapsto\nu_I(\pi), \tag{3.19}$$

not depending on the choice of uniformizer $\pi\in O$, under which we may regard the former as the subgroup of $\tilde{W}$ acting by translation on the standard apartment, see also [HR08, Proposition 13].

Definition 3.11. Let μ be a geometric conjugacy class of cocharacters with reflex field E. The μ-admissible locus is the Schubert perfect k_E-scheme

$$\mathcal{A}_{\mathcal{G},\mu}=\mathcal{F}\ell_{\mathcal{G},\{\lambda_I(\pi)\}}, \tag{3.20}$$

where $\lambda\in X_*(T)$ runs over all representatives of μ and $\lambda_I\in X_*(T)_I$ denotes the associated coinvariant under I.

Note that $\mathcal{A}_{\mathcal{G},\mu}$ is geometrically connected because the finite Weyl group acts trivially on $\pi_1(G)_I$. It does not depend on the choice of T. By a result of Haines [Hai18, Theorem 4.2], we know that $\mathcal{A}^{\circ}_{\mathcal{G},\mu}:=\mathcal{F}\ell^{\circ}_{\mathcal{G},\{\lambda_I(\pi)\}}$, where λ now runs over the rational conjugates of μ, that is, all those which are contained in a closed Weyl chamber attached to $w_0Bw_0^{-1}$ for some $w_0\in W_0$, the finite Weyl group of $G_{\breve{F}}$ with respect to $S_{\breve{F}}$.

It will turn out that $\mathcal{A}_{\mathcal{G},\bar{k},\mu}$ is functorial in $(\mathcal{G},\mu)$, as soon as we develop a theory of local models $\mathcal{M}_{\mathcal{G},\mu}$, see Definition 4.11, and calculate their special fibers, confer Theorem 6.16. The admissible locus also admits the following representation-theoretic interpretation in terms of representations of the Langlands dual group $\widehat{G}$ (here, taken over any algebraically closed field) with dual torus $\widehat{T}$:

Lemma 3.12. *Let $\widehat{\lambda}^I\in X^*(\widehat{T}^I)\cong X_*(T)_I$ be running through the set of restrictions of all weights $\widehat{\lambda}$ for $\widehat{T}$ occurring in a finite dimensional algebraic representation of $\widehat{G}$ with fixed highest weight $\widehat{\mu}=\mu$. Then $\mathcal{A}_{\mathcal{G},\mu}=\mathcal{F}\ell_{\mathcal{G},\{\widehat{\lambda}^I(\pi)\}}$.*

[14] This is false for general Bruhat–Tits subgroups, for example, consider $L^+_{\bar{k}}\mathcal{T}\cdot R_u(L^+_{\bar{k}}\mathcal{G})$ where $R_u(\text{-})$ denotes the pro-unipotent radical.

Proof. Being a $\widehat{G}$-representation, V contains all the weights $\widehat{\lambda}$ conjugate to $\widehat{\mu}$ under the absolute Weyl group with the same non-zero multiplicity. Under $X^*(\widehat{T}) \cong X_*(T)$, these correspond to the conjugates of μ compatibly with the projection to $X^*(\widehat{T}^I) \cong X_*(T)_I$. Hence, the lemma follows from the definition of the admissible locus. $\square$

Example 3.13. The basic example of the admissible locus occurs for $G = \mathrm{GL}_2$, $\mu = (1,0)$ and $\mathcal{G} = \mathcal{I}$ an Iwahori. In this case, $\mathcal{A}_{\mathcal{G},\mu}$ is the union of two copies of $\mathbb{P}^{1,\mathrm{perf}}_k$ intersecting transversally at a point. More generally, one can enumerate the Iwahori–Schubert orbits of the translated to the neutral component admissible locus $\mathcal{A}_{\mathcal{I},\mu}$ in terms of alcoves in the standard apartment $\mathscr{A}(G,S,F)$. For pictures in the case of unitary groups of split rank 2, the reader is referred to the introduction of [PR09]. For further examples, see the survey [PRS13].

3.3. **Canonical deperfections.** Now, we wish to introduce equivariant deperfections of the Schubert perfect schemes $\mathcal{F}\ell_{\mathcal{G},W}$ following Proposition 3.3 and discuss their geometry, at least for certain W. We are especially interested in admissible loci $\mathcal{A}_{\mathcal{G},\mu}$ for μ minuscule.

First, recall that the congruence quotient $L^{\leq n}_k\mathcal{G}$ of $L^+_k\mathcal{G}$ has a deperfection $\mathrm{Gr}_n\mathcal{G}$, given by $(n+1)$-truncated Witt vectors and which is called the Greenberg realization. We denote by $L^{>n}_k\mathcal{G}$ the kernel of $L^+_k\mathcal{G} \to L^{\leq n}_k\mathcal{G}$.

Definition 3.14. Let n be the smallest nonnegative integer such that $L^{>n}_{\bar{k}}\mathcal{G}$ acts trivially on $\mathcal{F}\ell_{\mathcal{G},\bar{k},W}$ and call it the associated depth. The canonical deperfection[15] $\mathcal{F}\ell^{\mathrm{can}}_{\mathcal{G},W}$ of the perfect Schubert scheme $\mathcal{F}\ell_{\mathcal{G},W}$ is the finite type k_W-scheme with $\mathrm{Gr}_n\mathcal{G}$-action determined by Proposition 3.3.

Assume the $L^+_{\bar{k}}\mathcal{G}$-action on $\mathcal{F}\ell_{\mathcal{G},\bar{k},W}$ factors through $L^0_{\bar{k}}\mathcal{G} = \mathcal{G}^{\mathrm{perf}}_{\bar{k}}$. For $V \leq W$, we get a deperfection

$$\mathcal{F}\ell^{\mathrm{can}}_{\mathcal{G},\bar{k},V} \to \mathcal{F}\ell^{\mathrm{can}}_{\mathcal{G},\bar{k},W} \tag{3.21}$$

of the closed immersion of perfect Schubert schemes, because the image is a finite type deperfection with smaller function fields, as it carries a $\mathcal{G}_{\bar{k}}$-action.

However, it is not clear that the finite type morphism is a closed immersion. To know more about the geometry of $\mathcal{F}\ell^{\mathrm{can}}_{\mathcal{G},W}$, we exploit the picture in equicharacteristic.

Assume G is adjoint, and also Assumption 1.9 for G over $\breve{F}$, that is, if $p = 2$, then G has no odd unitary factors over $\breve{F}$. Then, for every parahoric $\breve{O}$-group $\mathcal{G}$ attached to a facet in $\mathscr{A}(G,S,\breve{F})$, we find smooth, affine, fiberwise connected $\breve{O}[\![t]\!]$-lifts $\underline{\mathcal{G}}$ in the sense of [FHLR22, Proposition 2.8]. Note that the $\bar{k}[\![t]\!]$-reductions $\mathcal{G}'$ are parahoric models of some adjoint connected reductive $\bar{k}(\!(t)\!)$-group G' attached to a facet in some appartment $\mathscr{A}(G',S',\bar{k}(\!(t)\!)) \cong \mathscr{A}(G,S,F)$, see [FHLR22, Lemma 2.7].

In particular, these come with isomorphisms

$$\mathcal{G} \otimes_{\breve{O}} \bar{k} \cong \mathcal{G}' \otimes_{\bar{k}[\![t]\!]} \bar{k}, \tag{3.22}$$

that are functorial as we vary $\mathcal{G}$ among parahoric models attached to a facet in $\mathscr{A}(G,S,\breve{F})$, and which we now exploit to compare their Schubert schemes. Let us note that the loop groups $L^+_{\bar{k}}\mathcal{G}$ and its equicharacteristic cousin $L^+_{\bar{k}}(\mathcal{G}')$ admit natural surjections on $\mathcal{G}^{\mathrm{perf}}_{\bar{k}} \cong \mathcal{G}'^{,\mathrm{perf}}_{\bar{k}}$. Below, we use subscripts $(\text{-})'$ to denote the perfections of the equicharacteristic loop groups and Schubert varieties for $\mathcal{G}'$.

Lemma 3.15. *Under the above constraints, there are unique equivariant isomorphisms*

$$\mathcal{F}\ell^{\mathrm{can}}_{\mathcal{G},\bar{k},W} \cong \mathcal{F}\ell^{\mathrm{can}}_{\mathcal{G}',\bar{k},W'} \tag{3.23}$$

for all connected $\mathcal{F}\ell_{\mathcal{G},\bar{k},W}$ of depth 0, that is, whose $L^+_{\bar{k}}\mathcal{G}$-action factors through $\mathcal{G}^{\mathrm{perf}}_{\bar{k}}$.

Proof. As $\mathcal{G}_{\bar{k}} \cong \mathcal{G}'_{\bar{k}}$, it suffices by Proposition 3.3 to produce equivariant isomorphism $\mathcal{F}\ell_{\mathcal{G},\bar{k},W} \cong \mathcal{F}\ell_{\mathcal{G}',\bar{k},W'}$ of the perfect Schubert schemes. During the proof, we fix an auxiliary Iwahori $\mathcal{I}$ dilated from $\mathcal{G}$ and consider the corresponding Iwahori–Schubert perfect scheme $\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W}$.

First assume that $W = \{w\}$. The perfect variety $\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},w}$ can be resolved via a Demazure variety $\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}}$. If s is the first letter of the word $\dot{w}$ and $\dot{v}$ is the word obtained from deleting the first letter, we get

$$\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}} = \mathcal{F}\ell_{\mathcal{I},\bar{k},s}\tilde{\times}\mathcal{D}_{\mathcal{I},\bar{k},\dot{v}} \tag{3.24}$$

where $L^+_{\bar{k}}\mathcal{I} \subset L^+_{\bar{k}}\mathcal{P}$ is the minimal parahoric corresponding to s. We claim that the action of $L^+_{\bar{k}}\mathcal{I}$ on $\mathcal{D}_{\mathcal{I},\bar{k},\dot{v}}$ is trivial when restricted to the normal subgroup $L^{\geq 1}_{\bar{k}}\mathcal{P}$. Otherwise, let α be the negative simple affine root corresponding to s and observe that $L^+_{\bar{k}}\mathcal{U}_{\alpha+1}$ acts non-trivially on $\mathcal{D}_{\mathcal{I},\bar{k},\dot{v}}$. But conjugating by s yields that $L^+_{\bar{k}}\mathcal{U}_{-\alpha+1} \subset L^{\geq 1}_{\bar{k}}\mathcal{I}$ does not act trivially on $\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}}$.

Arguing inductively on $\dot{w}$, and exploting the above claim, we reach at an $\mathcal{I}^{\mathrm{perf}}_{\bar{k}}$-equivariant identification of the Demazure perfect varieties

$$\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}} \cong \mathcal{D}_{\mathcal{I}',\bar{k},\dot{w}'} \tag{3.25}$$

bounded by $\dot{w}$ resp. $\dot{w}'$ and attached to $\mathcal{I}$, respectively the $\bar{k}(\!(t)\!)$-reduction $\mathcal{I}'$ of the Iwahori $\breve{O}[\![t]\!]$-lift. In the case $l(\dot{w}) = 1$, then we get the unique equivariant identification of one-dimensional Iwahori–Schubert perfect varieties, which are perfected projective lines.

[15] In equicharacteristic, one recovers the weak normalization of classical Schubert schemes, which turn out to be the classical ones under Assumption 1.9 and also $p \nmid |\pi_1(G_{\mathrm{der}})|$, see [FHLR22, Section 3.1] and [HLR18].

As the Picard group of the Demazure varieties have already been determined, see Theorem 3.8, respectively [HZ20, Section 3.2] for the equicharacteristic case, the previous isomorphism descends uniquely by Proposition 3.2 to an $\mathcal{I}_{\bar{k}}^{\mathrm{perf}}$-equivariant identification

$$\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},w} \cong \mathcal{F}\ell_{(\mathcal{I}',\mathcal{G}'),\bar{k},w'} \tag{3.26}$$

of the perfect Schubert varieties, see Remark 3.9. If the left side is stable under $\mathcal{G}_{\bar{k}}$, then we need to show the map is not only $\mathcal{I}_{\bar{k}}^{\mathrm{perf}}$-equivariant, but furthermore $\mathcal{G}_{\bar{k}}^{\mathrm{perf}}$-equivariant.

Let $\bar{Q} \subset \mathcal{G}_{\bar{k}}$ denote the image of $\mathcal{I}_{\bar{k}}$. By assumption, the $\mathcal{I}_{\bar{k}}^{\mathrm{perf}}$-action on both sides factors through the perfection of $\bar{Q}$. Using the convolution product

$$\mathcal{G}_V^{\mathrm{perf}} \times^{\bar{Q}^{\mathrm{perf}}} \mathcal{F}\ell_{\mathcal{G},\bar{k},w}, \tag{3.27}$$

we get a perfect $\bar{k}$-variety mapping $\mathcal{I}_{\bar{k}}^{\mathrm{perf}}$-equivariantly to $\mathcal{F}\ell_{\mathcal{G},\bar{k},w}$ and that can be identified with its equicharacteristic analogue in a $\mathcal{G}_{\bar{k}}^{\mathrm{perf}}$-equivariant fashion. Since $\mathcal{G}_{\bar{k}}^{\mathrm{perf}}/\bar{Q}^{\mathrm{perf}} \subset \mathcal{F}\ell_{\mathcal{I},\bar{k}}$ is a Schubert perfect variety at Iwahori level, we know its Picard group by Theorem 3.8 and Remark 3.9. Applying again Proposition 3.2, we not only recover the original isomorphism, by Proposition 3.10, but also conclude it is $\mathcal{G}_{\bar{k}}^{\mathrm{perf}}$-equivariant and the unique such map.

For a general W as in the statement, we now can glue the above isomorphism to non-irreducible Schubert perfect schemes

$$\mathcal{F}\ell_{\mathcal{G},\bar{k},W} \cong \mathcal{F}\ell_{\mathcal{G}',\bar{k},W'} \tag{3.28}$$

appealing again to Proposition 3.10. □

From now on G is no longer assumed to be adjoint. We approach the canonical admissible locus $\mathcal{A}_{\mathcal{G},\mu}^{\mathrm{can}}$ for minuscule μ, that is, the canonical perfection of the admissible locus, with our comparison result, describing its singularities (thereby confirming [Zhu17a, Conjecture III] for Schubert varieties in the admissible locus) and computing its coherent cohomology.

Theorem 3.16. *Let μ be minuscule and assume Assumption 1.9. Then, $\mathcal{A}_{\mathcal{G},\mu}^{\mathrm{can}}$ is Cohen–Macaulay and Frobenius split compatibly with its $\mathcal{G}_{\bar{k}}$-stable reduced $\bar{k}$-subschemes.*

Moreover, for every ample line bundle $\mathcal{L}$ on $\mathcal{F}\ell_{\mathcal{G},\bar{k}}$ that descends to $\mathcal{A}_{\mathcal{G},\bar{k},\mu}^{\mathrm{can}}$, there is an equality

$$\dim_{\bar{k}} \mathrm{H}^0(\mathcal{A}_{\mathcal{G},\bar{k},\mu}^{\mathrm{can}}, \mathcal{L}) = \dim_C \mathrm{H}^0(\mathcal{F}_{G,C,\mu}, \mathcal{O}(c_{\mathcal{L}})). \tag{3.29}$$

Here, $\mathcal{F}_{G,\mu} = G_E/P_\mu^-$ is the classical flag variety attached to μ, the central charge $c_{\mathcal{L}} \in \mathbb{Z}^{\mathrm{rk}\,\Psi_G - \mathrm{rk}\,\Phi_G}$ is given by Kac–Moody coefficients, see [PR08, Section 10] and [BS17, Section 10], and the line bundle $\mathcal{O}(c_{\mathcal{L}})$ is the corresponding power of the ample generator of $\mathrm{Pic}(\mathcal{F}_{G,C,\mu})$.

Proof. We want to apply Lemma 3.15, in order to reduce the statements to equicharacteristic, where we refer to [FHLR22, Theorem 3.1, Theorem 4.1].

In order to do this, we first notice that there is an equivariant isomorphism $\mathcal{A}_{\mathcal{G},\bar{k},\mu}^{\mathrm{can}} \cong \mathcal{A}_{\mathcal{G}_{\mathrm{ad}},\bar{k},\mu_{\mathrm{ad}}}^{\mathrm{can}}$ via the natural map. Here, μ_{ad} denotes the composition of μ with $G_C \to G_{\mathrm{ad}C}$. Indeed, this can be checked on perfections and then at the level of geometric points, where it follows from the assertion that $\mathcal{F}\ell_{\mathcal{G},\bar{k}} \to \mathcal{F}\ell_{\mathcal{G}_{\mathrm{ad}},\bar{k}}$ is an open and closed immersion.

We still have to show that $\mathcal{A}_{\mathcal{G},\bar{k},\mu}^{\mathrm{can}}$ has minimal depth, that is, $L_{\bar{k}}^{+}\mathcal{G}$ acts via $\mathcal{G}_{\bar{k}}$. Since $L_{\bar{k}}^{\geq 1}\mathcal{G}$ is a normal subgroup, it suffices to check that it fixes each of the sections λ defining the admissible locus. By the combinatorial dictionary, see our proof of Proposition 3.10, it suffices to show that $|a(\lambda_I)| \leq 1$, that is, the translation λ_I moves every affine root to a parallel one at distance at most one. By definition, one has

$$a(\lambda_I) = [K : \breve{F}]^{-1} \sum_{\sigma \in \mathrm{Gal}(K/\breve{F})} \sigma\tilde{a}(\lambda), \tag{3.30}$$

where K is a finite Galois extension of $\breve{F}$ splitting $G_{\breve{F}}$, and $\tilde{a}$ is an absolute root restricting to a, so its absolute value is at most 1, since λ is minuscule. □

4. Affine Grassmannians and local models

In this section, we start by gathering several basic facts on the B_{dR}^{+}-affine Grassmannian over $\mathrm{Spd}\, C$. Most of them were established in [SW20, Lecture XIX] and [FS21, Chapters VI.2, VI.5], but our approach is sufficiently different and relevant to later sections that it merits some elaboration.

Then, we introduce the main objects of study of this article, to wit the local models

$$\mathcal{M}_{\mathcal{G},\mu} \subset \mathrm{Gr}_{\mathcal{G},O_E} \tag{4.1}$$

defined for every $\mu \in X_*(T)$ via v-closures of Schubert diamonds in a Beilinson–Drinfeld Grassmannian. We dedicate the rest of the section to showing that $\mathcal{M}_{\mathcal{G},\mu}$ is an $L_{O_E}^{+}\mathcal{G}$-stable flat, proper π-adic kimberlite with good finiteness properties. In particular, its special fiber will be shown to be representable by some connected Schubert perfect scheme $\mathcal{F}\ell_{\mathcal{G},W}$.

4.1. **The B_{dR}^+-affine Grassmannian.** In this section, we fix a complete discretely valued field $F/\mathbb{Q}_p$ with perfect residue field k, ring of integers O and uniformizer π, a complete algebraic closure C/F with ring of integers O_C and residue field $\bar{k} = k_C$. We denote by $\breve{F} \subset C$ the maximal unramified complete subextension with ring of integers $\breve{O}$ and the same residue field $\bar{k} = k_{\breve{F}}$. Further, we fix a (connected) reductive F-group G and a maximal $\breve{F}$-split F-torus $S \subset G$ containing a maximal F-split torus, see [BT84, Proposition 5.1.10]. As G is quasi-split over $\breve{F}$ by Steinberg's theorem, the centralizer T of S is a maximal torus. Also, we fix a Borel subgroup $B \subset G_{\breve{F}}$ containing $T_{\breve{F}}$.

For any affinoid perfectoid space $\mathrm{Spa}(R, R^+)$ in characteristic p equipped with a map to $\mathrm{Spd}\,\mathbb{Z}_p$, let $B_{\mathrm{dR}}^+(R^\sharp)$, respectively $B_{\mathrm{dR}}(R^\sharp)$, be the rings of de Rham periods formed using O-Witt vectors. For convenience, we set $B_{\mathrm{dR}}^+ := B_{\mathrm{dR}}^+(C)$ and $B_{\mathrm{dR}} := B_{\mathrm{dR}}(C)$. The B_{dR}^+-loop group of G is the group functor over $\mathrm{Spd}\,F$ given by

$$LG\colon (R, R^+) \mapsto G(B_{\mathrm{dR}}(R^\sharp)), \tag{4.2}$$

and the positive loop group is the subgroup functor

$$L^+G\colon (R, R^+) \mapsto G(B_{\mathrm{dR}}^+(R^\sharp)). \tag{4.3}$$

Their v-sheaf quotient

$$\mathrm{Gr}_G := LG/L^+G \tag{4.4}$$

is called the B_{dR}^+-affine Grassmannian. Similarly to Section 3.2, $\mathrm{Gr}_G(R, R^+)$ parametrizes G-torsors on the spectrum $\mathrm{Spec}(B_{\mathrm{dR}}^+(R^\sharp))$ with a trivialization over $\mathrm{Spec}(B_{\mathrm{dR}}(R^\sharp))$. Here, we are primarily interested in the geometry and work therefore over $\mathrm{Spd}\,C$. The base changes are denoted by L_CG, L_C^+G and $\mathrm{Gr}_{G,C}$, for convenience.

As an auxiliary first step, we study the affine flag variety and then translate the results to the affine Grassmannian. For this, the Iwahori group B_{dR}^+-model $\mathcal{I}$ is given as the dilatation of $G \otimes_F B_{\mathrm{dR}}^+$ along the subscheme $B_C \subset G_C$ of its special fiber. Define

$$L_C^+\mathcal{I}\colon (R, R^+) \mapsto \mathcal{I}(B_{\mathrm{dR}}^+(R^\sharp)) \tag{4.5}$$

which is a subgroup v-sheaf of L^+G. It gives rise to the B_{dR}^+-affine flag variety

$$\mathcal{F}\ell_{\mathcal{I},C} := L_CG/L_C^+\mathcal{I}, \tag{4.6}$$

viewed as a v-sheaf over $\mathrm{Spd}\,C$.

We recall that $\mathrm{Gr}_{G,C} \to \mathrm{Spd}\,C$ is an increasing union of proper, spatial diamonds by [SW20, Lecture XIX]. The same holds for $\mathcal{F}\ell_{\mathcal{I},C} \to \mathrm{Spd}\,C$, as the projection

$$\mathcal{F}\ell_{\mathcal{I},C} \to \mathrm{Gr}_{G,C} \tag{4.7}$$

is a proper, cohomologically smooth $(G_C/B_C)^\diamondsuit$-fibration. The following discussion is parallel to parts of Section 3.2 but simplified by the fact that we consider $G \otimes_F B_{\mathrm{dR}}^+$ which is a (split) reductive group over B_{dR}^+ (and not some parahoric group scheme). The geometry of the affine flag variety $\mathcal{F}\ell_{\mathcal{I},C}$ or, better, the v-stack quotient

$$\mathrm{Hk}_{\mathcal{I},C} := L_C^+\mathcal{I}\backslash\mathcal{F}\ell_{\mathcal{I},C} = L_C^+\mathcal{I}\backslash L_CG/L_C^+\mathcal{I} \tag{4.8}$$

is reflected in the Iwahori-Weyl group

$$\tilde{W} := N(B_{\mathrm{dR}})/T(B_{\mathrm{dR}}^+), \tag{4.9}$$

where N denotes the normalizer of T in G. There is a canonical map $\underline{\tilde{W}} \to \mathcal{F}\ell_{\mathcal{I},C}$ because $T(B_{\mathrm{dR}}^+) \subset \mathcal{I}(B_{\mathrm{dR}}^+)$.

Lemma 4.1. *The map $\underline{\tilde{W}} \to \mathcal{F}\ell_{\mathcal{I},C}$ induces a bijection*

$$\tilde{W} \cong |\mathrm{Hk}_{\mathcal{I},C}|. \tag{4.10}$$

Proof. Every point of $|\mathrm{Hk}_{\mathcal{I},C}|$ is represented by a map $\mathrm{Spa}(K, K^+) \to L_CG$ with K algebraically closed perfectoid. Two K-valued points have the same underlying element in $|\mathrm{Hk}_{\mathcal{I},C}|$ if, v-locally, they lie in the same double coset

$$\mathcal{I}(B_{\mathrm{dR}}^+(K^\sharp))\backslash G(B_{\mathrm{dR}}(K^\sharp))/\mathcal{I}(B_{\mathrm{dR}}^+(K^\sharp)). \tag{4.11}$$

The identification now follows from the Bruhat decomposition which is independent of $K^\sharp$. □

Let $\mathbf{a} \subset \mathscr{A}(G, S, F)$ be the alcove defined by $\mathcal{I}$ in the appartment fixed by S of the Bruhat–Tits building. Let $\mathbb{S} \subset \tilde{W}$ be the set of simple reflections along the walls bounding $\mathbf{a}$. The affine Weyl group $W_{\mathrm{af}} \subset \tilde{W}$ is the subgroup generated by the elements in $\mathbb{S}$. Then W_{af} is a Coxeter group, and depends only on the Bruhat–Tits building of G. There exists a canonical short exact sequence

$$1 \to W_{\mathrm{af}} \to \tilde{W} \to \pi_1(G) \to 1, \tag{4.12}$$

which is naturally split by taking the stabilizer $\Omega_{\mathbf{a}} \subset \tilde{W}$ of the alcove $\mathbf{a}$. Thus, we can write each $w \in \tilde{W}$ uniquely as $w = w_{\mathrm{af}}\tau$ with $\tau \in \Omega_{\mathbf{a}}$ and $w_{\mathrm{af}} \in W_{\mathrm{af}}$.

Lemma 4.2. *Equip $\pi_1(G)$ with the discrete topology. The morphism*

$$|\mathrm{Hk}_{\mathcal{I},C}| \to \pi_1(G) \tag{4.13}$$

is locally constant, thus underlies a morphism $\mathrm{Hk}_{\mathcal{I},C} \to \underline{\pi_1(G)}$ of small v-stacks.

Proof. Here, we follow the argument behind the proof of [PR08, Theorem 5.1]. If $G = G_{\mathrm{sc}}$ is simply connected, then we see that, for every algebraically closed perfectoid field K/C, the group $L_C G(K)$ is generated by its affine root subgroups $L_C U_a(K)$.

But, since $L_C U_a$ is connected (choose a pinning), we conclude that $L_C G$, hence also $\mathcal{F}\ell_{\mathcal{I},C}$ and $\mathrm{Hk}_{\mathcal{I},C}$ are connected. If $G = T$ is a torus, then we see easily that $\mathrm{Gr}_{T,C}$ equals the v-sheaf $\underline{X_*(T)}$ compatibly with the map above.

Now, suppose that $G_{\mathrm{der}} = G_{\mathrm{sc}}$. Then, $\pi_1(G)$ identifies with the fundamental group of the abelian quotient G/G_{der}, so the claim is clear. Finally, for a general group G, consider the z-extension

$$1 \to T_{\mathrm{sc}} \to \tilde{G} \to G \to 1, \tag{4.14}$$

where $\tilde{G}$ is given by the pushout of $(G_{\mathrm{sc}} \rtimes T)$ along the morphism $\ker(T_{\mathrm{sc}} \to T) \to T_{\mathrm{sc}}$, where T_{sc} is the preimage of the maximal torus $T \subset G$ under the map $G_{\mathrm{sc}} \to G$. Using the fact that T_{sc} is an induced torus, we see that the Hecke stacks and the π_1's lie in a similar exact sequence, which yields the claim. $\square$

For $\tau \in \pi_1(G)$, we denote by $\mathcal{F}\ell^{\tau}_{\mathcal{I},C}$ the fiber over τ of the morphism $\mathcal{F}\ell_{\mathcal{I},C} \to \underline{\pi_1(G)}$. We note that right translation by a representative of τ in $L_C G$ induces an isomorphism

$$\mathcal{F}\ell^{1}_{\mathcal{I},C} \xrightarrow{\cong} \mathcal{F}\ell^{\tau}_{\mathcal{I},C}. \tag{4.15}$$

Moreover, $\mathcal{F}\ell^{1}_{\mathcal{I},C}$ is canonically isomorphic to the affine flag variety $\mathcal{F}\ell_{\mathcal{I}_{\mathrm{sc}},C}$ of the simply connected cover G_{sc}. Namely, the transition morphism $\mathcal{F}\ell_{\mathcal{I}_{\mathrm{sc}},C} \to \mathcal{F}\ell^{1}_{\mathcal{I},C}$ is bijective by checking on geometric points ([Sch17, Lemma 12.5]) and using the Bruhat decomposition, hence must be an isomorphism as both $\mathcal{F}\ell_{\mathcal{I}_{\mathrm{sc}},C}, \mathcal{F}\ell^{1}_{\mathcal{I},C}$ are ind-proper over $\mathrm{Spd}\, C$.

Definition 4.3. Let $w \in \tilde{W}$. The Schubert cell $\mathcal{F}\ell^{\circ}_{\mathcal{I},C,w} \subset \mathcal{F}\ell_{\mathcal{I},C}$ is the v-sheaf-theoretic image of the orbit map

$$L^+_C\mathcal{I} \to \mathcal{F}\ell_{\mathcal{I},C},\ i \mapsto iw. \tag{4.16}$$

The Schubert variety is the v-closure $\mathcal{F}\ell_{\mathcal{I},C,w} := \mathcal{F}\ell^{\circ,\mathrm{cl}}_{\mathcal{I},C,w}$ in the sense of Section 2.1.

By Proposition 2.8, we know that the underlying topological space of $\mathcal{F}\ell_{I,w}$ is the weakly generalizing closure of $|\mathcal{F}\ell^{\circ}_{\mathcal{I},C,w}|$ inside $|\mathcal{F}\ell_{\mathcal{I},C}|$. But, $\mathcal{F}\ell^{\circ}_{\mathcal{I},C,w}$ is possibly ill-behaved because $L^+_C\mathcal{I}$ is not quasicompact. As we show in Proposition 4.5 and Corollary 4.6 for the affine Grassmannian, our definition is equivalent to the pointwise definition in [FS21, Definition VI.2.2]. We start with the case of simple reflections:

Lemma 4.4. *Let $s \in \mathbb{S}$ be a simple reflection. Then there is an isomorphism $\mathcal{F}\ell_{\mathcal{I},C,s} \simeq (\mathbb{P}^1_C)^{\diamondsuit}$ that restricts to $\mathcal{F}\ell^{\circ}_{\mathcal{I},C,s} \simeq (\mathbb{A}^1_C)^{\diamondsuit}$. In particular, $\mathcal{F}\ell^{\circ}_{\mathcal{I},C,s}$ is a topologically dense open subset of $\mathcal{F}\ell_{\mathcal{I},C,s}$.*

Proof. Let $\mathcal{P}_s$ be the parahoric group scheme over B^+_{dR} associated to the wall of $\mathbf{a}$ defining s. The maximal reductive quotient H of its special fiber over C has semisimple rank 1. Using [BT84, Théorème 4.6.33], we see that $L^+_C\mathcal{I}$ is the preimage of $Q^{\diamondsuit}$ under $L^+\mathcal{P}_s \to H^{\diamondsuit}$ for some Borel subgroup $Q \subset H$. Thus, there are isomorphisms $L^+_C\mathcal{P}_s/L^+_C\mathcal{I} \simeq (H/Q)^{\diamondsuit} \simeq (\mathbb{P}^1_C)^{\diamondsuit}$ which can be made explicit via the choice of a pinning. This implies that the monomorphism

$$L^+_C\mathcal{P}_s/L^+_C\mathcal{I} \subset \mathcal{F}\ell_{\mathcal{I},C} \tag{4.17}$$

is a closed embedding, as $\mathcal{F}\ell_{\mathcal{I},C}$ is separated and $(\mathbb{P}^1_C)^{\diamondsuit}$ is proper over $\mathrm{Spd}\, C$. The isomorphism $\mathcal{F}\ell^{\circ}_{\mathcal{I},C,s} \simeq (\mathbb{A}^1_C)^{\diamondsuit}$ is now clear, since this is the only non-trivial $Q^{\diamondsuit}$-orbit in $(\mathbb{P}^1_C)^{\diamondsuit}$. $\square$

In order to treat more general $w = w_{\mathrm{af}}\tau \in \tilde{W} \cong W_{\mathrm{af}} \rtimes \Omega_{\mathfrak{a}}$, we invoke Demazure resolutions as follows. Let $\dot{w} = s_1 \dots s_n$ be a reduced word for $w_{\mathrm{af}} = w\tau^{-1}$ with $s_i \in \mathbb{S}$ and consider the Demazure variety

$$\mathcal{D}_{C,\dot{w}} := L^+_C\mathcal{P}_1 \times^{L^+_C\mathcal{I}} \dots \times^{L^+_C\mathcal{I}} L^+_C\mathcal{P}_n/L^+_C\mathcal{I} \tag{4.18}$$

which will also be denoted by $\mathcal{F}\ell_{\mathcal{I},C,s_1}\tilde{\times}\dots\tilde{\times}\mathcal{F}\ell_{\mathcal{I},C,s_n}$. It is connected and cohomologically smooth over $\mathrm{Spd}\, C$ (being an iterated $\mathbb{P}^1_C$-fibration), and the twisted product

$$\mathcal{D}^{\circ}_{C,\dot{w}} = \mathcal{F}\ell^{\circ}_{\mathcal{I},C,s_1}\tilde{\times}\dots\tilde{\times}\mathcal{F}\ell^{\circ}_{\mathcal{I},C,s_n} \tag{4.19}$$

of the open cells is topologically dense by induction on n, starting with Lemma 4.4 and using that $L^+_C\mathcal{I}$ is pro-(cohomologically smooth) over $\mathrm{Spd}\, C$. It carries, moreover, a natural morphism (induced by multiplication)

$$\pi_{\dot{w}} \colon \mathcal{D}_{C,\dot{w}} \to \mathcal{F}\ell_{\mathcal{I},C} \tag{4.20}$$

which necessarily maps onto $\mathcal{F}\ell_{\mathcal{I},C,w_{\mathrm{af}}}$, by properness, $L^+_C\mathcal{I}$-equivariance and the fact that $\dot{w}$ maps to w. After translation by τ, we may regard this as a resolution of $\mathcal{F}\ell_{\mathcal{I},C,w}$, which is thus in particular connected.

For the next result, we note that $\tilde{W}$, in analogy to the discussion following (3.8), is equipped with a length function and Bruhat partial order induced from the quasi-Coxeter structure on $\tilde{W} \cong W_{\mathrm{af}} \rtimes \Omega_{\mathfrak{a}}$.

Proposition 4.5. *Let $w \in \tilde{W}$. Then $\mathcal{F}\ell^{\circ}_{\mathcal{I},C,w}$, respectively $\mathcal{F}\ell_{\mathcal{I},C,w}$, agrees with the subfunctor of all maps $S \to \mathcal{F}\ell_{\mathcal{I},C}$ such that for all geometric points $S' = \mathrm{Spa}(K, K^+) \to S$, the induced point $S' \to \mathcal{F}\ell_{\mathcal{I},C} \to \mathrm{Hk}_{\mathcal{I},C}$ is given by w, respectively by v for some $v \leq w$. In particular, $\mathcal{F}\ell^{\circ}_{\mathcal{I},C,w} \subset \mathcal{F}\ell_{\mathcal{I},C,w}$ is a topologically dense open.*

Proof. Observe that the first assertion cannot be verified at geometric points because $L_C^+\mathcal{I}$ is not quasicompact. However, we see from Lemma 4.4 that the result holds for simple reflections. Indeed, we even have by Bruhat–Tits combinatorics

$$L_C^+\mathcal{U}_{\alpha_s}\cdot s=\mathcal{F}\ell^{\circ}_{\mathcal{I},C,s}, \tag{4.21}$$

where α_s denotes the positive simple affine root associated to the simple reflection s, and $\mathcal{U}_{\alpha_s}$ is the corresponding B_{dR}^+-model of the affine root group. Pulling across the reflections s_i appearing in the convolution product of $\mathcal{D}_{C,\dot{w}}$, we see that $\mathcal{F}\ell^{\circ}_{\mathcal{I},C,w}$ surjects to the v-sheaf image of $\mathcal{D}^{\circ}_{C,\dot{w}}$ along $\pi_{\dot{w}}$. This v-sheaf image identifies with the pointwise description of $\mathcal{F}\ell^{\circ}_{\mathcal{I},C,w}$ by quasicompactness of $\pi_{\dot{w}}$ and bijectivity at geometric points.

Similarly, the v-sheaf image of $\mathcal{D}_{C,\dot{w}}$ along $\pi_{\dot{w}}$ is a proper closed sub-v-sheaf of $\mathcal{F}\ell_{\mathcal{I},C}$. By generalities of Tits system, see [BT72, 1.2.6], this v-sheaf image coincides with the desired pointwise description of $\mathcal{F}\ell_{\mathcal{I},C,w}$. Pulling back again via the quotient map $\pi_{\dot{w}}$, we see that $\mathcal{F}\ell^{\circ}_{\mathcal{I},C,w}\subset\mathcal{F}\ell_{\mathcal{I},C,w}$ is a topologically dense open of the closed v-sheaf image of $\pi_{\dot{w}}$. □

As a corollary, we get that the bijection $|\mathrm{Hk}_{\mathcal{I},C}|\cong\tilde{W}$ from Lemma 4.1 is a homeomorphism where $\tilde{W}$ is endowed with order topology via its Bruhat order, and also that $\pi_0(\mathcal{F}\ell_{\mathcal{I},C})=\pi_0(\mathrm{Gr}_{G,C})=\pi_1(G)$ via Lemma 4.2.

Now, we apply our results to the affine Grassmannian Gr_G. Note that there is the group isomorphism

$$X_*(T)\cong T(B_{\mathrm{dR}})/T(B_{\mathrm{dR}}^+),\ \chi\mapsto\chi(\xi) \tag{4.22}$$

which is independent of the choice of uniformizer $\xi\in B_{\mathrm{dR}}^+$. Then the Cartan decomposition induces a bijection

$$|\mathrm{Hk}_{G,C}|\simeq X_*(T)_+, \tag{4.23}$$

where $\mathrm{Hk}_{G,C}=L_C^+G\backslash L_CG/L_C^+G$ denotes the Hecke stack. Therefore, we get a Schubert cell $\mathrm{Gr}^{\circ}_{G,C,\mu}\subset\mathrm{Gr}_{G,C}$ defined as the v-sheaf-theoretic image of the orbit map and the Schubert cell $\mathrm{Gr}_{G,C,\mu}$ defined as its closure, for each $\mu\in X_*(T)_+$, compare with Definition 4.3.

Corollary 4.6. *Let $\mu\in X_*(T)_+$. Then $\mathrm{Gr}^{\circ}_{G,C,\mu}$, respectively $\mathrm{Gr}_{G,C,\mu}$ agrees with the subfunctor of all maps $S\to\mathrm{Gr}_{G,C,\mu}$ such that for all geometric points $S'=\mathrm{Spa}(K,K^+)\to S$, the induced point $S'\to\mathrm{Gr}_{G,C}\to\mathrm{Hk}_{G,C}$ is given by μ, respectively by some $\lambda\leq\mu$ in the dominance order on $X_*(T)_+$. In particular, $\mathrm{Gr}^{\circ}_{G,C,\mu}\subset\mathrm{Gr}_{G,C,\mu}$ is a topologically dense open.*

Proof. This formally follows from Proposition 4.5 by using the projection $\mathcal{F}\ell_{\mathcal{I},C}\to\mathrm{Gr}_{G,C}$ from (4.7) and noting that the dominance order on $X_*(T)_+$ is induced by the Bruhat order, see [Ric13, Corollaries 1.8, 2.10] for similar arguments. We leave the details to the reader. □

We also have the following fact which says that the $\mathrm{Spd}\,C$-valued points are dense in $\mathrm{Gr}_{G,C,\mu}$ even for the constructible topology.

Corollary 4.7. *Let $\mu\in X_*(T)_+$. The spatial diamond $\mathrm{Gr}_{G,C,\mu}$ has enough C-facets in the sense of* [Gle20, Definition 1.4.38].

Proof. Taking the preimage under the projection $\mathcal{F}\ell_{\mathcal{I},C}\to\mathrm{Gr}_{G,C}$ from (4.7), this reduces to the analogous assertion for $\mathcal{F}\ell_{\mathcal{I},C,w}$ for some $w\in\tilde{W}$. Since the Demazure resolution is a v-cover, it is enough to prove that $\mathcal{D}_{C,\dot{w}}$ has enough C-facets. This in turn can be proved inductively on the length of $\dot{w}$. If $\dot{w}=s\cdot\dot{v}$ then $\mathcal{D}_{C,\dot{w}}$ is a pro-étale $(\mathbb{P}^1_C)^{\diamondsuit}$-bundle over $\mathcal{D}_{C,\dot{v}}$. We may find a pro-étale cover

$$X\to\mathcal{D}_{C,\dot{v}} \tag{4.24}$$

with $X\times_{\mathcal{D}_{C,\dot{v}}}\mathcal{D}_{C,\dot{w}}=X\times_{\mathrm{Spd}\,C}(\mathbb{P}^1_C)^{\diamondsuit}$. Following the arguments given in [Gle20, Lemma 2.2.24, Proposition 2.2.30], we may even assume that X has enough facets over $\mathrm{Spd}\,C$. By [Gle20, Proposition 1.4.39, 2.], $\mathcal{D}_{C,\dot{w}}$ also has enough facets. □

We conclude with some motivation for our later discussion of representability.

Proposition 4.8. *Let $\mu\in X_*(T)_+$. The v-sheaf $\mathrm{Gr}_{G,C,\mu}$ is representable by a projective C-scheme $\mathcal{F}_{G,C,\mu}$ if and only if μ is minuscule.*

Proof. If μ is minuscule, then the L_C^+G-action factors through $G_C^{\diamondsuit}$ and the Bialynicki-Birula map gives an isomorphism $\mathrm{Gr}_{G,C,\mu}\simeq(G_C/P_\mu)^{\diamondsuit}$, see [SW20, Proposition 19.4.2].

Now suppose that μ is not minuscule, that is, $\langle\mu,\theta\rangle\geq 2$ for some root θ of G_C. Then the $L_C^+\mathcal{I}$-orbit of the point μ is not representable by a rigid space: This orbit can be filtered by successive smooth rigid spaces, so taking fiber products appropriately, we would eventually arrive at representability of the Banach-Colmez space $\mathcal{BC}(B_{\mathrm{dR}}^+/\xi^2)$. However, this is a non-split self-extension of $\mathbb{G}_a^{\diamondsuit}$ which does not even split étale locally, so cannot be representable. Indeed, if the extension were split étale locally, it would actually split on the nose, as $H^1(\mathbb{A}^1_C,\mathcal{O})$ is trivial. However, if

$$X:=\mathrm{Spa}(C\langle T^{\pm1}\rangle,O_C\langle T^{\pm1}\rangle)\subset\mathbb{A}^1_C \tag{4.25}$$

is the affinoid torus, the element $T \in \mathbb{A}^1_C(X)$ does not admit a lift to $\mathcal{BC}(B^+_{\mathrm{dR}}/\xi^2)(X)$ as we now show. Let $X' = \mathrm{Spa}(C\langle T^{\pm 1/p^\infty}\rangle, O_C\langle T^{\pm 1/p^\infty}\rangle)$ be the usual perfectoid $\mathbb{Z}_p(1)$-cover of X. Elements in $\mathcal{BC}(B^+_{\mathrm{dR}}/\xi^2)(X')$ can be represented by $[a] + [b]\xi$ with $a, b \in (C\langle T^{\pm 1/p^\infty}\rangle)^\flat$. Assume now that

$$x := [a] + [b]\xi$$

maps to T, that means, $a^\sharp = T$ in $\mathbb{A}^1_C(X')$ (cf. [SW20, Section 6.2] for the map $(-)^\sharp$), and assume that x is invariant under $\mathbb{Z}_p(1)$. Let $g \in \mathbb{Z}_p(1)$. Then g acts on $[a]$ via multiplication with $[g]$ if we identify $\mathbb{Z}_p(1) \subset C^\flat$. Using the (-)$^\sharp$-map, we get

$$a^\sharp \left(\frac{[g]-1}{\xi}\right)^\sharp = b^\sharp - g(b^\sharp). \tag{4.26}$$

But, if $g \in \mathbb{Z}_p(1)$ is a generator, then $a^\sharp(\frac{[g]-1}{\xi})^\sharp = cT$ with $c \in C$ non-zero. Writing $b^\sharp$ as a powerseries in the g-eigenvectors T^n with $n \in \mathbb{Z}[1/p]$, then shows that (4.26) can not hold because $g \in \mathbb{Z}_p(1)$ fixes T. This finishes the argument. □

4.2. **Local models.** We continue with the notation of Section 4.1, and additionally let $\mathcal{G}$ be a parahoric O-model of G.

We work with the moduli space $\mathrm{Gr}_{\mathcal{G}}$ of $\mathcal{G}$-torsors over $\mathrm{Spec}(B^+_{\mathrm{dR}})$ trivialized over $\mathrm{Spec}(B_{\mathrm{dR}})$, see [SW20, Definition 20.3.1], which is the Beilinson–Drinfeld Grassmannian over $\mathrm{Spd}\, O$. A crucial result of Scholze–Weinstein concerns its ind-properness, see [SW20, Theorems 19.3.4, 20.3.6, 21.2.1].

Theorem 4.9 (Scholze–Weinstein)**.** *The structure morphism of the Beilinson–Drinfeld Grassmannian*

$$\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Spd}\, O \tag{4.27}$$

is ind-proper and ind-representable in spatial diamonds.

In [Ans18, Section 12] and then later in [Gle20, Section 2.2], the first named and second named author respectively constructed and studied the specialization map for $\mathrm{Gr}_{\mathcal{G}}$, see also [Gle20, Proposition 2.2.5].

Again, we have natural loop groups at hand, namely

$$L_O\mathcal{G} \colon (R, R^+) \mapsto \mathcal{G}(B_{\mathrm{dR}}(R^\sharp)) \tag{4.28}$$

and

$$L^+_O\mathcal{G} \colon (R, R^+) \mapsto \mathcal{G}(B^+_{\mathrm{dR}}(R^\sharp)), \tag{4.29}$$

where $(R^\sharp, R^{\sharp +})$ denotes an untilt of (R, R^+) over (O, O) and $B^{(+)}_{\mathrm{dR}}(R^\sharp)$ the ring of de Rham periods formed using O-Witt vectors. These define v-sheaves over $\mathrm{Spd}\, O$ and the base changes to $\mathrm{Spd}\, F$, respectively $\mathrm{Spd}\, k$ recover the loop groups $L^+G \subset LG$ from Section 4.1, respectively the v-sheaves $(L^+_k\mathcal{G})^\diamondsuit \subset (L_k G)^\diamondsuit$ associated with the loop groups from Section 3.2. Their base changes to O_C are denoted $L_{O_C}\mathcal{G}$, $L^+_{O_C}\mathcal{G}$ and $\mathrm{Gr}_{\mathcal{G}, O_C}$.

Lemma 4.10. *There is a natural isomorphism*

$$L_O\mathcal{G}/L^+_O\mathcal{G} \cong \mathrm{Gr}_{\mathcal{G}}, \tag{4.30}$$

where the left side is a quotient for the étale topology. In particular, on geometric fibers

$$\mathrm{Gr}_G \cong \mathrm{Gr}_{\mathcal{G}} \times_{\mathrm{Spd}\, O} \mathrm{Spd}\, F, \qquad \mathcal{F}\ell^\diamondsuit_{\mathcal{G}} \cong \mathrm{Gr}_{\mathcal{G}} \times_{\mathrm{Spd}\, O} \mathrm{Spd}\, k, \tag{4.31}$$

where Gr_G is the affine Grassmannian from Section 4.1 and $\mathcal{F}\ell^\diamondsuit_{\mathcal{G}}$ the v-sheaf attached to the Witt vector partial affine flag variety from Section 3.2.

Proof. For the uniformization (4.30), see [SW20, Proposition 20.3.2]. The isomorphisms (4.31) are given by base change from (4.30) by unwinding the definitions. □

Let μ be a conjugacy class of cocharacters of G_C, with field of definition $E \subset C$. We denote by O_E its ring of integers with residue field k_E. We wish to construct a closed sub-v-sheaf

$$\mathcal{M}_{\mathcal{G},\mu} \subset \mathrm{Gr}_{\mathcal{G}}|_{\mathrm{Spd}\, O_E} \tag{4.32}$$

prolonging the Schubert diamonds $\mathrm{Gr}_{G,\mu}$ which are the descent to $\mathrm{Spd}\, E$ of the ones we studied in the previous subsection.

Definition 4.11. Let μ be a conjugacy class of cocharacters in G_C. The local model $\mathcal{M}_{\mathcal{G},\mu}$ is the v-closure of $\mathrm{Gr}_{G,\mu}$ inside $\mathrm{Gr}_{\mathcal{G}}|_{\mathrm{Spd}\, O_E}$.

A priori $\mathcal{M}_{\mathcal{G},\mu}$ does not admit a moduli problem description for general parahoric $\mathcal{G}$, so its structure could be harder to parse. Let us give some examples where the local model $\mathcal{M}_{\mathcal{G},\mu}$ is relatively well understood.

Example 4.12. If $\mathcal{G}$ is reductive, then $\mathcal{M}_{\mathcal{G},\mu}$ is the integral Schubert variety over $\mathrm{Spd}\, O_E$ and generalizes the objects introduced in Section 4.1, see [SW20, Proposition 20.3.6] and [FS21, VI.1]. If $G = T$ is a torus, then the explicit description of $\mathrm{Gr}_{\mathcal{T}}$ furnishes an identity $\mathcal{M}_{\mathcal{T},\mu} = \mathrm{Spd}\, O_E$, see [SW20, Proposition 21.3.1].

We need to show permanence of the local model under the $L^+_O\mathcal{G}$-action.

Proposition 4.13. *The natural action map $L^+_O\mathcal{G} \times \mathrm{Gr}_\mathcal{G} \to \mathrm{Gr}_\mathcal{G}$ restricts, after base change to $\mathrm{Spd}\, O_E$, to a group action on the closed sub-v-sheaf $\mathcal{M}_{\mathcal{G},\mu}$. Moreover, the generic fiber of $\mathcal{M}_{\mathcal{G},\mu}$ is topologically dense.*

Proof. By embedding $\mathcal{G}$ in GL_n, we may always find a quasi-compact closed subsheaf $X \subset \mathrm{Gr}_{\mathcal{G},O_E}$ with $\mathcal{M}_{\mathcal{G},\mu} \subset X$, stable under $L^+_{O_E}\mathcal{G}$ whose action factors through a congruence quotient $L^{\leq N}_{O_E}\mathcal{G}$, where N is a sufficiently large positive integer (necessarily at least $\langle 2\rho, \mu\rangle$ as one sees by restricting to $\mathrm{Gr}_{G,\mu}$).

The structure map $L^{\leq N}_{O_E}\mathcal{G} \to \mathrm{Spd}\, O_E$ is partially proper, cohomologically smooth and consequently universally open, see [Sch17, Proposition 23.11]. By Corollary 2.9,

$$L^{\leq N}_{O_E}\mathcal{G} \times_{\mathrm{Spd}\, O_E} \mathcal{M}_{\mathcal{G},\mu} = (L^{\leq N}_E G \times_{\mathrm{Spd}\, E} \mathrm{Gr}_{G,\mu})^{\mathrm{cl}} \tag{4.33}$$

as closed sub-v-sheaves of $L^{\leq N}_{O_E}\mathcal{G} \times_{\mathrm{Spd}\, O_E} X$. Now, we have seen that $L^+_E G$ respects $\mathrm{Gr}_{G,\mu}$, so the multiplication map $L^{\leq N}_E G \times_{\mathrm{Spd}\, E} \mathrm{Gr}_{G,\mu} \to X$ factors through $\mathrm{Gr}_{G,\mu}$. This also implies that the integral action map

$$L^{\leq N}_{O_E}\mathcal{G} \times_{\mathrm{Spd}\, O_E} \mathcal{M}_{\mathcal{G},\mu} \to X \tag{4.34}$$

factors through the closure $\mathcal{M}_{\mathcal{G},\mu}$, as we desired.

For the last claim, we consider the restricted variant of the Hecke v-stack

$$\mathrm{Hk}^{\leq N}_{\mathcal{G},\mu} := [L^{\leq N}_{O_E}\mathcal{G} \backslash \mathcal{M}_{\mathcal{G},\mu}]. \tag{4.35}$$

Its underlying topological space has the extra special property that every subset is weakly generalizing, since for every perfectoid affinoid field (K, K^+) the Bruhat decomposition over $B_{\mathrm{dR}}(K)$ is insensitive to variation of K^+. Now, the projection map

$$\mathrm{pr}\colon \mathcal{M}_{\mathcal{G},\mu} \to \mathrm{Hk}^{\leq N}_{\mathcal{G},\mu} \tag{4.36}$$

is cohomologically smooth and consequently open. Therefore, by the same argument of Corollary 2.9, we see that both the usual topological and the weakly generalizing closure commute with pullback along $|\mathrm{pr}|$. It results that $|\mathrm{Gr}_{G,\mu}|$ is a dense open of $|\mathcal{M}_{\mathcal{G},\mu}|$. ☐

We can now prove the following structural properties of $\mathcal{M}_{\mathcal{G},\mu}$.

Proposition 4.14. *With notation as in Definition 2.35, $\mathcal{M}_{\mathcal{G},\mu} \in \mathcal{K}$. More specifically, the local model $\mathcal{M}_{\mathcal{G},\mu}$ is a flat π-adic kimberlite over O_E with enough facets over C and O_C-formalizable C-sections. Moreover, the special fiber*

$$\mathcal{M}_{\mathcal{G},\mu,k_E} := \mathcal{M}_{\mathcal{G},\mu}|_{\mathrm{Spd}(k_E)} \tag{4.37}$$

is of the form $\mathcal{F}\ell^\diamondsuit_{\mathcal{G},W}$ for a connected perfect Schubert scheme $\mathcal{F}\ell_{\mathcal{G},W}$. In particular, $\mathcal{M}^{\mathrm{red}}_{\mathcal{G},\mu} = \mathcal{F}\ell_{\mathcal{G},W}$ is perfectly proper and perfectly finitely presented perfect k-scheme.

Proof. Choosing a closed embedding $\mathcal{G} \hookrightarrow \mathrm{GL}_n$, we may find a cocharacter ν of GL_n giving rise to a closed immersion

$$\mathcal{M}_{\mathcal{G},\mu} \hookrightarrow \mathcal{M}_{\mathrm{GL}_n,\nu}, \tag{4.38}$$

so that local models are proper v-sheaves and, in particular, quasi-compact. By [Gle20, Proposition 1.4.30.(3), Proposition 2.2.5] to prove $\mathcal{M}_{\mathcal{G},\mu}$ is a kimberlite, it suffices to prove it is π-adic. By [Gle20, Proposition 1.3.35], we may reduce to proving that the special fiber of $\mathcal{M}_{\mathcal{G},\mu}$ is represented by a v-sheaf of the form $X^\diamond$ for X a perfect scheme.

By Proposition 4.13, $\mathcal{M}_{\mathcal{G},\mu,\bar{k}}$, is of the form[16] $\cup_{i\in I}\mathcal{F}\ell^\diamond_{\mathcal{G},\bar{k},W_i}$ with finite subsets $W_i \subset \tilde{W}$, where we have used the fact that $\mathcal{F}\ell_{\mathcal{G},W}$ is perfectly proper, in order to deduce $\mathcal{F}\ell^\diamondsuit_{\mathcal{G},W} = \mathcal{F}\ell^\diamond_{\mathcal{G},W}$. By quasi-compactness, we get $\mathcal{M}_{\mathcal{G},\mu,k_E} = \mathcal{F}\ell^\diamond_{\mathcal{G},W}$ for some finite subset $W \subset \tilde{W}$, which finishes the proof that $\mathcal{M}_{\mathcal{G},\mu}$ is a π-adic kimberlite.

That $\mathcal{M}_{\mathcal{G},\mu}$ has O_C-formalizable C-sections is proved in [Gle20, Proposition 2.2.4], see also [Ans18, Section 12]. We explained in Corollary 4.7 that $\mathcal{M}_{\mathcal{G},\mu}$ has enough C-facets. Together with Proposition 4.13 and Lemma 2.37, flatness follows. By Proposition 2.34, $|\mathcal{F}\ell_{\mathcal{G},W}| = \mathrm{sp}(|\mathrm{Gr}_{G,\mu}|)$ and since $\mathrm{Gr}_{G,\mu}$ is connected $\mathcal{F}\ell_{\mathcal{G},W}$ is also connected. ☐

Remark 4.15. It follows that the base change $\mathcal{M}_{\mathcal{G},\mu}|_{\mathrm{Spd}\, O_C}$ is still topologically flat, and hence agrees with the v-closure $\mathcal{M}_{\mathcal{G},O_C,\mu}$ of $\mathrm{Gr}_{G,C,\mu}$ inside $\mathrm{Gr}_{\mathcal{G},O_C}$. Indeed, repeating the argument of Proposition 4.13 over $\mathrm{Spd}\, O_C$, we see that the special fiber of $\mathcal{M}_{\mathcal{G},O_C,\mu}$ is represented by a Schubert perfect scheme. But a $\mathrm{Spd}\, k$-valued point of $\mathcal{M}_{\mathcal{G},\mu}$ is a specialization of some $\mathrm{Spd}\, C$-valued point by Proposition 4.14, hence equality of both closures is clear.

Next, we analyse some functoriality behavior of $\mathcal{M}_{\mathcal{G},\mu}$ in the pair $(\mathcal{G}, \mu)$. Here, by definition, a map $(\mathcal{G}_1, \mu_1) \to (\mathcal{G}_2, \mu_2)$ is a morphism of O-group schemes $\mathcal{G}_1 \to \mathcal{G}_2$ such that the image of μ_1 in $G_{2,C}$ lies in the same conjugacy class as μ_2.

Proposition 4.16. *The association $(\mathcal{G}, \mu) \mapsto \mathcal{M}_{\mathcal{G},O_C,\mu}$, see Remark 4.15, is functorial, preserves closed embeddings and direct products, and induces isomorphisms $\mathcal{M}_{\mathcal{G},O_C,\mu} \cong \mathcal{M}_{\mathcal{G}_{\mathrm{ad}},O_C,\mu_{\mathrm{ad}}}$, where μ_{ad} denotes the composite of μ with $G_C \to G_{\mathrm{ad},C}$.*

[16] A priori this could be an infinite union.

Proof. Functoriality follows from that of $\mathrm{Gr}_{G,C,\mu}$ and the definition using v-closures, see Remark 4.15. For the claim regarding closed embeddings and central extensions, we refer to [Lou20, IV, Proposition 4.16, Corollary 4.17]: one checks injectivity at geometric points, using Lemma 4.17. As for direct products, it suffices to check equality at the level of $\mathrm{Spd}\, k$-valued points by Proposition 4.14, and this is easy because the generic fiber was already a product. □

Lemma 4.17. *The specialization map induces a bijection*

$$\pi_0(\mathrm{Gr}_{\mathcal{G},\breve{O}}) \xrightarrow{\cong} \pi_0(\mathcal{F}\ell_{\mathcal{G},\bar{k}}) \cong \pi_1(G)_I, \tag{4.39}$$

where I is the absolute Galois group of $\breve{F}$.

Proof. For the final bijection, see [Zhu17a, Proposition 1.21]. The first is a consequence of proper base change [Sch17, Theorem 19.2, Remark 19.3] applied to $f\colon \mathrm{Gr}_{\mathcal{G},\breve{O}} \to \mathrm{Spd}\,\breve{O}$ and the base change $i\colon \mathrm{Spd}\,\bar{k} \to \mathrm{Spd}\,\breve{O}$, using that the 0-th cohomology group computes connected components: for some coefficient ring, say, $\Lambda = \mathbb{Z}/\ell$ with $\ell \neq p$, we apply the proper base change $i^*R^0f_*\Lambda_X \cong R^0(f')_*(i')^*\Lambda_X$ where Λ_X is the constant sheaf supported on increasing closed $\breve{O}$-proper sub-v-sheaves $X \subset \mathrm{Gr}_{\mathcal{G},\breve{O}}$. Passing to global sections, the second computes $\Lambda^{\pi_0(X_{\bar{k}})}$ by definition whereas the first computes $\Lambda^{\pi_0(X)}$ by using the v-cover $\mathrm{Spd}\, O_C \to \mathrm{Spd}\,\breve{O}$. Finally, we use [Sch17, Section 27] to pass between $\mathcal{F}\ell^{\diamondsuit}_{\mathcal{G},\bar{k}}$ and $\mathcal{F}\ell_{\mathcal{G},\bar{k}}$. □

5. Geometry of multiplicative group actions

Our approach to the Scholze–Weinstein conjecture requires determining the special fiber of local models in terms of admissible loci. We follow the general strategy of Haines and the fourth named author [HR21] of calculating the support of nearby cycles using hyperbolic localization. This requires translating the results from [HR21, Section 5] to the v-sheaf Beilinson-Drinfeld Grassmannian. For basic facts pertaining to $\mathbb{G}_m^{\diamondsuit}$-actions on small v-stacks, the reader is referred to [FS21, Chapter IV.6].

5.1. **Over O.** As in Section 4.1, we continue to fix a complete discretely valued field $F/\mathbb{Q}_p$ with ring of integers O and perfect residue field k, a complete algebraic closure C/F and a connected reductive F-group G with parahoric model $\mathcal{G}$ over O containing the connected Néron model $\mathcal{S}$ of the maximally F-split maximal $\breve{F}$-split F-torus S.

Fix a cocharacter $\lambda\colon \mathbb{G}_m \to \mathcal{S} \subset \mathcal{G}$ defined over O. After base change to F, this induces a Levi $M = M_\lambda$ with Lie algebra $\mathrm{Lie}\, M = (\mathrm{Lie}\, G)_{\lambda=0}$, a parabolic subgroup $P = P_\lambda^+$ with Lie algebra $\mathrm{Lie}\, P = (\mathrm{Lie}\, G)_{\lambda\geq 0}$ and an unipotent subgroup $U = U_\lambda^+$ with Lie algebra $\mathrm{Lie}\, U = (\mathrm{Lie}\, G)_{\lambda>0}$ fitting in a semi-direct product decomposition $P = M \ltimes U$. Since λ is defined over O, the decomposition $P = M \ltimes U$ extends to O-models $\mathcal{P} = \mathcal{M} \ltimes \mathcal{U}$, admitting analogous descriptions for their Lie algebras and being equipped with homomorphisms

$$\mathcal{M} \longleftarrow \mathcal{P} \longrightarrow \mathcal{G}. \tag{5.1}$$

The O-group schemes $\mathcal{P}$, $\mathcal{M}$, $\mathcal{U}$ are smooth affine with connected fibers, and $\mathcal{M}$ is a parahoric O-model of the Levi subgroup M, see [HR21, Lemma 4.5] and also [CGP15, Section 2.1], [KP21, Section 6.2] for proofs of these claims.

By functoriality, (5.1) induces maps of ind-(spatial $\mathrm{Spd}\, O$-diamonds)

$$\mathrm{Gr}_{\mathcal{M}} \longleftarrow \mathrm{Gr}_{\mathcal{P}} \longrightarrow \mathrm{Gr}_{\mathcal{G}}, \tag{5.2}$$

where $\mathrm{Gr}_{\mathcal{M}} \to \mathrm{Spd}\, O$ and $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Spd}\, O$ are ind-proper by Theorem 4.9. On the other hand, the cocharacter λ induces a cocharacter

$$\mathbb{G}_m^{\diamondsuit} \xrightarrow{[\text{-}]} L_O^+\mathbb{G}_m \xrightarrow{L_O^+\lambda} L_O^+\mathcal{G}, \tag{5.3}$$

where [-] denotes the Teichmüller lift. Thus, we obtain a left action of $\mathbb{G}_m^{\diamondsuit}$ on $\mathrm{Gr}_{\mathcal{G}}$.

Lemma 5.1. *The $\mathbb{G}_m^{\diamondsuit}$-action on $\mathrm{Gr}_{\mathcal{G}}$ satisfies* [FS21, Hypothesis IV.6.1].

Proof. Choosing a closed immersion $\mathcal{G} \hookrightarrow \mathrm{GL}_{n,O}$ of group schemes, we reduce to the case $\mathcal{G} = \mathrm{GL}_{n,O}$, using Theorem 4.9 to see that the induced map $\mathrm{Gr}_{\mathcal{G}} \hookrightarrow \mathrm{Gr}_{\mathrm{GL}_{n,O}}$ is a closed immersion. Then the lemma is a special case of [FS21, Proposition VI.3.1]. □

Consequently, we obtain a $\mathbb{G}_m^{\diamondsuit}$-equivariant diagram

$$(\mathrm{Gr}_{\mathcal{G}})^0 \longleftarrow (\mathrm{Gr}_{\mathcal{G}})^+ \longrightarrow \mathrm{Gr}_{\mathcal{G}}, \tag{5.4}$$

where $(\mathrm{Gr}_{\mathcal{G}})^0 = (\mathrm{Gr}_{\mathcal{G}})^{\mathbb{G}_m^{\diamondsuit}}$ denotes the fixed points and $(\mathrm{Gr}_{\mathcal{G}})^+$ the attractor classifying $\mathbb{G}_m^{\diamondsuit}$-equivariant maps $(\mathbb{A}^1)^{\diamondsuit,+} \to \mathrm{Gr}_{\mathcal{G}}$ over $\mathrm{Spd}\, O$, see also (6.3) below. The map $(\mathrm{Gr}_{\mathcal{G}})^+ \to (\mathrm{Gr}_{\mathcal{G}})^0$ is the Bialynicki-Birula map given by evaluating at the zero section. Our aim is to understand the relation of (5.2) with (5.4). The following result is the analogue of [HR21, Theorem 5.6, Theorem 5.19] in the context of ind-schemes:

Theorem 5.2. *The maps* (5.2) *and* (5.4) *fit into a commutative diagram of ind-(spatial* $\mathrm{Spd}\, O$*-diamonds)*

$$\begin{tikzcd} \mathrm{Gr}_{\mathcal{M}} \arrow[d, "\iota^0"'] & \mathrm{Gr}_{\mathcal{P}} \arrow[l] \arrow[r] \arrow[d, "\iota^+"'] & \mathrm{Gr}_{\mathcal{G}} \arrow[d, "\mathrm{id}"'] \\ (\mathrm{Gr}_{\mathcal{G}})^0 & (\mathrm{Gr}_{\mathcal{G}})^+ \arrow[l] \arrow[r] & \mathrm{Gr}_{\mathcal{G}}, \end{tikzcd} \tag{5.5}$$

with the following properties.

(1) *The maps* ι^0, ι^+ *are open and closed immersions.*
(2) *Their base changes* ι^0_F, ι^+_F *are isomorphisms.*
(3) *If* $\mathcal{G}$ *is special parahoric (for example, reductive), then* ι^0, ι^+ *are isomorphisms.*

In particular, the complements $(\mathrm{Gr}_{\mathcal{G}})^0 \setminus \iota^0(\mathrm{Gr}_{\mathcal{M}})$, $(\mathrm{Gr}_{\mathcal{G}})^+ \setminus \iota^+(\mathrm{Gr}_{\mathcal{P}})$ *are concentrated over* $\mathrm{Spd}\, k$*, that is, their base change to* $\mathrm{Spd}\, F$ *is empty.*

If $\mathcal{G}$ is reductive, then Theorem 5.2 is proved in [FS21, Proposition VI.3.1]. Indeed, in this case ι^0, ι^+ are isomorphisms. In general, we follow the strategy of [HR21]:

Proof of Theorem 5.2. First, we construct the maps ι^0, ι^+. The closed subgroup $\mathcal{M} \hookrightarrow \mathcal{G}$ induces a closed immersion $\mathrm{Gr}_{\mathcal{M}} \hookrightarrow \mathrm{Gr}_{\mathcal{G}}$ (using Theorem 4.9) that is $\mathbb{G}_m^{\diamondsuit}$-equivariant for the trivial action on the source. Thus, the map factors though the fixed points, defining the necessarily closed immersion $\iota^0\colon \mathrm{Gr}_{\mathcal{M}} \hookrightarrow (\mathrm{Gr}_{\mathcal{G}})^0$. The construction of ι^+ is more delicate and proceeds as follows. Pick a closed immersion $\mathcal{G} \hookrightarrow \mathrm{GL}_{n,O}$ of O-group schemes such that the fppf quotient $\mathrm{GL}_{n,O}/\mathcal{G}$ is quasi-affine, see [PR08, Proposition 1.3]. The cocharacter $\lambda'\colon \mathbb{G}_m \to^{\lambda} \mathcal{G} \hookrightarrow \mathrm{GL}_{n,O}$ induces parabolic subgroup $\mathcal{P}'$ with Lie algebra $\mathrm{Lie}\, \mathcal{P}' = (\mathrm{Lie}\, \mathrm{GL}_{n,O})_{\lambda' \geq 0}$. The induced map on fppf quotients $\mathcal{P}'/\mathcal{P} \to \mathrm{GL}_{n,O}/\mathcal{G}$ is a monomorphism between finite type O-schemes, thus quasi-affine by Zariski's main theorem. By functoriality of the construction $\mathcal{G} \mapsto \mathrm{Gr}_{\mathcal{G}}$, we obtain a commutative diagram of ind-(spatial $\mathrm{Spd}\, O$-diamonds):

$$\begin{tikzcd} \mathrm{Gr}_{\mathcal{P}} \arrow[dr, dotted, "\iota^+"'] \arrow[drr, bend left] \arrow[dd] & & \\ & (\mathrm{Gr}_{\mathcal{G}})^+ \arrow[r] \arrow[d] & \mathrm{Gr}_{\mathcal{G}} \arrow[d] \\ \mathrm{Gr}_{\mathcal{P}'} \arrow[r, "\cong"] & (\mathrm{Gr}_{\mathrm{GL}_{n,O}})^+ \arrow[r] & \mathrm{Gr}_{\mathrm{GL}_{n,O}} \end{tikzcd} \tag{5.6}$$

Using Theorem 4.9, the map $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Gr}_{\mathrm{GL}_{n,O}}$ is a closed immersion so that the square is Cartesian. This proves the existence of ι^+. Furthermore, the displayed map $\mathrm{Gr}_{\mathcal{P}'} \to (\mathrm{Gr}_{\mathrm{GL}_{n,O}})^+$ is an isomorphism by [FS21, Proposition VI.3.1]. As $\mathcal{P}'/\mathcal{P}$ is quasi-affine, so $\mathrm{Gr}_{\mathcal{P}} \to \mathrm{Gr}_{\mathcal{P}'}$ is a locally closed immersion (compare with the proof of [Zhu17b, Proposition 1.2.6] and [SW20, Lemma 19.1.5]), the map ι^+ is necessarily a locally closed immersion as well.

Now, part (2) is immediate from our construction and [FS21, Proposition VI.3.1] applied over $\mathrm{Spd}\, F$. For part (3), we observe that ι^0, ι^+ are bijective on geometric points if $\mathcal{G}$ is special parahoric: by (2) for geometric points lying over $\mathrm{Spd}\, F$ and by the Iwasawa decomposition [KP21, Section 3.3] for geometric points lying over $\mathrm{Spd}\, k$. As ι^0, ι^+ are locally closed immersions, they must be isomorphisms, so (3) follows.

For (1), it remains to prove that ι^0, ι^+ are open immersions for general parahoric group schemes $\mathcal{G}$. For this, we may and do assume that k is algebraically closed and observe that there are bijections of connected components

$$\pi_0((\mathrm{Gr}_{\mathcal{G}})^+) \xrightarrow{\cong} \pi_0((\mathrm{Gr}_{\mathcal{G}})^0) \xrightarrow{\cong} \pi_0((\mathcal{F}\ell_{\mathcal{G}})^0), \tag{5.7}$$

where the first holds by general properties of Bialynicki-Birula maps (see the proof of [Ric19, Corollary 1.12]) and the second by proper base change as in the proof of Lemma 4.17. The fixed points $(\mathcal{F}\ell_{\mathcal{G}})^0$ in the Witt vector partial affine flag variety can be analyzed in analogy to [HR21, Section 4]: concretely, if $\mathcal{P}_{\mathrm{sc}} = \mathcal{M}_{\mathrm{sc}} \ltimes \mathcal{U}$ for $\mathcal{M}_{\mathrm{sc}}$ being the corresponding parahoric model of M_{sc}, then there is a disjoint union (on points) into connected locally closed sub-ind-schemes

$$\mathcal{F}\ell_{\mathcal{G}} = \bigcup_{[w]} \mathcal{S}_w, \ \mathcal{S}_w = L_k \mathcal{P}_{\mathrm{sc}} \cdot w \tag{5.8}$$

where $[w]$ runs through the double coset $W_{M,\mathrm{af}} \backslash \tilde{W}/W_{\mathcal{G}}$ and w denotes the image of a representative under the embedding $\tilde{W}/W_{\mathcal{G}} \hookrightarrow \mathcal{F}\ell_{\mathcal{G}}$. The image of $\mathcal{F}\ell_{\mathcal{P}} \hookrightarrow \mathcal{F}\ell_{\mathcal{G}}$ consists of those $\mathcal{S}_w$ for $[w]$ lying in $W_{M,\mathrm{af}} \backslash \tilde{W}_M/W_{\mathcal{M}}$. Passing to fixed points, the image of $\mathcal{F}\ell_{\mathcal{M}} \hookrightarrow (\mathcal{F}\ell_{\mathcal{G}})^0$ is the union of the $L_k\mathcal{M}_{\mathrm{sc}}$-orbits for these $[w]$. So the map $\pi_0(\mathcal{F}\ell_{\mathcal{M}}) \to \pi_0((\mathcal{F}\ell_{\mathcal{G}})^0)$ identifies with the injection

$$W_{M,\mathrm{af}} \backslash \tilde{W}_M / W_{\mathcal{M}} \hookrightarrow W_{M,\mathrm{af}} \backslash \tilde{W} / W_{\mathcal{G}}. \tag{5.9}$$

We let $\mathcal{C}^0_{\mathcal{G}}$, respectively $\mathcal{C}^+_{\mathcal{G}}$, be the open and closed sub-v-sheaf of $(\mathrm{Gr}_{\mathcal{G}})^0$, respectively of $(\mathrm{Gr}_{\mathcal{G}})^+$, consisting of those components belonging to $\mathrm{im}(\pi_0(\mathcal{F}\ell_{\mathcal{M}}) \hookrightarrow \pi_0(\mathcal{F}\ell_{\mathcal{G}}))$ under (5.7). Then the maps ι^0, ι^+ factor through $\mathcal{C}^0_{\mathcal{G}}$, respectively $\mathcal{C}^+_{\mathcal{G}}$ inducing locally closed immersions

$$\mathrm{Gr}_{\mathcal{M}} \hookrightarrow \mathcal{C}^0_{\mathcal{G}}, \ \mathrm{Gr}_{\mathcal{P}} \hookrightarrow \mathcal{C}^+_{\mathcal{G}}, \tag{5.10}$$

that are bijective on geometric points, hence isomorphisms. The theorem is thus proven. □

5.2. **Semi-infinite orbits.** We end this section with a study of the stratification (5.8). Throughout, we assume that $k = \bar{k}$ is algebraically closed so that $F = \breve{F}$ and $O = \breve{O}$. Note that the torus S is then maximal F-split. The following lemma simplifies some arguments of [HR21, Theorem 6.12] and is used in the proof of Theorem 6.16 given in Section 6.5.

Lemma 5.3. *For every $w \in \tilde{W}/W_{\mathcal{G}}$, there is an O-cocharacter $\mathbb{G}_m \to \mathcal{S} \subset \mathcal{G}$ such that for the induced strata $\mathcal{S}_w \cap \mathcal{F}\ell_{\mathcal{G},w} = \{w\}$.*

Proof. Up to changing the Iwahori $L_k^+\mathcal{I} \subset L_k^+\mathcal{G}$, we may and do assume that the Iwahori–Schubert variety $\mathcal{F}\ell^\circ_{(\mathcal{I},\mathcal{G}),w}$ is a dense open of $\mathcal{F}\ell_{\mathcal{G},w}$. Notice that the closed complement of that dense open is stable under the $\mathbb{G}_m^{\mathrm{perf}}$-action, so it follows that the connected component of the fixed point w in the attractor $\mathcal{F}\ell_{\mathcal{G}}^+$ cuts $\mathcal{F}\ell_{\mathcal{G},w}$ inside $\mathcal{F}\ell^\circ_{(\mathcal{I},\mathcal{G}),w}$.

The reduced word $\dot{w} = s_1 \dots s_n$ determines a minimal gallery $\Gamma = (\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_n)$, where $\mathbf{a}_i = s_1 \dots s_i(\mathbf{a})$, going from the alcove $\mathbf{a}$ fixed by $\mathcal{I}(O)$ to its $\dot{w}$-conjugate. Let α_i be the unique positive affine root such that $\partial\alpha_i$ is the wall separating $\mathbf{a}_{i-1}$ and $\mathbf{a}_i$. We claim that

$$\mathcal{F}\ell^\circ_{(\mathcal{I},\mathcal{G}),w} = L_k^+\mathcal{U}_{\alpha_1} \cdot \dots \cdot L_k^+\mathcal{U}_{\alpha_n} w. \tag{5.11}$$

This follows by expanding the Demazure twisted product, pulling across the simple reflections to the right, compare with Proposition 4.5. Indeed, $s_{i-1} \dots s_1(\alpha_i)$ is by construction the positive simple affine root attached to s_i. We need to produce an O-cocharacter $\mathbb{G}_m \to \mathcal{S}$ whose induced $\mathbb{G}_m^{\mathrm{perf}}$-action repels every affine root group $L^+\mathcal{U}_{\alpha_i}$, because then it would also repel $\mathcal{F}\ell^\circ_{(\mathcal{I},\mathcal{G}),w}$ by (5.11). This follows now by [HN02, Corollary 5.6] but we give below a quick proof for the reader's convenience.

Consider the subset $\Phi_{G,w} \subset \Phi_G$ of all euclidean roots $a = \nabla\alpha$ which are gradients of the prescribed affine root α_i attached to Γ. If b denotes the barycenter of $\mathbf{a}$, then by definition $a(\dot{w}b - b) < 0$ is stricly negative for $a \in \Phi_{G,w}$. In particular, $\Phi_{G,w}$ consists entirely of negative B-roots where $S \subset B \subset G$ is a Borel subgroup whose closed Weyl chamber contains the vector $\dot{w}b - b$. So we may take any B-dominant regular coweight $\mathbb{G}_m \to S$ which uniquely extends to the desired $\mathbb{G}_m \to \mathcal{S}$ because S is F-split. □

Now assume that λ is regular. Then $M_\lambda = T$ is a maximal torus, $P_\lambda = B$ a Borel subgroup with unipotent radical U defined over F. The stratification (5.8) becomes

$$\mathcal{F}\ell_{\mathcal{G}} = \bigcup_{w \in \tilde{W}/W_{\mathcal{G}}} \mathcal{S}_w, \ \mathcal{S}_w = L_kU \cdot w \tag{5.12}$$

and the strata $\mathcal{S}_w$ are called semi-infinite orbits, compare with [FS21, Proposition VI.3.1]. Recall that there is a semi-infinite Bruhat order $\leq^{\frac{\infty}{2}}$ on $\tilde{W}/W_{\mathcal{G}}$ defined by:

$$v \leq^{\frac{\infty}{2}} w \quad\Longleftrightarrow\quad \forall\, \nu_I \gg 0\colon\ \nu_I(\pi) \cdot v \leq \nu_I(\pi) \cdot w \tag{5.13}$$

where $X_*(T)_I \subset \tilde{W}$, $\nu_I \mapsto \nu_I(\pi)$ is viewed as a subgroup using the Kottwitz morphism, see (3.19), and where $\nu_I \gg 0$ means that ν_I is sufficiently B-dominant. This order was first introduced by Lusztig in [Lus80] and depends on the F-Borel subgroup B attracted by λ.

Proposition 5.4. *The ind-closure of $\mathcal{S}_w$ inside $\mathcal{F}\ell_{\mathcal{G}}$ is given by the perfect sub-ind-scheme whose geometric points factor through some $\mathcal{S}_v$ with $v \leq^{\frac{\infty}{2}} w$.*

Proof. Let $\mathcal{I} \to \mathcal{G}$ be an auxiliary Iwahori model, fixed for the remainder of the proof. Suppose there is a curve $\mathcal{C} \subset \mathcal{F}\ell_{\mathcal{G}}$ containing v and whose complement $\mathcal{C}^\circ := \mathcal{C} \setminus \{v\}$ is contained in $\mathcal{S}_w = L_kU \cdot w$. Now, notice that, for sufficiently dominant ν_I, one gets the inclusion

$$\nu_I(\pi) \cdot \mathcal{C}^\circ \subset L_k^+\mathcal{I} \cdot \nu_I(\pi) \cdot w, \tag{5.14}$$

because conjugation by $\nu_I(\pi)$ moves any given perfect k-subscheme of L_kU inside the Iwahori loop group $L_k^+\mathcal{I} \subset L_k^+\mathcal{G}$. This implies the inequality $\nu_I(\pi)v \leq \nu_I(\pi)w$, and therefore $v \leq^{\frac{\infty}{2}} w$.

Conversely, assume that the inequality $v \leq^{\frac{\infty}{2}} w$ holds. By definition, for all sufficiently dominant translations $\nu_I \in X_*(T)_I$, the inequality $\nu_I(\pi) \cdot w \leq \nu_I(\pi) \cdot v$ holds in the Bruhat order. By enlarging ν_I if necessary, we may assume the $\mathcal{F}\ell^\circ_{(\mathcal{I},\mathcal{G}),\nu_I(\pi)\cdot w}$ is of the form $\prod_{\alpha \in \Gamma} L_k^+\mathcal{U}_\alpha \cdot \nu_I(\pi) \cdot w$ where all of the $\alpha \in \Gamma$ have positive gradient. There is a curve $\mathcal{C}$ in $\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\nu_I(\pi)\cdot w}$ joining $\nu_I(\pi) \cdot w$ to $\nu_I(\pi) \cdot v$ since $\nu_I(\pi) \cdot v \leq \nu_I(\pi) \cdot w$. By our assumption on $\nu_I(\pi)$, we have $\mathcal{C}^\circ \subset \mathcal{S}_{\nu_I(\pi)\cdot w}$ for $\mathcal{C}^\circ := \mathcal{C} \cap \mathcal{F}\ell^\circ_{(\mathcal{I},\mathcal{G}),\nu_I(\pi)\cdot w}$. Now, the map $t_{\nu_I} \colon \mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),w} \to \mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\nu_I(\pi)\cdot w}$ induced by left translation with $\nu_I(\pi)$ is an isomorphism and hence induces $\mathcal{S}_w \cong \mathcal{S}_{\nu_I(\pi)\cdot w}$. Then the curve $t_{\nu_I}^{-1}(\mathcal{C})$ joins w to v, and $t_{\nu_I}^{-1}(\mathcal{C}^\circ) \subset \mathcal{S}_w$. □

Next, we extend the equi-dimensionality of Mirković–Vilonen cycles [MV07, Theorem 3.2] (see also [Zhu17a, Corollary 2.8] and [FS21, Corollary VI.3.8]) from split groups to twisted groups as follows. We continue to assume that λ is regular and, additionally, that $\mathcal{G}$ is special parahoric. Then $X_*(T)_I = \tilde{W}/W_{\mathcal{G}}$ and (5.12) becomes

$$\mathcal{F}\ell_{\mathcal{G}} = \bigcup_{\nu_I \in X_*(T)_I} \mathcal{S}_{\nu_I}, \ \mathcal{S}_{\nu_I} = L_kU \cdot \nu_I(\pi). \tag{5.15}$$

If $\mathcal{G}$ is reductive, then $\mathcal{F}\ell_{\mathcal{G}}$ is the Witt vector affine Grassmannian studied in [Zhu17a]. So, in general, $\mathcal{F}\ell_{\mathcal{G}}$ can be regarded as a twisted version when $\mathcal{G}$ is special parahoric. Indeed, $\mathcal{F}\ell_{\mathcal{G}} = \operatorname{colim} \mathcal{F}\ell_{\mathcal{G},\mu_I}$ where μ_I runs through $X_*(T)_{I,+}$, the image of the B-dominant cocharacters under the projection $X_*(T) \to X_*(T)_I$ equipped with the induced dominance order and $\mathcal{F}\ell_{\mathcal{G},\mu_I} := \mathcal{F}\ell_{\mathcal{G},\mu_I(\pi)}$, as is usual notation for (twisted) affine Grassmannians. Also, the semi-infinite Bruhat order on $X_*(T)_I$ specializes to the dominance relation, that is, $\nu_I' \leq^{\frac{\infty}{2}} \nu_I$ if and only if $\nu_I - \nu_I'$ is a sum of coinvariants of positive roots with non-negative coefficients.

Lemma 5.5. *For any $\nu_I \in X_*(T)_I$, the intersection $\mathcal{S}_{\nu_I} \cap \mathcal{F}\ell_{\mathcal{G},\mu_I}$ is non-empty if and only if ν_I lies in the $W_{\mathcal{G}}$-orbit of some $\mu_I' \in X_*(T)_{I,+}$ with $\mu_I' \leq \mu_I$. In this case, it is affine and equidimensional of dimension $\langle \rho_G, \nu + \mu \rangle$.*

Here $\rho_G \in X^*(T)$ denotes the half sum of the B-positive roots. We note that the pairing $\langle \rho_G, \nu + \mu \rangle$ is well-defined independently of the choice of lifts $\nu, \mu \in X_*(T)$ of ν_I, μ_I because ρ_G is I-invariant.

Proof of Lemma 5.5. The map $(\mathcal{F}\ell_{\mathcal{G},\mu_I})^+ \to \mathcal{F}\ell_{\mathcal{G},\mu_I}$ induces an isomorphism (see Theorem 5.2)

$$(\mathcal{F}\ell_{\mathcal{G},\mu_I})^+ \xrightarrow{\cong} \coprod_{\nu_I \in X_*(T)_I} \mathcal{S}_{\nu_I} \cap \mathcal{F}\ell_{\mathcal{G},\mu_I}. \tag{5.16}$$

Under $(\mathcal{F}\ell_{\mathcal{G},\mu_I})^+ \to (\mathcal{F}\ell_{\mathcal{G},\mu_I})^0$, the component $\mathcal{S}_{\nu_I} \cap \mathcal{F}\ell_{\mathcal{G},\mu_I}$ contracts to $\underline{\{\nu_I\}} = \operatorname{Spec}(k)$. Thus, it is affine because Bialynicki-Birula maps for schemes are affine by [Ric19, Corollary 1.12]. Also, $(\mathcal{F}\ell_{\mathcal{G},\mu_I})^0$ identifies with the constant scheme associated with the subset of $\nu_I \in X_*(T)_I$ lying in the $W_{\mathcal{G}}$-orbit of some $\mu_I' \in X_*(T)_{I,+}$ with $\mu_I' \leq \mu_I$. So only such ν_I contribute to (5.16), and as the Bialynicki-Birula map has a section, the non-emptiness criterion for $\mathcal{S}_{\nu_I} \cap \mathcal{F}\ell_{\mathcal{G},\mu_I}$ holds true. Furthermore, the union over all $\mathcal{S}_{\nu_I'} \cap \mathcal{F}\ell_{\mathcal{G},\mu_I}$ with $\nu_I' \leq^{\frac{\infty}{2}} \nu_I$ is a closed perfect subscheme by Proposition 5.4.

As noted in [FS21, Corollary VI.3.8], the affineness implies the dimension formula once we show that $\mathcal{S}_{\mu_I^{\text{anti}}} \cap \mathcal{F}\ell_{\mathcal{G},\mu_I}$ is a point where μ_I^{anti} is the antidominant element in the $W_{\mathcal{G}}$-orbit of μ_I. This follows from the proof of Lemma 5.3. □

6. Nearby cycles of étale sheaves

6.1. **Recollections.** In [Sch17], Scholze constructs a category of étale sheaves

$$\mathrm{D}(X, \Lambda) := \mathrm{D}_{\text{ét}}(X, \Lambda) \tag{6.1}$$

for all small v-stacks X. As coefficients Λ, we allow prime-to-p torsion rings or, by the adic formalism of [Sch17, Section 26], an ℓ-torsion free, complete ℓ-adic ring for $\ell \neq p$, or a ring of the form $\Lambda = \Lambda_0[\ell^{-1}]$ where Λ_0 is as in the previous case. In the final case, as this is not covered in [Sch17], we define the triangulated category

$$\mathrm{D}(X, \Lambda) := \mathrm{D}(X, \Lambda_0) \otimes_{\Lambda_0} \Lambda, \tag{6.2}$$

in analogy to the classical definition for schemes, for example, see [KW01, Appendix A]. The adic formalism of [Sch17, Section 26] carries over to the categories (6.2). Finally, we also allow Λ to be a filtered colimit of the aforementioned rings, with the obvious definition for the categories. This includes algebraic field extensions $L/\mathbb{Q}_\ell$ and their rings of integers O_L.

The categories of étale sheaves are equipped with the usual six functors formalism: the endofunctors $\otimes^{\mathbb{L}}$, $R\mathscr{H}om$ and functors Rf_*, f^* for a morphism $f\colon X \to Y$ of small v-stacks. If f is compactifiable and representable in locally spatial diamonds with $\dim.\mathrm{tr} f < \infty$, we dispose of the functors $Rf_!$, $Rf^!$, completing the six functor formalism.

In general, the categories $\mathrm{D}(X, \Lambda)$ and the six functors are rather inexplicit, constructed through v-descent using Lurie's ∞-categorical machinery. Nevertheless, whenever $f\colon X \to Y$ is a morphism between locally spatial diamonds, then X and Y admit a well-defined étale site and Scholze's operations are very closely related to the operations that one can construct site-theoretically, see [Sch17, Proposition 14.15, Section 17].

When X and Y are locally spatial diamonds we say that an object $A \in \mathrm{D}(X, \Lambda)$ is ULA (=universally locally acyclic) with respect to f if, for all locally spatial diamonds $Y' \to Y$, the pullback $A' \in \mathrm{D}(X', \Lambda)$ is overconvergent along the fibers of $f' : X' = X \times_Y Y' \to Y'$ and $R(f' \circ j')_! j'^* A$ is perfect-constructible for all separated étale neighborhoods $j'\colon U' \to X'$ for which $f' \circ j'$ is quasi-compact, see [FS21, Definition IV.2.1]. If Λ is ℓ-adic as above, then a complex $A \in \mathrm{D}(X, \Lambda)$ is called perfect-constructible if $A \otimes^{\mathbb{L}}_\Lambda \Lambda/\ell$ is étale locally perfect-constant after passing to a constructible stratification, equivalently $A \otimes^{\mathbb{L}}_\Lambda \Lambda/\ell^n$ are so for all $n \geq 1$. Finally, if $\Lambda = \Lambda_0[\ell^{-1}]$ is as in (6.2), then an object in $\mathrm{D}(X, \Lambda)$ is called perfect-constructible if it admits a Λ_0-lattice which is so. For X and Y more general v-stacks (and $f\colon X \to Y$ representable in locally spatial diamonds), we call A ULA if it is ULA after any base change $S \to Y$ with S a locally spatial diamond.

Suppose X is a small v-stack proper and representable in spatial diamonds over a base S, and that X is equipped with an action by $\mathbb{G}_{m,S}^\diamondsuit$ satisfying the conditions [FS21, Hypothesis IV.6.1.]. One can consider the v-stacks

$$X^\pm = \operatorname{Hom}_{\mathbb{G}_m^\diamondsuit}\left((\mathbb{A}^1)^{\pm,\diamondsuit}, X\right) \tag{6.3}$$

which (by hypothesis) are represented by a finite partition of X into locally closed subsets. This also induces a partition of the fixed-point v-stack $X^0 = X^{\mathbb{G}_m^\diamond}$ into closed and open subsets. We have inclusion maps $q^\pm \colon X^\pm \to X$ and projection maps $p^\pm \colon X^\pm \to X^0$, from that we obtain the hyperbolic localization functor

$$L_{X/S} \colon \mathrm{D}(X/\mathbb{G}_{m,S}^\diamond, \Lambda) \to \mathrm{D}(X^0, \Lambda), \tag{6.4}$$

which can be expressed as $R(p^+)_!(q^+)^*$ or equivalently as $R(p^-)_* R(q^-)^!$ by [FS21, Theorem IV.6.5]. This functor enjoys plenty of compatibilities, in analogy to [Ric19], which we will exploit to compute nearby cycles, see [FS21, Propositions IV.6.12, IV.6.13, IV.6.14].

6.2. **Over C.** We continue with the notation and denote by $F/\mathbb{Q}_p$ a complete discretely valued field with ring of integers O and perfect residue field k of characteristic $p > 0$. Also, we fix a complete algebraic closure C/F, and a connected reductive F-group G.

In this section, we recall the structure of the categories of monodromic sheaves with bounded support $\mathrm{D}(\mathrm{Hk}_{G,C}, \Lambda)^{\mathrm{bd}}$ and $\mathrm{D}(\mathrm{Gr}_{G,C}, \Lambda)^{\mathrm{mon,bd}}$ studied in [FS21, Section VI]. As in Section 5, for any cocharacter $\lambda \colon \mathbb{G}_m \to G_C$, we have the induced $\mathbb{G}_m^\diamond$-action on $\mathrm{Gr}_{G,C}$, whose attractors only depend on the attracting parabolic $P \subset G_C$.

In particular, hyperbolic localization gives a constant terms functor

$$\mathrm{CT}_P \colon \mathrm{D}(\mathrm{Hk}_{G,C}, \Lambda)^{\mathrm{bd}} \to \mathrm{D}(\mathrm{Gr}_{G,C}, \Lambda)^{\mathrm{mon,bd}} \xrightarrow{L_{\mathrm{Gr}_{G/C}}} \mathrm{D}(\mathrm{Gr}_{M,C}, \Lambda) \tag{6.5}$$

providing the main tool to effectively study the category of derived étale sheaves on $\mathrm{Hk}_{G,C}$ as in [FS21, Corollary VI.3.5]. One of the crucial techniques is the following conservativity lemma [FS21, Proposition VI.4.2] whose proof we sketch for convenience.

Lemma 6.1. *Let $T \subset B \subset G_C$ be an arbitrary maximal torus and a Borel containing it. Then $A \in \mathrm{D}(\mathrm{Hk}_{G,C}, \Lambda)^{\mathrm{bd}}$ vanishes if and only if $\mathrm{CT}_B(A) \in \mathrm{D}(\mathrm{Gr}_{T,C}, \Lambda)$ does.*

Proof. The proof is done by considering a maximal strata where A is concentrated. This strata is of the form $[\mathrm{Spd}\, C/(L_C^+ G)_\mu]$ for the stabilizer of $\mu \in X_*(T)_+$. The attractor of $\mathrm{Gr}_{G,C,\mu}$ at the anti-dominant coweight $-\mu$ with respect to B is an isolated point. Using the $R(p^+)_!(q^+)^*$-version of hyperbolic localization, we see that the fiber of CT_B over $\mu \in \mathrm{Gr}_T$ agrees with pullback to this point. □

This allows us to localize several properties of derived objects in $\mathrm{D}(\mathrm{Hk}_{G,C}, \Lambda)^{\mathrm{bd}}$. For instance, A is ULA if and only if $\mathrm{CT}_B(A)$ which, in turn, is equivalent to $[\mu]^* A$ being a perfect object for all maps $[\mu] \colon \mathrm{Spd}\, C \to \mathrm{Hk}_G$ with $\mu \in X_*(T)$, see [FS21, Propositions VI.6.4, VI.6.5].

Next, we move to the natural perverse t-structure on $\mathrm{D}(\mathrm{Hk}_{G,C,\mu}, \Lambda)^{\mathrm{bd}}$, see [FS21, Definition/Proposition VI.7.1]. This is given in terms of the following subcategory

$${}^p\mathrm{D}^{\leq 0}(\mathrm{Hk}_{G,C}, \Lambda)^{\mathrm{bd}} = \{A \in \mathrm{D}(\mathrm{Hk}_{G,C}, \Lambda)^{\mathrm{bd}} \colon j_\mu^* A \in \mathrm{D}^{\leq -\langle 2\rho, \mu \rangle}\}, \tag{6.6}$$

which determines ${}^p\mathrm{D}^{\geq 0}(\mathrm{Hk}_{G,C}, \Lambda)$ uniquely. Intersecting these two, we get the category of perverse sheaves $\mathrm{Perv}(\mathrm{Hk}_{G,C}, \Lambda)$.

Thanks to [FS21, Proposition VI.7.4], the t-structure is preserved under and detected by $\mathrm{CT}_B[\deg_B]$. Here, for any C-parabolic $B \subset P$ with Levi quotient M, the degree is the locally constant function on $\mathrm{Gr}_{P,C}$ induced by

$$\deg_P(\lambda) = \langle 2\rho_G - 2\rho_M, \lambda \rangle, \tag{6.7}$$

where $\lambda \in X_*(T)$ is a coweight and ρ_M is the half-sum of all B-positive M-roots. The main geometric fact used in the proof of [FS21, Proposition VI.7.4] is the equidimensionality of semi-infinite orbits.

When working with torsion coefficients, it is convenient to single out flat perverse sheaves, which are those objects A such that for every Λ-module M the complex $A \otimes_\Lambda^{\mathbb{L}} M$ is perverse.

Definition 6.2. The Satake category $\mathrm{Sat}(\mathrm{Hk}_{G,C}, \Lambda)$ is the full subcategory of flat ULA objects in $\mathrm{Perv}(\mathrm{Hk}_{G,C}, \Lambda)$.

The Satake category is endowed with a monoidal product

$$\star \colon \mathrm{Sat}(\mathrm{Hk}_{G,C}, \Lambda) \times \mathrm{Sat}(\mathrm{Hk}_{G,C}, \Lambda) \to \mathrm{Sat}(\mathrm{Hk}_{G,C}, \Lambda) \tag{6.8}$$

arising from the convolution Hecke stack $\mathrm{Hk}_{G,C} \widetilde{\times} \mathrm{Hk}_{G,C}$, see [FS21, Proposition VI.8.1]. Due to the fusion interpretation [FS21, Definition/Proposition VI.9.4], the monoidal structure is naturally symmetric monoidal.

Taking cohomology of the affine Grassmannian furnishes a fiber functor

$$F \colon \mathrm{Sat}(\mathrm{Hk}_{G,C}, \Lambda) \to \mathrm{Rep}(\Lambda) \tag{6.9}$$

to the category of Λ-finite locally free modules and the Tannakian formalism gives us an interpretation of these categories in terms of a group of automorphisms.

Theorem 6.3 (Fargues–Scholze)**.** *The automorphism group of F is naturally isomorphic to the Langlands dual group $\widehat{G}_\Lambda$ formed over Λ.*

One may regard the dual group $\widehat{G}_\Lambda$ as combinatorially defined in terms of root data (a priori), or as its correct natural definition (a posteriori). The cyclotomic twist in [FS21, Theorem VI.11.1] is not needed, because we are working over the algebraically closed field C.

6.3. **Over $\bar{k}$.** Let $\mathcal{G}$ be a parahoric O-model of G. In this section, we look at what happens with the geometric special fiber Hecke v-stack $\mathrm{Hk}_{\mathcal{G},\bar{k}} = L^+_{\bar{k}}\mathcal{G}^\diamondsuit \backslash L_{\bar{k}}G^\diamondsuit / L^+_{\bar{k}}\mathcal{G}^\diamondsuit$. We note that the categories of étale sheaves compare well to their scheme-theoretic companions, see Proposition A.5.

We continue with the notation and, in addition, fix a maximal $\breve{F}$-split F-torus $S \subset G$ containing a maximal F-split torus (see [BT84, Proposition 5.1.10]) with centralizer T, which is a maximal F-torus inside G, such that their connected Néron O-models $\mathcal{S} \subset \mathcal{T}$ embed into $\mathcal{G}$. Each parabolic subgroup $P \subset G_{\breve{F}}$ with Levi M containing $S_{\breve{F}}$ extends to a diagram of O-group schemes $\mathcal{M} \leftarrow \mathcal{P} \rightarrow \mathcal{G}_{\breve{O}}$ by taking flat closures. Again, choosing a cocharacter $\lambda\colon \mathbb{G}_m \to \mathcal{S}_{\breve{O}} \subset \mathcal{G}_{\breve{O}}$ with $M = M_\lambda$ and $P = P_\lambda^+$, the formalism of Section 5 applies to define $\mathbb{G}_m^\diamondsuit$-actions on the pro-smooth cover $\mathrm{Gr}_{\mathcal{G},\bar{k}} = \mathcal{F}\ell^\diamondsuit_{\mathcal{G},\bar{k}}$. This gives rise to constant term functors

$$\mathrm{CT}_{\mathcal{P}}\colon \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}}, \Lambda)^{\mathrm{bd}} \to \mathrm{D}(\mathcal{F}\ell^{0,\diamondsuit}_{\mathcal{G},\bar{k}}, \Lambda)^{\mathrm{bd}}, \tag{6.10}$$

not depending on the choice of λ such that $M = M_\lambda$ and $P = P_\lambda^+$. Here, the fixed points $\mathcal{F}\ell^0_{\mathcal{G},\bar{k}}$ contain $\mathcal{F}\ell_{\mathcal{M},\bar{k}}$ as an open and closed sub-ind-scheme by Theorem 5.2, but are strictly bigger unless $\mathcal{G}_{\breve{O}}$ is special parahoric.

As in the previous section, we analyse the key properties ULA, flatness and perversity. The crucial step is the following conservativity result.

Proposition 6.4. *An object $A \in \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}}, \Lambda)^{\mathrm{bd}}$ vanishes if and only if $\mathrm{CT}_{\mathcal{B}}(A)$ does for every $\breve{F}$-Borel $S_{\breve{F}} \subset B \subset G_{\breve{F}}$.*

Proof. Just like in Lemma 6.1, we argue on a maximal strata of $\mathrm{Hk}_{\mathcal{G},\bar{k}}$ where A does not vanish, say one indexed by some w. In this case, by Lemma 5.3 there is a choice of $\breve{F}$-Borel B for which the associated attractor intersects $\mathcal{F}\ell_{\mathcal{G},\bar{k},w}$ in an isolated point. In this case, $\mathrm{CT}_{\mathcal{B}}$ agrees with pullback to this point. □

Definition 6.5. An object $A \in \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}}, \Lambda)^{\mathrm{bd}}$ is ULA whenever its pullback to $\mathcal{F}\ell^\diamondsuit_{\mathcal{G},\bar{k},w}$ is ULA over $\mathrm{Spd}\,\bar{k}$.

A priori, this notion depends on the choice of left or right trivialization, but it follows a posteriori from Proposition 6.7 that it does not, see [FS21, Proposition VI.6.2]. The ULA property interacts very well with constant terms:

Proposition 6.6. *If $A \in \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}}, \Lambda)^{\mathrm{bd}}$ is ULA, then so is $\mathrm{CT}_{\mathcal{P}}(A)$. Conversely, if $\mathrm{CT}_{\mathcal{B}}(A)$ is ULA for all Borel subgroups $S_{\breve{F}} \subset B \subset G_{\breve{F}}$, then so is A.*

Proof. Abusing notation, we also call A the pullback of this object to $\mathcal{F}\ell^\diamondsuit_{\mathcal{G},\bar{k}}$. By [FS21, Theorem IV.2.23], to prove that for $B = A$ or $B = \mathrm{CT}_{\mathcal{P}}(A)$, the object B is ULA it is enough to show that

$$p_1^*\mathbb{D}(B) \otimes p_2^*B \to R\mathscr{H}\mathrm{om}(p_1^*B, Rp_2^!B) \tag{6.11}$$

is an isomorphism. Now, for any pair of flat closures of parabolics $\mathcal{P}_1$ and $\mathcal{P}_2$ a direct computation (using properties of hyperbolic localization, cf. the proof of [FS21, Proposition VI.6.4]) shows that

$$\mathrm{CT}_{\mathcal{P}_1\times\mathcal{P}_2}(p_1^*\mathbb{D}(A) \otimes p_2^*A) = p_1^*\mathbb{D}(\mathrm{CT}_{\mathcal{P}_1^-}(A)) \otimes p_2^*\,\mathrm{CT}_{\mathcal{P}_2}(A) \tag{6.12}$$

and that

$$\mathrm{CT}_{\mathcal{P}_1\times\mathcal{P}_2}(R\mathscr{H}\mathrm{om}(p_1^*A, Rp_2^!A)) = R\mathscr{H}\mathrm{om}(p_1^*(\mathrm{CT}_{\mathcal{P}_1^-}(A), Rp_2^!\,\mathrm{CT}_{\mathcal{P}_2}(A))) \tag{6.13}$$

where $\mathcal{P}_1^-$ is opposite to $\mathcal{P}_1$. In the forward direction, it is enough to use this for $\mathcal{P}_1 = \mathcal{P}^-$ and $\mathcal{P}_2 = \mathcal{P}$. For the converse, we let K denote the cone of Equation (6.11). By the conservativity of Proposition 6.4, it is enough to prove $\mathrm{CT}_{\mathcal{B}_1,\mathcal{B}_2}(K) = 0$ for all $\mathcal{B}_1$, $\mathcal{B}_2$ since these exhaust the Borel subgroups of of $G_{\breve{F}} \times G_{\breve{F}}$. But this follows from the computation above, [FS21, Proposition IV.2.19] and the hypothesis that $\mathrm{CT}_{\mathcal{B}_1}(A)$ is ULA. □

We prove that ULA sheaves admit an easy description in terms of restrictions to Schubert strata:

Proposition 6.7. *The following are equivalent for an object $A \in \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}}, \Lambda)^{\mathrm{bd}}$:*

(1) *A is ULA.*
(2) *For all strata of A pullback along $[w]\colon \mathrm{Spd}\,\bar{k} \to \mathrm{Hk}_{\mathcal{G},\bar{k}}$ is a perfect complex*[17] *in*

$$\mathrm{D}(\mathrm{Spd}\,\bar{k}, \Lambda) = \mathrm{D}(\Lambda). \tag{6.14}$$

(3) *The pullback to $\mathcal{F}\ell^\diamondsuit_{\mathcal{G},\bar{k}}$ lies in*

$$\mathrm{D}_{\mathrm{cons}}(\mathcal{F}\ell_{\mathcal{G},\bar{k}}, \Lambda)^{\mathrm{bd}} \subset \mathrm{D}(\mathcal{F}\ell^\diamondsuit_{\mathcal{G},\bar{k}}, \Lambda)^{\mathrm{ula,bd}}, \tag{6.15}$$

where $\mathrm{D}_{\mathrm{cons}}(\mathcal{F}\ell_{\mathcal{G},\bar{k}}, \Lambda)^{\mathrm{bd}}$ is the category of perfect-constructible Λ-sheaves with bounded support on the ind-scheme $\mathcal{F}\ell_{\mathcal{G},\bar{k}}$ and the inclusion is the one constructed in [Sch17, Section 27].

[17] If $\Lambda = \Lambda_0[\ell^{-1}]$ as in (6.2), then we require the pullback to arise as the ℓ localization of a perfect complex over Λ_0.

Proof. The equivalence of (2) and (3) follows from Proposition A.5, and the fact that the equivalence in Proposition A.5 is compatible with pullback to $\mathcal{F}\ell_{\mathcal{G}}$, respectively to $\mathcal{F}\ell_{\mathcal{G}}^{\diamondsuit}$ or to strata.

Let j_w denote the inclusion of the strata in $\mathrm{Hk}_{\mathcal{G},\bar{k}}$ corresponding to w. For proving that (2) implies (1) one can, as in [FS21, Proposition VI.6.5], reduce to showing that $R(j_w)_!\Lambda$ is ULA (for each w). But their pullback to $\mathcal{F}\ell_{\mathcal{G},\bar{k}}$ are clearly algebraic and by Proposition A.1 they are also ULA, see also [FS21, Proposition IV.2.30]. Alternatively one can use the Demazure resolution, compare with [FS21, Proposition VI.5.7].

For the converse implication, we induct on the number of strata where A does not vanish and consider the cone of $R(j_w)_!j_w^*A \to A$ for a maximal strata w. Indeed, pullback by open immersion preserves being ULA by [FS21, Proposition VI.2.13.(i)]. Then we apply [FS21, Proposition VI.4.1] to see that $j_w^*A \in \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k},\mu},\Lambda)$ has a perfect stalk, and use that $R(j_w)_!(j_w^*A)$ is ULA, by the proven (2) implies (1). Then we can conclude by induction. □

Arguing as in [FS21, Definition/Proposition VI.7.1], we can define a perverse t-structure.

Definition 6.8. The perverse t-structure on $\mathrm{Hk}_{\mathcal{G},\bar{k}}$ is the only such that

$$ {}^p\mathrm{D}^{\leq 0}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda) = \{A \in \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda) \colon j_w^*A \in \mathrm{D}^{\leq -l(w)}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda)\}, \tag{6.16}$$

respectively

$$ {}^p\mathrm{D}^{\geq 0}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda) = \{A \in \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda) \colon Rj_w^!A \in \mathrm{D}^{\geq -l(w)}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda)\}. \tag{6.17}$$

Perverse sheaves

$$ \mathrm{Perv}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda) = {}^p\mathrm{D}^{\leq 0}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda) \cap {}^p\mathrm{D}^{\geq 0}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda) \tag{6.18}$$

are the heart of the t-structure. Such an A is flat perverse if in addition $A \otimes_\Lambda^{\mathbb{L}} M$ is in $\mathrm{Perv}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda)$ for all Λ-modules M.

We note that, in general, there cannot be any degree shifts such that $\mathrm{CT}_{\mathcal{P}}[\deg_{\mathcal{P}}]$ preserves the perverse t-structure, due to lack of parity. But, we define

$$ \deg_{\mathcal{P}}(\lambda_I) = \langle 2\rho_G - 2\rho_M, \lambda\rangle \tag{6.19}$$

for translation elements $\lambda_I \in X_*(T)_I$. This is useful for the following result:

Proposition 6.9. *Assume that $\mathcal{G}_{\breve{O}}$ is special parahoric, and let $A \in \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda)$. Then A is perverse if and only if $\mathrm{CT}_{\mathcal{B}}(A)[\deg_{\mathcal{B}}] \in \mathrm{Perv}(\mathcal{F}\ell_{\mathcal{T},\bar{k}}^{\diamondsuit},\Lambda)$ for all Borel subgroups $S_{\breve{F}} \subset B \subset G_{\breve{F}}$. The same applies to the flat objects.*

Proof. It suffices to follow the proof of [FS21, Proposition VI.7.4]. For preserving the t-structure, we use the fact that the non-empty intersections $\mathcal{S}_{\bar{k},\lambda_I} \cap \mathcal{F}\ell_{\mathcal{G},\bar{k},\nu_I}$ are equidimensional of dimension $\langle 2\rho_G, \lambda+\nu\rangle$, see Lemma 5.5. The converse then follows from Proposition 6.4. □

In particular, it is now permitted to introduce the Satake category at special level. Notice that there is no hope of such a well-behaved class of objects to exist at arbitrary level, because the quotient $\widetilde{W}/W_{\mathcal{G}}$ carries no natural abelian structure.

Definition 6.10. Let $\mathcal{G}_{\breve{O}}$ be special parahoric. Then the Satake category $\mathrm{Sat}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda)$ is the full subcategory of $\mathrm{Perv}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda)$ comprised of flat ULA objects.

This category lies within the category of perverse sheaves $\mathrm{Perv}(\mathrm{Hk}_{\mathcal{G},\bar{k}}^{\mathrm{sch}},\Lambda)$ on the schematic Hecke stack $\mathrm{Hk}_{\mathcal{G},\bar{k}}^{\mathrm{sch}} = L_{\bar{k}}^+\mathcal{G}\backslash L_{\bar{k}}G/L_{\bar{k}}^+\mathcal{G}$ by Proposition 6.7 and Section A for the comparison with sheaves on schematic v-stacks. It carries moreover a monoidal structure given by convolution $\star$.

6.4. **Over O_C.** Let $f\colon X \to \mathrm{Spec}(O_C)$ be a scheme of finite presentation over O_C and denote by j the inclusion $X_\eta \hookrightarrow X$ of the generic fiber. In [HS21, Theorem 1.7], Hansen and Scholze prove that the pullback functor

$$ j^*\colon \mathrm{D}(X,\Lambda) \to \mathrm{D}(X_\eta,\Lambda) \tag{6.20}$$

restricts to an equivalence between f-ULA and f_η-ULA objects. In the setup of diamonds, the argument for full faithfulness is the same as was explained to us by Scholze, and it consists of proving the adjunction map $A \to Rj_*j^*A$ is an isomorphism.

Lemma 6.11. *Let X be a small v-sheaf over $\mathrm{Spd}(O_C)$ representable in locally spatial diamonds, compactifiable and of finite trascendence degree. Let $A \in \mathrm{D}(X,\Lambda)$ be ULA for the structure map to $\mathrm{Spd}(O_C)$. Then $A \to Rj'_*j'^*A$ is a isomorphism, where $j'\colon X_\eta \to X$ and $j\colon \mathrm{Spd}(C) \to \mathrm{Spd}(O_C)$ denote the inclusion of generic fibers.*

Proof. By hypothesis j'^*A is ULA with respect to $\mathrm{Spd}(C)$. In particular, by [FS21, Proposition IV.2.19] the map

$$j'^*A \otimes^{\mathbb{L}}_{\Lambda} \Lambda \cong R\mathscr{H}\mathrm{om}(\mathbb{D}_{X_\eta/\,\mathrm{Spd}(C)}(j'^*A), Rf_\eta^!\Lambda) \tag{6.21}$$

is an isomorphism. Since j' is an open immersion $j'^* = Rj'^!$ and $\mathbb{D}_{X_\eta/\,\mathrm{Spd}(C)}(j'^*A) = j'^*\mathbb{D}_{X/\,\mathrm{Spd}(O_C)}(A)$ as follows from [Sch17, Theorem 1.8.(v)]. We get

$$\begin{aligned} Rj'_*j'^*A \cong\ & Rj'_*\, R\mathscr{H}\mathrm{om}(j'^*\mathbb{D}_{X/\,\mathrm{Spd}(O_C)}(A), Rf_\eta^!\Lambda) \\ \cong\ & R\mathscr{H}\mathrm{om}(\mathbb{D}_{X/\,\mathrm{Spd}(O_C)}(A), Rj'_*Rf_\eta^!\Lambda) \\ \cong\ & R\mathscr{H}\mathrm{om}(\mathbb{D}_{X/\,\mathrm{Spd}(O_C)}(A), Rf^!Rj_*\Lambda) \end{aligned} \tag{6.22}$$

the result now follows from the identity $\Lambda = Rj_*\Lambda$ and double duality for ULA sheaves. □

In particular, a f_η-ULA object A comes from a f-ULA object if and only if Rj_*j^*A is f-ULA.

Below, we prove essential surjectivity for $\mathrm{Hk}_{\mathcal{G},O_C}$, the Hecke stack over $\mathrm{Spd}\, O_C$. For hyperspecial parahoric $\mathcal{G}$, this is [FS21, Corollary VI.6.7]. Before doing this, recall that hyperbolic localization allows us to define again a constant term functors

$$\mathrm{CT}_{\mathcal{P}} \colon \mathrm{D}(\mathrm{Hk}_{\mathcal{G},O_C}, \Lambda)^{\mathrm{bd}} \to \mathrm{D}(\mathrm{Gr}^0_{\mathcal{G},O_C}, \Lambda)^{\mathrm{bd}}. \tag{6.23}$$

By [FS21, Proposition IV.6.12], there is a natural equivalence

$$\mathrm{CT}_{\mathcal{P}} \circ Rj_{\mathcal{G},*} \cong Rj_{\mathcal{M},*} \circ \mathrm{CT}_P, \tag{6.24}$$

with $j_{\mathcal{G}}, j_{\mathcal{M}}$ denoting the inclusion of the respective generic fibers. Now, we can probe integral ULA objects.

Proposition 6.12. *Consider the inclusion of Hecke stacks $j\colon \mathrm{Hk}_{G,C} \to \mathrm{Hk}_{\mathcal{G},O_C}$. There is an equivalence*

$$j^* \colon \mathrm{D}(\mathrm{Hk}_{\mathcal{G},O_C}, \Lambda)^{\mathrm{bd,ula}} \to \mathrm{D}(\mathrm{Hk}_{G,C}, \Lambda)^{\mathrm{bd,ula}}, \tag{6.25}$$

whose inverse functor is Rj_*.

Proof. Suppose $A \in \mathrm{D}(\mathrm{Hk}_{G,C}, \Lambda)^{\mathrm{bd,ula}}$, it suffices to prove $Rj_*A \in \mathrm{D}(\mathrm{Hk}_{\mathcal{G},O_C}, \Lambda)^{\mathrm{bd,ula}}$. Let B denote the pullback of A to $\mathrm{Gr}_{\mathcal{G},O_C}$, which by definition is ULA. By smooth base change, Rj_*A pulls back to Rj_*B (here we implicitly use [FS21, Proposition VI.4.1]). By [FS21, Theorem IV.2.23], we must show

$$p_1^*\mathbb{D}(Rj_*B) \otimes p_2^*Rj_*B \to R\mathscr{H}\mathrm{om}(p_1^*Rj_*B, Rp_2^!Rj_*B) \tag{6.26}$$

is an isomorphism. Let K denote the cone of this map. By assumption, and since $j^* = Rj^!$, this map is an isomorphism on the generic fiber. Consequently, $K = i_*L$ for some $L \in D(\mathrm{Hk}_{\mathcal{G},\bar{k}}, \Lambda)^{\mathrm{bd}}$ and the inclusion $i\colon \mathrm{Hk}_{\mathcal{G},\bar{k}} \to \mathrm{Hk}_{\mathcal{G},O_C}$. We may use the conservativity result Proposition 6.4 to prove $L = 0$. This reduces us to proving that $\mathrm{CT}_{\mathcal{B}}(Rj_*A) = Rj_*\,\mathrm{CT}_{\mathcal{B}}(A)$ is ULA for all Borel subgroups $S_{\breve{F}} \subset B \subset G_{\breve{F}}$. Now, the fixed-point locus $\mathrm{Gr}^0_{\mathcal{G},O_C}$ of the action induced by $\mathcal{B}$ is ind-representable by a locally finite type scheme over O_C of relative dimension 0. We call this scheme X and let $h\colon X_\eta \to X$ the inclusion of generic fibers. By inspection, $\mathrm{D}(\mathrm{Gr}^0_{\mathcal{G},O_C}, \Lambda) \cong \mathrm{D}(X, \Lambda)$ and $\mathrm{D}(\mathrm{Gr}^0_{\mathcal{G},C}, \Lambda) \cong \mathrm{D}(X_\eta, \Lambda)$. In particular $Rj_* \cong c_X^*Rh_*Rc_{X_\eta,*}$ with notation as in [Sch17, Section 27]. By [HS21, Theorem 1.7], Rh_* preserves ULA objects, which allows us to conclude the same holds for Rj_*. □

6.5. **Nearby cycles.** We can now look at the nearby cycles functor

$$\Psi_{\mathcal{G}} := i^*Rj_* \colon \mathrm{D}(\mathrm{Hk}_{G,C}, \Lambda)^{\mathrm{bd}} \to \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}}, \Lambda)^{\mathrm{bd}}, \tag{6.27}$$

Arising from the diagram

$$\mathrm{Hk}_{G,C} \xrightarrow{j} \mathrm{Hk}_{\mathcal{G},O_C} \xleftarrow{i} \mathrm{Hk}_{\mathcal{G},\bar{k}} \tag{6.28}$$

of geometric fibers inclusions of the integral Hecke stack.

Proposition 6.13. *The functor of nearby cycles lies in a natural equivalence*

$$\mathrm{CT}_{\mathcal{P}}[\deg_{\mathcal{P}}] \circ \Psi_{\mathcal{G}} \cong \Psi_{\mathcal{M}} \circ \mathrm{CT}_P[\deg_P], \tag{6.29}$$

that is, it commutes with shifted constant term functors.

Proof. Without the shift, this is a direct consequence of [FS21, Proposition IV.6.12]. Using Theorem 5.2, this also shows that $\mathrm{CT}_{\mathcal{P}} \circ \Psi_{\mathcal{G}}$ is supported on the open and closed sub-v-sheaf $\mathcal{F}\ell_{\mathcal{M},\bar{k}} \subset \mathcal{F}\ell^0_{\mathcal{G},\bar{k}}$. So the shifts agree by definition. □

Surprisingly, this commutativity property delivers us a lot of control on the values assumed by $\Psi_{\mathcal{G}}$ on the Satake category.

Corollary 6.14. *Nearby cycles $\Psi_{\mathcal{G}}$ restrict to a functor*

$$\mathrm{D}(\mathrm{Hk}_{G,C}, \Lambda)^{\mathrm{bd,ula}} \to \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}}, \Lambda)^{\mathrm{bd,ula}} \tag{6.30}$$

and, if $\mathcal{G}$ is furthermore special parahoric, then it even restricts to

$$\mathrm{Sat}(\mathrm{Hk}_{G,C}, \Lambda) \to \mathrm{Sat}(\mathrm{Hk}_{\mathcal{G},\bar{k}}, \Lambda) \tag{6.31}$$

between the Satake categories.

Proof. This follows from Proposition 6.6 and Proposition 6.9. □

Let us examine the nearby cycles $\Psi_{\mathcal{G}}(\mathrm{Sat}(V))$ applied to a Satake object $\mathrm{Sat}(V) \in \mathrm{Sat}(\mathrm{Hk}_{G,C}, \Lambda)$ corresponding to a $\widehat{G}_\Lambda$-representation V with μ as its highest weight. Given an F-Borel $B \subset G$, the commutativity of Proposition 6.13 yields

$$\mathrm{CT}_{\mathcal{B}}[\deg_{\mathcal{B}}]\big(\Psi_{\mathcal{G}}(\mathrm{Sat}(V))\big) = \bigoplus_{\lambda_I} V(\lambda_I) \cdot \lambda_I, \tag{6.32}$$

where now the $\widehat{G}_\Lambda$-representation is regarded as a $\widehat{T}^I_\Lambda$-representation by restriction. Here, we use that (by construction) the constant term functor corresponds via geometric Satake to restriction of representations, see [FS21, Section VI.11]. In particular, we get:

Corollary 6.15. *For a $\widehat{G}_\Lambda$-representation V with highest weight μ, the compactly supported cohomology groups*

$$\mathrm{H}^l_c\big(\mathcal{S}_{\bar{k},w}, \Psi_{\mathcal{G}}(\mathrm{Sat}(V))\big) \tag{6.33}$$

vanish for all $l \in \mathbb{Z}$ unless $\mathcal{F}\ell_{\mathcal{G},\bar{k},w} \subset \mathcal{A}_{\mathcal{G},\bar{k},\mu}$.

Proof. This follows from Lemma 3.12. □

We are now finally ready to compute the special fiber of the local model.

Theorem 6.16. *There is an equality $\mathcal{A}^\diamondsuit_{\mathcal{G},\mu} = \mathcal{M}_{\mathcal{G},\mu,k_E}$ as sub-v-sheaves of $\mathcal{F}\ell^\diamondsuit_{\mathcal{G},k_E}$.*

Proof. By specializing the orbit of μ under the finite Weyl group, it is easy to see that $\mathcal{A}^\diamondsuit_{\mathcal{G},\mu}$ is contained in the special fiber of $\mathcal{M}_{\mathcal{G},\mu}$. By Corollary 6.15, it is thus enough to prove that for a maximal stratum in $\mathcal{M}_{\mathcal{G},\bar{k},\mu}$ enumerated by w, we have $\mathrm{H}^l_c(\mathcal{S}_{\bar{k},w}, A) \neq 0$ for some $l \in \mathbb{Z}$ and for $A := \Psi_{\mathcal{G}}(\mathrm{Sat}(V))$ for some $\widehat{G}_\Lambda$-representation V with highest weight μ. Using induction on μ, we may and do assume that w lies in the open complement of the closed union of $\mathcal{M}_{\mathcal{G},O_C,\lambda}$ for all $\lambda < \mu$. Indeed, if $\lambda < \mu$, then $\mathcal{A}^\diamondsuit_{\mathcal{G},\lambda} \subset \mathcal{A}^\diamondsuit_{\mathcal{G},\mu}$. By Lemma 5.3, our Borel subgroup $S_{\breve{F}} \subset B \subset G_{\breve{F}}$ can always be chosen such that w is an isolated point of the attractor $\mathcal{F}\ell^+_{\mathcal{G},\bar{k},w}$. Since w enumerates a maximal stratum, we also see that w is an isolated point of $\mathcal{M}^+_{\mathcal{G},\bar{k},\mu}$, so that $\mathrm{H}^*_c(\mathcal{S}_{\bar{k},w}, A) = \mathrm{H}^*(\{w\}, A) =: A_w$ is the stalk of A at w.

Consider $X = \mathcal{M}_{\mathcal{G},O_C,\mu} \times_{\mathrm{Spd}\, O_C} U$ where U denotes the analytic locus of the open unit ball $\mathbb{D}^\diamondsuit_{O_C}$. Let $g\colon X_C \hookrightarrow X$ be the inclusion of the generic fiber. Let K denote a completed algebraic closure of $k(\!(t)\!)$. We may choose a $\mathrm{Spd}(K)$-valued point $\overline{w}$ of X that lies over w. It suffices to prove $A_{\overline{w}}$ is not identically 0. Since U is smooth over $\mathrm{Spd}\, O_C$, the smooth base-change theorem and our inductive assumption on w allows us to compute

$$A_{\overline{w}} = (Rg_*\Lambda[\langle 2\rho, \mu\rangle])_{\overline{w}}, \tag{6.34}$$

where V is chosen to have weight multiplicity 1 at μ. Since X_C, X and $K^\diamondsuit$ are locally spatial diamonds, we may compute the right-side term site-theoretically. Letting $l := -\langle 2\rho, \mu\rangle$, we have

$$\mathrm{H}^l(A_{\overline{w}}) = \lim_W \mathrm{H}^0(W_C, \Lambda) \tag{6.35}$$

where W ranges over étale neighborhoods of $\overline{w}$ in X. By Remark 4.15 and openness of $X \to \mathcal{M}_{\mathcal{G},O_C,\mu}$, the generic fiber X_C is dense in X which proves that the above expression does not vanish. □

6.6. **Centrality of nearby cycles.** In the classical theory, say, over function fields, it is known that nearby cycles on Hecke stacks give central perverse sheaves on partial affine flag varieties, see [Gai01]. Centrality holds true in our context as well:

Proposition 6.17. *For every $A \in \mathrm{D}(\mathrm{Hk}_{G,C}, \Lambda)$ and $B \in \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}}, \Lambda)$ with bounded support, there is a canonical isomorphism*

$$\Psi_{\mathcal{G}}(A) \star B \cong B \star \Psi_{\mathcal{G}}(A) \tag{6.36}$$

in $\mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}}, \Lambda)$.

Proof. We can repeat the proof of [Zhu14, Proposition 7.4] in our context:

Similar to [FS21, Definition/Proposition VI.9.4], we work with the convolution integral Hecke stack

$$\mathrm{Hk}^{I;I_1,\ldots,I_k}_{\mathcal{G}} \to (\mathrm{Spd}\, O)^I, \tag{6.37}$$

where $I = I_1 \sqcup \ldots \sqcup I_k$ is a finite partitioned index set. It parametrizes $\mathcal{G}$-bundles $\mathcal{E}_0, \ldots, \mathcal{E}_k$ over B^+_{dR} together with isomorphisms of $\mathcal{E}_{j-1}$ and $\mathcal{E}_j$ outside the union of the divisors ξ_i for all $i \in I_j$. We fix $I := \{1,2\}$ and drop it from the notation. There are three ordered partitions $\{1\} \sqcup \{2\}$, $\{2\} \sqcup \{1\}$ and $\{1,2\}$, leading to the diagram of v-sheaves:

$$\begin{tikzcd} \mathrm{Hk}^{\{1\},\{2\}}_{\mathcal{G}}|_{\mathrm{Spd}\, O_C} \arrow[r, "m"] \arrow[dr, "p"'] & \mathrm{Hk}^{\{1,2\}}_{\mathcal{G}}|_{\mathrm{Spd}\, O_C} & \mathrm{Hk}^{\{2\},\{1\}}_{\mathcal{G}}|_{\mathrm{Spd}\, O_C} \arrow[l, "n"'] \arrow[dl, "q"] \\ & \mathrm{Hk}_{\mathcal{G},O_C} \times \mathrm{Hk}_{\mathcal{G},\bar{k}} & \end{tikzcd} \tag{6.38}$$

The diagram arises by base change along the map $\mathrm{Spd}\, O_C \to (\mathrm{Spd}\, O)^2$ induced by the divisor $\pi = 0$ in the second coordinate. The maps m, n are the natural projections given by remembering $\mathcal{E}_0$ and $\mathcal{E}_2$, and are ind-proper, as one sees by pulling back to the convolution affine Grassmannian, combine Theorem 4.9 with the proof of [SW20, Proposition 20.4.1]. The maps p, q are given by sending $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$ to the ordered pair $((\mathcal{E}_0, \mathcal{E}_1), (\mathcal{E}_1, \mathcal{E}_2))$, respectively $((\mathcal{E}_1, \mathcal{E}_2), (\mathcal{E}_0, \mathcal{E}_1))$, and are pro-(cohomologically smooth) because $L^+_O\mathcal{G} \to \operatorname{Spd} O$ is so. More precisely, one passes to a bounded part and factors the action of $L^+_O\mathcal{G}$ through a congruence quotient.

Furthermore, the maps m, n are convolution maps in the special fiber and induce isomorphisms over the generic fiber such that (6.38) commutes. The commutativity yields a canonical isomorphism by using adjunctions

$$Rm_{\eta,*}p^*_\eta(A \boxtimes B) \cong Rn_{\eta,*}q^*_\eta(A \boxtimes B), \tag{6.39}$$

which will induce the desired isomorphism (6.36) upon applying the nearby cycles for the family $\mathrm{Hk}_{\mathcal{G}}^{\{1,2\}}|_{\operatorname{Spd} O_C}$. Indeed, since $(Rj_*A) \boxtimes B$ is ULA by Proposition 6.12 and [FS21, Corollary IV.2.25] for outer tensor products, it is still ULA after cohomologically smooth pullback along p, q and proper pushforward along m, n (here, we use that the support of A, B is bounded). Thus, (6.39) canonically extends integrally yielding (6.36) after restriction to the special fiber. □

The following would be a natural reinforcement of the previous proposition to also preserving perversity. For schemes, nearby cycles always preserve perversity [HS21, Lemma 6.3]. In our setting, this is not immediate and would transport Gaitsgory's central functor [Gai01] to the p-adic context.

Conjecture 6.18. *For every $A \in \mathrm{D}(\mathrm{Hk}_{G,C}, \Lambda)$, the Λ-flat central sheaf $\Psi_{\mathcal{G}}(A) \in \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}}, \Lambda)^{\mathrm{bd,ula}}$ is perverse.*

By a combination of Corollary 6.14 and Proposition 6.17, we know that Conjecture 6.18 holds true whenever $\mathcal{G}_{\breve{O}}$ is special parahoric. Also, using the representability in Theorem 1.1 and a comparison with schematic nearby cycles (see [Sch17, Proposition 27.6.]), the conjecture holds true whenever μ is minuscule. In general, we lack tools to verify Conjecture 6.18 – shifted constant terms appear to be insufficient – but one still expects some form of Artin vanishing to hold in this very particular context of the Hecke stack.

7. Minuscule means representable

Our goal in this section is to prove the Scholze–Weinstein conjecture on minuscule[18] local models [SW20, Conjecture 21.4.1] as stated in Theorem 1.1. Its main feature is the representability part, see [Lou20, Conjecture IV.4.18], which we verify without any assumption on the prime p or the pair $(\mathcal{G}, \mu)$, thereby showing the existence of weakly normal projective O_E-schemes $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ with natural, equivariant isomorphisms $(\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu})^\diamondsuit \cong \mathcal{M}_{\mathcal{G},\mu}$ in all cases.

As for their geometry, we show under Assumption 1.9 and Assumption 1.13 that the special fiber is given by $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}$, in particular reduced and even weakly normal. This implies the geometry part of the Scholze–Weinstein conjecture under those assumptions, see [Lou20, Conjecture IV.4.19].

Recall that our strategy for representability involves specializations triples. Since explicitly calculating the specialization map seems very hard, we need to consider convolutions of local models, so as to partially resolve $\mathcal{M}_{\mathcal{G},\mu}$ and understand their integral sections better.

7.1. Convolution.

We continue to denote by $F/\mathbb{Q}_p$ a complete discretely valued field with ring of integers O and perfect residue field k of characteristic $p > 0$. Fix a completed algebraic closure C/F, and a connected reductive F-group G with parahoric O-model $\mathcal{G}$. Also, we fix an auxiliary maximal $\breve{F}$-split F-torus $S \subset G$ whose connected Néron model $\mathcal{S}$ embeds in $\mathcal{G}$, see [BT84, Proposition 5.10], and denote by T its centralizer with connected Néron model $\mathcal{T} \subset \mathcal{G}$. Additionally, we fix an auxiliary $\breve{F}$-Borel $T_{\breve{F}} \subset B \subset G_{\breve{F}}$.

When proving the representability of the v-sheaf local models, it is not difficult to reduce to the case that G is the Weil restriction of a split group, see proof of Theorem 7.21. In this case, it will be helpful to partially resolve the local model via convolution. We recall that the Beilinson–Drinfeld Grassmannian admits the following convolution variant

$$\mathrm{Gr}_{\mathcal{G}} \widetilde{\times} \dots \widetilde{\times} \mathrm{Gr}_{\mathcal{G}} := L_O\mathcal{G} \times^{L^+_O\mathcal{G}} \dots \times^{L^+_O\mathcal{G}} \mathrm{Gr}_{\mathcal{G}} \to \operatorname{Spd} O, \tag{7.1}$$

which, in terms of torsors, parametrizes successive modifications of $\mathcal{G}$-torsors together with a generic trivialization of the last. It admits natural closed sub-v-sheaves

$$\mathcal{M}_{\mathcal{G},\mu_\bullet} := \mathcal{M}_{\mathcal{G},\mu_1} \widetilde{\times} \dots \widetilde{\times} \mathcal{M}_{\mathcal{G},\mu_n}, \tag{7.2}$$

for any sequence $\mu_\bullet = (\mu_1, \dots, \mu_n)$ of B_C-dominant coweights μ_i of T_C, after base change to $\operatorname{Spd} O_E$, where E is the reflex field of $\mu_\bullet$. We will still call them (convolution) local models for simplicity. More precisely, denote by $\widetilde{\mathcal{M}_{\mathcal{G},O_C,\mu_i}}$ the preimage in $L_{O_C}\mathcal{G}$ of $\mathcal{M}_{\mathcal{G},O_C,\mu_i} \subset \mathrm{Gr}_{\mathcal{G},O_C}$, which is an $L^+_{O_C}\mathcal{G}$-torsor over $\mathcal{M}_{\mathcal{G},O_C,\mu_i}$. Then

$$\mathcal{M}_{\mathcal{G},O_C,\mu_\bullet} = \widetilde{\mathcal{M}_{\mathcal{G},O_C,\mu_1}} \times^{L^+_{O_C}\mathcal{G}} \dots \times^{L^+_{O_C}\mathcal{G}} \widetilde{\mathcal{M}_{\mathcal{G},O_C,\mu_2}} \times^{L^+_{O_C}\mathcal{G}} \mathcal{M}_{\mathcal{G},\mu_n}. \tag{7.3}$$

This presentation is not "minimal" in the following sense: Namely, given a contracted product $X \times^H Y$ in any topos and a normal subgroup $N \subset H$ acting trivially on Y, then the natural map

$$X \times^H Y \to X/N \times^{H/N} Y$$

[18] Recall that this is sharp, due to the theory of Banach-Colmez spaces, see Proposition 4.8.

is an isomorphism. Hence, in (7.3) we may replace $L^+_{O_C}\mathcal{G}$ by some sufficiently large congruence quotient, and accordingly the torsors $\widetilde{\mathcal{M}_{\mathcal{G},O_C,\mu_i}}$ by their pushforwards to these congruence quotients. Let us note that the multiplication

$$\mathcal{M}_{\mathcal{G},O_C,\mu_\bullet} \to \mathrm{Gr}_{\mathcal{G},O_C}$$

has image $\mathcal{M}_{\mathcal{G},O_C,|\mu_\bullet|}$ with $|\mu_\bullet| := \mu_1 + \ldots + \mu_n$, and can therefore be regarded as a (partial) resolution of the latter. Regarding the structure of the convoluted local models, we can record the following.

Lemma 7.1. *The convoluted local model $\mathcal{M}_{\mathcal{G},\mu_\bullet}$ is a proper, flat π-adic kimberlite over $\mathrm{Spd}\, O_E$ with topologically dense generic fiber.*

Proof. In order to prove that $\mathcal{M}_{\mathcal{G},\mu_\bullet}$ is a proper flat π-adic kimberlite, we must first show that this proper v-sheaf is v-formalizing. Replace the universal $L^+_{O_C}\mathcal{G}$-torsors by the corresponding $W^+_{O_C}\mathcal{G}$-torsors, see [Gle20, Definition 2.2.14] and denote by $\mathcal{M}^{\mathrm{for}}_{\mathcal{G},O_C,\mu_\bullet}$ the corresponding convolution. The natural map

$$\mathcal{M}^{\mathrm{for}}_{\mathcal{G},O_C,\mu_\bullet} \to \mathcal{M}_{\mathcal{G},O_C,\mu_\bullet} \tag{7.4}$$

is an isomorphism because both v-sheaves are qcqs and have the same geometric points, compare with [Gle20, Proposition 2.2.27]. It is now easy to show that it is formally separated and formally adic, with representable special fiber, see Proposition 7.2. Via projection to the first factor we have a map

$$\mathcal{M}_{\mathcal{G},O_C,\mu_\bullet} \to \mathcal{M}_{\mathcal{G},O_C,\mu_1}, \tag{7.5}$$

which splits after pullback to $\widetilde{\mathcal{M}_{\mathcal{G},O_C,\mu_1}}$ into the projection of the product $\widetilde{\mathcal{M}_{\mathcal{G},O_C,\mu_1}} \times_{\mathrm{Spd}\, O_C} \mathcal{M}_{\mathcal{G},O_C,(\mu_2,\ldots,\mu_n)}$. Hence, after replacing $L^+_{O_C}\mathcal{G}$ (and hence $\widetilde{\mathcal{M}_{\mathcal{G},O_C,\mu_1}}$) by a sufficiently large congruence quotient, we can deduce that $\mathcal{M}_{\mathcal{G},O_C,\mu_\bullet}$ has dense generic fiber by induction on n, Proposition 4.14 and preservation of closures under open maps. □

From now on, we will always consider sequences of minuscule dominant coweights $\mu_\bullet = (\mu_1, \ldots, \mu_n)$ whose sum

$$|\mu_\bullet| = \mu_1 + \cdots + \mu_n \tag{7.6}$$

is still minuscule. Basically, this means that the support of each μ_i lies in disjoint irreducible components of the Dynkin diagram. We say that a coweight is *tiny* if it is minuscule and its support is contained in at most one irreducible component.

Proposition 7.2. *Both fibers of $\mathcal{M}_{\mathcal{G},O_C,\mu_\bullet} \to \mathrm{Spd}\, O_C$ are representable. More precisely, we have isomorphisms*

$$\mathcal{M}_{\mathcal{G},O_C,\mu_\bullet}|_{\mathrm{Spd}\, C} \cong \mathcal{F}^{\diamondsuit}_{G,C,\mu_\bullet} \cong \mathcal{F}^{\diamondsuit}_{G,C,\mu_1} \times \cdots \times \mathcal{F}^{\diamondsuit}_{G,C,\mu_n}, \tag{7.7}$$

and also

$$\mathcal{M}_{\mathcal{G},O_C,\mu_\bullet}|_{\mathrm{Spd}\, \bar{k}} \cong \mathcal{A}^{\diamondsuit}_{\mathcal{G},\mu_\bullet}, \tag{7.8}$$

where on the right we mean the convolution $\mathcal{A}_{\mathcal{G},\mu_1} \widetilde{\times} \ldots \widetilde{\times} \mathcal{A}_{\mathcal{G},\mu_n}$.

Proof. The description of the generic fiber via convolution is formal, and it is formal (using Theorem 6.16) that $\mathcal{M}_{\mathcal{G},O_C,\mu_\bullet}|_{\mathrm{Spd}\, k}$ is the convolution of the $\mathcal{A}^{\diamondsuit}_{\mathcal{G},\mu_i}$. Using Lemma A.3 this convolution identifies with $\mathcal{A}^{\diamondsuit}_{\mathcal{G},\mu_\bullet}$. □

We are aiming to carefully write down certain "minimal" $\mathcal{G}^{>i,\diamondsuit}_{\mathrm{ad},O_C}$-torsors $\mathcal{M}^{\mathrm{tor}}_{\mathcal{G},O_C,\mu_i} \to \mathcal{M}_{\mathcal{G},O_C,\mu_i}$ for some associated smooth connected groups $\mathcal{G}^{>i}_{\mathrm{ad},O_C}$, such that

$$\mathcal{M}^{\mathrm{tor}}_{\mathcal{G},O_C,\mu_1} \times^{\mathcal{G}^{>1,\diamondsuit}_{\mathrm{ad},O_C}} \cdots \times^{\mathcal{G}^{>i-1,\diamondsuit}_{\mathrm{ad},O_C}} \mathcal{M}^{\mathrm{tor}}_{\mathcal{G},O_C,\mu_i} \times^{\mathcal{G}^{>i,\diamondsuit}_{\mathrm{ad},O_C}} \cdots \times^{\mathcal{G}^{>n-1,\diamondsuit}_{\mathrm{ad},O_C}} \mathcal{M}^{\mathrm{tor}}_{\mathcal{G},O_C,\mu_n} \tag{7.9}$$

recovers the convolution local model $\mathcal{M}_{\mathcal{G},O_C,\mu_\bullet}$. We begin by introducing the group schemes $\mathcal{G}^{>i}_{\mathrm{ad},O_C}$.

Lemma 7.3. *Let $\mu_{>i} = \mu_{i+1} + \cdots + \mu_n$ and let $G^{>i}_{\mathrm{ad},C}$ be the quotient of G_C by the intersection of all conjugates of $P^-_{\mu_{>i}}$. Then, $G^{>i}_{\mathrm{ad},C}$ acts faithfully on $\mathcal{F}_{G,C,\mu_{>i}}$.*

Furthermore, if we let $\mathcal{G}^{>i}_{\mathrm{ad},O_C}$ be the unique fppf quotient[19] *of $\mathcal{G}_{\mathrm{ad},O_C}$ with generic fiber $G^{>i}_{\mathrm{ad},C}$, then $\mathcal{G}^{>i}_{\mathrm{ad},O_C}$ acts on $\mathcal{M}_{\mathcal{G},O_C,\mu_{>i}}$, and its fibers are smooth, affine, connected with trivial center.*

Proof. The claims on $G^{>i}_{\mathrm{ad},C}$ follow from Lemma 7.1. The smooth group scheme quotient with the asserted properties obviously exist, due to [BT84, Proposition 1.7.6], and it clearly inherits connected fibers. The generic fiber is clearly adjoint: apply semi-simplicity of G_{ad}. However, it is more delicate to show that the special fiber is adjoint.

Assume without loss of generality (Proposition 4.16) that $G = \mathrm{Res}_{F'/F} G'$ is an adjoint F-simple group with G' absolutely simple, and F' a finite field extension of F. Let $\mathcal{G}'$ be that parahoric over F' associated with $\mathcal{G}$. Then, a simple calculation reveals that

$$\mathcal{G}^{>i} = \mathrm{Res}_{A_i/O_C} \mathcal{G}'_{A_i}, \tag{7.10}$$

where A_i is the finite O_C-algebra obtained as the image of $O_{F'} \otimes_O O_C$ in the product of those copies of C indexed by the support of $\mu_{>i}$. Indeed, this smooth connected group scheme has the desired universal property

[19] Beware that the morphism of parahoric group schemes $\mathcal{G} \to \mathcal{G}_{\mathrm{ad}}$ is not always an fppf surjection.

and its special fiber is adjoint by [CGP15, Proposition A.5.15 (1)]. (Notice that the reducedness hypothesis is superfluous for calculating the center.) $\square$

Now, we come to the definition of the torsors.

Definition 7.4. The v-sheaf $\mathcal{M}^{\mathrm{tor}}_{\mathcal{G},O_C,\mu_i}$ is defined as the pushforward to $(\mathcal{G}^{>i}_{\mathrm{ad},O_C})^\diamondsuit$ of the natural $L^+_{O_C}\mathcal{G}$-torsor $\widetilde{\mathcal{M}_{\mathcal{G},O_C,\mu_i}}$ over $\mathcal{M}_{\mathcal{G},O_C,\mu_i}$.

One might expect, just like in Theorem 7.21, that these torsors have natural algebraic models over the ring of integers of some reflex field and this will be an integral part of our strategy. Since the fibers are relatively easy to understand by the Demazure resolution, we now focus on the behavior over a certain $\mathcal{G}_{O_C}$-semi-orbit.

Recall that for any $\lambda \in W_0 \cdot \mu$, where W_0 is the Weyl group of (G, S), the induced point $[\lambda]\colon \operatorname{Spd} C \to \mathrm{Gr}_{G,C,\mu}$ uniquely extends to a point $[\lambda]\colon \operatorname{Spd} O_C \to \mathcal{M}_{\mathcal{G},O_C,\mu}$ by properness of v-sheaf local models.

Definition 7.5. Let $\mathcal{M}^\circ_{\mathcal{G},\mu} \subset \mathcal{M}_{\mathcal{G},\mu}$ be the unique sub-v-sheaf whose base change $\mathcal{M}^\circ_{\mathcal{G},O_C,\mu} \subset \mathcal{M}_{\mathcal{G},O_C,\mu}$ to $\operatorname{Spd} O_C$ is given by the finite (non-disjoint) union

$$\mathcal{M}^\circ_{\mathcal{G},O_C,\mu} = \bigcup_{\lambda \in W_0\cdot\mu} \mathcal{G}^\diamondsuit_{O_C} \cdot [\lambda]. \tag{7.11}$$

We recall that the elements $\lambda \in W_0 \cdot \mu$ are the rational conjugates of μ in $X_*(T)$ and correspond to the open Schubert orbits in the μ-admissible locus, see the discussion after Definition 3.11. Also, it is easy to see and left to the reader that the definition of $\mathcal{M}^\circ_{\mathcal{G},\mu}$ does not depend on the choice of the auxiliary maximal $\breve{F}$-split F-torus $S \subset G$ whose connected Néron model $\mathcal{S}$ embeds in $\mathcal{G}$. The stabilizer of rational conjugates $[\lambda]$ is actually representable and well behaved.

Lemma 7.6. *Let $\mathcal{P}^-_\lambda$ be the flat closure in $\mathcal{G}_{O_C}$ of the repeller parabolic $P^-_\lambda \subset G_C$ defined by λ. Then*

1. *$\mathcal{P}^{-,\diamondsuit}_\lambda$ is the $\mathcal{G}^\diamondsuit_{O_C}$-fixer of λ inside $\mathcal{M}_{\mathcal{G},O_C,\mu}$.*
2. *$\mathcal{P}^-_\lambda \to \operatorname{Spec}(O_C)$ is smooth affine with connected fibers.*

Proof. By topological flatness, it is clear that $\mathcal{P}^{-,\diamondsuit}_\lambda$ fixes λ. For dimension reasons, the special fiber of $\mathcal{P}^-_\lambda$ is equal to the fixer in $\mathcal{G}_{O_C}$ of λ_I in the affine flag variety $\mathcal{F}\ell_{\mathcal{G},\bar{k}}$, which in particular shows that the special fiber of $\mathcal{P}^-_\lambda$ is connected. Having described the special fiber of $\mathcal{P}^-_\lambda$, we see that all (K, K^+)-valued points of the $\mathcal{G}^\diamondsuit_{O_C}$-fixer of λ inside $\mathcal{M}_{\mathcal{G},O_C,\mu}$ actually belong to the closed subgroup $\mathcal{P}^{-,\diamondsuit}_\lambda$, so it necessarily lies in that closed subgroup.

It suffices now to verify smoothness[20] of $\mathcal{P}^-_\lambda$. By [BT84, Corollaire 2.2.5], we can do this by restricting to the flat closures of a-root groups of P^-_λ with respect to the (non-maximal) split torus S_C. Using the structure of $\mathcal{U}_a \subset \mathcal{G}$ defined over O, we see that this amounts to check that the morphism

$$\mathrm{Res}_{B/A}\mathbb{A}^1 \to \mathrm{Res}_{C/A}\mathbb{A}^1 \tag{7.12}$$

induced by a surjection $B \to C$ of finite free A-algebras is a smooth cover. Indeed, the a-root group of $\mathcal{P}^-_\lambda$ decomposes scheme-theoretically as a product of fibers of such morphisms. $\square$

We want to uniquely characterize the left $\mathcal{G}^\diamondsuit_{O_C}$-equivariant right $\mathcal{G}^{>i,\diamondsuit}_{O_C}$-torsor

$$\mathcal{M}^{\mathrm{tor},\circ}_{\mathcal{G},O_C,\mu_i} := \mathcal{M}^{\mathrm{tor}}_{\mathcal{G},O_C,\mu_i} \times_{\mathcal{M}_{\mathcal{G},O_C,\mu_i}} \mathcal{M}^\circ_{\mathcal{G},O_C,\mu_i}. \tag{7.13}$$

For this, we use the following abstract statement.

Lemma 7.7. *Let $\mathfrak{X}$ be any topos, and let $J, A \in \mathfrak{X}$ be group objects. Let $Z := J/P$ be an orbit for J.*

1. *The groupoid of left J-equivariant right A-torsors $\mathcal{T}$ over Z is equivalent to the groupoid of right A-torsors $\mathcal{S}$ over a terminal object equipped with a morphism of groups $\varphi_\mathcal{T}\colon P \to \mathrm{Aut}_A(\mathcal{S})$, with equivalence given by sending a $\mathcal{T}$ to the fiber $\mathcal{S} := \mathcal{T}_{1\cdot P}$ of $1 \cdot P \in Z$ with its action by P.*
2. *If P is self-normalizing, then for each left J-equivariant right A-torsor $f\colon \mathcal{T} \to Z$ each morphism $\sigma\colon \mathcal{T} \to \mathcal{T}$, which is equivariant for J and A, is automatically a morphism of left J-equivariant right A-torsors, that is, $f \circ \sigma = f$. If furthermore $\varphi_\mathcal{T}$ has trivial centralizer, then $\sigma = \mathrm{id}$.*

Note that if $\mathcal{S}$ is trivial, then $\mathrm{Aut}_A(\mathcal{S}) \cong A$.

Proof. For (1), it suffices to note that an inverse is given by sending a pair $(\mathcal{S}, P \to \mathrm{Aut}_A(\mathcal{S}))$ to the contracted product $J \times^P \mathcal{S}$. For (2), one notes that by A-equivariance σ descents to an J-equivariant morphism of Z. As P is self-normalizing, this morphism must be the identity. $\square$

[20]Observe that the conjugation action of λ on G_C does not always extend to $\mathcal{G}_{O_C}$, so we are not in the presence of a repeller subgroup.

We can therefore conclude that the left $\mathcal{G}_{O_C}^{\diamondsuit}$-equivariant $\mathcal{G}_{O_C}^{>i,\diamondsuit}$-right torsor $\mathcal{M}_{\mathcal{G},O_C,\mu_i}^{\mathrm{tor},\circ}$ is governed by certain morphisms of groups

$$\mathcal{P}_{\lambda_i}^{\diamondsuit} \to (\mathcal{G}_{\mathrm{ad},O_C}^{>i})^{\diamondsuit} \tag{7.14}$$

for all rational conjugates λ_i of μ_i by applying Lemma 7.7. In our situation, the generic fiber of $\mathcal{M}_{\mathcal{G},O_C,\mu_i}^{\mathrm{tor},\circ}$ is quite well-understood, and thus we will apply the following general result describing extensions of a given left equivariant right torsor.

Lemma 7.8. *We use the notation of Lemma 7.7. Furthermore, let $Y \in \mathfrak{X}$ be any object, and denote by a subscript $(\text{-})_Y$ the base change to Y. Assume that $\widetilde{\mathcal{T}} \to Z_Y$ is a left J_Y-equivariant right A_Y-torsor with associated tuple $(\widetilde{\mathcal{S}}, \varphi_{\widetilde{\mathcal{T}}}\colon P_Y \to \mathrm{Aut}_{A_Y}(\widetilde{\mathcal{S}}))$. Then the groupoid of pairs of a left J-equivariant right A-torsors $\mathcal{T}$ over Z with an isomorphism $\mathcal{T}_Y \cong \widetilde{\mathcal{T}}$ identifies with the groupoid of following data: $\mathcal{S}$ an A-torsor over a terminal object of $\mathfrak{X}$, an identification $\gamma\colon \mathcal{S}_Y \cong \widetilde{\mathcal{S}}$, and a morphism of groups $\varphi\colon P \to \mathrm{Aut}_A(\mathcal{S})$ such that φ_Y agrees with $\varphi_{\widetilde{\mathcal{T}}}$ under the identification $\mathrm{Aut}_{A_Y}(\mathcal{S}_Y) \cong \mathrm{Aut}_{A_Y}(\widetilde{\mathcal{S}})$ induced by γ.*

Proof. This follows from Lemma 7.7. □

In our case, the A-torsors $\widetilde{\mathcal{S}} \cong A_{Y\times Z}, \mathcal{S} \cong A_Z$ we are interested in are trivial, and the morphism $A(\mathfrak{X}) \to A(Y)$ is injective. Then φ is determined by γ (and $\varphi_{\widetilde{\mathcal{T}}}$), and after fixing a section $z \in \widetilde{\mathcal{S}}(Y) \subset \widetilde{\mathcal{T}}(Y)$ we get thus an injection from isomorphism classes of pairs $(\mathcal{T}, \mathcal{T}_Y \cong \widetilde{\mathcal{T}})$ to $A(Y)/A(\mathfrak{X})$ by sending $(\mathcal{T}, \mathcal{T}_Y \cong \widetilde{\mathcal{T}})$ to the class of $z^{-1}y_Y \in A(Y)/A(\mathfrak{X})$, where $y \in \mathcal{S}(\mathfrak{X}) \subset \mathcal{T}(Y)$ is any section. If $\varphi_{\widetilde{\mathcal{T}}}$ has trivial centralizer, the groupoid of such pairs is equivalent to the groupoid of left J-equivariant right A-torsors $\mathcal{T}$ over Z, which are isomorphic to $\widetilde{\mathcal{T}}$ over Y (but we do not fix such an isomorphism).

Applying these considerations to the orbits in $\mathcal{M}_{\mathcal{G},O_C,\mu_i}^{\circ}$ with its left $\mathcal{G}_{O_C}^{\diamondsuit}$-equivariant right $\mathcal{G}_{\mathrm{ad},O_C}^{>i}$-torsor $\mathcal{M}_{\mathcal{G},\mu_i}^{\mathrm{tor},\circ}$ having generic fiber $\mathcal{F}_{G,\mu_i,C}^{\mathrm{tor}}$, we need therefore to

- fix base points over O_C in the orbits in $\mathcal{M}_{\mathcal{G},O_C,\mu_i}^{\circ}$,
- fix C-points $\tilde{\lambda}_j \in \mathcal{F}_{G,C,\mu_i}^{\mathrm{tor},\diamondsuit}$ lying over the generic fibers of the chosen base points in $\mathcal{M}_{\mathcal{G},O_C,\mu_i}^{\circ}$,
- find sections $y_j \in \mathcal{M}_{\mathcal{G},O_C,\mu_i}^{\circ,\mathrm{tor}}(O_C)$ lying over the chosen base points in $\mathcal{M}_{\mathcal{G},O_C,\mu_i}^{\circ}$,
- calculate the difference of $y_j^{-1}\tilde{\lambda}_j \in G_{\mathrm{ad},C}^{>i}(C)$, which yields the desired n classes modulo $\mathcal{G}_{\mathrm{ad},O_C}^{>i}(O_C)$.

The first point is easy as we can take the $[\lambda]\colon \mathrm{Spd}\, O_C \to \mathcal{M}_{\mathcal{G},O_C,\mu_i}^{\circ}$ with λ running through the rational Weyl group conjugates of μ_i. For the second point, we can fix a (suitable) uniformizer $\xi \in B_{\mathrm{dR}}^+(C)$ and consider the images of the $\lambda(\xi) \in LG(C)$ in $\mathcal{F}_{\mathcal{G},\mu_i,C}^{\mathrm{tor},\diamondsuit}$. To state the outcome, we have to make the following definition.

Definition 7.9. Let $\nu \in X_*(T)$. The different $\delta_G(\nu)$ is the class in $T(C)/\mathcal{T}(O_C)$ of

$$\prod_{\sigma \neq 1} \nu^{\sigma}(\pi_E^{\sigma} - \pi_E), \tag{7.15}$$

where $F \subset E \subset C$ is the reflex field of ν, $\pi_E \in O_E$ some uniformizer and σ varies over the non-trivial cosets in the quotient $\mathrm{Gal}_F/\mathrm{Gal}_E$ of the absolute Galois groups.

For a uniformizer $\pi_E \in O_E$ for a finite field extension E/F, contained in C, we denote by $\pi_E^\flat \in O_C^\flat$ a chosen sequence of compatible p^n-roots of π_E. Recall that $\xi_E := \pi_E - [\pi_E^\flat] \in W_{O_E}(O_C^\flat)$ maps to a uniformizer of $B_{\mathrm{dR}}^+(C)$. We can now define the $\tilde{\lambda}$ as the images of $\lambda(\xi_F) \in \mathcal{F}_{\mathcal{G},C,\mu_i}^{\mathrm{tor},\diamondsuit}(C)$ for any $\lambda \in W_0 \cdot \mu_i$.

Proposition 7.10. *The v-sheaf $\mathcal{M}_{\mathcal{G},,O_C,\mu_i}^{\mathrm{tor},\circ}$ is the unique left $\mathcal{G}_{O_C}$-equivariant right $\mathcal{G}_{\mathrm{ad},O_C}^{>i}$-torsor over $\mathcal{M}_{\mathcal{G},O_C,\mu_i}^{\circ}$ with generic fiber isomorphic to $\mathcal{F}_{\mathcal{G},C,\mu_i}^{\mathrm{tor},\diamondsuit}$ determined by the images of $\delta_G(\lambda)$ in $G_{\mathrm{ad},C}^{>i}(C)/\mathcal{G}_{\mathrm{ad},O_C}^{>i}(O_C)$ for $\lambda \in W_0 \cdot \mu_i$ (and the above choices for the $\tilde{\lambda}$'s).*

Proof. Let us fix some $\lambda \in W_0 \cdot \mu_i$. Consider the morphism

$$T' := \mathrm{Res}_{E/F}\mathbb{G}_m \to G \tag{7.16}$$

of algebraic groups induced by λ as follows: compose its Weil restriction $T' := \mathrm{Res}_{E/F}\mathbb{G}_m \to \mathrm{Res}_{E/F}T_E$ with the norm map $\mathrm{Res}_{E/F}T_E \to T$. Note that $\lambda\colon \mathbb{G}_{m,E} \to G_E$ can be reconstructed from this composition by restricting its base change to E to the first factor. Set $\mathcal{T}' := \mathrm{Res}_{O_E/O}\mathbb{G}_m$. We will now construct a section $y \in L_O\mathcal{T}'(O_C)$, whose image in $\mathrm{Gr}_{\mathcal{G}}$ is the section $[\lambda]$, and then calculate $y^{-1}\tilde{\lambda} \in LT(C)$ as necessary.

For this we claim that the element $\xi_E = \pi_E - [\pi_E^\flat]$ becomes a unit after inverting $\xi_F = \pi_F - [\pi_F^\flat]$, thus giving rise to an element $y \in \mathbb{G}_m(W_{O_E}(O_C^\flat)) \subset L_O\mathcal{T}'(O_C)$. Indeed, let $P(X) = X^d + a_1X^{d-1} + \ldots + a_d$ be the minimal polynomial of π_E over F, which is Eisenstein as E/F is totally ramified. Then the norm of ξ_E in $W_{O_F}(O_C^\flat)$ equals

$$P([\pi_E^\flat]) = [\pi_E^\flat]^d + a_1[\pi_E^\flat]^{d-1} + \cdots + a_d. \tag{7.17}$$

Reducing modulo ξ_F, this element certainly vanishes because $[\pi_E^\flat] \equiv [\pi_E^\flat]^\sharp = \pi_E$ modulo ξ_F. On the other hand, $P([\pi_E^\flat])$ is clearly a primitive element of degree 1 inside $W_O(O_C^\flat)$, as $a_d \in \pi_F O^\times$. Hence, $P([\pi_E^\flat])$ and ξ_F generate the same principal ideal, see [BS19, Lemma 2.24].

Let us now consider the generic fiber of the point y. For this we must pass to

$$B_{\mathrm{dR}}(C) \otimes_F E = \prod_{\sigma} B^{\sigma}_{\mathrm{dR}}(C). \tag{7.18}$$

Here, notice that we are conjugating the natural E-structure on the corresponding factor of the right side by σ. Then the coordinate of ξ_E for $\sigma \neq 1$ is a unit in $B^+_{\mathrm{dR}}(C)$ which reduces modulo the uniformizer ξ_F to $\pi_E^{\sigma} - \pi_E$. As for its coordinate on the $\sigma = 1$ factor, it must be a prime element, because so is the norm in $B_{\mathrm{dR}}(C)$, as seen in the previous calculation. We can conclude that the generic fiber of the section y maps to λ, and that $\tilde{\lambda}^{-1}y$ is $\delta_G(\lambda)$. □

The remark below will not be needed in the continuation.

Remark 7.11. Inspecting Proposition 7.10, one sees that $\mathcal{M}^{\mathrm{tor},\circ}_{\mathcal{G},O_C,\mu_i}$ is representable by a smooth O_E-scheme. Indeed, the defining maps $\varphi_\lambda \colon \mathcal{P}^{-,\diamondsuit}_{\lambda} \to \mathcal{G}^{>i,\diamondsuit}_{\mathrm{ad},O_C}$ are naturally algebraic over a finite unramified extension of E and so are their integral models by [BT84, Proposition 1.7.6]. The considerations after Lemma 7.8 then furnishes an algebraic space, and it is a scheme because it is a torsor over a scheme under an affine group scheme.

7.2. **Specialization maps.** The aim of this subsection is to characterize the specialization map. We are going to see that it is already determined by the semi-orbit. In the following, we consider all pairs $(\mathcal{G}, \mu_I)$ where $\mathcal{G}$ is a parahoric O-model of some reductive F-group G, I some finite index set and $\mu_I = (\mu_i)_{i \in I}$ is a sequence of minuscule coweights in G_C such that $\sum_{i \in I} \mu_i$ is still minuscule (and so are all subsums). Morphisms of such pairs $(\mathcal{G}, \mu_I) \to (\mathcal{G}', \mu'_J)$ are given by morphisms of O-group schemes $\mathcal{G} \to \mathcal{G}'$, surjections of sets $\mathrm{pr} \colon I \twoheadrightarrow J$ such that, for all $j \in J$, the image of $\sum_{i \in \mathrm{pr}^{-1}(j)} \mu_i$ in G'_C lies in the conjugacy class of μ'_j. This generalizes the functoriality considered in Proposition 4.16. We denote $(\mathcal{G}, \mu_I)$ also by $(\mathcal{G}, \mu_\bullet)$ if the index set I is understood.

Theorem 7.12. *The specialization morphisms for all pairs $(\mathcal{G}, \mu_\bullet)$ as above*

$$\mathrm{sp}_{\mathcal{G},\mu_\bullet} \colon |\mathcal{F}_{G,C,\mu_\bullet}| \to |\mathcal{A}_{\mathcal{G},\bar{k},\mu_\bullet}| \tag{7.19}$$

are the only functorial collection of continuous and spectral maps, whose restrictions to $\mathcal{M}^{\circ}_{\mathcal{G},O_C,\mu_\bullet}(O_C)$ agree with the natural maps.

Proof. We are going to uniquely determine the values taken by $\mathrm{sp}_{\mathcal{G},\mu_\bullet}$ on the subset $\mathcal{F}_{G,\mu_\bullet}(K)$ for a cofinal set of finite extensions K/F given by those Galois extensions that split G. This characterizes the map $\mathrm{sp}_{\mathcal{G},\mu_\bullet}$ by continuity with respect to the constructible topology. Indeed, $\mathcal{F}_{G,\mu_\bullet}$ is a smooth rigid space defined over E, see [Gle20, Theorem 1.4.35.].[21]

Having fixed a Galois extension K/F splitting G, we are however allowed to enlarge the parahoric group $\mathcal{G}$ in order to compute these values. In particular, we may and do assume that $G = \mathrm{Res}_{K/F} H$, where $H = G_K$ is a split reductive group. Refining $\mu_\bullet$ so that every element in the sequence is tiny allows us to conclude that every K-valued point of $\mathcal{F}_{G,\mu_\bullet}$ extends to an O_K-valued point of the semi-homogeneous $\mathcal{M}^{\circ}_{\mathcal{G},\mu_\bullet}$ by Lemma 7.13 below. □

Lemma 7.13. *Suppose K/F is Galois, $G = \mathrm{Res}_{K/F} H$ and all $\mu_i \in \mu_\bullet$ are tiny. Then, we have an equality*

$$\mathcal{M}^{\circ}_{\mathcal{G},\mu_\bullet}(O_K) = \mathcal{F}_{G,\mu_\bullet}(K) \tag{7.20}$$

of sets induced by the natural morphism.

Proof. Let us assume first that $K = F$. Then $\mathcal{G} = \mathcal{H}$ is a parahoric model of a split reductive F-group and the right side can be given by the Iwasawa decomposition

$$\mathcal{F}_{G,\mu}(K) = \bigcup_{\lambda} \mathcal{H}(O_K) \cdot \lambda, \tag{7.21}$$

see [BT72, Proposition 4.4.3]. Now, obviously the points of the form λ extend to integral points of $\mathcal{M}^{\circ}_{\mathcal{H},\mu}$, due to the splitness assumption, and thus the same holds for its $\mathcal{H}(O_K)$-orbits.

Now, consider an arbitrary finite Galois extension K/F. We get the result immediately for tiny coweights, as the $\mathcal{G}(O_K)$-action on $\mathcal{M}_{\mathcal{G},\mu}$ is via the parahoric subgroup $\mathcal{H}(O_K) \subset H(K)$. In general, we use this to show the claim by an inductive procedure. Suppose one is given an element $(x_1, \dots, x_n) \in \mathcal{F}_{G,\mu_\bullet}(K)$, such that each of the representatives $x_j \in \mathcal{F}^{\mathrm{tor}}_{G,\mu_j}(K)$ lies in $\mathcal{M}^{\mathrm{tor},\circ}_{\mathcal{G},\mu_j}(O_K)$ for $j < i$. Now note that x_i is in the $\mathcal{G}^{>i}(O_K)$-orbit of $\mathcal{M}^{\mathrm{tor},\circ}_{\mathcal{G},\mu_i}(O_K)$, due to the $n = 1$ case and the fact that the O_C-section of Proposition 7.10 descends to O_K, see also Remark 7.11. So we may replace it in the expression, and now the assumption holds for all $j < i + 1$. After finitely many steps of this iteration, we get the claim. □

Remark 7.14. It follows by inspecting the proof of Theorem 7.12 that in order to compute the specialization mapping, it is enough to consider Weil restrictions of split groups and their Iwahori models, see Assumption 7.16.

[21] If μ is not minuscule it is not true that $\bar{\mathbb{Q}}_p$-points of $\mathrm{Gr}_{G,\mu}$ are dense for the constructible topology. Indeed, Bialynicki-Birula maps give a bijection between $\bar{\mathbb{Q}}_p$-points.

Remark 7.15. He–Pappas–Rapoport [HPR20, Conjecture 2.12] conjecture that, for any fixed pair $(\mathcal{G},\mu)$, there is at most one flat projective O_E-scheme equipped with an $\mathcal{G}_{O_E}$-action having the correct fibers, identified $\mathcal{G}$-equivariantly. This is much stronger than Theorem 7.12 above, since it makes no reference to convolution or functoriality. Our approach is inspired by their conjecture in applying equivariant methods to pin down the specialization map.

7.3. **Comparison isomorphisms.** In this subsection, we use our work from the previous ones to establish certain comparison isomorphisms between (at least some of) our local models and those that have appeared elsewhere, see [PZ13, Lev16, Lou19, FHLR22]. During this subsection, we shall work under the following:

Assumption 7.16. Given a pinned split simple adjoint group (H, T_H, B_H, e_H), let $G = \mathrm{Res}_{K/F} H$ with K/F an arbitrary finite extension, with K_0/F being the maximal unramified subextension. Also let $\mathcal{I}$ be the standard Iwahori model with respect to the chosen pinning.

In order to prove Theorem 7.21, we need to compare $\mathcal{M}_{\mathcal{I},\mu}$ to certain candidates $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{I}},\mu}$ constructed in [FHLR22, Definition 4.7], which are variations on the work of Levin [Lev16]. Here, $\underline{\mathcal{I}}$ is a $O[\![t]\!]$-lift of $\mathcal{I}$ along $t\mapsto\pi$, obtained by taking restriction of scalars along an ad hoc lift $O[\![t]\!]\to O_0[\![u]\!]$ of $O\to O_K$ of the dilatation of $H\otimes O_0[\![u]\!]$ along $B_H\otimes O_0$ concentrated in the u-divisor, see [PZ13, Theorem 4.1] and [MRR20, Definition 2.1, Example 3.3]. The various lifts $O[\![t]\!]\to O_0[\![u]\!]$ defined in [FHLR22, Subsection 2.2] are given by choosing uniformizers and lifting Eisenstein polynomials over O_0 in such a way that they remain separable Eisenstein over both $k_0[\![t]\!]$ and $K_0[\![t]\!]$.

One has a schematic Beilinson–Drinfeld Grassmannian $\mathrm{Gr}^{\mathrm{sch}}_{\underline{\mathcal{I}}}$ defined in terms of power series rings, classifying $\underline{\mathcal{I}}$-torsors over $R[\![t-\pi]\!]$ trivialized over $R(\!(t-\pi)\!)$, and admitting uniformization via loop groups $L^{\mathrm{sch}}_O\underline{\mathcal{I}}/L^{\mathrm{sch},+}_O\underline{\mathcal{I}}$. The generic fiber is equivariantly isomorphic to the schematic affine Grassmannian $\mathrm{Gr}^{\mathrm{sch}}_G$ over F, see [FHLR22]. So we get an embedding $\mathcal{F}_{G,\mu}\subset \mathrm{Gr}^{\mathrm{sch}}_G|_{\mathrm{Spec}\,E}$ for a minuscule coweight μ.

Definition 7.17. The O_E-scheme $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{I}},\mu}$ is defined as the absolute weak normalization of the flat closure of $\mathcal{F}_{G,\mu}$ inside $\mathrm{Gr}^{\mathrm{sch}}_{\underline{\mathcal{I}},O_E}$. For a minuscule sequence $\mu_\bullet$ of dominant coweights, we set $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{I}},O_E,\mu_\bullet}$ as the convolution product of the $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{I}},O_E,\mu_i}$. We define the $\mathcal{I}^{>i}_{O_C}$-torsor $\mathcal{N}^{\mathrm{sch,tor}}_{\underline{\mathcal{I}},O_C,\mu_i}$ by pushing forward the universal $L^{\mathrm{sch},+}_{O_C}\underline{\mathcal{I}}$-torsor under the natural projection.

The $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{I}},\mu}$ are normal O_E-schemes by [FHLR22, Theorem 4.10], so their formation commutes with base change to O_C. They also come with transition morphisms

$$\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{I}},O_C,\mu}\to\mathcal{N}^{\mathrm{sch}}_{\underline{\tilde{\mathcal{I}}},O_C,\tilde{\mu}},\tag{7.22}$$

which are closed immersions, where $\underline{\tilde{\mathcal{I}}}$ is a further $O[\![t]\!]$-lift defined with respect to the extension $\tilde{K}/K/F$, see [FHLR22, Corollary 2.10], and $\tilde{\mu}$ is the image of μ in $\tilde{G}$. In what follows, we shall simply say they are functorial in $(\underline{\mathcal{I}},\mu)$.

Our next goal is to compare the v-sheaves associated with $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{I}},O_C,\mu_\bullet}$ to $\mathcal{M}_{\mathcal{G},O_C,\mu_\bullet}$. We start by recording what happens in the generic fiber.

Lemma 7.18. *There are unique equivariant isomorphisms*

$$\mathcal{N}^{\mathrm{sch,tor}}_{\underline{\mathcal{I}},\mu}|_{\mathrm{Spec}\,C}\cong\mathcal{F}^{\mathrm{tor}}_{G,C,\mu_i}\tag{7.23}$$

for each term μ_i of the sequence $\mu_\bullet$. They yield canonical equivariant isomorphisms

$$\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{I}},\mu_\bullet}|_{\mathrm{Spec}\,C}\cong\mathcal{F}_{G,C,\mu_\bullet}\tag{7.24}$$

functorially in $(\underline{\mathcal{I}},\mu_\bullet)$.

Proof. This follows by definition and uniqueness is ensured by Lemma 7.7. □

Next, we need to take care of the special fiber:

Proposition 7.19. *There are unique equivariant isomorphisms*

$$\left(\mathcal{N}^{\mathrm{sch,tor}}_{\underline{\mathcal{I}},\mu_i}|_{\mathrm{Spec}\,\bar{k}}\right)^{\mathrm{perf}}\cong\mathcal{A}^{\mathrm{tor}}_{\mathcal{I},\mu_i,\bar{k}}\tag{7.25}$$

for each term μ_i of the sequence $\mu_\bullet$. They yield canonical equivariant isomorphisms

$$\left(\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{I}},\mu_\bullet}|_{\mathrm{Spec}\,\bar{k}}\right)^{\mathrm{perf}}\cong\mathcal{A}_{\mathcal{I},\mu_\bullet,\bar{k}}\tag{7.26}$$

functorially in $(\underline{\mathcal{I}},\mu_\bullet)$.

Proof. Set $\mathcal{I}'=\underline{\mathcal{I}}\otimes k[\![t]\!]$, a standard Iwahori model of the connected reductive group $G'=\mathrm{Res}_{k_0(\!(u)\!)/k(\!(t)\!)}H$. By [FHLR22, Theorem 4.10], we have $\mathcal{N}^{\mathrm{sch,perf}}_{\underline{\mathcal{I}},\mu,\bar{k}}=\mathcal{A}_{\mathcal{I}',\mu',\bar{k}}$. Hence, the statement above is just a generalization of Lemma 3.15 to convolution products.

Let $w \in \tilde{W}$ be an element such that $\mathcal{F}\ell_{\mathcal{I},\bar{k},w} \subset \mathcal{A}_{\mathcal{I},\mu_i,\bar{k}}$ and choose a Demazure resolution $\pi_{\dot{w}} \colon \mathcal{D}_{\mathcal{I},\bar{k},\dot{w}} \to \mathcal{F}\ell_{\mathcal{I},\bar{k},w}$. We have to compare the following pullback square

$$\begin{tikzcd} \mathcal{D}^{\mathrm{tor}}_{\mathcal{I},\bar{k},\dot{w}} \arrow[r, "\pi^{\mathrm{tor}}_{\dot{w}}"] \arrow[d, "p_{\mathcal{D}}"] & \mathcal{F}\ell^{\mathrm{tor}}_{\mathcal{I},\bar{k},w} \arrow[d, "p_{\mathcal{F}\ell}"] \\ \mathcal{D}_{\mathcal{I},\bar{k},\dot{w}} \arrow[r, "\pi_{\dot{w}}"] & \mathcal{F}\ell_{\mathcal{I},\bar{k},w} \end{tikzcd} \tag{7.27}$$

with its equicharacteristic counterpart. The bottow arrow was dealt with in Lemma 3.15 and the second paragraph there applies verbatim to comparing the left arrow. We claim that these suffice to recover the remainder of the diagram.

Note that the vertical arrows affine morphisms, so they can be written via relative spectra, i.e. we have $\mathcal{D}^{\mathrm{tor}}_{\mathcal{I},\bar{k},\dot{w}} = \mathrm{Spec}(p_{\mathcal{D},*}\mathcal{O}_{\mathcal{D}^{\mathrm{tor}}})$ and $\mathcal{F}\ell^{\mathrm{tor}}_{\mathcal{I},\bar{k},w} = \mathrm{Spec}(p_{\mathcal{F}\ell,*}\mathcal{O}_{\mathcal{F}\ell^{\mathrm{tor}}})$. On the other hand, we know that $\pi_{\dot{w},*}\mathcal{O}_{\mathcal{D}} = \mathcal{O}_{\mathcal{F}\ell}$, so the same equality holds for $\pi^{\mathrm{tor}}_{\dot{w}}$ by flat base change, as the vertical arrows are perfectly smooth. But this means $p_{\mathcal{F}\ell,*}\mathcal{O}_{\mathcal{F}\ell^{\mathrm{tor}}} = \pi_{\dot{w},*}p_{\mathcal{D},*}\mathcal{O}_{\mathcal{D}^{\mathrm{tor}}}$, just as asserted.

Next, we show that the isomorphisms constructed above are unique. Without torsors, this has been verified in Proposition 3.10, so any automorphism respects the orbit part $\mathcal{F}\ell^{\circ,\mathrm{tor}}_{\mathcal{I},w}$. So we only have to verify the conditions of Lemma 7.7 on centralizers of the transfer homomorphism φ_w. In this case, it is given by

$$\mathrm{int}(w^{-1}) \colon L^+\mathcal{I} \cap wL^+\mathcal{I}w^{-1} \mapsto w^{-1}L^+\mathcal{I}w \cap L^+\mathcal{I}. \tag{7.28}$$

We see that the image of the right side in $(\mathcal{I}^{>i}_{\bar{k}})^{\mathrm{perf}}$ contains the image of $\mathcal{B}^{\mathrm{perf}}_{\bar{k}}$, where $\mathcal{B} \subset \mathcal{G}$ is the flat closure of some Borel $B \subset G$. However, inspecting the description of $\mathcal{I}^{>i}_{\bar{k}}$ given in Lemma 7.3, we see that the centralizer must be trivial: indeed, it is contained in $\mathcal{T}^{>i}_{\bar{k}}$, which itself decomposes as a product of groups indexed by positive simple $\mathcal{S}^{>i}_{\bar{k}}$-roots a, acting faithfully on the corresponding a-root groups.

Finally, we must show that the isomorphisms just constructed are functorial with respect to $(\underline{\mathcal{I}}, \mu_\bullet)$. This is easy for $\underline{\mathcal{I}}$, by uniqueness of equivariant automorphisms. As for $\mu_\bullet$, we appeal to Proposition 3.2 and the calculation of Picard groups in Theorem 3.8 and Remark 3.9 to recover the Stein factorization of the proper surjection

$$\mathcal{A}_{\mathcal{I},\mu_\bullet} \to \mathcal{A}_{\mathcal{I},\mu}. \tag{7.29}$$

In equicharecteristic, the Stein factorization is already $\mathcal{A}_{\mathcal{I}',\mu'}$ due to Zariski's connectedness theorem applied to $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{I}},O_C,\mu}$. Therefore, we get a new equivariant surjection $\mathcal{A}_{\mathcal{I}',\mu'} \to \mathcal{A}_{\mathcal{I},\mu}$, which becomes the identity after composing with the isomorphism $\mathcal{A}_{\mathcal{I},\mu} \cong \mathcal{A}_{\mathcal{I}',\mu'}$ of Lemma 3.15, by the uniqueness proved in Proposition 3.10. □

The last comparison involves the semi-orbits.

Proposition 7.20. *There are unique equivariant isomorphisms*

$$(\mathcal{N}^{\mathrm{sch},\circ,\mathrm{tor}}_{\mathcal{I},O_C,\mu_i})^\diamondsuit \simeq \mathcal{M}^{\circ,\mathrm{tor}}_{\mathcal{I},O_C,\mu_i} \tag{7.30}$$

for each term μ_i of the sequence $\mu_\bullet$. They yield canonical equivariant isomorphisms

$$(\mathcal{N}^{\mathrm{sch},\circ}_{\mathcal{I},O_C,\mu_\bullet})^\diamondsuit \simeq \mathcal{M}^{\circ}_{\mathcal{I},O_C,\mu_\bullet}, \tag{7.31}$$

compatibly with those of Lemma 7.18 and Proposition 7.19 in the obvious sense.

Proof. We have already identified the generic fibers of these v-sheaves, see Lemma 7.18. By Lemma 7.8, we reduce to calculating $\mathrm{Spd}\, O_C$-valued points of the left side torsor and compare their residue to that of Proposition 7.10. The resulting isomorphism will then reduce to the expected isomorphisms over $\mathrm{Spd}\, k$ obtained in Proposition 7.19, by uniqueness of equivariant automorphisms.

Now, we repeat the same calculation of Proposition 7.10, that already went back to Zhu, see [Zhu14] and [Lev16, Proposition 4.2.8]. Here, we work with the power series loop group $L^{\mathrm{sch}}_{O_C}\underline{\mathcal{T}}$ in the setting of [FHLR22, Subsection 2.2]. After refining $\mu_\bullet$, we may and do assume that each term $\mu_i \in \mu_\bullet$ is concentrated in a single component of the Dynkin diagram of G. There is a natural map

$$\mathrm{Res}_{O[\![u]\!]/O[\![t]\!]}\mathbb{G}_m \to \underline{\mathcal{T}} \tag{7.32}$$

induced by λ_i via taking the norm of restriction of scalars. Hence, we may and do assume that $\underline{\mathcal{T}} = \mathrm{Res}_{O[\![u]\!]/O[\![t]\!]}\mathbb{G}_m$. Note that here $O[\![u]\!]$ is a finite $O[\![t]\!]$-algebra, where u satisfies here an Eisenstein–Teichmüller type polynomial

$$u^n + a_1(t)u^{n-1} + \cdots + a_n(t) = 0 \tag{7.33}$$

in t based on some fixed choices of uniformizers π_K for K and π for F, see [FHLR22, Subsection 2.2]. Now, we claim for any $\sigma \in \mathrm{Gal}_F$, the element $\sigma z_u = u - \sigma\pi_K$ is a unit in $O_C[\![u]\!][z_t^{-1}]$ with $z_t = t - \pi$. Notice that its norm in $O_C[\![t]\!]$ equals

$$P(t) = \sigma\pi_K^n + a_1(t)\sigma\pi_K^{n-1} + \cdots + a_n(t) \tag{7.34}$$

which is the product of z_t with a unit of $O_C[\![t]\!]$. Indeed, we calculate the value $P(\pi) = 0$, also of the first derivative $P'(\pi) \in O_C^\times$ and apply the Taylor series inside $C[\![t]\!]$. Finally, we notice that σz_u reduces to the unit

$\tau\pi_K - \sigma\pi_K$ for all $\tau \neq \sigma$ of $\mathrm{Gal}_F/\mathrm{Gal}_K$ of the semi-field $C[\![u]\!][z_t^{-1}]$, and to a prime element in the factor indexed by σ due to norm considerations. The desired claim has been shown. □

7.4. **The Scholze–Weinstein conjecture.** In this subsection, we finally prove the Scholze–Weinstein conjecture, see Theorem 7.21 and Theorem 7.23 below.

We start by adressing the representability problem as in [Lou20, Conjecture IV.4.18], which is one half of [SW20, Conjecture 21.4.1]. Recall that $F/\mathbb{Q}_p$ is a complete non-archimedean field with perfect residue field k, G is an arbitrary (connected) reductive F-group, μ is a dominant coweight of G_C and $\mathcal{G}$ an arbitrary parahoric O-model of G.

Theorem 7.21. *Let μ be minuscule. Then, there is a unique (up to unique isomorphism) flat, projective and weakly normal O_E-model $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ of the E-scheme $\mathcal{F}_{G,\mu}$ endowed with a $\mathcal{G}_{O_E}$-action for which*

$$\mathcal{M}^{\mathrm{sch},\diamondsuit}_{\mathcal{G},\mu} \cong \mathcal{M}_{\mathcal{G},\mu}, \tag{7.35}$$

prolonging $\mathcal{F}^{\diamondsuit}_{G,\mu} \cong \mathrm{Gr}_{G,\mu}$ equivariantly under $\mathcal{G}^{\diamondsuit}_{O_E}$.

Proof. First of all, let us work under Assumption 7.16. We know that the geometric fibers of $\mathcal{N}^{\mathrm{sch},\diamondsuit}_{\underline{\mathcal{I}},O_C,\mu}$ and $\mathcal{M}_{\mathcal{I},O_C,\mu}$ are uniquely equivariantly isomorphic by Lemma 7.18, Proposition 7.19. By uniqueness, this commutes with the Galois action, so it descends to the fibers over $\mathrm{Spd}\, O_E$.

Furthermore, thanks also to Proposition 7.20, Theorem 7.12, and Remark 7.14, we know that the specialization maps

$$\mathrm{sp}\colon \mathcal{F}_{G,\mu}(C) \to \mathcal{A}_{\mathcal{I},\mu}(\bar{k}), \tag{7.36}$$

arising respectively from the π-adic kimberlite $\mathcal{M}_{\mathcal{I},O_C,\mu}$ and $\mathcal{N}^{\mathrm{sch},\diamondsuit}_{\underline{\mathcal{I}},O_C,\mu}$ must coincide. By continuity for the constructible topology, we obtain an equivariant isomorphism of specialization triples:

$$\big(\mathcal{N}^{\mathrm{sch},\diamondsuit}_{\underline{\mathcal{I}},E,\mu}, \mathcal{N}^{\mathrm{sch},\diamondsuit}_{\underline{\mathcal{I}},\mu,k_E}, \mathrm{sp}_{\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{I}},\mu}}\big) \cong \big(\mathcal{M}_{\mathcal{I},E,\mu}, \mathcal{M}_{\mathcal{I},\mu,k_E}, \mathrm{sp}_{\mathcal{M}_{\mathcal{I},\mu}}\big) \tag{7.37}$$

associated with v-sheaves over $\mathrm{Spd}\, O_E$. Observing that both v-sheaves satisfy the hypothesis of Theorem 2.36, we may directly appeal to it in order to get a necessarily equivariant isomorphism

$$\mathcal{N}^{\mathrm{sch},\diamondsuit}_{\underline{\mathcal{I}},\mu} \cong \mathcal{M}_{\mathcal{I},\mu}. \tag{7.38}$$

Now maintain the part of Assumption 7.16 that refers to G, but suppose $\mathcal{G}$ is now an arbitrary parahoric model such that $\mathcal{I} \to \mathcal{G}$. We get a v-cover

$$\mathcal{M}_{\mathcal{I},\mu} \to \mathcal{M}_{\mathcal{G},\mu} \tag{7.39}$$

and, paralelly, a scheme-theoretic projective cover

$$\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{I}},\mu} \to \mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{G}},\mu} \tag{7.40}$$

by virtue of [FHLR22, Section 4.2]. Therefore, it is enough to verify that the v-sheaf-theoretic equivalence relations coincide along the left side identification.

By construction, this reduces to $\mathcal{I}_k^{\mathrm{perf}}$-equivariantly compare the surjection

$$\mathcal{A}_{\mathcal{I},\mu} \cong \mathcal{A}_{\mathcal{G},\mu} \tag{7.41}$$

to the one obtained in equicharacteristic. This is entirely similar to what was done in Lemma 3.15 and Proposition 7.19, so we omit it.

Finally, suppose that G is arbitrary. Thanks to Proposition 4.16, $\mathcal{M}_{\mathcal{G},\mu}$ is isomorphic to $\mathcal{M}_{\mathcal{G}_{\mathrm{ad}},\mu_{\mathrm{ad}}}$ after base change to $\mathrm{Spd}\, O_E$, and decomposes into products, hence we may assume G is simple and adjoint. We can find a locally closed immersion

$$\mathcal{G} \to \tilde{\mathcal{G}}, \tag{7.42}$$

where $\tilde{\mathcal{G}}$ is a parahoric model of a Weil-restricted split form of G, which was treated in the previous paragraph. Since we have an inclusion $\mathcal{M}_{\mathcal{G},\mu} \subset \mathcal{M}_{\tilde{\mathcal{G}},\tilde{\mu}}$, it now suffices to take the absolute weak normalization of the flat closure of $\mathcal{F}_{G,\mu}$ inside the scheme-theoretic local model attached to $(\tilde{\mathcal{G}}, \tilde{\mu})$. □

Remark 7.22. Let us explain how representability can be proved for classical groups without resorting to the characterization of the specialization map found in Theorem 7.12. Indeed, for those groups we can directly understand the v-sheaves $\mathcal{M}^{\mathrm{tor}}_{\mathcal{G},\mu_i}$ by embedding them in a similar torsor attached to Weil-restricted PGL_n. Those had been studied already by Pappas–Rapoport, see [PR05, Proposition 5.2], and a careful analysis of the map in [SW20, Proposition 21.6.9] reveals that all proposed definitions coincide. The result follows by v-descent.

We have found certain finite type O_E-schemes $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ representing $\mathcal{M}_{\mathcal{G},\mu}$, but we still do not know a lot about the geometry of its special fiber, which is the second part of [SW20, Conjecture 21.4.1], see also [Lou20, Conjecture IV.4.19]. We recall the canonical deperfection $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}$ of the μ-admissible locus introduced in Definition 3.11 and Definition 3.14.

Theorem 7.23. *Under Assumption 1.9 and Assumption 1.13, the special fiber of the O_E-scheme $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ of Theorem 7.21 is uniquely $\mathcal{G}_{k_E}$-equivariantly isomorphic to the canonical deperfection of the μ-admissible locus:*

$$\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}|_{\operatorname{Spec} k_E} \cong \mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu} \tag{7.43}$$

In particular, $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is normal, Cohen–Macaulay and has a reduced, weakly normal, Frobenius split special fiber.

Proof. During the proof of Theorem 7.21, we already saw that the algebraic local models $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ are actually the $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{G}},\mu}$ constructed in [FHLR22, Definition 4.7] by a variation on the techniques of Pappas–Zhu [PZ13], Levin [Lev16] and also the third author [Lou19, Lou20]. For this, we may pass to a finite unramified extension of F, so G is quasi-split and residually split, so that $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{G}},\mu}$ is defined (under Assumption 1.9). Then, it embeds in a local model associated with a Weil-restricted split group, confer [FHLR22, Corollary 2.10] (this is where the Assumption 1.13 comes from). We conclude under the given hypothesis that the $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ are indeed normal, Cohen–Macaulay and have a Frobenius split special fiber by [FHLR22, Theorem 4.10]. Indeed, the special fiber of $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{G}},\mu}$ is reduced equal to an equicharacteristic admissible locus $\mathcal{A}^{\mathrm{can}}_{\mathcal{G}',\mu'}$, which equivariantly identifies with $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}$ by Lemma 3.15. □

Remark 7.24. More generally, we may incorporate in Theorem 7.23 the additional cases found in [Lou19, Lou20] in order to get the statement of Theorem 7.23 in slightly bigger generality, see Theorem 1.1. We leave the details to the reader.

To conclude, let us only use Theorem 7.21 –and not resort to the construction of local models in [FHLR22]– in order to study the geometry of the special fiber of $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$.

First of all, we know that the perfection of $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu,k_E}$ equals $\mathcal{A}_{\mathcal{G},\mu}$ by Theorem 6.16 and fully faithfulness of $\diamondsuit$ on perfect schemes, see [SW20, Proposition 18.3.1]. By the weak normality property and Lemma 7.6, we conclude that $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ admits a smooth open subscheme $\mathcal{M}^{\mathrm{sch},\circ}_{\mathcal{G},\mu}$ descending

$$\mathcal{M}^{\mathrm{sch},\circ}_{\mathcal{G},O_{\breve{E}},\mu} = \bigcup_{\lambda} \mathcal{G}_{O_{\breve{E}}}/\mathcal{P}^{-}_{\lambda}, \tag{7.44}$$

compare with the argument in [Ric16, Corollary 2.14]. It follows that we have a natural morphism

$$\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu} \to \mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu,k_E}. \tag{7.45}$$

The following conjecture is then the full p-adic coherence conjecture:

Conjecture 7.25. *The map (7.45) is always an isomorphism.*

In order to prove this in the very few cases left, see Assumption 1.9 and Assumption 1.13 and Remark 7.24, we envision the following steps. First, one should remove Assumption 1.9 from Theorem 3.16. Then, it would suffice to show that (7.45) is a closed immersion, removing also Assumption 1.13. This property can be verified after mapping to some suitable larger $\mathcal{A}^{\mathrm{can}}_{\tilde{\mathcal{G}},\tilde{\mu}}$. In particular, we see that both steps take place entirely in the special fiber.

Alternatively, we can give a purely v-sheaf-theoretic criterion for $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu,k_E}$ being reduced:

Lemma 7.26. *Suppose $(\widehat{\mathcal{M}_{\mathcal{G},O_C,\mu/\bar{x}}})_\eta$ is connected for every $\bar{k}$-valued point $\bar{x}$ of $\mathcal{A}_{\mathcal{G},\mu}$. Then $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu,k_E}$ is geometrically reduced. Under Assumption 1.9, Conjecture 7.25 holds.*

Proof. By Proposition 2.38, we know that $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is normal. In particular, $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu,k_E}$ is S1, but it must also be R0, as it contains a smooth dense open $\mathcal{A}^{\mathrm{can},\circ}_{\mathcal{G},\mu}$. So Serre's criterion for reducedness furnishes the claim. As for identifying the special fiber with $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}$ as per Conjecture 7.25, we appeal to Theorem 3.16, which computes the dimension of the vector spaces of global sections of ample line bundles. □

Finally, let us also mention the following conjecture, arising from [FHLR22], on the singularities of $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$:

Conjecture 7.27. *The local model $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ has pseudo-rational singularities.*

8. The test function conjecture

Throughout this section, we let $F/\mathbb{Q}_p$ be a finite field extension with ring of integers O and finite residue field k of cardinality q. We fix an algebraic closure $\bar{\mathbb{Q}}_p$, an embedding $F \hookrightarrow \bar{\mathbb{Q}}_p$ and denote by $\Gamma = \mathrm{Gal}(\bar{\mathbb{Q}}_p/F)$ the absolute Galois group of F with inertia subgroup I. Let G be a reductive F-group with parahoric O-model $\mathcal{G}$.

Furthermore, fix a square root $\sqrt{q}$, an auxiliary prime $\ell \nmid q$ and put $\Lambda = \mathbb{Q}_\ell(\sqrt{q})$. We let ${}^LG = \widehat{G}_\Lambda \rtimes \Gamma$ be the Langlands dual group viewed as a pro-algebraic Λ-group scheme. Each algebraic representation V of ${}^LG$ furnishes, by choosing a quasi-inverse to the geometric Satake equivalence, a semi-simple perverse Λ-sheaf $\mathrm{Sat}(V)$ of "weight zero" on the B^+_{dR}-affine Grassmannian $\mathrm{Gr}_G \to \operatorname{Spd} F$. Here $\sqrt{q}$ is needed to define a square root of the ℓ-adic cyclotomic character used when Tate twisting irreducible perverse sheaves supported on components of Gr_G of odd parity to be of "weight zero". More precisely, for a dominant coweight μ defined over F, we have

$$\mathrm{Sat}(V_\mu) = i_{\mu,*} j_{\mu,!*} \Lambda_{\mathrm{Gr}^\circ_\mu}\big(\tfrac{\langle 2\rho,\mu\rangle}{2}\big), \tag{8.1}$$

where $\mathrm{Gr}^\circ_{G,\mu} \xrightarrow{j_\mu} \mathrm{Gr}_{G,\mu} \xrightarrow{i_\mu} \mathrm{Gr}_G$ and V_μ is the irreducible representation of ${}^LG$ of highest weight μ. Every simple object is of this form, up to taking a finite Galois orbit of μ's and tensoring with simple Λ-local systems on $\mathrm{Spd}\, F$ of weight zero (corresponding to irreducible representations of Γ factoring through a finite quotient).

As in Section 6.5, we consider the functor of nearby cycles

$$\Psi_{\mathcal{G}} := i^* Rj_*(\text{-})|_{\mathrm{Spd}\,\mathbb{C}_p} \colon \mathrm{D}(\mathrm{Hk}_G, \Lambda) \to \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}}, \Lambda), \tag{8.2}$$

where $\mathrm{Hk}_{G,\mathbb{C}_p} \xrightarrow{j} \mathrm{Hk}_{\mathcal{G},O_{\mathbb{C}_p}} \xleftarrow{i} \mathrm{Hk}_{\mathcal{G},\bar{k}}$ are the inclusions of the geometric fibers.

Lemma 8.1. *For every finite dimensional algebraic ${}^LG$-representation V, the sheaf of nearby cycles $\Psi_{\mathcal{G}}(\mathrm{Sat}(V))$ naturally defines an object in the category*

$$\mathrm{D}_{\mathrm{cons}}\big([\underline{\Gamma}\backslash \mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G},\bar{k}}], \Lambda\big)^{\mathrm{bd}} \tag{8.3}$$

of constructible Λ-sheaves with bounded support on the v-stack $[\underline{\Gamma}\backslash \mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G},\bar{k}}]$ where $\underline{\Gamma}$ denotes the associated group v-sheaf and the action on the schematic Hecke stack $\mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G},\bar{k}}$ is induced by the quotient map $\Gamma \to \mathrm{Gal}(\bar{k}/k)$. In particular, the cohomology sheaves $\mathrm{R}^n\Psi_{\mathcal{G}}(\mathrm{Sat}(V))$, $n \in \mathbb{Z}$ define $L^+_{\bar{k}}\mathcal{G}$-equivariant, constructible Λ-sheaves with bounded support on $\mathcal{F}\ell_{\mathcal{G},\bar{k}}$ equipped with an equivariant continuous Γ-action as defined in [SGA73, Exposé XIII] *compatibly with the $L^+_{\bar{k}}\mathcal{G}$-action.*

Proof. The group Γ is identified with the group of continuous automorphisms of $\mathbb{C}_p$ over F. Since the geometric fiber inclusions i and j are $\underline{\Gamma}$-equivariant, we obtain maps of v-stacks

$$\mathrm{Hk}_G = [\underline{\Gamma}\backslash \mathrm{Hk}_{G,\mathbb{C}_p}] \xrightarrow{[\underline{\Gamma}\backslash j]} [\underline{\Gamma}\backslash \mathrm{Hk}_{\mathcal{G},O_{\mathbb{C}_p}}] \xleftarrow{[\underline{\Gamma}\backslash i]} [\underline{\Gamma}\backslash \mathrm{Hk}_{\mathcal{G},\bar{k}}], \tag{8.4}$$

and define the Galois equivariant nearby cycles functor $[\underline{\Gamma}\backslash\Psi_{\mathcal{G}}] := [\underline{\Gamma}\backslash i]^* R[\underline{\Gamma}\backslash j]_*(\text{-})$ in analogy to (8.2). Consider the quotient map $v\colon \mathrm{Hk}_{\mathcal{G},\bar{k}} \to [\underline{\Gamma}\backslash \mathrm{Hk}_{\mathcal{G},\bar{k}}]$. We claim that the map $v^*[\underline{\Gamma}\backslash\Psi_{\mathcal{G}}](\mathrm{Sat}(V)) \to \Psi_{\mathcal{G}}(\mathrm{Sat}(V))$ induced by base change is an isomorphism.

Note that we can not apply the base change theorem directly because j is not quasi-compact and Γ is only profinite. Instead, we apply constant term functors, which commute with arbitrary base change: By finite étale descent, we may and do assume that G is quasi-split and residually split, so that every $\breve{F}$-Borel descends to F. Using the conservativity of constant term functors, see Proposition 6.4, we see again as in Proposition 6.12 that the equivariant integral extension $R[\underline{\Gamma}\backslash j]_*\,\mathrm{Sat}(V)$ is ULA over $[\underline{\Gamma}\backslash \mathrm{Spd}\, O_{\mathbb{C}_p}]$. In particular, so is its pullback to $\mathrm{Spd}\, O_{\mathbb{C}_p}$, which implies by Proposition 6.12 that it equals $Rj_*(\mathrm{Sat}(V)|_{\mathrm{Spd}\,\mathbb{C}_p})$. Restricting to geometric special fibers implies the claim.

It formally follows from Proposition A.5 and the construction of derived categories of Λ-sheaves that the comparison functor (A.3) induces an equivalence

$$\mathrm{D}([\underline{\Gamma}\backslash \mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G},\bar{k}}], \Lambda)^{\mathrm{bd}} \cong \mathrm{D}([\underline{\Gamma}\backslash \mathrm{Hk}_{\mathcal{G},\bar{k}}], \Lambda)^{\mathrm{bd}} \tag{8.5}$$

under which constructible sheaves correspond to ULA sheaves, see Proposition 6.7. Note that both properties are preserved and detected under the functor v^*, respectively its schematic counterpart. Therefore, $[\underline{\Gamma}\backslash\Psi_{\mathcal{G}}](\mathrm{Sat}(V))$ naturally defines an object in the category (8.3) and its underlying sheaf is $\Psi_{\mathcal{G}}(\mathrm{Sat}(V))$.

For the final statement on the comparison with [SGA73, Exposé XIII], we reduce to the case where Λ is a finite ring by the construction of categories of ℓ-adic sheaves, see also (6.2). Then, for any qcqs k-scheme X, the category of abelian Λ-sheaves on $(X_{\bar{k}})_{\text{ét}}$ equipped with a continuous Γ-action as in [SGA73, Exposé XIII, Définition 1.1.2] embeds fully faithfully into the category of abelian Λ-sheaves on $[\underline{\Gamma}\backslash X_{\bar{k}}]$ inducing an equivalence on full subcategories of constructible sheaves. Applying this to closed subschemes $X \subset \mathcal{F}\ell_{\mathcal{G}}$ implies the lemma. ☐

For every $\Phi \in \Gamma$, we define a function $\mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G}}(k) \to \Lambda$ by the formula

$$\tau^{\Phi}_{\mathcal{G},V}(x) := (-1)^{d_V} \sum_{n\in\mathbb{Z}} (-1)^n \,\mathrm{trace}\,\big(\Phi \mid \mathrm{R}^n\Psi_{\mathcal{G}}\mathrm{Sat}(V)_{\bar{x}}\big) \tag{8.6}$$

whenever V is irreducible and extend the definition to general V by linearity. Here, $d_V = \langle 2\rho, \mu\rangle$ with μ being the highest weight of V. So the sign $(-1)^{d_V}$ in (8.6) only depends on the parity of the connected component of Gr_G that supports $\mathrm{Sat}(V)$.

Lemma 8.2. *For every finite dimensional algebraic ${}^LG$-representation V and every $\Phi \in \Gamma$, the function $\tau^{\Phi}_{\mathcal{G},V}$ naturally lies in the center of the parahoric Hecke algebra $\mathcal{H}(G(F), \mathcal{G}(O))_\Lambda$.*

Proof. Lang's lemma together with an approximation argument [RS20, Lemma A.3] implies that $\mathrm{H}^1_{\text{ét}}(k, L^+\mathcal{G})$ vanishes, so $\mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G}}(k) = \mathcal{G}(O)\backslash G(F)/\mathcal{G}(O)$. As the function $\tau^{\Phi}_{\mathcal{G},V}$ is supported on finitely many double cosets, it lies in $\mathcal{H}(G(F), \mathcal{G}(O))_\Lambda$. Centrality follows from Proposition 6.17 and the usual sheaf function dictionary. ☐

On the other hand, the theory of Bernstein centers defines another function: Namely, for every choice of lift $\Phi \in \Gamma$ of geometric Frobenius, we let $z^{\Phi}_{\mathcal{G},V}$ be the unique function in the center of $\mathcal{H}(G(F), \mathcal{G}(O))_\Lambda$ that acts on every smooth irreducible $\mathcal{G}(O)$-spherical representation π over Λ by the scalar

$$\mathrm{trace}\Big(s^{\Phi}(\pi) \mid V\Big), \tag{8.7}$$

where $s^{\Phi}(\pi) \in [\widehat{G}^I \rtimes \Phi]_{\mathrm{ss}}/\widehat{G}^I$ is the Satake parameter for π with respect to Φ constructed in [Hai15].

Theorem 8.3. *For every finite dimensional algebraic ${}^LG$-representation V and every choice of lift Φ of geometric Frobenius, there is an equality*

$$\tau^{\Phi}_{\mathcal{G},V} = z^{\Phi}_{\mathcal{G},V} \tag{8.8}$$

of functions in the parahoric Hecke algebra.

Proof. As both sides of (8.8) are additive in V, we may freely assume that V is irreducible, and even further that $V|_{\widehat{G}\rtimes I}$ is irreducible: otherwise both sides in (8.8) are zero (hence, equal) by elementary considerations, see [HR21, Lemma 7.7].

Fix a maximal F-split torus $A \subset G$ whose Néron model embeds in $\mathcal{G}$ and a regular cocharacter $\lambda\colon \mathbb{G}_m \to A$. Then λ induces a minimal F-Levi M, respectively F-parabolic P in G. Denote by $\mathcal{M} \subset \mathcal{P}$ their flat closures in $\mathcal{G}$. Then the constant terms morphism [Hai14, Section 11.11] induces an injective morphism on the centers of the parahoric Hecke algebras

$$\mathrm{ct}_{\mathcal{P}}\colon \mathcal{Z}(G(F), \mathcal{G}(O))_\Lambda \hookrightarrow \mathcal{Z}(M(F), \mathcal{M}(O))_\Lambda. \tag{8.9}$$

As in [HR21, Lemma 7.8, Equation (7.15)], one checks the formulas

$$\mathrm{ct}_{\mathcal{P}}\left(\tau^{\Phi}_{\mathcal{G},V}\right) = \tau^{\Phi}_{\mathcal{M},V|_{{}^LM}}, \quad \mathrm{ct}_{\mathcal{P}}\left(z^{\Phi}_{\mathcal{G},V}\right) = z^{\Phi}_{\mathcal{M},V|_{{}^LM}}, \tag{8.10}$$

where ${}^LM = \widehat{M} \rtimes \Gamma$ is viewed as a closed subgroup of ${}^LG$. The second formula in (8.10) is straight forward. The first formula in (8.10) is based on the isomorphism

$$\mathrm{CT}_{\mathcal{P}}[\deg_{\mathcal{P}}] \circ \Psi_{\mathcal{G}} \cong \Psi_{\mathcal{M}} \circ \mathrm{CT}_P[\deg_P]\colon \mathrm{Sat}(\mathrm{Hk}_G, \Lambda) \to \mathrm{D}_{\mathrm{cons}}([\underline{\Gamma}\backslash \mathcal{F}\ell_{\mathcal{M},\bar{k}}], \Lambda)^{\mathrm{bd}}, \tag{8.11}$$

see Proposition 6.13, using that $\mathrm{CT}_P[\deg_P]$ corresponds to the restriction of representations $V \mapsto V|_{{}^LM}$ under the geometric Satake equivalence [FS21, Section VI]. (We note that the sign $(-1)^{d_V}$ in (8.6) appears when comparing $\mathrm{CT}_{\mathcal{P}}[\deg_{\mathcal{P}}]$ and $\mathrm{ct}_{\mathcal{P}}$ under the sheaf function dictionary, see also [HR21, Lemma 7.2].)

Hence, we reduce to the case where $G = M$ is a minimal F-Levi, so anisotropic modulo center, and $V|_{\widehat{G}\rtimes I}$ is irreducible. Let $\mathcal{M}_{\mathcal{G},V}$ be the v-sheaf theoretic closure of the support of $\mathrm{Sat}(V)$ in $\mathrm{Gr}_{\mathcal{G}}$, a finite union of $\mathcal{M}_{\mathcal{G},\mu}$ for μ ranging over the highest weights of V. The proof of [HR21, Lemma 7.13] is based on Iwahori-Weyl group combinatorics, hence applies to show that $\mathcal{M}_{\mathcal{G},V}$ has only a single $\mathrm{Spd}\, k$-valued point x_V. As Φ lifts the geometric Frobenius, we can apply the Grothendieck-Lefschetz trace formula to $\Psi_{\mathcal{G}}\mathrm{Sat}(V)$ viewed as an object in $\mathrm{D}_{\mathrm{cons}}([\underline{\Gamma}\backslash \mathcal{F}\ell_{\mathcal{G},\bar{k}}], \Lambda)^{\mathrm{bd}}$ to compute

$$\mathrm{trace}\left(\Phi \mid \Psi_{\mathcal{G}}\mathrm{Sat}(V)_{\overline{x_V}}\right) = \mathrm{trace}\left(\Phi \mid \mathrm{H}^*(\mathcal{F}\ell_{\mathcal{G},\bar{k}}, \Psi_{\mathcal{G}}\mathrm{Sat}(V))\right) \tag{8.12}$$

Since $Rj_*(\mathrm{Sat}(V)|_{\mathrm{Spd}\ \mathbb{C}_p})$ is ULA by Proposition 6.12, the latter cohomology group is Γ-equivariantly isomorphic to

$$\mathrm{H}^*(\mathrm{Gr}_{G,\mathbb{C}_p}, \mathrm{Sat}(V)) = \mathrm{H}^*(\mathrm{Gr}_{G^*,\mathbb{C}_p}, \mathrm{Sat}(V)), \tag{8.13}$$

where G^* is the unique quasi-split inner form of G. We note that there is a canonical identification ${}^LG = {}^LG^*$ so that on Satake categories $\mathrm{Sat}(\mathrm{Hk}_G, \Lambda) \cong \mathrm{Sat}(\mathrm{Hk}_{G^*}, \Lambda)$ by [FS21, Section VI]. Let $\mathcal{G}^*$ denote the parahoric corresponding to $\mathcal{G}$ (necessarily, an Iwahori) and $\mathcal{M}_{\mathcal{G}^*,V}$ the associated v-sheaf local model. On the other hand, we know [Hai14, Proposition 11.12.6] that $z^{\Phi}_{\mathcal{G},V}$ is supported at x_V with value

$$z^{\Phi}_{\mathcal{G},V}(x_V) = \sum_{x \in \mathcal{M}_{\mathcal{G}^*,V}(\mathrm{Spd}\ k)} z^{\Phi}_{\mathcal{G}^*,V}(x). \tag{8.14}$$

Now assuming the test function conjecture for the pair $(\mathcal{G}^*, V)$, that is, assuming $z^{\Phi}_{\mathcal{G}^*,V} = \tau^{\Phi}_{\mathcal{G}^*,V}$, we can apply the Grothendieck-Lefschetz trace formula again to see that (8.14) equals the trace of Φ on (8.13), up to the sign $(-1)^{d_V}$. So $z^{\Phi}_{\mathcal{G}^*,V} = \tau^{\Phi}_{\mathcal{G}^*,V}$ implies $z^{\Phi}_{\mathcal{G},V} = \tau^{\Phi}_{\mathcal{G},V}$.

Hence, we reduce to the case where $G = G^*$ is quasi-split. Now, the minimal Levi $M = T$ is a maximal torus, so (8.10) reduces us to the case where $G = T$ is a torus and $\mathcal{G} = \mathcal{T}$ its connected locally finite type Néron model. Without loss of generality, we assume that $V|_{\widehat{T}\rtimes I}$ is irreducible. Evidently, T is anisotropic modulo center so that both functions $\tau^{\Phi}_{\mathcal{T},V}, z^{\Phi}_{\mathcal{T},V}$ are supported at x_V. Using (8.12), the ULA property of $Rj_*(\mathrm{Sat}(V)|_{\mathrm{Spd}\ \mathbb{C}_p})$ and $\mathrm{H}^0(\mathrm{Gr}_{T,\mathbb{C}_p}, \mathrm{Sat}(V)) = V$, we see

$$\tau^{\Phi}_{\mathcal{T},V}(x_V) = (-1)^{d_V}\,\mathrm{trace}\left(\Phi \mid V\right) \tag{8.15}$$

which equals $z^{\Phi}_{\mathcal{T},V}(x_V)$ because $d_V = 0$. This finishes the proof. □

Lemma 8.4. *Theorem 8.3 implies Theorem 1.2.*

Proof. Let μ be a conjugacy class of geometric cocharacters in G. Denote by $E \subset \bar{\mathbb{Q}}_p$ its reflex field with maximal unramified subextension E_0/F. Their rings of integers are denoted by $O_E \supset O_{E_0}$ with residue fields $k_E = k_{E_0}$ and absolute Galois groups $\Gamma_E \subset \Gamma_{E_0}$. For every $\Phi \in \Gamma_E$ and $x \in \mathrm{Gr}_{\mathcal{G}}(k_E)$, there is an equality

$$\mathrm{trace}\left(\Phi \mid \Psi_{\mathcal{G},O_E}\mathrm{Sat}(V_\mu)_{\bar{x}}\right) = \mathrm{trace}\left(\Phi \mid \Psi_{\mathcal{G},O_{E_0}}\mathrm{Sat}(I_E^{E_0}(V_\mu))_{\bar{x}}\right), \tag{8.16}$$

where $I_E^{E_0}(V_\mu)$ is the induction to $\widehat{G} \rtimes \Gamma_{E_0}$ of the $\widehat{G} \rtimes \Gamma_E$-representation V_μ and $\mathrm{Sat}(V_\mu)$, $\mathrm{Sat}(I_E^{E_0}(V_\mu))$ the corresponding Satake sheaves on $\mathrm{Gr}_G|_{\mathrm{Spec}\, E}$, respectively $\mathrm{Gr}_G|_{\mathrm{Spec}\, E_0}$. Indeed, (8.16) follows from the commutation of nearby cycles with proper pushforward applied to the finite morphism $\mathrm{Gr}_{\mathcal{G}}|_{\mathrm{Spd}\, O_E} \to \mathrm{Gr}_{\mathcal{G}}|_{\mathrm{Spd}\, O_{E_0}}$, noting that it induces the induction of representations on Satake categories.

Now, we apply (8.16) to the pair $\mathcal{G}_0 := \mathcal{G}_{O_{E_0}}, V_{\mu,0} := I_E^{E_0}(V_\mu)$ and any choice of lift $\Phi \in \Gamma_E \subset \Gamma_{E_0}$ of geometric Frobenius to obtain

$$\tau^{\Phi}_{\mathcal{G}_0,V_{\mu,0}} = z^{\Phi}_{\mathcal{G}_0,V_{\mu,0}}. \tag{8.17}$$

The left hand side of (8.17) is equal to the function from Theorem 1.2 by (8.16) and so is the right hand side of (8.17) by a similar equality [Hai18, Lemma 8.1] for inductions of representations along the totally ramified extension E/E_0. That the function (8.17) takes, after multiplying by $(\sqrt{q_E})^{\langle 2\rho,\mu\rangle}$, values in $\mathbb{Z}$ independently of the choice of $\ell \neq p$, $\sqrt{q_E}$ and $E \hookrightarrow \bar{\mathbb{Q}}_p$ follows from [HR21, Theorem 7.15] where the statement is verified for the semi-simplified version of the right hand side of (8.17) without any assumptions on $(\mathcal{G},\mu)$. The same arguments apply here. $\square$

A. Étale cohomology for v-stacks on schemes

In this section, we extend some parts of [Sch17, Section 27] to v-stacks on perfect schemes, see also [Wu21, Appendix A]. Let k be a perfect field of characteristic $p > 0$. Let X be a perfect scheme over k, and let $X^\diamondsuit$ be the associated v-sheaf from Section 3.1 (under the tilting equivalence), that is, if $\mathrm{Spa}(R,R^+)$ is an affinoid perfectoid space over $\mathrm{Spa}(k,k)$ and X affine, then

$$X^\diamondsuit(\mathrm{Spa}(R,R^+)) = X(\mathrm{Spec}(R)), \tag{A.1}$$

while for X general $X^\diamondsuit$ is the analytic sheafification of the presheaf $\mathrm{Spa}(R,R^+) \mapsto X(\mathrm{Spec}(R))$. We note that the functor $(\text{-})^\diamondsuit$ does depend on k.

Fix a torsion ring Λ with $p \in \Lambda^\times$. We let $\mathrm{D}(X,\Lambda) := \widehat{\mathrm{D}}(X_{\text{ét}},\Lambda)$ be the left-completed étale derived ∞-category of X, see [Sch17, Section 27]. By [Sch17, Section 27] there is a morphism

$$c_X\colon (X^\diamondsuit)_v \to X_{\text{ét}} \tag{A.2}$$

of sites (even to the proétale site of X), and the induced functor of ∞-categories

$$c_X^*\colon \mathrm{D}(X,\Lambda) \to \mathrm{D}(X^\diamondsuit,\Lambda) \tag{A.3}$$

is fully faithful, [Sch17, Proposition 27.2.]. In general the functor c_X^* is not essentially surjective, for example, on topological spaces $|X^\diamondsuit| \to |X|$ is surjective, but very often not injective.

The functor c_X^* enjoys many compatibilities. If $f\colon Y \to X$ is a map of schemes, then $c_X^* \circ f^* \cong (f^\diamondsuit)^* \circ c_X^*$ and $c_X^*(A) \otimes^{\mathbb{L}}_\Lambda c_X^*(B) \cong c_X^*(A \otimes^{\mathbb{L}}_\Lambda B)$, see [Sch17, Proposition 27.1.]. If $f\colon Y \to X$ is separated perfectly of finite type, then $c_X^* \circ Rf_! \cong Rf_!^\diamondsuit \circ c_X^*$, see [Sch17, Proposition 27.4.]. As we now justify c_X^* also preserves ULA-objects.

Proposition A.1. *Let S be a qcqs perfect scheme in characteristic p, and let $f\colon X \to S$ be a separated perfect scheme perfectly of finite presentation over S. Let $A \in \mathrm{D}(X,\Lambda)$ such that A is ULA with respect to f. Then c_X^*A is ULA with respect to $f^\diamondsuit$, and $c_X^*\mathbb{D}_{X/S}(A) \cong \mathbb{D}_{X^\diamond/S^\diamond}(c_X^*A)$ for the Verdier duals.*

Proof. As in [HS21, Section 3], we let $\mathcal{C}_S$ denote the 2-category whose objects are schemes over S as in the hypothesis, and where morphisms from X to Y are given by objects in $\mathrm{D}(X \times_S Y,\Lambda)$. Given two maps $A \in \mathrm{Hom}_{\mathcal{C}_S}(X_1,X_2)$ and $B \in \mathrm{Hom}_{\mathcal{C}_S}(X_2,X_3)$, we define their composition $A * B \in \mathrm{Hom}_{\mathcal{C}_S}(X_1,X_3)$ by the formula

$$A * B := R\pi_{1,3!}(\pi_{1,2}^*A \otimes^{\mathbb{L}}_\Lambda \pi_{2,3}^*B). \tag{A.4}$$

By [HS21, Proposition 3.4, Definition 3.2], the object $A \in \mathrm{Hom}_{\mathcal{C}_S}(X,S)$ is ULA with respect to f if and only if A is a left adjoint in $\mathcal{C}_S$. Analogously, let $\mathcal{C}_{S^\diamond}$ denote the category considered in [FS21, Section IV.2.3.3.]. By [FS21, Theorem IV.2.23.], the object $c_X^*A \in \mathrm{Hom}_{\mathcal{C}_S}(X^\diamondsuit,S^\diamondsuit)$ is ULA with respect to $f^\diamondsuit$ if it is a left adjoint in $\mathcal{C}_{S^\diamond}$. Now, we observe that the functors c_X^* can be promoted to a functor of 2-categories $c^*\colon \mathcal{C}_S \to \mathcal{C}_{S^\diamond}$ by the rule $c^*X = X^\diamondsuit$ and $c^*(A) = c^*_{X\times_S Y}(A)$ for $A \in \mathrm{Hom}_{\mathcal{C}_S}(X,Y)$. Here, we use that c^* commutes with the required operations by [Sch17, Propositions 27.1, 27.4]. But functors between 2-categories preserve the adjunctions between 1-morphisms which finishes the proof as the right adjoints are given by Verdier duals. $\square$

We move on to study stacks. Let Ani be the category of anima (also called spaces, ∞-groups or Kan complexes). By left Kan extension along the Yoneda embedding

$$\mathrm{SchPerf}_k \to \mathrm{Fun}(\mathrm{SchPerf}_k^{\mathrm{op}},\mathrm{Ani}), \tag{A.5}$$

we can extend[22] the functors $\mathrm{D}(\text{-}, \Lambda), \mathrm{D}((\text{-})^\diamondsuit, \Lambda)$ using $*$-pullbacks and the natural transformation $c^*_{(\text{-})}$ to contravariant functors $\mathrm{D}(\text{-}, \Lambda), \mathrm{D}((\text{-})^\diamondsuit, \Lambda)^{\mathrm{Kan}}$ on $\mathrm{Fun}(\mathrm{SchPerf}_k^{\mathrm{op}}, \mathrm{Ani})$ with values in symmetric monoidal stable ∞-categories, sending colimits to limits. More concretely, if a functor (also known as, higher prestack)

$$\mathfrak{X} \cong \operatorname*{colim}_i X_i \in \mathrm{Fun}(\mathrm{SchPerf}_k^{\mathrm{op}}, \mathrm{Ani}) \tag{A.6}$$

is written as a colimit of representables, then

$$\mathrm{D}(\mathfrak{X}, \Lambda) \cong \lim_i \mathrm{D}(X_i, \Lambda) \tag{A.7}$$

and similarly for $\mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda)^{\mathrm{Kan}}$. By [Sch17, Proposition 27.2] and [BN19, Lemma B.6] (more precisely, its proof of 1.), the natural transformation

$$c^*_{\mathfrak{X}} \colon \mathrm{D}(\mathfrak{X}, \Lambda) \to \mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda)^{\mathrm{Kan}} \tag{A.8}$$

is fully faithful. Note that at this moment the right hand side is not the left completed derived étale category of some (higher) v-stack "$\mathfrak{X}^\diamondsuit$" on Perf_k, but depends on $\mathfrak{X}$ and is defined abstractly (therefore, we have added the superscript Kan).

Assume now that $\mathfrak{X}$ is a stack (in 1-groupoids) with representable diagonal such that there exists a v-cover by a perfect scheme $X \to \mathfrak{X}$. Then

$$\mathfrak{X} \cong \mathrm{colim}_{\Delta^{\mathrm{op}}} X^{\bullet/\mathfrak{X}} \tag{A.9}$$

with the colimit of the Čech nerve of $X \to \mathfrak{X}$ taken in Ani-valued v-sheaves on $\mathrm{SchPerf}_k$. Using [HS21, Theorem 5.7], we get

$$\mathrm{D}(\mathfrak{X}, \Lambda) \cong \lim_{\Delta} \mathrm{D}(X^{\bullet/\mathfrak{X}}, \Lambda). \tag{A.10}$$

Indeed, by definition

$$\mathrm{D}(\mathfrak{X}, \Lambda) \cong \lim_{U \to \mathfrak{X}} \mathrm{D}(U, \Lambda) \tag{A.11}$$

where the limit is taken over all perfect schemes with a morphism to $\mathfrak{X}$, and thus

$$\begin{array}{cc} & \mathrm{D}(\mathfrak{X}, \Lambda) \\ = & \lim_{U \to \mathfrak{X}} \mathrm{D}(U, \Lambda) \\ \cong & \lim_{U \to \mathfrak{X}} \lim_{\Delta} \mathrm{D}(U \times_{\mathfrak{X}} X^{\bullet/\mathfrak{X}}, \Lambda) \\ \cong & \lim_{\Delta} \lim_{U \to \mathfrak{X}} \mathrm{D}(U \times_{\mathfrak{X}} X^{\bullet/\mathfrak{X}}, \Lambda) \\ \cong & \lim_{\Delta} \mathrm{D}(\mathfrak{X} \times_{\mathfrak{X}} X^{\bullet/\mathfrak{X}}, \Lambda) \\ \cong & \lim_{\Delta} \mathrm{D}(X^{\bullet/\mathfrak{X}}, \Lambda), \end{array} \tag{A.12}$$

where the first isomorphism comes from [HS21, Theorem 5.7] applied to the covering $X \times_{\mathfrak{X}} U \to U$, the second isomorphism just commutes two inverse limits, and the last two isomorphisms use that $\mathrm{D}(\text{-}, \Lambda)$ sends (by definition) colimits to limits and that $\mathrm{colim}_{U \to \mathfrak{X}} U \times_{\mathfrak{X}} X^{n/\mathfrak{X}} \cong X^{n/\mathfrak{X}}$.

Let $X^{\diamondsuit, \bullet/\mathfrak{X}}$ be the simplicial v-sheaf obtained by applying the functor $(\text{-})^\diamondsuit$ to $X^{\bullet/\mathfrak{X}}$. Now assume additionally that the projection maps in $X^{\diamondsuit, \bullet/\mathfrak{X}}$ are v-covers, and let $\mathfrak{X}^\diamondsuit$ be the colimit of $X^{\diamondsuit, \bullet/\mathfrak{X}}$ in v-stacks on Perf_k. Then $\mathfrak{X}^\diamondsuit$ is a small v-stack with well-defined $\mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda)$, and actually the v-stackification of $\mathrm{Spa}(R, R^+) \mapsto \mathfrak{X}(\mathrm{Spec}(R))$. By [Sch17, Proposition 17.3], we can conclude that

$$\mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda) \cong \lim_{\Delta^{\mathrm{op}}} \mathrm{D}(X^{\diamondsuit, \bullet/\mathfrak{X}}, \Lambda). \tag{A.13}$$

Moreover, note that there exists a canonical morphism

$$\mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda)^{\mathrm{Kan}} \to \mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda), \tag{A.14}$$

which probably need not be an equivalence in general, but whose composite with $c^*_{\mathfrak{X}}$ is still fully faithful.

In general, it need however not be true that for a v-cover $Y \to X$ of (perfect) schemes the map $Y^\diamondsuit \to X^\diamondsuit$ is a v-cover of small v-sheaves.

Example A.2. Let C be a perfect, non-archimedean field over $\mathbb{F}_p$, and fix a pseudo-uniformizer $\pi_C \in C$. Let

$$Z := \{\tfrac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\} \subset [0, 1] \tag{A.15}$$

with its subspace topology, which is profinite. We consider the space of continuous functions

$$X := \mathrm{Spec}(C^0(Z, C_{\mathrm{disc}})), \tag{A.16}$$

where C_{disc} is the field C equipped with the discrete topology. Note that $|X| \cong Z$. For $n \in \mathbb{N}$, let $g_n, h_n \colon Z \to C_{\mathrm{disc}}$ be the locally constant functions with

$$g_n(1/m) = 1/\pi_C^m, h_n(1/m) = 1 \tag{A.17}$$

for $m \leq n$ and $g_n(z) = h_n(z) = 0$ otherwise. Let

$$Y_n \subset \mathbb{A}^1_X = \mathrm{Spec}(C^0(Z, C_{\mathrm{disc}})[T]) \tag{A.18}$$

be the vanishing locus of $h_n T - g_n$. Then $Y_{n+1} \subset Y_n$, and we can set $Y := \lim_n Y_n$ which is a v-cover of X. Indeed, each Y_n is a v-cover, and inverse limits of v-covers between affine schemes are v-covers. More concretely,

[22] We take care of the set-theoretic issues by fixing a suitable cardinal, large enough to allow all the examples that we are interested in.

each map $\mathrm{Spec}(V) \to X$ with V a valuation ring must factor through a (closed) point of X, and then it suffices to see that $Y \to X$ is surjective (it is bijective over $X \setminus \{0\}$, and $\mathbb{A}^1_{C_{\mathrm{disc}}}$ over 0).

We claim that $Y^\diamondsuit \to X^\diamondsuit$ is not a v-cover. Set $R := C^0(Z, C)$ (now C given its valuation topology), and $R^+ = C^0(Z, \mathcal{O}_C)$. The canonical map $C^0(Z, C_{\mathrm{disc}}) \to R$ defines a map

$$S := \mathrm{Spa}(R, R^+) \to X^\diamondsuit, \tag{A.19}$$

which does not v-locally factor through $Y^\diamondsuit \subset (\mathbb{A}^1_X)^\diamondsuit$. Indeed, assume that $S' \to S$ is a v-cover with S' affinoid and $S' \to Y^\diamondsuit$ a lift of $S \to X^\diamondsuit$. Then the image of $S' \to Y^\diamondsuit \times_{X^\diamondsuit} S \subset \mathbb{A}^{1,\mathrm{ad}}_S$ must factor through some quasi-compact subset. But over each point $z_n := 1/n \in Z \cong |S|$ with $n \in \mathbb{N}$, we have that $Y^\diamondsuit \times_{X^\diamondsuit} \{z_n\}$ is the point $1/\pi_C^n \in \mathbb{A}^{1,\mathrm{ad}}_{z_n}$, and in $\mathbb{A}^{1,\mathrm{ad}}_S$ this set of points does not lie in a quasi-compact open.

Adding (perfect) finite presentation such examples cannot occur.

Lemma A.3. *If $f\colon Y \to X$ is a (perfectly) finitely presented map of perfect schemes and v-cover, then $f^\diamondsuit\colon Y^\diamondsuit \to X^\diamondsuit$ is a v-cover.*

Proof. The functor $(\text{-})^\diamondsuit$ preserves open covers, so we may assume that $Y \to X$ are affine. In this case, a (perfectly) finite presented v-cover is a cofiltered limit of v-covers between (perfect) affine schemes of (perfect) finite presentation over $\mathrm{Spec}(k)$, see [BS17, Lemma 2.12]. As f is of (perfectly) finite presentation, we may assume that f is the base change of a v-cover between (perfect) affine schemes of (perfect) finite presentation over k. The functor $(\text{-})^\diamondsuit$ preserves fiber products, and base changes of v-cover of v-sheaves on Perf_k are again v-covers. Thus, we may reduce to the case that Y, X are of (perfect) finite presentation over k. Then the statement follows from [Gle20, Proposition 2.2.9]. □

We get the following consequence.

Lemma A.4. *Assume $\mathfrak{X}, \mathfrak{Y}$ are v-stacks on $\mathrm{SchPerf}_k$ with representable diagonal of perfectly finite presentation, and that $\mathfrak{X}, \mathfrak{Y}$ admit a (perfectly) finitely presented v-covers by a perfect schemes, which are of (perfect) finite presentation over $\mathrm{Spec}(k)$. Let $\mathfrak{X}^\diamondsuit$ be the v-stackification of the functor $\mathrm{Spa}(R, R^+) \mapsto \mathfrak{X}(\mathrm{Spec}(R))$ on Perf_k, and similarly for $\mathfrak{Y}$. Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of v-stacks.*

(1) *Then the canonical morphism*

$$\mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda)^{\mathrm{Kan}} \to \mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda) \tag{A.20}$$

is an equivalence, and the composite (still denoted $c^*_{\mathfrak{X}}$)

$$\mathrm{D}(\mathfrak{X}, \Lambda) \xrightarrow{c^*_{\mathfrak{X}}} \mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda)^{\mathrm{Kan}} \cong \mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda) \tag{A.21}$$

is fully faithful.

(2) *The functors $c^*_{\mathfrak{X}} \circ f^*, (f^\diamondsuit)^* \circ c^*_{\mathfrak{Y}}$ are naturally isomorphic.*

(3) *If $\mathfrak{X} \cong [\mathrm{Spec}(k)/H]$ for some perfectly finitely presented group scheme H over k, then the functor*

$$\mathrm{D}(\mathfrak{X}, \Lambda) \to \mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda) \tag{A.22}$$

is an equivalence.

Proof. We prove the first point. From the assumptions on $\mathfrak{X}^\diamondsuit$ we can conclude that the morphisms $U \to \mathfrak{X}$ of perfectly finite presentation such that U is of perfectly finite presentation over k are cofinal among all maps $V \to \mathfrak{X}$ with V a perfect scheme. In particular, in the definition of $\mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda)^{\mathrm{Kan}}$ one can replace the limit over all $V's$ with a morphism to $\mathfrak{X}$ by the limit over all $U's$ with morphism to $\mathfrak{X}$. Using Lemma A.3 and the same argument as for $\mathrm{D}(\mathfrak{X}, \Lambda)$, we can then conclude that $\mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda)^{\mathrm{Kan}} \cong \mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda)$. Fully faithfulness follows from fully faithfulness of $c^*_{\mathfrak{X}}$. The second point follows by expressing the categories as limits over Δ by choosing v-covers $X \to \mathfrak{X}, Y \to \mathfrak{Y}$ of perfectly finite presentation with X, Y of perfectly finite presentation over k, such that $X \to \mathfrak{X} \xrightarrow{f} \mathfrak{Y}$ factors over $Y \to \mathfrak{Y}$. For the last point, we note that $\mathrm{D}(\mathrm{Spec}(k), \Lambda) \cong \mathrm{D}(\mathrm{Spa}(k, k)^\diamond, \Lambda)$ as both identify naturally with the derived category of Λ-modules, see [FS21, Theorem V.1.1]. Computing both sides via the Čech nerve of the covering $\mathrm{Spec}(k) \to [\mathrm{Spec}(k)/H]$, the statement follows from [BN19, Lemma B.6]. □

We want to apply the results of this appendix to the schematic Hecke stack. Thus, assume that O is a complete discrete valuation ring with residue field k, $\mathcal{G}$ a parahoric group scheme over O with generic fiber G and $\mathcal{F}\ell_{\mathcal{G}} = L_k G / L_k^+ \mathcal{G}$ the partial affine flag variety for $\mathcal{G}$ as in Section 3.2. Let

$$\mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G},k} := [L_k^+ \mathcal{G} \backslash L_k G / L_k^+ \mathcal{G}] \tag{A.23}$$

be the (schematic) Hecke stack where the quotient is taken in v-stacks on $\mathrm{SchPerf}_k$. Then $\mathrm{Hk}_{\mathcal{G},k} := (\mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G},k})^\diamondsuit$ is the small v-stack on Perf_k considered in Section 6. We denote by

$$\mathrm{D}(\mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G},k}, \Lambda)^{\mathrm{bd}},\ \mathrm{D}(\mathrm{Hk}_{\mathcal{G},k}, \Lambda)^{\mathrm{bd}} \tag{A.24}$$

the categories of objects with bounded support.

Proposition A.5. *The categories* $\mathrm{D}(\mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G},k},\Lambda)^{\mathrm{bd}}$, $\mathrm{D}(\mathrm{Hk}_{\mathcal{G},k},\Lambda)^{\mathrm{bd}}$ *are equivalent.*

Proof. Consider a closed substack

$$[L^+_k\mathcal{G}\backslash X]\subset \mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G},k} \tag{A.25}$$

with $X\subset \mathcal{F}\ell_{\mathcal{G}}$ a closed $L^+_k\mathcal{G}$-stable subscheme, that is, a union of Schubert varieties. By the argument of [FS21, Proposition VI.4.1], the vanishing of the cohomology of the affine line over k implies that

$$\mathrm{D}([L^+_k\mathcal{G}\backslash X],\Lambda)\cong \mathrm{D}([H\backslash X],\Lambda) \tag{A.26}$$

for any perfectly finitely presented quotient H of $L^+_k\mathcal{G}$ by some congruence subgroup whose action on X is trivial. By Lemma A.4, we have, abusing notation, a natural fully faithful functor

$$c^*_{[H\backslash X]}\colon \mathrm{D}([H\backslash X],\Lambda)\to \mathrm{D}([H\backslash X]^\diamondsuit,\Lambda)\cong \mathrm{D}([H^\diamondsuit\backslash X^\diamondsuit],\Lambda), \tag{A.27}$$

and we claim that this functor is an equivalence. We prove this by induction on the number of Schubert strata contained in X. Let $i\colon Y\subset X$ be a closed H-stable (perfect) subscheme with non-empty open complement $j\colon U\to X$, for which $[H\backslash U]$ is a disjoint union of classifying stacks for closed subgroups of H. By Lemma A.4, we have

$$\mathrm{D}([H\backslash U],\Lambda)\cong \mathrm{D}([H^\diamondsuit\backslash U^\diamondsuit],\Lambda), \tag{A.28}$$

and by induction induction hypothesis

$$\mathrm{D}([H\backslash Y],\Lambda)\cong \mathrm{D}([H^\diamondsuit\backslash Y^\diamondsuit],\Lambda). \tag{A.29}$$

Let us note that as in [Sch17, Section 27] the functors $c^*_{[H\backslash X]}, c^*_{[H\backslash U]}, c^*_{[H\backslash Y]}$ admit right adjoints $Rc_{[H\backslash X],*}$, $Rc_{[H\backslash U],*}$, $Rc_{[H\backslash Y],*}$, and it suffices to see that $Rc_{[H\backslash X],*}$ is conservative. It is formal that

$$[H\backslash j]^*\circ Rc_{[H\backslash X],*}\cong Rc_{[H\backslash U],*}\circ [H\backslash j]^{\diamondsuit,*}, \tag{A.30}$$

where $[H\backslash j]\colon [H\backslash U]\to [H\backslash X]$ denotes the morphism induced by j. More precisely, there exists a natural morphism from the left hand side to the right hand side, and it suffices to see that the morphims induced on the left adjoints is an isomorphism. If $T\to [H\backslash X]$ is a v-cover with $T\to \mathrm{Spec}(k)$ of morphism of schemes of finite type, then it suffices (by Lemma A.3) to prove the statement on the isomorphism of left adjoints after pullback to $T^\diamondsuit$. Here, the functor c^*_T on the étale derived categories is induced by a morphism of topoi, and then (A.30) follows by general base change to open subtopoi.

Let $A\in \mathrm{D}([H\backslash X]^\diamondsuit,\Lambda)$ such that $Rc_{[H\backslash X],*}(A)=0$. Then we deduce $[H\backslash j]^{\diamondsuit,*}(A)=0$ because $Rc_{[H\backslash U],*}$ is an equivalence. In particular, $A\cong [H\backslash i]^\diamondsuit_*[H\backslash i]^{\diamondsuit,*}(A)$. Now note that

$$[H\backslash i]_*\circ Rc_{[H\backslash Y],*}\cong Rc_{[H\backslash X],*}\circ [H\backslash i]^\diamondsuit \tag{A.31}$$

as follows by adjunction from $[H\backslash i]^*\circ c^*_{[H\backslash X]}\cong c^*_{[H\backslash Y]}\circ [H\backslash i]^{\diamondsuit,*}$. We can conclude that

$$[H\backslash i]_* Rc_{[H\backslash Y],*}([H\backslash i]^{\diamondsuit,*}A)=0, \tag{A.32}$$

which implies $[H\backslash i]^{\diamondsuit,*}(A)=0$ because $[H\backslash i]_*$ is conservative and $Rc_{[H\backslash Y],*}$ an equivalence. This implies that $A=0$ as desired.

The equivalence $c^*_{[H\backslash X]}$ is natural with respect to inclusions $X\to X'$ of H-stable subsets, and morphism $H'\to H$ of quotients of $L^+_k\mathcal{G}$ (which act trivially on X). More precisely, from $c^*_{[H\backslash X]}$ we get an equivalence

$$\mathrm{D}([L^+_k\mathcal{G}\backslash X],\Lambda)\cong \mathrm{D}([(L^+_k\mathcal{G})^\diamondsuit\backslash X^\diamondsuit],\Lambda) \tag{A.33}$$

using [FS21, VI.4.1], and then we can pass to the colimits of both sides along closed $L^+_k\mathcal{G}$-stable subschemes of $\mathcal{F}\ell_{\mathcal{G}}$. Then the left hand side is $\mathrm{D}(\mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G},k},\Lambda)^{\mathrm{bd}}$ while the right hand side is $\mathrm{D}(\mathrm{Hk}_{\mathcal{G},k},\Lambda)^{\mathrm{bd}}$. □

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