The equivalence of Heegaard Floer homology and embedded contact homology via open book decompositions

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Main result

Theorem

Let M be a closed, oriented 3-manifold. Then

$$\widehat{HF}(-M) \simeq \widehat{ECH}(M).$$

Here HF refers to Heegaard Floer homology, due to Ozsváth-Szabó, and ECH refers to embedded contact homology, due to Hutchings.

We work over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$-coefficients.

Both HF and ECH are topological invariants of M:

- HF a priori depends on the choice of a Heegaard diagram of M;
- ECH a priori depends on the choice of a contact form α on M.

Main result

Remark

An alternate proof of the same result is due to Kutluhan-Lee-Taubes.

Theorem

$$\widehat{SWF}(-M) \xrightarrow{\simeq,\ Taubes} \widehat{ECH}(M)$$

$$\widehat{SWF}(-M) \xrightarrow{\simeq,\ KLT} \widehat{HF}(-M) \xrightarrow{\simeq,\ CGH} \widehat{ECH}(M)$$

Here *SWF* refers to Seiberg-Witten Floer homology, developed by Kronheimer-Mrowka. The top isomorphism is due to Taubes, and is proved using the circle of ideas involved in his proof of the Weinstein conjecture in dimension three.

Contact structures

Let M be a closed, oriented 3-manifold. A contact form α is a 1-form on M which satisfies $\alpha \wedge d\alpha > 0$. The corresponding contact structure is the 2-plane field $\xi = \ker \alpha$.

To any contact 1-form α we can assign its Reeb vector field $R = R_\alpha$ defined by:

$$i_R d\alpha = 0,$$

$$i_R \alpha = \alpha(R) = 1.$$

Equivalently, a Reeb vector field R is a contact vector field which is everywhere transverse to ξ.

Open book decompositions

Let S be a compact oriented surface with boundary and $h : S \xrightarrow{\sim} S$ be a diffeomorphism satisfying $h|_{\partial S} = id$.

Definition

An open book decomposition (S, h) for M is an identification

$$M \simeq S \times [0,1]/ \sim,$$

where

- $(x, 1) \sim (h(x), 0)$ *for all* $x \in S$; *and*
- $(x, t) \sim (x, t')$ *for all* $x \in \partial S$ *and* $t, t' \in [0, 1]$.

$S \times \{t\}$ is called a "page" and ∂S is called the "binding".

Open book decompositions

Definition

An open book decomposition (S, h) is adapted to (M, ξ) if there is a contact form α for ξ such that the Reeb vector field R_α is transverse to the interiors of the pages $S \times \{t\}$ and ∂S is a closed orbit of R_α.

The starting point of this work is the following fundamental result:

Theorem (Thurston-Winkelnkemper,Torisu,...,Giroux)

There is a one-to-one correspondence between contact structures (M, ξ) up to isomorphism and open book decompositions (S, h) up to "positive stabilization".

Embedded contact homology

ECH, due to Hutchings, is a variant of symplectic field theory, due to Eliashberg-Givental-Hofer.

Let α be a contact form with non-degenerate Reeb vector field R_α.

The generators of the ECH chain complex are orbit sets, i.e., finite multisets $\Gamma = \{(\gamma_i, m_i)\}$, where γ_i is a simple (i.e., embedded) orbit of R_α and m_i is a positive integer. (We use multiplicative notation $\Gamma = \prod_i \gamma_i^{m_i}$.)

To define the differentials, consider the symplectization $(\mathbb{R} \times M, d(e^s\alpha))$. An adapted almost complex structure J on $\mathbb{R} \times M$ satisfies the following:

- J takes ∂_s to R_α;
- J takes ξ to itself;
- $d\alpha(v, Jv) > 0$ for all nonzero v.

Embedded contact homology

Let $\Gamma = \prod_i \gamma_i^{m_i}$ and $\Gamma' = \prod_j (\gamma_j')^{m_i'}$ be orbit sets.

The differential $\langle \partial \Gamma, \Gamma' \rangle$ counts "isolated" embedded J-holomorphic curves C in $\mathbb{R} \times M$ which are asymptotic to cylinders over periodic orbits at both ends. The total multiplicity of γ_i at the positive end is m_i and the total multiplicity of γ_j' at the negative end is m_j'.

The proof of $\partial^2 = 0$ rather intricate and is due to Hutchings-Taubes. $ECH(M, \alpha, J)$ is independent of the choice of α and J by Taubes' isomorphism theorem with SWF; in particular there is currently no direct proof.

Reformulation of ECH

Let R_α be a Reeb vector field which is adapted to the open book (S, h). Recall that the binding γ_0 is a closed orbit of R_α. We want to get rid of the binding and express $\widehat{ECH}(M, \alpha)$ in terms of data on

$$N = (S \times [0,1])/(x,1) \sim (h(x), 0).$$

Since $h|_{\partial S} = id$, ∂N is foliated by an S^1-family of closed orbits. This Morse-Bott family can be perturbed into the pair e, h.

Let $ECH_i(S, h)$ be ECH on N whose orbit sets intersect $S \times \{t\}$ exactly i times.

Reformulation of ECH

Consider the map induced by inclusion:

$$ECH_i(S,h) \to ECH_{i+1}(S,h),$$

$$\Gamma \mapsto e\Gamma.$$

Theorem

$$\widehat{ECH}(M) \simeq \lim_{i\to\infty} ECH_i(S,h).$$

Remark

We are viewing e as the unit 1. In ECH, the contact invariant for $(M, \ker\alpha)$ is given by the empty set $\emptyset$, written multiplicatively as 1.

Reformulation of ECH

Ideas of proof:

1. Effectively eliminate the binding by taking the slopes of Reeb orbits near γ_0 to approach the meridian slope and use a direct limit argument.
2. An isolated holomorphic curve in $\mathbb{R} \times M$ which intersects the cylinder $\mathbb{R} \times \gamma_0$ over the binding once is equivalent to a holomorphic curve in $\mathbb{R} \times N$ which has e at the negative end.

Reformulation of ECH

Remark

There is a symplectic/holomorphic fibration

$$\pi : \mathbb{R} \times N \to \mathbb{R} \times S^1$$

with fiber S, and we are now counting holomorphic multisections of π.

Heegaard Floer homology (à la Eliashberg-Lipshitz)

Let Σ be a Heegaard surface of M, i.e., Σ splits M into two handlebodies H_α and H_β of genus k.

Let $\alpha = \{\alpha_1, \dots, \alpha_k\}$ be a pairwise disjoint collection of embedded curves such that the α_i bound disks in H_α and $\Sigma - \cup_i \alpha_i$ is connected. Similarly define $\beta = \{\beta_1, \dots, \beta_k\}$. Also pick a basepoint $z \in \Sigma - \cup_i \alpha_i - \cup_i \beta_i$.

Consider the 3-manifold $[0,1] \times \Sigma$. The chain complex $\widehat{CF}(\Sigma, \alpha, \beta, z)$ is generated by k-tuples of "Reeb chords" $\{[0,1] \times \{x_i\}, i = 1, \dots, k\}$, where $x_i \in \alpha_i \cap \beta_{\sigma(i)}$ for some permutation σ. We will often write $\mathbf{x} = \{x_1, \dots, x_k\}$ for a k-tuple of Reeb chords.

Heegaard Floer homology

Remark

There is a symplectic/holomorphic fibration

$$\pi : \mathbb{R} \times [0,1] \times \Sigma \to \mathbb{R} \times [0,1]$$

with fiber Σ.

The differential $\langle \partial \mathbf{x}, \mathbf{y} \rangle$ counts "isolated" embedded degree k holomorphic multisections C of π which satisfy the following:

- C is asymptotic to $[0,1] \times \mathbf{x}$ at the positive end and to $[0,1] \times \mathbf{y}$ at the negative end;
- ∂C is a subset of $(\mathbb{R} \times \{1\} \times \alpha) \cup (\mathbb{R} \times \{0\} \times \beta)$, and uses each component of $\mathbb{R} \times \{0\} \times \alpha$ and $\mathbb{R} \times \{0\} \times \beta$ exactly once; and
- C does not intersect $\mathbb{R} \times [0,1] \times \{z\}$.

HF and open books à la H.-Kazez-Matić

A compatible Heegaard diagram for an open book is given as follows:

1. Let $\Sigma = (S \times \{1/2\}) \cup -(S \times \{0\})$.
2. Pick a basis $\{a_1, \dots, a_{2g}\}$ of properly embedded arcs on S such that $S - \cup_i a_i$ is a polygon. Here $g = g(S)$.
3. Let b_i be a pushoff of a_i in the direction of ∂S so that $a_i \cap b_i$ is one point x_i.
4. Finally, let

$$\begin{aligned} \alpha_i &= \partial(a_i \times [0, 1/2]), \\ \beta_i &= \partial(b_i \times [1/2, 1]) \\ &= (b_i \times \{1/2\}) \cup (h(b_i) \times \{0\}). \end{aligned}$$

HF and open books

Remark

The $2g$-tuple $\{x_1, \dots, x_{2g}\}$ gives rise to the contact invariant

$$c(M,\xi) \in \widehat{HF}(\beta,\alpha) = \widehat{HF}(-M)$$

of the contact structure (M,ξ) corresponding to the open book (S,h), and all of the interesting activity occurs on $S \times \{0\}$.

If we place two copies of x_i on the boundary of $S \times \{0\}$, then:

$$\widehat{CF}(\Sigma,\beta,\alpha) = \widehat{CF}(S \times \{0\}, \mathbf{a}, h(\mathbf{a})).$$

HF and open books

Remark

On the ECH side, the contact invariant for (M, ξ) is generated by the empty set. This suggests that

$$\mathbf{y} = \{y_1, \dots, y_k, x_{i_1}, \dots, x_{i_{2g-k}}\}$$

corresponds to an orbit set which intersects a page k times.

HF and open books

Remark

We are now counting "isolated" embedded holomorphic degree $2g$ multisections of $\pi : \mathbb{R} \times [0,1] \times S \to \mathbb{R} \times [0,1]$.

Broken closed strings

As an intermediary between $\mathbf{y} = \{y_1, \dots, y_{2g}\}$ and Γ, we consider a broken closed string $\gamma_{\mathbf{y}}$, i.e., a collection of closed curves in N, obtained by concatenating:

- $[0,1] \times y_i$, $i = 1, \dots, 2g$, and
- $\{0\} \times c_i$, $i = 1, \dots, 2g$, where c_i is a subarc of $h(a_i)$ which connects $h(x_{\sigma(i)})$ to x_i.

Defining the chain map

We now define a chain map

$$\Phi : \widehat{CF}(S, \mathbf{a}, h(\mathbf{a})) \to ECC_{2g}(S, h),$$

obtained by counting "isolated" degree $2g$ holomorphic multisections in a certain symplectic fibration $\pi : W_+ \to B_+$ which approach $\mathbf{y}$ at the positive end and Γ at the negative end. The chain map is motivated by the work of Seidel and Donaldson-Smith.

Defining the chain map

The fibration $\pi : W_+ \to B_+$ is the restriction of $\pi : \mathbb{R} \times N \to \mathbb{R} \times S^1$ to the base B_+, given below:

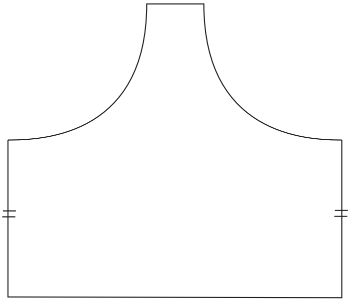

Figure: The base B_+.

In particular, the fibers are $\simeq S$.

Defining the chain map

Lagrangian boundary condition: Place a copy of a_i on one fiber $\pi^{-1}(p)$, where $p \in \partial B_+$. Apply parallel transport to a_i along ∂B_+. Then it sweeps out a Lagrangian L_i which restricts to (a subset of) $\mathbb{R} \times \{1\} \times a_i$ and $\mathbb{R} \times \{0\} \times h(a_i)$ at the positive end.

The isomorphism

Theorem

Φ *is a chain map which induces an isomorphism on the level of homology.*

Proof.

Define an inverse map

$$\Psi : ECC_{2g}(S,h) \to \widehat{CF}(S,\mathbf{a},h(\mathbf{a})),$$

in a similar manner. Then $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are chain homotopic to the identity by degenerating the base. □

Happy Birthday, Mike!