Fibered knots and potential counterexamples to the Property 2R and Slice-Ribbon Conjectures

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June 2011
Berkeley FreedmanFest

Theorem (Gabai 1987)

If surgery on a knot $K \subset S^3$ gives $S^1 \times S^2$, then K is the unknot.

Question: If surgery on a link L of n components gives $\#_n(S^1 \times S^2)$, what is L?

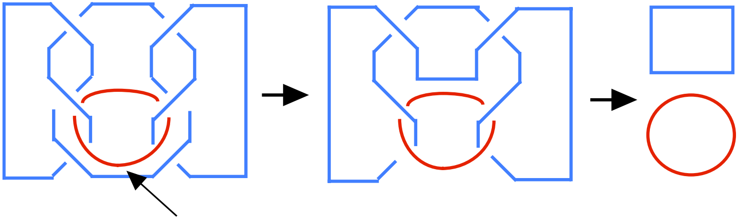
Homology argument shows that

- each pair of components in L is algebraically unlinked and
- the surgery framing on each component of L is the 0-framing.

Conjecture (Naive)

If surgery on a link L of n components gives $\#_n(S^1 \times S^2)$, then L is the unlink.

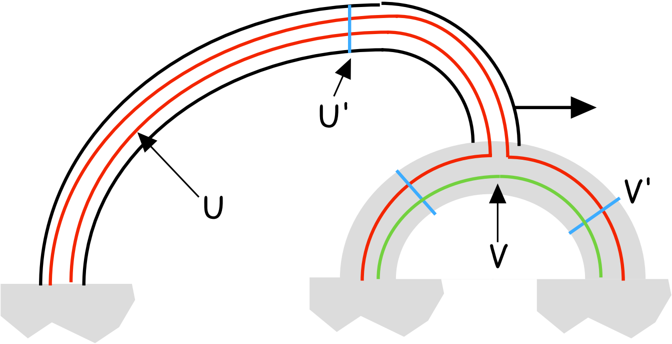
Why naive?
The result of surgery is unchanged when one component of L is replaced by a band-sum to another. So here's a counterexample:



The 4-dimensional view of the band-sum operation:

Integral surgery on $L \subset S^3 \leftrightarrow$ 2-handle addition to ∂B^4.

Band-sum operation corresponds to a 2-handle slide



Effect on dual handles: U slid over $V \leftrightarrow V'$ slid over U'.

The fallback:

Conjecture (Generalized Property R)

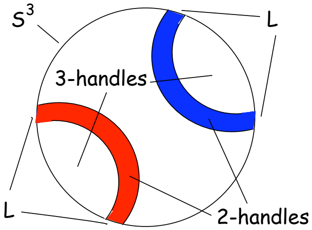
If surgery on an n component link $L \subset S^3$ gives $\#_n(S^1 \times S^2)$, then, perhaps after some handle-slides, L becomes the unlink.

Conjecture is unknown even for $n = 2$.

Questions: If it's *not* true, what's the simplest counterexample?

What's the simplest knot that could be part of a counterexample?

A potential counterexample must be slice in some homotopy 4-ball:



Slice complement is built from link complement by:

- attaching copies of $(D^2 - \{0\}) \times D^2$ to $(D^2 - \{0\}) \times S^1$, i. e. paired 2- and 3-handles
- attaching some more 3-handles
- attaching a 4-handle.

Like a ribbon disk complement, via descending radius function:

- at saddle points of ribbon disk, 2-handles attached
- at minima of ribbon disk, 3-handles attached
- at center of ball, 4-handle attached

So $\pi_1(\textit{link complement}) \to \pi_1(\textit{disk complement})$ surjective.

Since Freedman tells us the homotopy 4-ball is topological 4-ball, might say the link is "topologically ribbon".

First 2-component result

Theorem (Reid)

If surgery on a 2-component link that has tunnel number one *gives* $\#_2(S^1 \times S^2)$*, then L is the unlink. [No handle-slides required.]*

Proof.

Free groups are Hopfian, so the epimorphism $\mathbb{Z} * \mathbb{Z} \to \mathbb{Z} * \mathbb{Z}$ given by

$$\pi_1(S^3 - (L \cup tunnel)) \to \pi_1(S^3 - L) \to \pi_1(\#_2(S^1 \times S^2))$$

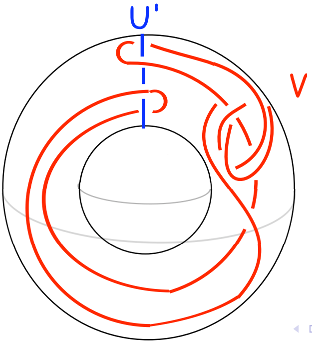
shows that $\pi_1(S^3 - L) \cong \mathbb{Z} * \mathbb{Z}$. □

Question: Can *the unknot* be part of a counterexample?
Answer: No

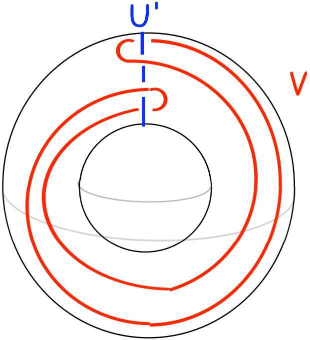
Theorem

If U is the unknot *and surgery on $L = U \cup V$ gives $\#_2(S^1 \times S^2)$, then some handle-slides of V over U change L to the unlink.*

Idea: Since U is the unknot, 0-surgery on U gives $S^1 \times S^2 \supset V$, and $S^1 \times S^2 \supset$ the dual knot $U' = S^1 \times (point)$.



This means that two different fillings of $\eta(V)$ (with and without surgery) give reducible manifolds. Deep theorem of Gabai says this means V lies in a ball, so V is the unknot in $S^1 \times S^2$. Can get it into a ball in the complement of U' by letting V cross U', i. e. V is handle-slide over U. Once inside a ball then Gabai's Property R means V is the unknot, so L becomes the unlink.

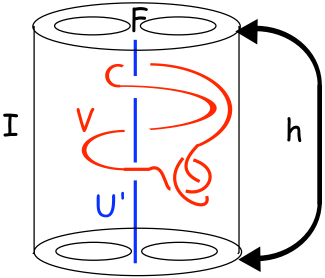


Can ramp this argument up, to consider what is the simplest knot that could occur in a 2-component counterexample.

Theorem

If U is a fibered knot *and surgery on $L = U \cup V$ gives $\#_2(S^1 \times S^2)$, then a sequence of handle-slides (back and forth) gives a link with a component of genus lower than U.*

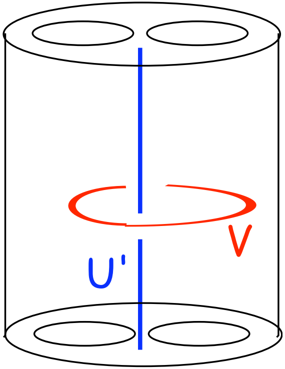
Idea: Since U is fibered, surgery on U gives a fibered manifold $M \supset U' \cup V$ with fiber F. Surgery on V in M gives $\#_2(S^1 \times S^2)$.



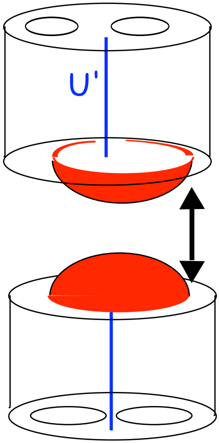
Theorem (S-Thompson 2008)

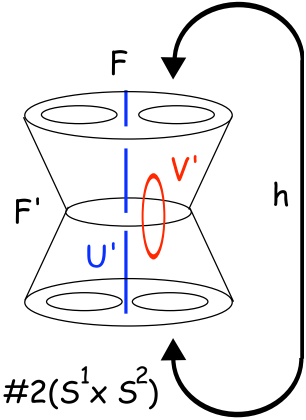
If surgery on V in fibered M gives a reducible manifold, then either V lies in a ball, or V is cabled, or V lies in a fiber.

So here V lies in F after passing through U', i. e. after V is changed by handle-sliding over U multiple times



What 0-framed surgery on $V \subset M$ now looks like: boundary union of two compression-bodies, each $\partial_+ = F$ and $\partial_- = F' =$, the surface obtained from F by compressing along V.

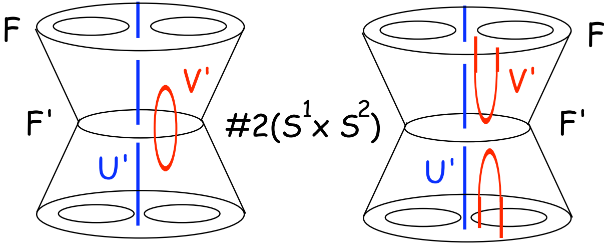




Then two handle-slides of V' over U' makes V' disjoint from F'.

$$\updownarrow$$

two handle-slides of U over V makes U fibered with fiber F'



Corollary

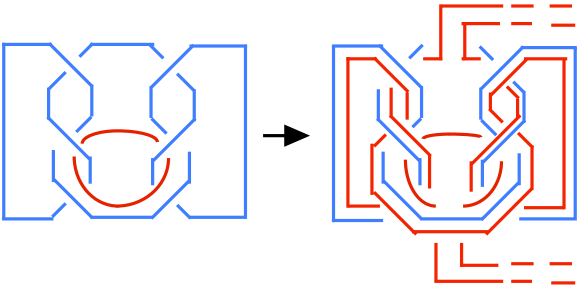
Both trefoil and figure 8 knots do not appear in a counterexample

But we knew that before: neither is slice in a homotopy 4-ball.

Genus 2 fibered knots: Can any be part of a 2-component counterexample? Has to be slice in a homotopy 4-sphere.

Maybe the simplest: $Q = K\# - K = \mathit{squareknot}$, K the trefoil.

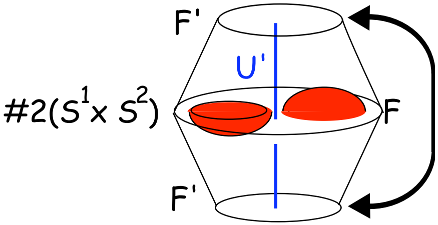
Examples of links $L = Q \cup V$ so surgery gives $\#_2(S^1 \times S^2)$:



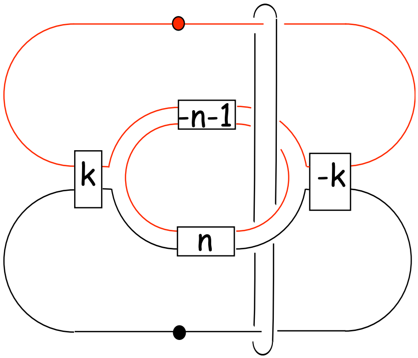
Know from earlier that none of these are counterexamples.

The square knot is so simple, can completely characterize (up to handle-slides over Q) all V so surgery on $Q \cup V$ gives $\#_2(S^1 \times S^2)$. In fact $\{such\ V\} \leftrightarrow \mathbb{Q}$.

Proof uses specific information about monodromy of Q and Heegaard analysis:



Reasonable idea: Figure out a way to show that each possible V is not a counterexample.

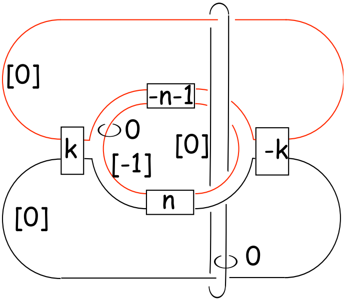


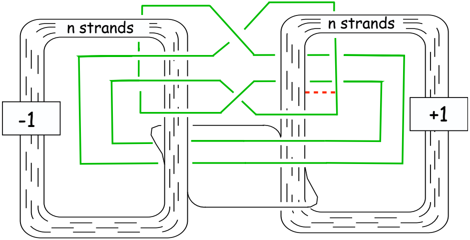
Theorem (Gompf 1991)

If this handle description of a homotopy 4-sphere can be handle-slid to standard, then this presentation is Andrews-Curtis trivial

$$< x, y | y = (yx)^{-1} x (yx), x^{n+1} = y^n >$$

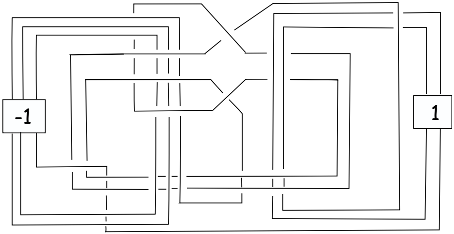
Handle description: Attach two 2-handles to $\natural_2(S^1 \times D^3)$; Gompf explicitly shows that result has boundary S^3. Dually, attach two 2-handles to D^4 and get manifold with boundary $\#_2(S^1 \times S^2)$.





When $k = 1$ can get picture on the right, combination of Q and $V_n = T_{n,n+1} \# - T_{n,n+1}$. Counterexample to Generalized Prop. R?

Link on right is ribbon but not obviously slice. Gives rise (via band-sum) to slice knot not obviously ribbon:

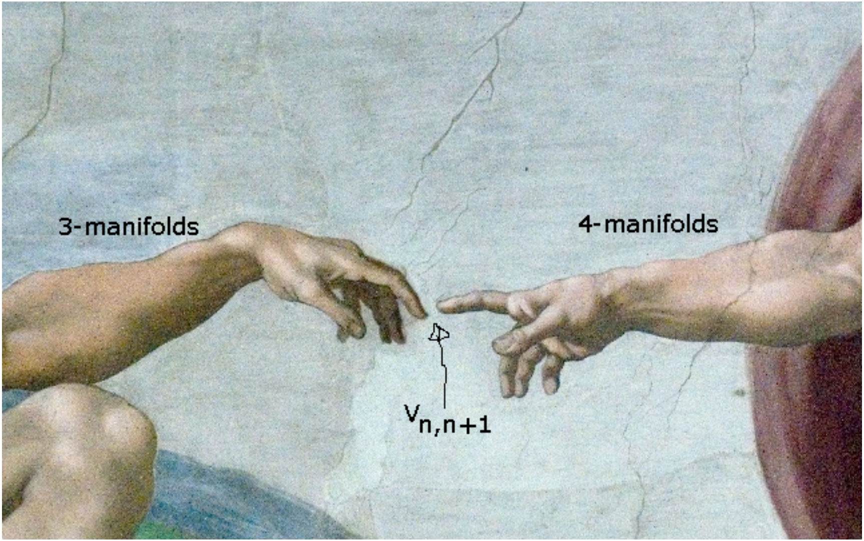


Question 1: How do the examples $Q \cup T_{n,n+1} \# - T_{n,n+1}$ fit into the classification scheme above?

In particular, how to get $V_{n,n+1} = T_{n,n+1} \# - T_{n,n+1}$ into a fiber?

Question 2: If you succeed, what is the genus 1 example arising from the theorem above?
[If Q is a fibered knot and surgery on $L = Q \cup V$ gives $\#_2(S^1 \times S^2)$, then a sequence of handle-slides gives a link with a component of genus lower than Q.]

Question 3: If NOT a counterexample to Generalized Property R, then could handle-slides in this description give an unexpected way of Andrews-Curtis trivializing the group presentation above?


3-manifolds
4-manifolds
$V_{n,n+1}$