Categorification, Lie algebras and Topology

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This talk is online at

`http://pages.uoregon.edu/bwebster/Mike.pdf`

References:

- Mikhail Khovanov and Aaron Lauda, *A diagrammatic approach to categorification of quantum groups III.*, arXiv:0807.3250.
- Raphaël Rouquier, *2-Kac-Moody algebras*, arXiv:0812.5023.
- Ben Webster, *Knot invariants and higher representation theory I & II*, arXiv:1001.2020 & 1005.4559.

Categorification

It is an old observation that some numbers are really sets in disguise, and some sets are categories in disguise. Of course, this added structure is a choice, but we know of oodles of instances where it "feels right."

You can linearize:

- Sometimes, a number doesn't seem to be the size of any particular set, but is the dimension of a vector space.
- Abelian groups can be gotten as the Grothendieck group of a category with some notion of exact sequence.

What I want to talk about today is how some very important and popular abelian groups, the semi-simple Lie algebras and their representations, managed to be the Grothendieck groups of categories for 100 years without anyone noticing.

Universal enveloping algebras

So, given your favorite abelian group (with extra structure), you can ask if there is some category (also with extra structure) whose GG it is.

What about universal enveloping algebras? Let $\mathfrak{g} = \mathfrak{sl}_2$ (actually any simple Lie algebra over $\mathbb{C}$ will do). This is the Lie algebra of 2×2 trace 0 matrices with the usual commutator.

If we let $E = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, $F = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ and $H = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, then this algebra has a presentation of the form

$$[H, E] = 2E \qquad [H, F] = -2F \qquad [E, F] = H$$

By definition, the universal enveloping algebra is the *associative* algebra generated by the symbols E, F, H subject to the relations above (where $[-, -]$ means commutator).

Universal enveloping algebras

We actually want a slightly bigger algebra $\dot{U}$, with some extra idempotents $\mathbb{1}_n$ for $n \in \mathbb{Z}$, which are projections to H-eigenspaces. These satisfy the relations

$$\mathbb{1}_m \mathbb{1}_n = \delta_n^m \mathbb{1}_m \qquad H\mathbb{1}_n = \mathbb{1}_n H = n\mathbb{1}_n.$$

Note that

$$\mathbb{1}_n E = E\mathbb{1}_{n-2} \qquad \mathbb{1}_n F = F\mathbb{1}_{n+2}.$$

Why? Can't have basis with positive structure coefficients in $U(\mathfrak{g})$, but we can in $\dot{U}$.

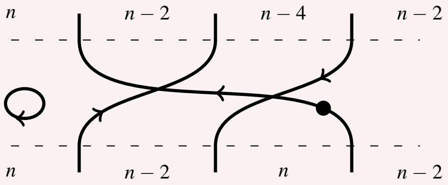
We can represent elements of $\dot{U}$ as pictures on a line

$\mathbb{1}_n EFE\mathbb{1}_{n-2} =$ 

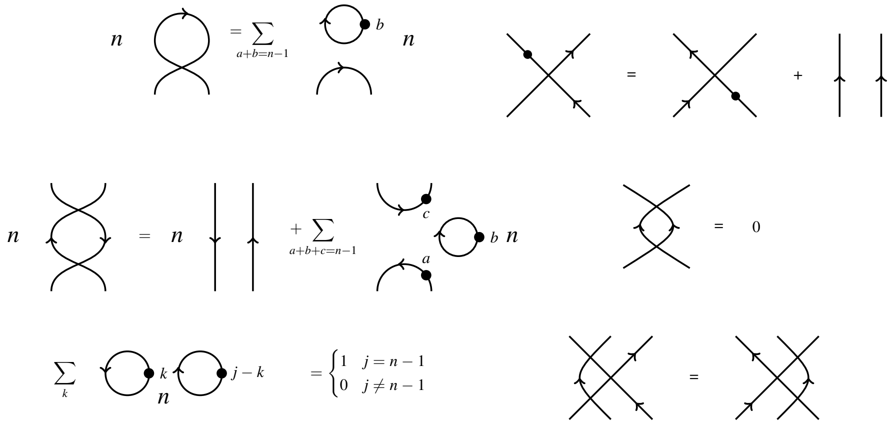
Categorifying UEAs

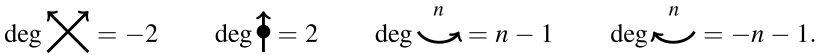
The remarkable insight of Khovanov and Lauda was that one could make these into the objects of a category $\mathcal{U}$, with morphisms given by pictures in the plane (Chuang and Rouquier had the same idea first, but never drew the pictures).

The morphisms of $\mathcal{U}$ are given by oriented 1-manifolds decorated with dots, whose boundaries are the given objects (with orientations and labels), modulo certain relations.



Relations in $\mathcal{U}$





The category $\mathcal{U}$

Those relations may look inscrutable, but actually every single one of them can be guessed by looking at the geometry of Grassmannians (for higher rank, quiver varieties).

We let $\mathcal{U}$ be the idempotent completion of the category whose

- objects are diagrams on a line shown above and
- morphisms are the pictures in the plane, modulo the relations of the previous slide.

Idempotent completion means adding a new object for each idempotent which is the image of that idempotent as a projection.

Equivalently one can think of the formal sums of diagrams described previously as a big algebra $\mathfrak{U}$. Then $\mathcal{U}$ is just projective modules over that algebra.

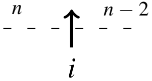
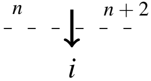
The monoidal structure

The category $\mathcal{U}$ is monoidal; it has a tensor product. Visually, it's quite simple. You just put diagrams next to each other if the label at the edges match, and get 0 if they don't.

$$\overset{n_1}{-\!-}\boxed{A}\overset{n_2}{-\!-}\ \otimes\ \overset{m_1}{-\!-}\boxed{B}\overset{m_2}{-\!-}\ =\ \overset{n_1}{-\!-}\boxed{A}\overset{n_2=m_1}{-\!-\!-\!-}\boxed{B}\overset{m_2}{-\!-}$$

This prescription works both for objects and for morphisms, since all the relations are local.

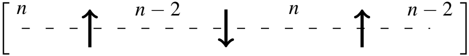
The Grothendieck group

Let $\mathcal{E}^n =$  and $\mathcal{F}^n =$ 

Theorem (Khovanov-Lauda)

The GG of $\mathcal{U}$ is $\dot{U}$, via the isomorphism $[\mathcal{E}^n] \mapsto \mathbb{1}_n E, [\mathcal{F}^n] \mapsto \mathbb{1}_n F$.

For example,

 $\mapsto \mathbb{1}_n EFE\mathbb{1}_{n-2}$

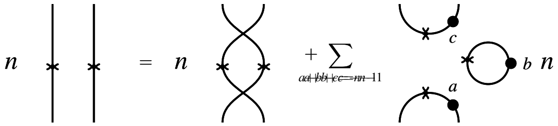
Note: I never imposed any of the relations of $\dot{U}$! They all follow (non-obviously) from the relations.

Relations in $\mathcal{U}$

So, how does one take the relations I wrote down, and find the relations of the universal enveloping algebra inside of them?

The thing to look for is writing the identity element of any object as maps factoring through another; this is how we find direct sum decompositions.

$$\mathbb{1}_n EE = \mathbb{1}_n EE + n \cdot \mathbb{1}_n \qquad (n \lessgtr 0)$$



Grading

As I indicated on the relations slide, the relations are homogeneous for a particular grading; the category $\mathcal{U}$ actually has a graded version $\tilde{\mathcal{U}}$.

Theorem (K.-L.)

The GG of $\tilde{\mathcal{U}}$ is $\dot{U}_q$, the quantized universal enveloping corresponding to $\mathfrak{sl}_2$.

I feel like I've written down enough relations in this talk, so let me take the above as a definition.

As a general rule, it's never harder to work with quantum groups in this picture (sometimes, it even makes things easier); you just pay attention to the grading.

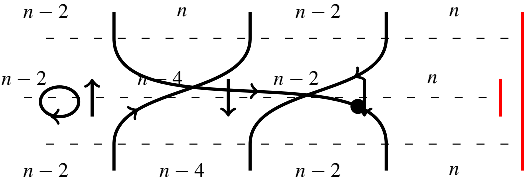
Representations

One of the reasons people like $\mathfrak{sl}_2$ is that it has a nice representation theory. Every finite dimensional irrep is generated by a unique line killed by all E, and the representation V_n is determined by the weight n of this line.

So, we can construct a representation $\mathcal{L}_n$ of $\mathcal{U}$ by starting with a single object $\mathbb{V}$ of weight n with boring endomorphisms, and letting $\mathcal{U}$ act by horizontal composition, subject to $\mathcal{E} \otimes \mathbb{V} = 0$.

Objects:

Morphisms:



Representations

Theorem (Rouquier/Khovanov-Lauda)

The GG of $\mathcal{L}_n$ is the irreducible representation of $\dot{U}$ with highest weight n, and $\mathcal{L}_n$ is essentially the unique such module category for $\mathcal{U}$.

(Small miracle: you might think that this would give you the Verma module; it doesn't!).

One advantage of such a description is that indecomposable modules give a basis of the GG; since $\mathcal{E} \otimes -$ or $\mathcal{F} \otimes -$ applied to an indecomposable is a sum of indecomposables, E, F manifestly have positive integral structure coefficients.

Theorem (Vasserot-Varagnolo)

The basis of indecomposables coincides with Lusztig's canonical basis for $\mathfrak{g}$ with symmetric Cartan matrix.

The big picture

You might ask: are there any applications? Well, quantum groups have applications in topology...

HAVE

quantum groups $U_q(\mathfrak{g})$ ←- - - - - - - - - - - - Khovanov-Lauda/Rouquier categories $\mathcal{U}$

↓ ↓

ribbon category of $U_q(\mathfrak{g})$-reps ←——— ribbon 2-category of $\mathcal{U}$-reps?

categorifications of tensor products of simples

???

↓ ↓ ?? !

quantum knot polynomials for each rep ← - - - - - quantum knot homologies

(Jones polynomial, etc.)

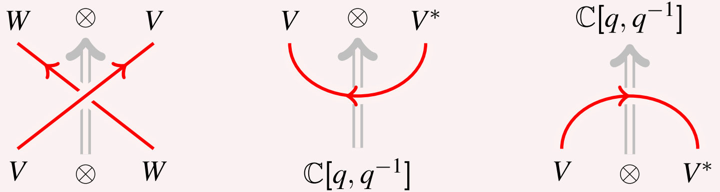
HAVE

WANT

Reshetikhin-Turaev invariants

Let me briefly indicate how the left side of the diagram works.

One labels each component of the link with a representation of $U_q(\mathfrak{g})$, and chooses a projection of the link. The theory of quantum groups attaches maps to small diagrams like:



These are called the **braiding**, the **quantum trace** and the **coevaluation**.

Composing these together for a given link results in a scalar: the **Reshetikhin-Turaev invariant** for that labeling.

A historical interlude

Progress has been made on categorifying these in a piecemeal fashion for a while

- Khovanov ('99): Jones polynomial ($\mathbb{C}^2$ for $\mathfrak{sl}_2$).
- Oszvath-Szabo, Rasmussen ('02): Alexander polynomial (which is actually a $\mathfrak{gl}(1|1)$ invariant, and doesn't fit into our general picture).
- Khovanov ('03): $\mathbb{C}^3$ for $\mathfrak{sl}_3$.
- Khovanov-Rozansky ('04): $\mathbb{C}^n$ for $\mathfrak{sl}_n$.
- Stroppel-Mazorchuk, Sussan ('06-'07): $\wedge^i\mathbb{C}^n$ for $\mathfrak{sl}_n$.
- Cautis-Kamnitzer ('06): $\wedge^i\mathbb{C}^n$ for $\mathfrak{sl}_n$.
- Khovanov-Rozansky('06): $\mathbb{C}^n$ for $\mathfrak{so}_n$.

p Khovanov ('99): Jones polynomial ($\mathbb{C}^2$ for $\mathfrak{sl}_2$).
? Oszvath-Szabo, Rasmussen ('02): Alexander polynomial (which is actually a $\mathfrak{gl}(1|1)$ invariant, and doesn't fit into our general picture).
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c Khovanov-Rozansky ('04): $\mathbb{C}^n$ for $\mathfrak{sl}_n$.
p Stroppel-Mazorchuk, Sussan ('06-'07): $\wedge^i\mathbb{C}^n$ for $\mathfrak{sl}_n$.

Tensor products

Now, some of you might think: “Wait, can’t you just take tensor product of the categories?”

There are a host of reasons why this is a bad idea. For one,

the whole point of quantum groups is that they treat the two sides of the tensor product inequitably. We shouldn’t expect a “democratic” construction, but one slanted toward one tensor factor or another.

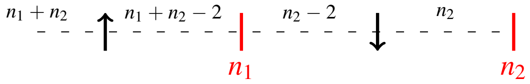
Also, the canonical bases give us hints of the structure of the categorifications of things, and the canonical basis of the tensor product is not the tensor product of canonical bases.

Tensor products

As with irreducibles or the UEA, we can introduce a graphical calculus for elements of $V_{\mathbf{n}} = V_{n_1} \otimes \cdots \otimes V_{n_\ell}$.

- A upward (downward) black line on the left means acting by E (F).
- A red line at the left labeled by n corresponds to $v_n \otimes -$, where v_n is the highest weight vector of V_n.

So, we obtain a spanning set of $V_{\mathbf{n}}$ consisting of vectors like

$E(v_{n_1} \otimes F v_{n_2}) \leftrightarrow$ 

Exactly as with $\mathcal{U}$ and $\mathcal{L}_n$, we can make these the objects of a category $\mathcal{L}_{\mathbf{n}}$, with morphisms given by diagrams.

Tensor products

Theorem (W.)

The GG of $\mathcal{L}_{\mathbf{n}}$ is $V_{\mathbf{n}}$.

Theorem (W.)

The classes of indecomposables give Lusztig's canonical basis of $V_{\mathbf{n}}$. More generally this holds for any $\mathfrak{g}$ with symmetric Cartan matrix.

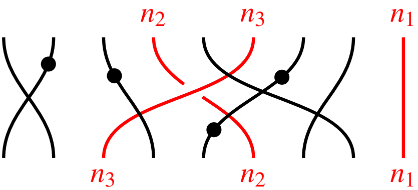
Braiding and duals

In order to get all the structures of a ribbon category (the braiding and duality) we should replace $\mathcal{L}_{\mathbf{n}}$ with the category $\mathcal{V}^{\mathbf{n}}$ of complexes in $\mathcal{L}_{\mathbf{n}}$ up to homotopy (same GG).

Theorem (W.)

Given any sequence $\mathbf{n}$*, for any* ℓ*-strand braid* σ*, we have a functor* $\mathcal{V}^{\mathbf{n}} \to \mathcal{V}^{\sigma\mathbf{n}}$ *which induces the usual braided structure on the GG.*

It's actually tensor product/Hom with a bimodule that looks like:

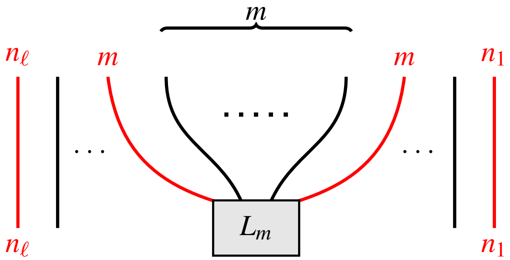


Braiding and duals

Theorem (W.)

Given any sequence $\mathbf{n}$ *and* $\mathbf{n}^+$ *given by adding an adjacent pair* (m, m)*, we have functors* $\mathcal{V}^{\mathbf{n}^+} \to \mathcal{V}^{\mathbf{n}}$ *inducing evaluation and quantum trace on GG, and dually for coevaluation and quantum cotrace (but for a funny ribbon structure!).*

It's actually tensor product/Hom with a bimodule that looks like:



Knot invariants

Now, we start with a picture of our knot (in red), cut it up into these elementary pieces, and compose these functors in the order the elementary pieces fit together.

For a link L, we get a functor $F_L : \mathcal{V}^{\emptyset} \cong D(\mathsf{gVect}) \to \mathcal{V}^{\emptyset} \cong D(\mathsf{gVect})$. So $F_L(\mathbb{C})$ is a complex of graded vector spaces.

Theorem (W.)

The cohomology of $F_L(\mathbb{C})$ is a knot invariant. The graded Euler characteristic of this complex is $J_{V,L}(q)$.

Something about affine Grassmannians

These categories seem to have connections to affine Grassmannians and quiver varieties.

Theorem (Braden-Licata-Proudfoot-W.)

These categories appear as modules over a deformation quantization of quiver varieties (in a funny derived way).

Some enterprising person should prove this for Fukaya categories of quiver varieties.

They seem to make a dual appearance (which is not derived) in modules over deformation quantizations of affine Grassmannians; hopefully this can hook up with Witten's talk from yesterday.

Open questions

There are lots of places to go with this. Two painfully obvious (and obviously painful) directions are

- Functoriality (maps exist for miniscule, but relations are unchecked).
- Generalizations of the s-invariant (Lobb: $\mathfrak{sl}_n$).

Also, there are many other interesting constructions in Lie theory that one can try to categorify:

- The q-Fock space (Stroppel-W.); this turns out to coincide with an earlier ungraded categorification (cyclotomic q-Schur algebra).
- Quantum groups at a root of unity? Khovanov has set-up a framework for categorifying modules over the cyclotomic numbers, but the diagrammatic picture doesn't fit in it yet.
- The Reshetikhin-Turaev 3-manifold invariants?
- What happened to category $\mathcal{O}$?

Happy Birthday!