Khovanov Homology And Gauge Theory

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Freedman Celebration, Berkeley, June 7, 2011

I will be proposing the use of certain elliptic differential equations in four and five dimensions for a new description of the Jones polynomial and Khovanov homology. The goal is a gauge theory alternative to a previous physics-based interpretation of Khovanov homology (Gukov, Vafa, and Schwarz, hep-th/0412243). So let us begin by describing the cast of characters.

We start by remembering the relation of the Chern-Simons function in three dimensions to the instanton equation in four dimensions.

In this talk, G is a compact simple Lie group and A is a connection on a G-bundle $E \longrightarrow W$, where W is some manifold. We write $G_{\mathbb{C}}$ for the complexification of G, and $\mathcal{A}$ for a connection on a $G_{\mathbb{C}}$ bundle, such as the complexification $E_{\mathbb{C}}$ of E. We write $\mathcal{U}$ and $\mathcal{U}_C$ for the spaces of, respectively, connections on E or on $E_{\mathbb{C}}$. Finally, by an elliptic equation, we mean an equation that is elliptic modulo the action of the gauge group.

If W is a three-manifold, then a connection A on the G-bundle $E \to W$ has a Chern-Simons invariant

$$\mathrm{CS}(A) = \frac{1}{4\pi} \int_W \mathrm{Tr} \left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right).$$

To arrive at the instanton equations, we pick a Riemannian metric on W and then place the obvious Riemannian metric on the space $\mathcal{U}$ of connections:

$$\mathrm{d}s^2 = -\int_W \mathrm{Tr}\, \delta A \wedge \star \delta A.$$

Then, viewing $-\mathrm{CS}(A)$ as a Morse function on $\mathcal{U}$, we write the equation of gradient flow:

$$\frac{\mathrm{d}A}{\mathrm{d}s} = \nabla \mathrm{CS}(A).$$

Something nice happens; the equation of gradient flow turns out to have four-dimensional symmetry. It is equivalent to the instanton equation on the four-manifold $M = W \times \mathbb{R}$:

$$F^+ = 0.$$

This fact is the starting point for Floer cohomology of three-manifolds and its relation to Donaldson theory of four-manifolds.

We want to do the same thing, roughly speaking, for the complex Lie group $G_{\mathbb{C}}$. To begin with, a connection $\mathcal{A}$ on a $G_{\mathbb{C}}$ bundle $E_{\mathbb{C}} \to W$ has a Chern-Simons function:

$$\mathrm{CS}(\mathcal{A}) = \frac{1}{4\pi}\int_W \mathrm{Tr}\left(\mathcal{A}\wedge \mathrm{d}\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right).$$

To do Morse theory, we have to make two immediate changes. First, a Morse function is supposed to be real, but $\mathrm{CS}(\mathcal{A})$ is actually complex-valued. So we pick a complex number $e^{i\alpha}$ of modulus 1 and define our Morse function to be (provisionally)

$$h_0 = -\mathrm{Re}\,(e^{i\alpha}\mathrm{CS}(\mathcal{A})).$$

Second, there is not any convenient metric on the space $\mathcal{U}_{\mathbb{C}}$ of complex connections that has the full $G_{\mathbb{C}}$ gauge symmetry. So we pick a Kahler metric on $\mathcal{U}_{\mathbb{C}}$ that is invariant only under G, not $G_{\mathbb{C}}$:

$$\mathrm{d}s^2 = -\int_W \mathrm{Tr}\,\delta\mathcal{A} \wedge \star\delta\overline{\mathcal{A}}.$$

Now we can write a gradient flow equation:

$$\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}s} = -\nabla h_0.$$

However, we are really usually interested in a complex connection $\mathcal{A}$ up to complex-valued gauge transformations, but here we have written an equation that is only invariant under unitary gauge transformations. To compensate for this, we should set the moment map to zero and consider the previous equation only in the space of zeroes of the moment map.

In other words, if we decompose $\mathcal{A}$ in real and imaginary parts as $\mathcal{A} = A + i\phi$, where A is a real connection and $\phi \in \Omega^1(W) \otimes \mathrm{ad}(E)$, then the Kahler manifold $\mathcal{U}_{\mathbb{C}}$ has a Kahler form

$$\omega = \int_W \mathrm{Tr}\, \delta A \wedge \star\delta\phi.$$

The moment map for the action of G-valued gauge transformations is

$$\mu = \mathrm{d}_A \star \phi$$

and we should really consider the previous gradient flow equations in the space of zeroes of the moment map.

However, it is somewhat better to introduce another field ϕ_0 as a sort of Lagrange muliplier. ϕ_0 is a section of the real adjoint bundle $\mathrm{ad}(E)$ and we write an extended Morse function

$$h = h_0 + \int_W \mathrm{d}^3 x \sqrt{g}\, \mathrm{Tr}\, \phi_0 \mu$$

whose critical points are all at $\mu = 0$. On the space of ϕ_0 fields we place the obvious metric

$$\mathrm{d}s^2 = -\int_W \mathrm{d}^3 x \sqrt{g}\, \mathrm{Tr}\, \delta\phi_0^2$$

and now, writing Φ for the pair $(\mathcal{A}, \phi_0)$, we write the gradient flow equations

$$\frac{\mathrm{d}\Phi}{\mathrm{d}s} = -\nabla h(\Phi).$$

Something nice happens, just like what happened in the real case. The flow equations in this sense are elliptic partial differential equations with a full four-dimensional symmetry. They can be written

$$(F - \phi \wedge \phi)^+ = t(\mathrm{d}_A \phi)^+$$
$$(F - \phi \wedge \phi)^- = -t^{-1}(\mathrm{d}_A \phi)^-$$
$$\mathrm{d}_A \star \phi = 0,$$

with

$$t = \frac{1 - \cos\alpha}{\sin\alpha}.$$

These are elliptic differential equations modulo the action of the gauge group, for each $t \in \mathbb{RP}^1 = \mathbb{R} \cup \infty$ (for $t \to 0$ or ∞, multiply the second equation by t or the first by t^{-1}).

In writing the equations, I combined the imaginary part of the connection, $\phi \in \Omega^1(W) \otimes \mathrm{ad}(E)$, with the Lagrange multiplier $\phi_0 \in \mathrm{ad}(E)$, to a field (also called ϕ) that takes values in $\Omega^1(M) \otimes \mathrm{ad}(E)$. We are using the fact that for $M = W \times \mathbb{R}$, we have $\Omega^1(M) = \Omega^1(W) \oplus \mathbb{R}$.

These equations can be the starting point for developing a Floer-like theory for the complex Lie group $G_{\mathbb{C}}$. What about other real forms of G?

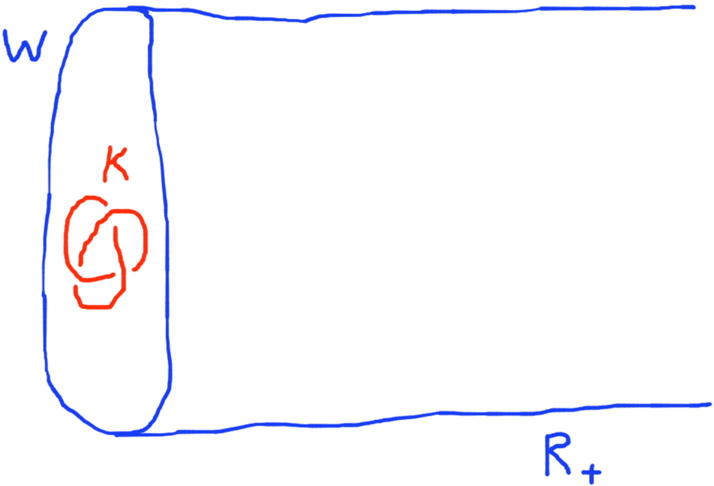
If $t = 0$ or ∞, the equations admit the involution $\phi \to -\phi$. Imposing invariance under this involution combined with an involution of the compact Lie group G, we get a reduced set of equations appropriate to constructing a Floer-like theory for any real form of G, not necessarily compact. (In a sense, the physical interpretation seems less natural than for other things I am describing.)

The equations that I have indicated were actually first studied in another context – by A. Kapustin and me in our work on gauge theory and geometric Langlands (hep-th/0604151). Roughly speaking, we considered a family of four-dimensional topological field theories just like Donaldson theory except based on these equations instead of the instanton equations, and we showed that geometric Langlands duality is naturally formulated as an equivalence between the theories that arise at two different values of t. From that vantage point, geometric Langlands duality is a consequence of S-duality of $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions. (I am leaving out a lot of details, one of which is that it is natural to consider complex values of t.)

I do not believe that this theory gives interesting four-manifold invariants (and there may actually be a technical problem in defining them, analogous to Donaldson theory for $b_2^+ = 1$). In geometric Langlands duality, four-manifold invariants are not the point. One is mainly interested not in invariants of four-manifolds but in the structures that this same topological field theory attaches to two-manifolds (categories of boundary conditions) and to three-manifolds (spaces of physical states or "morphisms"). These are essentially not affected by the technical difficulties that may affect the four-manifold invariants.

Although this theory probably does not give interesting four-manifold invariants, the math literature gives a reason to believe in hindsight, and maybe even in foresight, that it does give interesting knot invariants. "Quantum" geometric Langlands is related to quantum groups (D. Gaitsgory, arXiv:0705.4571) which in turn are related to the Jones polynomial and similar invariants of knots. So possibly this should have made us think that the topological field theory related to geometric Langlands can be used to compute the Jones polynomial. (I should also mention the work of Cautis and Kamnitzer, interpreting Seidel and Smith, for a possible clue that Khovanov homology is related to geometric Langlands.)

Anyway, the picture is like so: Take the four-manifold to be $M = W \times \mathbb{R}_+$, where W is a three-manifold and $\mathbb{R}_+$ is a half-line $y \geq 0$. For $y \to \infty$, require the fields to approach a chosen critical point of the Morse function. The simplest case is $W = S^3$ (or $\mathbb{R}^3$) where there is only one possible critical point, $A = \phi = 0$ up to a unitary gauge transformation. The knot K lives instead at $y = 0$, that is at the endpoint of $\mathbb{R}_+$. The picture is shown in the next slide.


W
K
R+

The boundary condition at $y = 0$ is an elliptic boundary condition that is a little involved to explain, and we will postpone it a bit. But this boundary condition depends on the knot K, and on the choice of a representation R of the Langlands or GNO dual group $G^\vee$ of G. In this description, this is the only way that K (or R) enters.

For the next step, we imitate Donaldson theory. Let a_n be the "number" of solutions of our equations, with instanton number n. Instanton number is defined in the usual way as a multiple of

$$\int_M \mathrm{Tr}\, F \wedge F,$$

where $F = \mathrm{d}A + A \wedge A$ is the curvature, and is a topological invariant here even though $M = W \times \mathbb{R}_+$ has "ends" at $y = 0, \infty$, because the boundary conditions at $y = 0$ and $y = \infty$ give trivializations of the bundle E. Conjecturally, a_n vanishes for large enough $|n|$, but this has not been proved.

Then we introduce a variable q and define the knot invariant

$$J(q; K, R) = \sum_n a_n q^n.$$

For $G^\vee = SU(2)$ and R the two-dimensional representation, this is supposed to be the Jones polynomial. In general, we expect to get the usual knot invariants associated to quantum groups and Chern-Simons gauge theory.

However, our goal is Khovanov homology, not the Jones polynomial. This means we are supposed to "categorify" the situation, and associate to a knot a vector space rather than a number. A suitable trace in the vector space will give back the number. In plain words, this means that the picture just described has to be derived from a picture in one more dimension. (An explanation for physicists: viewing the extra dimension as "time," quantization gives a physical "Hilbert" space, which will be the Khovanov homology, and then if we compactify the extra dimension on a circle, we get a trace leading back to the original theory.)

Let us practice by "categorifying" the Casson invariant. The Casson invariant is an invariant of a three-manifold W. It is defined by "counting" (up to gauge transformation) the flat connections A on a G-bundle $E \to W$. A flat connection is a solution of the equation

$$F = 0. \quad (*)$$

This is not an elliptic equation, but is part of a nonlinear elliptic complex. Just as for linear elliptic complexes, it is convenient to "fold" the complex and reduce to the case of an ordinary elliptic equation, rather than a complex. In the present example, this is done by introducing a section ϕ_0 of $\mathrm{ad}(E)$ and replacing the equation $(*)$ with the Bogomolny equation

$$F + \star \mathrm{d}_A \phi_0 = 0.$$

The count of solutions is the same, since a simple vanishing theorem says that (for a smooth solution on a compact manifold W) any solution has $\phi_0 = 0$. But now the equation is elliptic.

In the present case, categorification just means replacing ϕ_0 by the covariant derivative with respect to a new coordinate, $\phi_0 \to D/Ds$. Thus we replace the three-manifold W by the four-manifold $M = W \times \mathbb{R}$, where $\mathbb{R}$ is parametrized by the "time" s, and we substitute $\phi_0 \to D/Ds$. This makes sense, in that the substitution gives a differential equation on M (rather than a differential operator), because we started with an equation in which ϕ_0 only appears inside the commutator $\mathrm{d}_A\phi_0 = [\mathrm{d}_A, \phi_0]$. This commutator is now replaced by $[\mathrm{d}_A, D/Ds]$, which is a component of the four-dimensional curvature.

Normally, a procedure like this, even if it gives a differential equation, won't give an elliptic one, let alone one with four-dimensional symmetry. In this case, however, we actually get back the instanton equation $F^+ = 0$, in a slightly different way from the way we got it before. What follows from this is that the Casson invariant – a numerical invariant computed by counting solutions of the original equation $F = 0$ – can be categorified to Floer cohomology, in which a more subtle invariant, a vector space, is constructed starting with a chain complex that has a basis corresponding to those same solutions.

Now we want to categorify the Jones polynomial, which from the point of view of the present lecture is the invariant associated to counting solutions of the equations

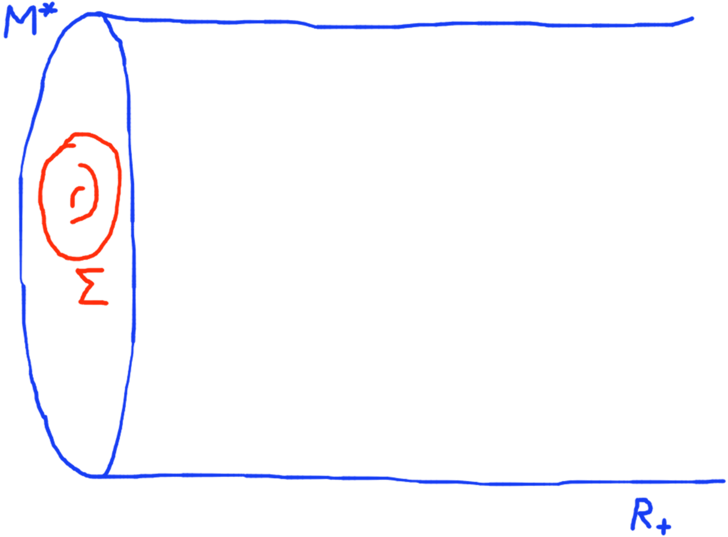
$$\begin{gathered}(F - \phi \wedge \phi)^+ = t(\mathrm{d}_A\phi)^+ \\ (F - \phi \wedge \phi)^- = -t^{-1}(\mathrm{d}_A\phi)^- \\ \mathrm{d}_A \star \phi = 0,\end{gathered}$$

with certain boundary conditions. On a generic four-manifold M, we would have no way to proceed as there is no candidate for a field ϕ_0 that will be replaced by D/Ds.

However, if $M = W \times \mathbb{R}_+$, which is the case if we are studying the Jones polynomial in the way I suggested, then we have $\Omega^1(M) = \Omega^1(W) \oplus \Omega^1(\mathbb{R}_+)$, where the part of ϕ associated to the second summand is the field ϕ_0 that we originally introduced as a Lagrange multiplier. We categorify by introducing a new dimension and replacing $\phi_0 \to D/Ds$, as before.

In this way, we get a partial differential equation on the five-manifold $X = \mathbb{R} \times W \times \mathbb{R}_+$. Moreover, this turns out to be an elliptic equation. And if we set $t = 1$, we get a full four-dimensional symmetry; that is, the five-dimensional equation (which also was obtained in A. Haydys, arXiv:1010.2353) can be naturally formulated on $X = M^* \times \mathbb{R}_+$ for any four-manifold M^*.

The four-dimensional boundary condition (which I didn't explain yet) that we have to use to get the Jones polynomial can be "lifted" to five dimensions, roughly by $\phi_0 \to D/Ds$. The boundary is now a four-manifold M^* rather than a three-manifold W. Instead of modifying the boundary condition along a knot $K \subset W$, we now modify it along a two-manifold $\Sigma \subset M^*$, as in the next picture.



M^*
Σ
R_+

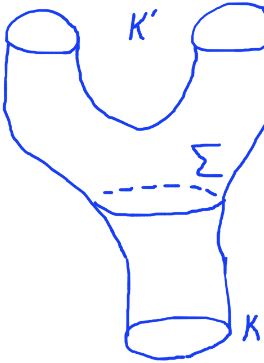
To get the candidate for Khovanov homology, we specialize to the time-independent case $M^* = \mathbb{R} \times W$, $\Sigma = \mathbb{R} \times K$. Then, following Floer, we define a chain complex which has a basis given by the time-independent solutions, that is the solutions of the four-dimensional equations

$$\begin{aligned} (F - \phi \wedge \phi)^+ &= t(\mathrm{d}_A \phi)^+ \\ (F - \phi \wedge \phi)^- &= -t^{-1}(\mathrm{d}_A \phi)^- \\ \mathrm{d}_A \star \phi &= 0. \end{aligned}$$

The differential in the chain complex is constructed in a standard fashion by counting certain time-dependent solutions. (Here we use the fact that the five-dimensional equations can themselves be interpreted in terms of gradient flow.) The cohomology of this differential is the candidate for Khovanov homology.

The candidate Khovanov homology is $\mathbb{Z} \times \mathbb{Z}$-graded, like the real thing, where one grading is the cohomological grading, and the second grading, sometimes called the q-grading, is the instanton number, integrated over $W \times \mathbb{R}_+$. (Because of the fact that $W \times \mathbb{R}_+$ is not compact and has a boundary, the definition of the q-grading has subtleties that match the framing anomalies of Chern-Simons theory.)

We are not limited to the time-independent case, and, considering a more general Σ, we get candidates for the "knot cobordisms" of Khovanov homology.



Next I would like to describe the boundary conditions at least away from knots. It is essentially enough to describe the boundary condition in four dimensions rather than five (once one understands it, the lift to five dimensions is fairly obvious), and as the boundary condition is local, we assume initially that the boundary of the four-manifold is just $\mathbb{R}^3$. So we work on $M = \mathbb{R}^3 \times \mathbb{R}_+$. (This special case is anyway the right case for the Jones polynomial, which concerns knots in $\mathbb{R}^3$ or equivalently S^3.)

Now I need to tell you about one of the important equations in gauge theory, which is Nahm's equations. Nahm's equation is a system of ordinary differential equations for a triple X_1, X_2, X_3 valued in $\mathfrak{g}^3$, where $\mathfrak{g}$ is the Lie algebra of G. The equations read

$$\frac{\mathrm{d}X_1}{\mathrm{d}y} + [X_2, X_3] = 0$$

and cyclic permutations. On a half-line $y \geq 0$, Nahm's equations have the special solution

$$X_i = \frac{t_i}{y},$$

where the t_i are elements of $\mathfrak{g}$ that obey the $\mathfrak{su}(2)$ commutation relations $[t_1, t_2] = t_3$, etc. We are mainly interested in the case that the t_i define a "principal $\mathfrak{su}_2$ subalgebra" of $\mathfrak{g}$, in the sense of Kostant.

This sort of singular solution of Nahm's equations was important in the work of Nahm on monopoles, and in later work of Kronheimer and others. We will use it to define an elliptic boundary condition for our equations.

In fact, Nahm's equations can be embedded in our four-dimensional equations on $\mathbb{R}^3 \times \mathbb{R}_+$. If we look for a solution that is (i) invariant under translations of $\mathbb{R}^3$, (ii) has the connection $A = 0$, (iii) has $\phi = \sum_{i=1}^3 \phi_i \, \mathrm{d}x_i + 0 \cdot \mathrm{d}y$ (where x_1, x_2, x_3 are coordinates on $\mathbb{R}^3$ and y is the normal coordinate) then our four-dimensional equations reduce to Nahm's equations

$$\frac{\mathrm{d}\phi_1}{\mathrm{d}y} + [\phi_2, \phi_3] = 0,$$

and cyclic permutations. So the "Nahm pole" gives a special solution of our equations

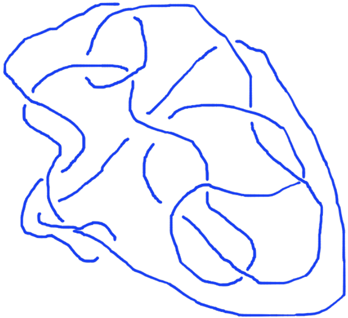
$$\phi_i = \frac{t_i}{y}.$$

We define an elliptic boundary condition by declaring that we will allow only solutions that are asymptotic to this one for $y \to 0$.

This is the boundary condition that we want at $y = 0$, in the absence of knots. For the most obvious boundary condition for getting Khovanov homology, we require that $A, \phi \to 0$ for $y \to \infty$. It is plausible (but unproved) that with these conditions, the special solution with the Nahm pole is the only one. (This would correspond to Khovanov homology of the unknot being of rank 1.)

In recent work with D. Gaiotto (to appear soon), we've made considerable progress towards understanding directly – rather than by invoking the original arguments which involved quantum field theory – why the counting of four-dimensional solutions gives the Jones polynomial. (Therefore the five-dimensional equations will give a categorification of the Jones polynomial. But this remains to be explored.)

First let us recall that standard approaches to the Jones polynomial and Khovanov homology often begin by considering a projection of a knot to two dimensions.



There is a very nice way to incorporate a knot projection by modifying the boundary conditions at infinity on $\mathbb{R}^3 \times \mathbb{R}_+$. Instead of requiring that $A, \phi \to 0$ for $y \to \infty$, we keep that condition on A, we change the condition on ϕ. We pick a triple c_1, c_2, c_3 of commuting elements of $\mathfrak{t}$, the Lie algebra of a maximal torus $T \subset G$, and we ask for

$$\phi \to \sum_i c_i \cdot \mathrm{d}x^i$$

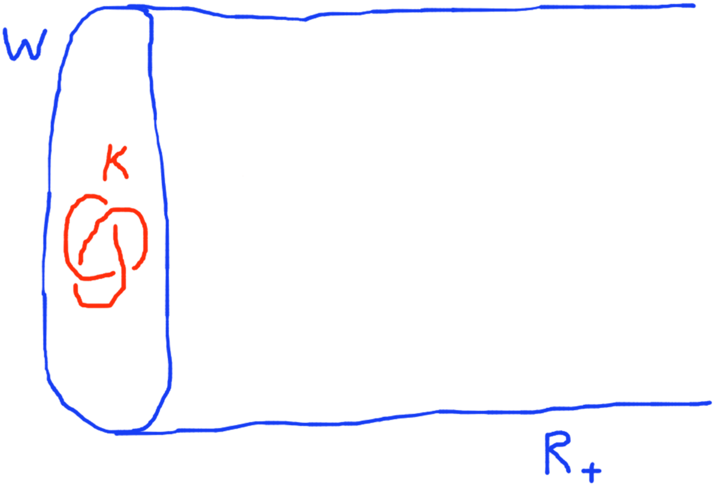
for $y \to \infty$. (x^1, x^2, x^3 are Euclidean coordinates on $\mathbb{R}^3$.) We use the fact that the equations have an exact solution for $A = 0$ and ϕ of the form I indicated.

The counting of solutions of an elliptic equation is constant under continuous variations (provided certain conditions are obeyed) so one expects that the Jones polynomial can be computed with this more general asymptotic condition, for an arbitrary choice of $\vec{c} = (c_1, c_2, c_3)$.

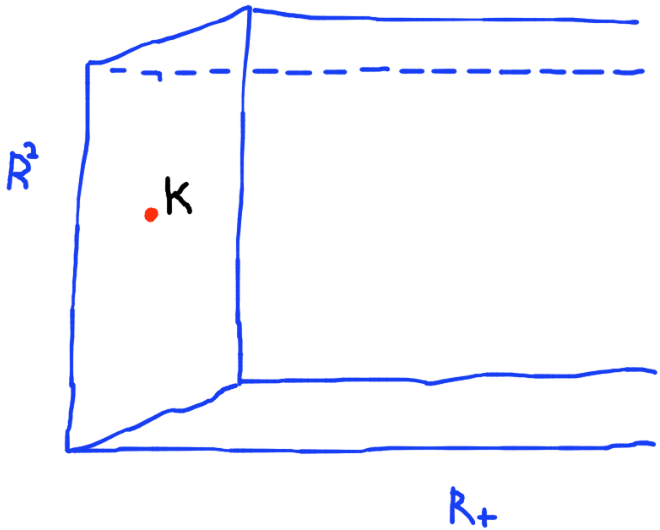
If $G = SU(2)$, then $\mathfrak{t}$ is one-dimensional. So if $\vec{c}$ is non-zero, it has the form $\vec{c} = c \cdot \vec{a}$ where c is a fixed (nonzero) element of $\mathfrak{t}$ and $\vec{a}$ is a vector in three-space. So picking $\vec{c}$ essentially means picking a vector $\vec{a}$ pointing in some direction in three-space. The choice of $\vec{a}$ determines a projection of $\mathbb{R}^3$ to a plane, so this is now built into the construction. For G of higher rank, one could do something more general, but it seems sufficient to take $\vec{c} = c\vec{a}$ with c a regular element of $\mathfrak{t}$.

Taking $\vec{c} \neq 0$ is described by physicists as "gauge symmetry breaking" or "moving on the Coulomb branch." A closely parallel construction is important in the theory of weak interactions and the theory of superconductivity. There actually is a somewhat similar idea in Taubes's proof that "SW=GW." Taking $\vec{c}$ sufficiently generic gives a drastic simplification because the equations become quasi-abelian in a certain sense. On a length scale large than $1/|\vec{c}|$, the solutions can be almost everywhere approximated by solutions of an abelian version of the same equations. There is an important locus where this fails, but it can be understood.

We scale up our knot until the quasi-abelian description is everywhere valid:



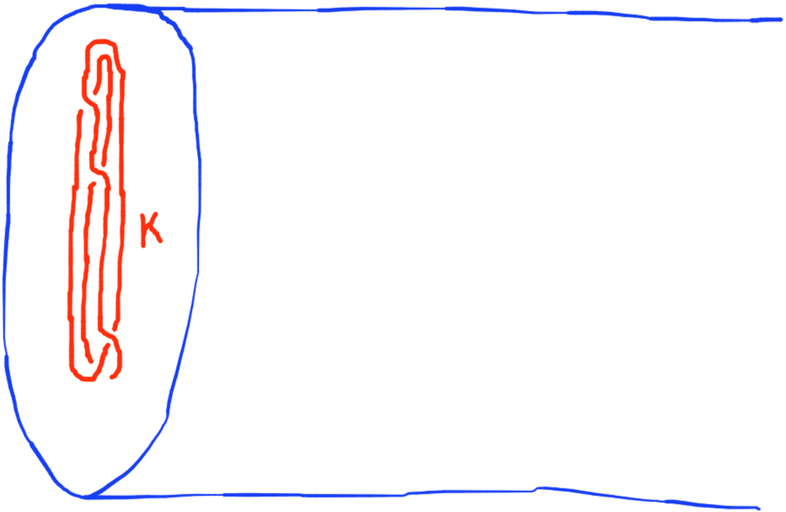
To go into more detail, I should explain how the boundary condition is modified along a knot K. The local model is that the boundary is $\mathbb{R}^3$, and K is a copy of $\mathbb{R} \subset \mathbb{R}^3$. The boundary condition is described by giving a singular model solution on $\mathbb{R}^3 \times \mathbb{R}_+$ that along the boundary has the now-familiar Nahm pole away from K, but has some other behavior along K. The model solution is invariant under translations along K, so it can be obtained by solving some reduced equations on $\mathbb{R}^2 \times \mathbb{R}_+$.



R^2
K
R_+

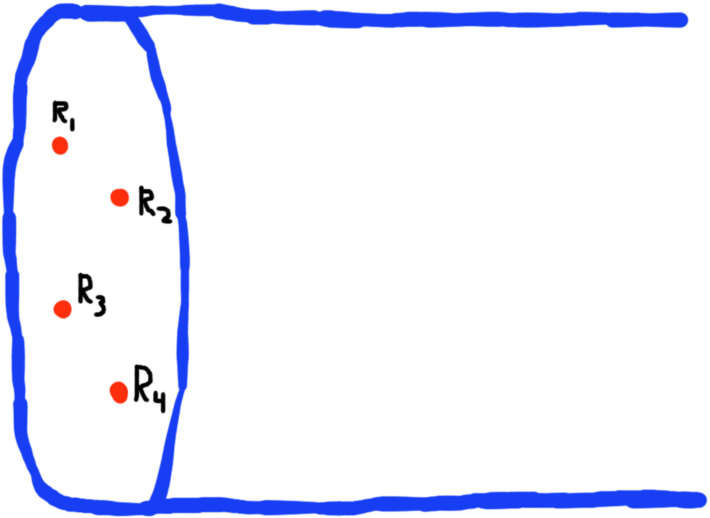
So to explain what is the boundary condition in the presence of a knot, we need to describe some special solutions of reduced equations in three dimensions – in fact, we need to give one solution for each irreducible representation R of the dual group $G^\vee$, since this is the data by which the knots are labeled.

There is another reason that it is important to describe the reduced equations in three dimensions. To compute the Jones polynomial, we need to count certain solutions in four dimensions; knowledge of these solutions is also the first step in constructing the candidate for Khovanov homology. How are we supposed to describe four-dimensional solutions? A standard strategy, often used in Floer theory and its cousins, involves "stretching" the knot in one direction, in the hope of reducing to a piecewise description by solutions in one dimension less.


K

Another way to make the point is as follows. Most mathematical definitions of Khovanov homology proceed, directly or implicitly, by defining a category of objects associated to a two-sphere (or in some versions, a copy of $\mathbb{C} = \mathbb{R}^2$) with marked points that are suitably labeled.

In the present approach, this category should be the A-model category of the moduli space of solutions of the reduced three-dimensional equations in the appropriate geometry, sketched in the next picture. (There is also a mirror approach that we haven't had time for today that involves a B-model category of almost the same space rather than an A-model.)



R_1
R_2
R_3
R_4

The equations when reduced to three dimensions have a really simple structure. After getting a simplification via a small vanishing theorem for some of the fields, the equations can be schematically described as follows. There are there operators $\mathcal{D}_i$ (constructed from A and ϕ) that commute,

$$[\mathcal{D}_i, \mathcal{D}_j] = 0, \quad i, j = 1, 2, 3.$$

And they obey a "moment map" constraint

$$\sum_{i=1}^{3} [\mathcal{D}_i, \mathcal{D}_i{}^\dagger] = 0.$$

The construction of the $\mathcal{D}_i$ in terms of A and ϕ depends on t. At $t = 1$, the equations that I just described coincide with what Kapustin and I called the "extended Bogomolny equations." They describe the Hecke transformations of the geometric Langlands correspondence. Khovanov homology has been described in terms of a B-model category of moduli spaces of geometric Hecke transformations by Cautis and Kamnitzer, and parts of a description in terms of an A-model of the same spaces have been given by Kamnitzer, following Seidel and Smith. The connection with Hecke transformations enables us to find a modification of the boundary condition for every representation $R^\vee$ of the dual group $G^\vee$.

The reason that Gaiotto and I were able to get a reasonable understanding of how the Jones polynomial emerges is that a more transparent structure arises for generic t. In this case the equations $[\mathcal{D}_i, \mathcal{D}_j] = 0 = \sum_i [\mathcal{D}_i, \mathcal{D}_i^\dagger]$ are actually more familiar. They describe a flat $G_{\mathbb{C}}$ bundle $E \to \mathbb{R}^3 \times \mathbb{R}_+$ endowed with a hermitian metric that obeys a moment map condition. For a special value of t, the moment map condition is the one studied long ago by K. Corlette. As far as we know, the precise moment map isn't important.

Since $\mathbb{R}^2 \times \mathbb{R}_+$ is simply-connected, how can we get anything interesting from a flat connection? The answer is that there is additional structure in the behavior at $y = 0$ (and ∞).

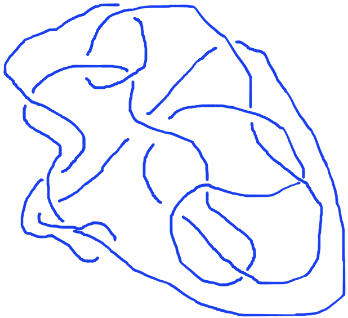
A flat bundle over $\mathbb{R}^2 \times \mathbb{R}_+$ is, of course, the pullback of a flat bundle on $\mathbb{R}^2$, which we will think of as $\mathbb{C}$. The boundary conditions at $y = 0$ gives the flat bundle $E \to \mathbb{C}$ the structure of an "oper," in the language of geometric Langlands. At the points corresponding to the knots, the oper has singularities, but the flat bundle has no monodromy around these singularities. Such oper singularities are classified again by representations of the dual group. See E. Frenkel, arXiv:math/0407524 for a review of these concepts.

Additionally, if we have taken $\vec{c} \neq 0$, the flat bundle has an irregular singularity at infinity (the connection has a pole of order 2). See Feigin, Frenkel, and Rybnikov arXiv:0712.1183.

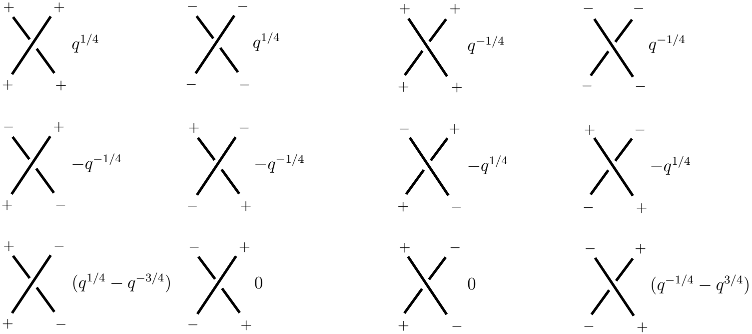
Opers of this sort are related to a variety of known and solved systems of mathematical physics, including the Gaudin spin chain and what are known as degenerate conformal blocks of the Virasoro algebra. It is known that the Jones polynomial can be expressed in terms of the monodromies associated to those conformal blocks. (References for that statement go back to work in the 1980's and early 1990's by, among others, Tsuchiya–Kanie, Dotsenko-Fateev, Felder, Lawrence, and Schechtman-Varchenko.)

So finally we were able to make contact with a known description of the Jones polynomial in a "vertex model." This is a description of the Jones polynomial via a sort of discrete statistical mechanics associated to a knot projection. (For example, see L. Kauffman, *Knots and Physics*.)

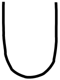
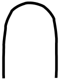
A summary of the vertex model: Given a knot projection with only simple crossings and only simple maxima and minima of the height

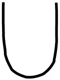
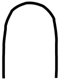


one labels the intervals between crossings, maxima, and minima by symbols $+$ or $-$. One sums over all such labelings with a suitable factor for each crossing



and for each creation or annihilation event

$+$  $-$ $iq^{-1/4}$ $+$  $-$ $iq^{-1/4}$

$-$  $+$ $-iq^{1/4}$ $-$  $+$ $-iq^{1/4}$