Smooth 4-Manifolds: 2011

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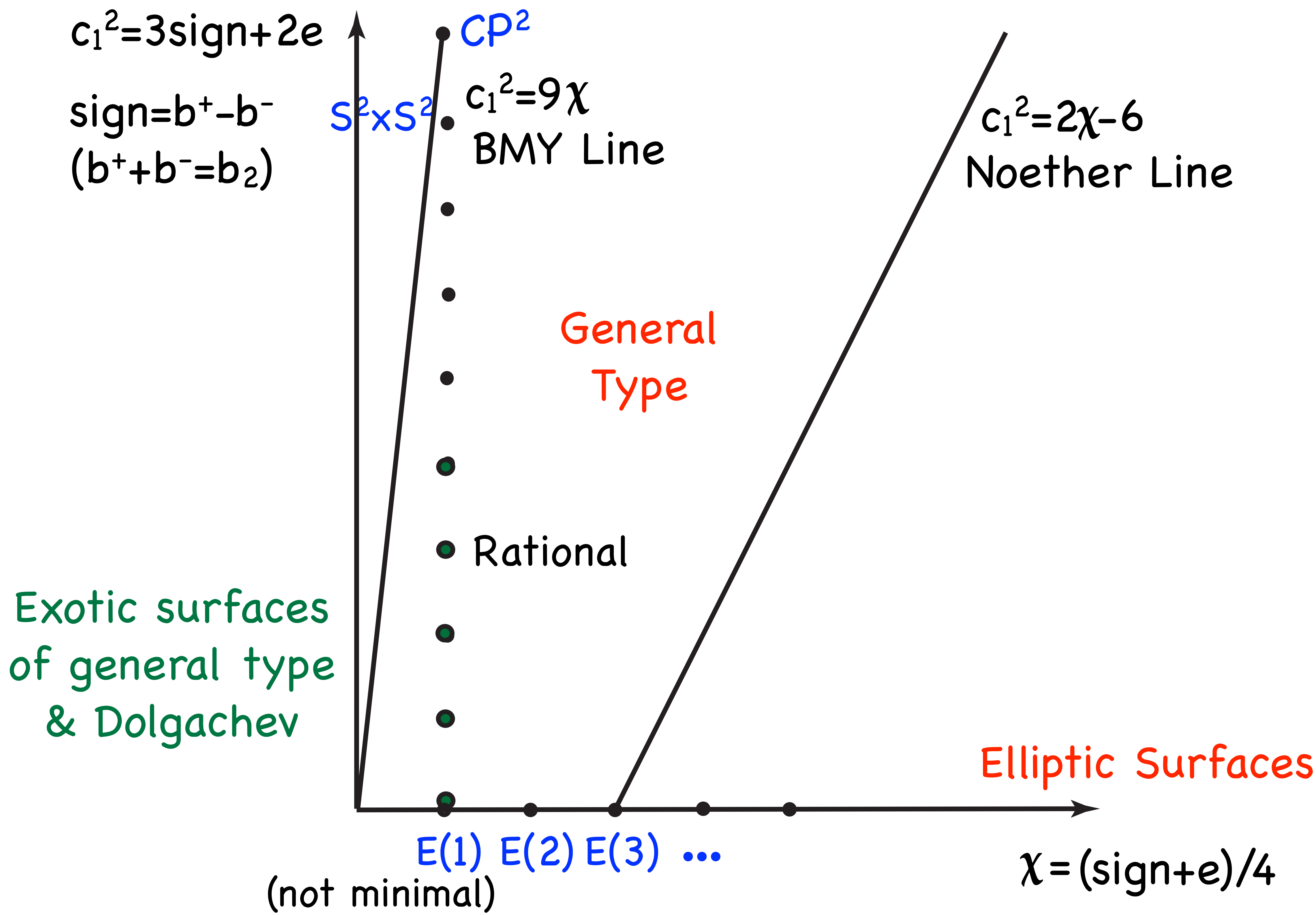
Geography of S.C. Minimal Complex Surfaces



Topological Classification

Smooth simply connected 4-manifolds with $b^+=1$ classified up to homeomorphism by Freedman's Th'm:

Odd intersection form: $CP^2 \# n\overline{CP}^2$

Even intersection form: $S^2 \times S^2$

$\Rightarrow$ Homeo type of s.c. smooth 4-mfd w/ b+=1 determined by Type (even, odd)

rank of H_2

Elliptic Surfaces

$E(1)=CP^2\#9\overline{CP}^2$ $\qquad$ $\begin{array}{ccc} T^2 & \to & E(1) \\ & & \downarrow \pi \\ & & CP^1 \end{array}$ $\qquad$ Elliptic fibration

Fiber sums: $E(n)=E(1)\#_F\ldots\#_F E(1)$ $\qquad$ $b^+=2n-1$

Log transform:

Remove nbd $T^2\times D^2=N_F$ of generic fiber and reglue

Multiplicity = degree of $\begin{array}{ccc} \partial D^2 & \to & \partial N_F \\ & & \downarrow \pi \\ & & \pi(\partial N_F)=S^1 \end{array}$

e.g. $S^1\times(p/q)$-Dehn surgery has mult = p

For elliptic surfaces with cusp fibers the result of a log transform depends only on the multiplicity.
True (up to diffeo) if simply connected.

Exotic Complex Surfaces with $b^+=1$

$b^-=9$ Dolgachev surfaces

$E(1)_{p,q}$ = result of mult. p and q log transforms (p,q rel prime and ≥ 2)

$E(1)_{p,q}$ s.c., odd & $b_2=10 \Rightarrow$ homeo to $E(1)$

Thm. (Donaldson, 1985) $E(1)_{2,3}$ not diffeo to $E(1)$.

Friedman, Morgan: $E(1)_{p,q} \cong E(1)_{p',q'} \Leftrightarrow \{p,q\}=\{p',q'\}$

$b^-=8$ Barlow surface

homeo to $CP^2\#8\overline{CP}^2$

not diffeo Kotschick, 1989

Seiberg-Witten Invariants

X : s.c. smooth 4-mfd, $b^+ \geq 1$, $SW_X \in \mathbf{Z}H_2(X)$ diffeo inv't

Only characteristic homology classes can have $\neq 0$ coefficients

Ex. $SW_{E(2)}=1$, $SW_{E(3)}=t-t^{-1}$

- If X admits +'ve scalar curv metric then $SW_X=0$

$\Rightarrow$ $SW_{CP^2\#n\overline{CP}^2}=0$ and $SW_{S^2\times S^2}=0$

- Adj$\neq$ If coeff of k in $SW_X \neq 0$ and Σ emb, $g(\Sigma)>0$ w/ $\Sigma.\Sigma \geqq 0$,

$2g(\Sigma)-2 \geqq \Sigma.\Sigma+|k.\Sigma|$

- For Kahler surfaces (Witten) and symplectic mfds (Taubes)

$b^+>1 \Rightarrow SW_X \neq 0$ (canonical class has coeff ± 1)

For $b^+=1$, minor complications arising from reducible solutions to SW eq'ns for some metrics. Get inv'ts $SW^\pm$ and these determine SW.

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Log transform formula: $SW_{X_p}= SW_X\cdot(t^{p-1}+t^{p-3}+\ldots+t^{3-p}+t^{1-p})$

where $t=$ multiple fiber; so $t^p=$fiber

Works for $SW^{\pm}$ when $b^+=1$. Can use to compute SW.

E.g. $SW_{E(1)_{2,3}} = t^{-1}-t$

Knot Surgery

T: homologically $\neq 0$, square 0 torus $\subset X$

K: knot in S^3 $\quad X_K = (X-(T\times D^2)) \cup (S^1 \times (S^3 - N_K))$

glued so that (long. of K) $\leftrightarrow \partial D^2$

- X_K: same int. form as X
- $\pi_1(X)=1$ and $\pi_1(X-T)=1 \Rightarrow \pi_1(X_K)=1 \quad \Rightarrow X_K$ homeo to X

<u>Knot Surgery Th'm</u> (F-Stern). $\quad SW_{X_K} = SW_X \cdot \Delta_K(t^2) \quad (b^+>1)$

- Works for $SW^{\pm}$ when $b^+=1$

Consequence: K=n-twist knot, $SW_{E(1)_K} = n(t^{-1}-t)$

$\Rightarrow$ no two diffeo, all homeo to E(1)

Rational Blowdown

Usual blowdown: $S^2 \subset X$, square -1, $N_{S^2} = \overline{CP}^2$-ball

$\partial N_{S^2} = S^3$ Trade N_{S^2} for B^4, get $\overline{X}$ $b^-_{\overline{X}} = b^-_X - 1$

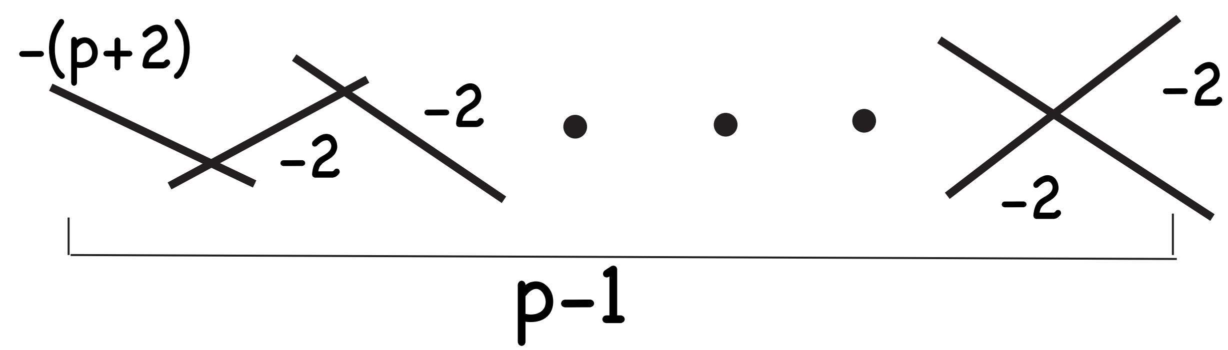
$SW_X = SW_{\overline{X}} \cdot (\varepsilon + \varepsilon^{-1})$

S^2: -4 sphere $\subset X$, $N_{S^2} = \overline{CP}^2 - N_{RP^2}$

Blowdown -4 sphere: replace N_{S^2} with $N_{RP^2 \subset CP^2}$

N_{RP^2} has $\pi_1 = Z_2$ and is a Q-homology ball

In general,



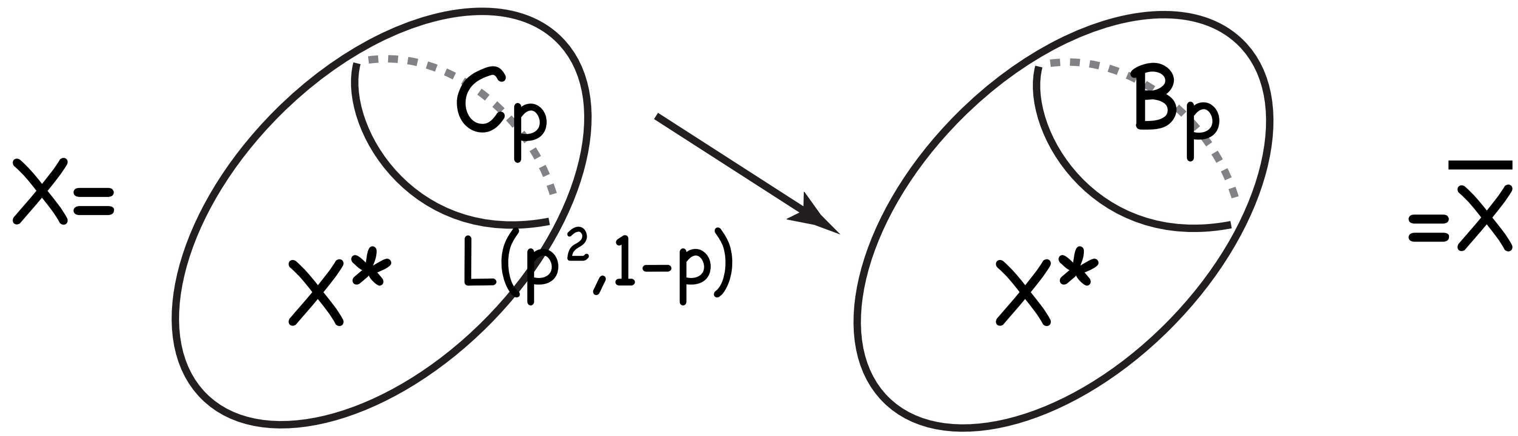
$= C_p$

has $\partial C_p = L(p^2, 1-p) = \partial B_p$ B_p is a Q-ball w/ $\pi_1 = Z_p$

Rational blowdown - remove C_p, glue in B_p. Get $\overline{X}$

Lowers b^- by $p-1$.

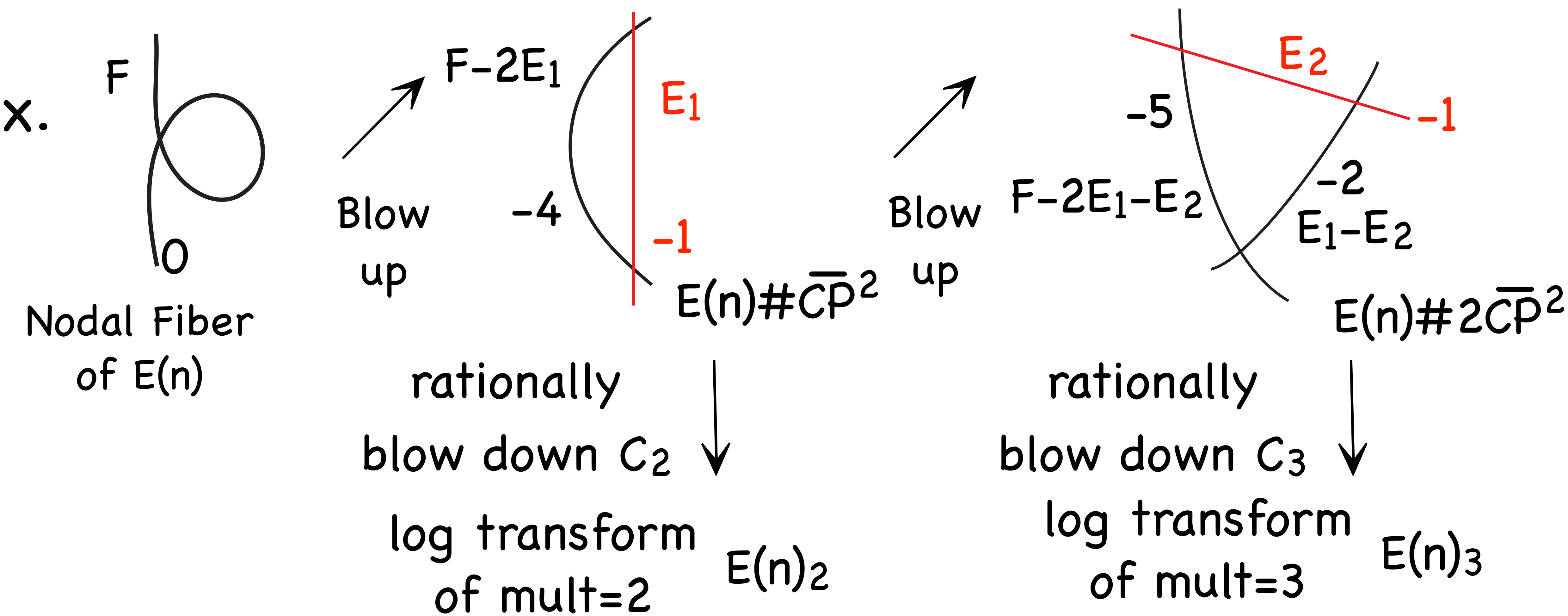
Rational Blowdowns, II



If $\bar{k}$=char homology class in $\bar{X}$, $\exists$ lift k in $X \ni$ PD's agree on X^*

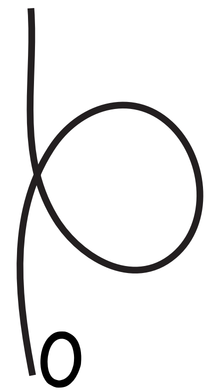
Thm. (F-Stern) Coeff of $\bar{k}$ in $SW_{\bar{X}}$ = Coeff of k in SW_X

Ex.

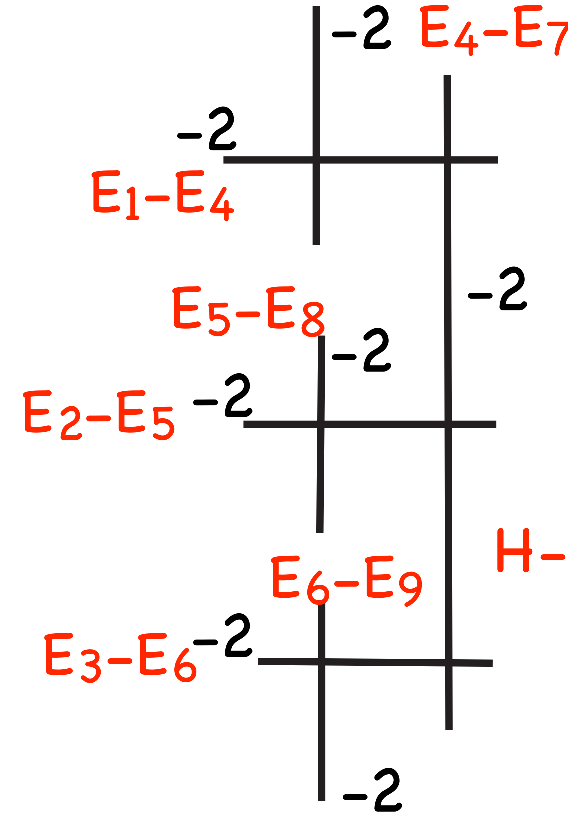


Jongil Park's Idea (2004)

Some singular elliptic fibers

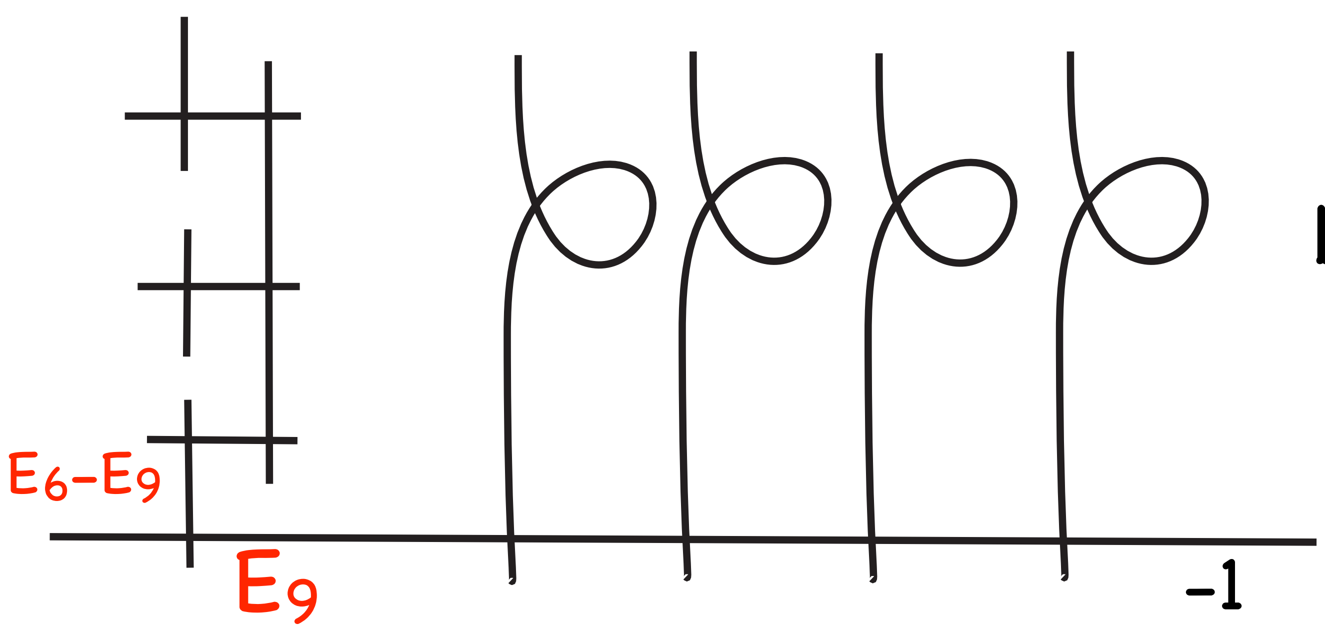


Nodal Fiber



$\tilde{E}_6$ fiber

E(1) has fibration w/ 4 nodal fibers + $\tilde{E}_6$



$b^+=1$, $b^-=9$

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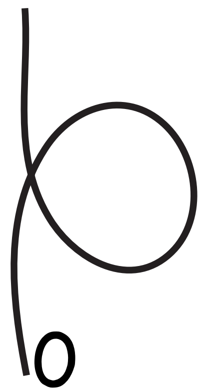

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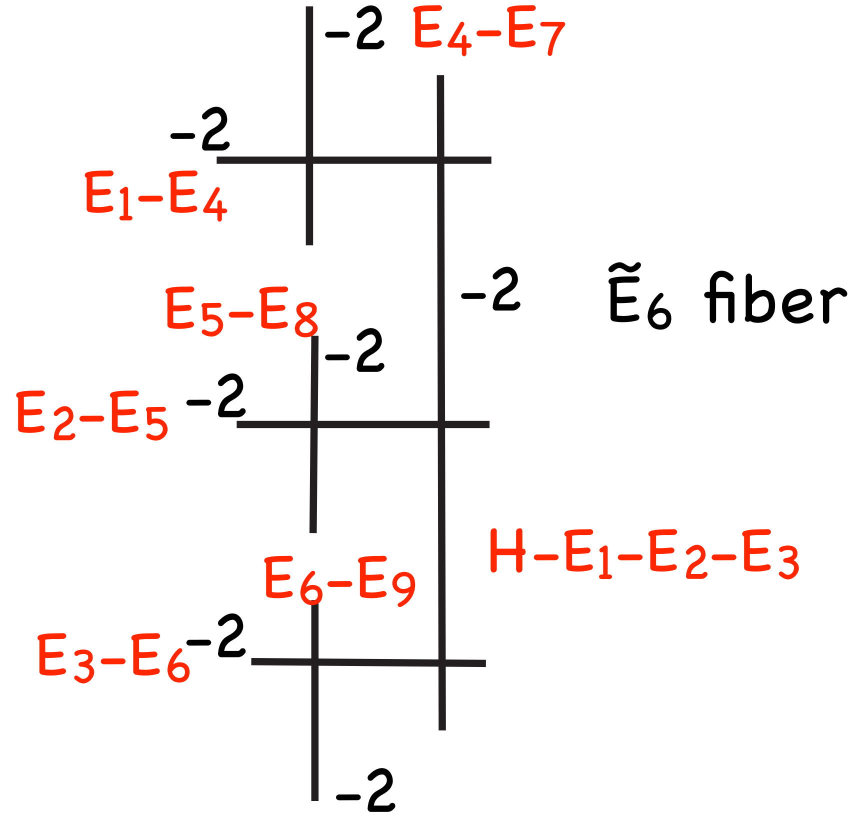

E(1) has fibration w/ 4 nodal fibers + $\tilde{E}_6$

Blow up 4 times

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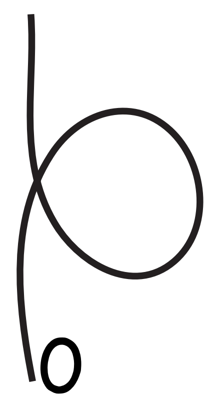
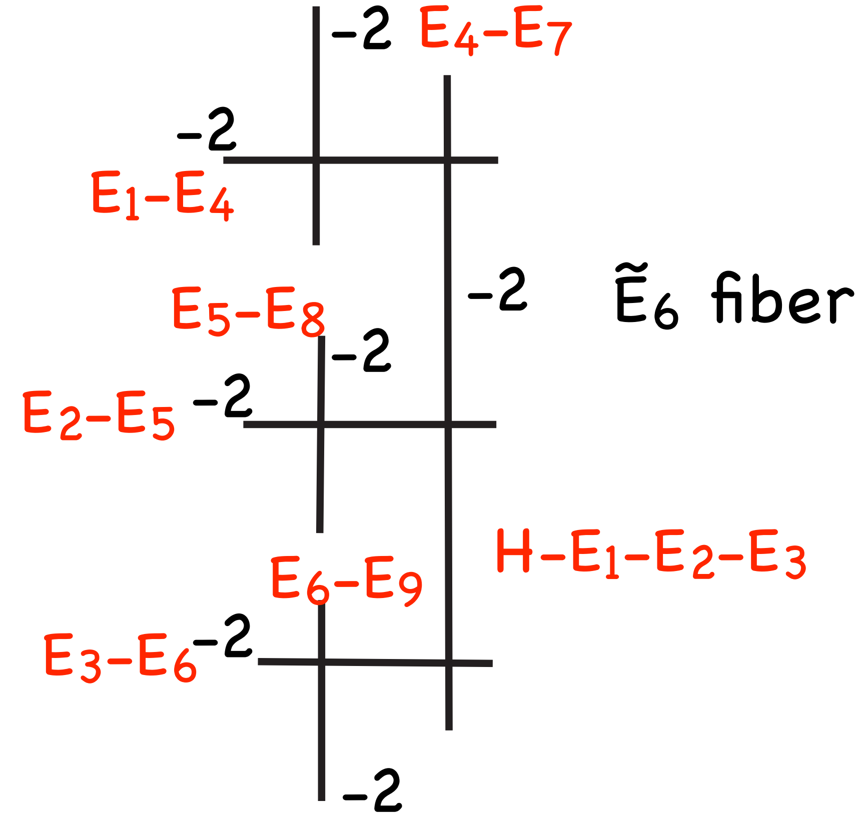
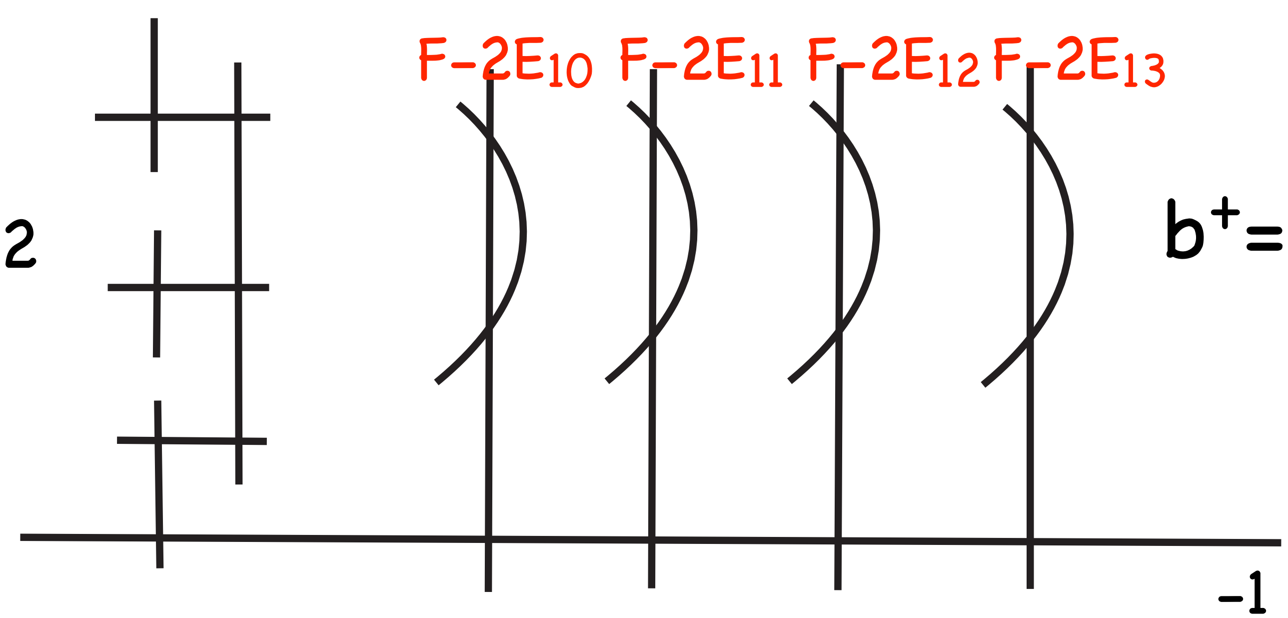


Nodal Fiber



$E(1)$ has fibration w/ 4 nodal fibers + $\tilde{E}_6$

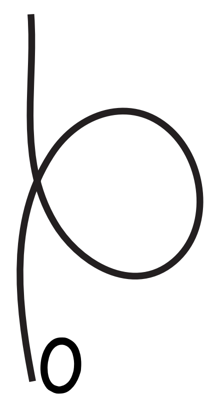
$E(1)\#4\overline{CP}^2$



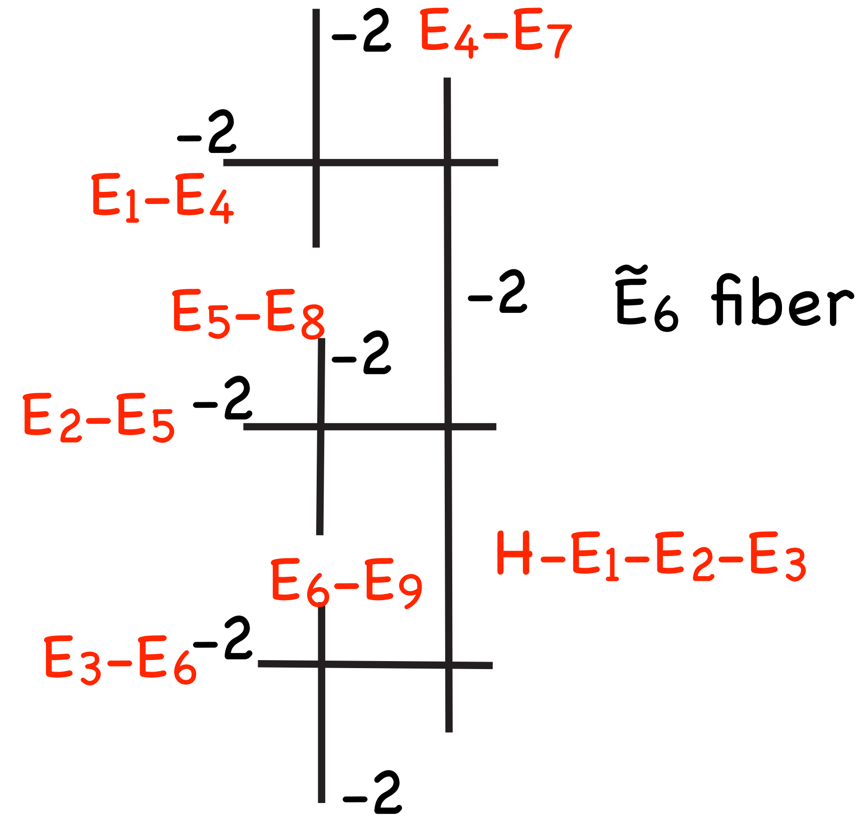
$b^+=1$, $b^-=13$

Jongil Park's Idea (2004)

Some singular elliptic fibers



Nodal Fiber

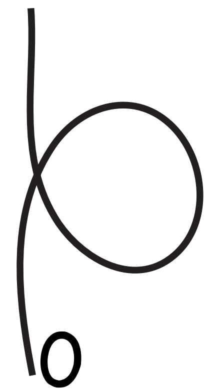


E(1) has fibration w/ 4 nodal fibers + $\tilde{E}_6$

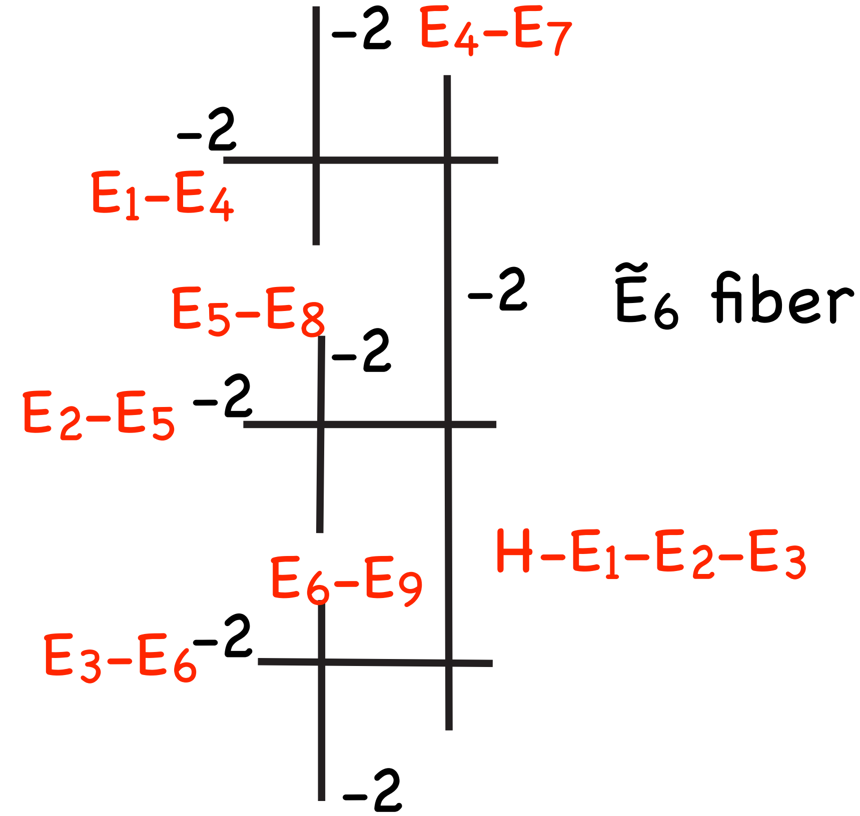
Resolve double points

Jongil Park's Idea (2004)

Some singular elliptic fibers

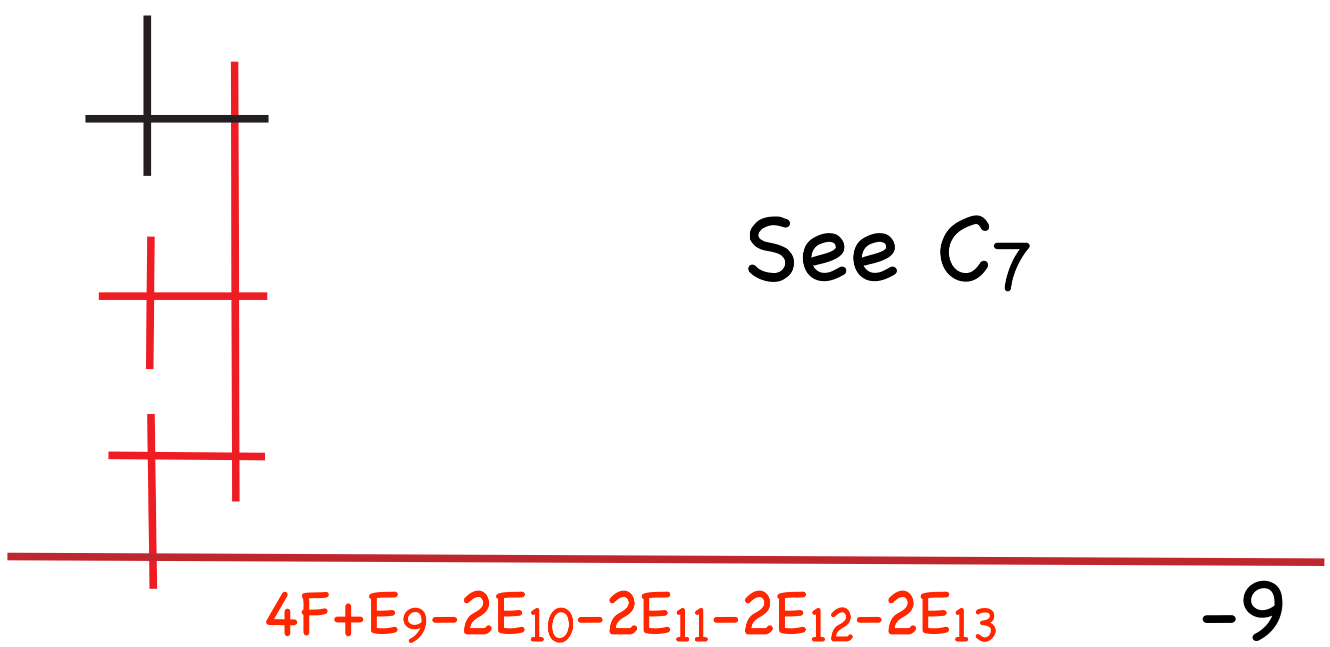


Nodal Fiber



E(1) has fibration w/ 4 nodal fibers + $\tilde{E}_6$

$E(1)\#4\mathbb{CP}^2$



See C_7

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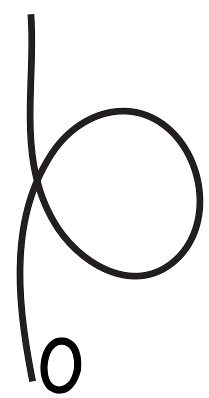

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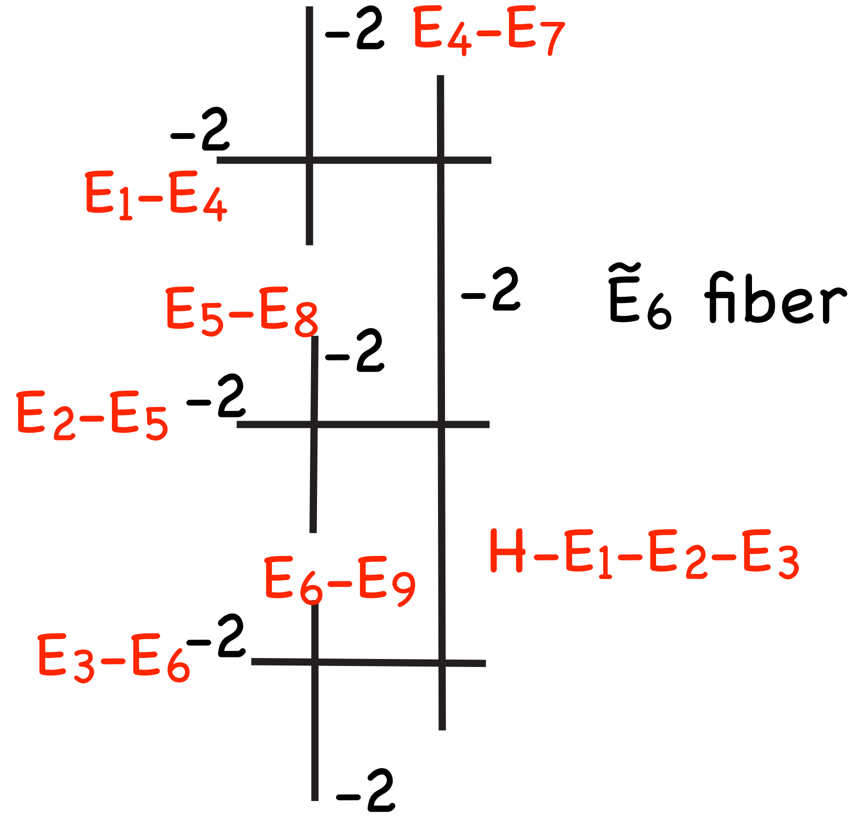

E(1) has fibration w/ 4 nodal fibers + $\tilde{E}_6$

Rationally blow down to get Park mfd P

$b^+=1$, $b^-=13-6=7$ and simply connected &

SW≠0 - an exotic $CP^2 \# 7\overline{CP}^2$

Simply Conn. 4-Mfds w/ $b^+=1$ & $b^-=5,6,7$

- P homeo to $CP^2\#7\overline{CP}^2$ (& symplectic), $SW_P=t-t^{-1}$

  (This $\Rightarrow$ minimality by blowup formula)

- (Stipsicz-Szabo) $\exists$ sympl 4-mfd homeo not diffeo to $CP^2\#6\overline{CP}^2$ (similar technique)

- (F-Stern) All these examples $b^+=1$, $b^-=6,7,8$ admit ∞'ly many smooth str's

- (J.Park-Stipsicz-Szabo) Same for $b^+=1$, $b^-=5$

- Cx Surfaces of general type (J. Park & coauthors)

- These techniques do not seem to work for $b^-<5$

Simply Conn. 4-Mfds w/ $b^+=1$ & $b^-\leq 4$

back to surgery on tori, but -

Prop. If $b^+_X=1$, $b^-_X\leq 8$, $SW_X\neq 0$, $\nexists$ essential torus of square 0 in X.

If k has a nonzero coeff in SW_X, adjunction $\neq \Rightarrow$ $k.T=0$. But $T^2=0$, $k^2\geq c_1{}^2>0$ gives contra. by Cauchy-Schwarz $\neq$.

If we could surger an essential torus (log transf or knot surgery)of square 0 in a mfd with $b^+=1$ and $b^-\leq 8$ to get $SW\neq 0$, we would be in the situation above.

So we need to work with nullhomologous tori.

The Morgan, Mrowka, Szabo Surgery Formula

$T \subset X$: torus of square 0

N_T=nbd, $\partial N_T = T^3$ basis $\alpha, \beta, \gamma = \partial D^2$ for $H_1(\partial N_T)$

$X_{\{p,q,r\}} = (X - N_T) \cup_{\varphi_{pqr}} (T^2 \times D^2)$

where $(\varphi_{pqr})_* [\partial D^2] = p\alpha + q\beta + r\gamma$

Write $SW_{X_{\{p,q,r\}}}(k_{pqr})$

for the coefficient of k_{pqr} in $SW_{X_{\{p,q,r\}}}$

M-M-Sz Formula:

$$\Sigma SW_{X_{\{p,q,r\}}}(k_{pqr}) =$$

$$p \Sigma SW_{X_{\{1,0,0\}}}(k_{100}) + q \Sigma SW_{X_{\{0,1,0\}}}(k_{010}) + r \Sigma SW_{X_{\{0,0,1\}}}(k_{001})$$

"$SW_{X_{\{p,q,r\}}} = p SW_{X_{\{1,0,0\}}} + q SW_{X_{\{0,1,0\}}} + r SW_{X_{\{0,0,1\}}}$"

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$$\text{“}SW_{X_{\{p,q,r\}}} = pSW_{X_{\{1,0,0\}}} + qSW_{X_{\{0,1,0\}}} + rSW_{X_{\{0,0,1\}}}\text{”}$$

- $X_{\{0,0,1\}} = X$, and $X_{\{1,0,0\}}$ and $X_{\{0,1,0\}}$ are results of $S^1 \times$ 0-surgeries

Using the M-M-Sz Formula

Ex: Want to construct exotic smooth str's on $CP^2\#n\overline{CP}^2$

Need to find useful nullhomologous torus T

in mfd with $b^+=1$, $b^-=n$, $\pi_1=0$

(0,k,1)-surgery = S^1x(1/k-Dehn surgery)

has same homology as X if $\beta=0$ in $H_1(X\text{-}T)$, and

$$SW_{X_{\{0,k,1\}}} \sim k\,SW_{X_{\{0,1,0\}}} + SW_{X_{\{0,0,1\}}=X}$$

0-surgery

$\Rightarrow$ If 0-surgery on T wrt correct circle has SW$\neq$0,

we get ∞'ly many distinct mfds

How to achieve this?

Surgery on Tori

(a)	(b)
$T' \subset X' \quad \alpha', \beta', \gamma' = \partial D^2$	$T \subset X \quad \alpha, \beta, \gamma = \partial D^2$
T' primitive	T nullhomologous
$\gamma' = 0$ in $H_1(X' - N_{T'})$	$\gamma \neq 0$ in $H_1(X - N_T)$
$\beta' \neq 0$ in $H_1(X' - N_{T'})$	$\beta = 0$ in $H_1(X - N_T)$

$$\xrightarrow{(0,1,1)\text{ surgery}}$$

$\beta' + \gamma' \leftrightarrow \gamma$

$\gamma' \leftrightarrow \beta$

$$\xleftarrow[(0,1,0)\text{ surgery}]{}$$

$\beta' \leftrightarrow -\beta + \gamma$

$\gamma' \leftrightarrow \beta$

$b_1(X) = b_1(X') - 1 \qquad SW_{X_{\{0,k,1\}}} = kSW_{X'} + SW_X$

Provides ∞-family in case $SW_{X'} \neq 0$

Lagrangian Surgery

X': sympl 4-mfd, T': Lagrangian torus in X'

Preferred framing for T': Lagrangian framing

$1/k$-surgeries w.r.t. this framing are again symplectic.
(Auroux, Donaldson, Katzarkov)

(Sometimes referred to as "Luttinger surgery")

This is how we can assure $SW_{X'} \neq 0$.

Reverse Engineering

Want to construct exotic smooth str's on $\hat{X}$.

(1) Find 'model mfd' M which is sympl w/ same e and sign as $\hat{X}$, but with $b_1 \neq 0$.

(2) Find b_1 disjoint Lagrangian tori in M containing gen's of H_1

(3) Perform Luttinger surgeries on these tori, killing these gen's of H_1.

(4) Result is sympl. mfd X with same e, sign as M, but with $b_1 = 0$, H_2 reduced by b_1 hyp pairs, and X contains a useful nullhomologous torus T.

(5) Get lucky, and compute $\pi_1(X) = 0$.

Get ∞ family if all $1/k$-surgeries are s.c.

$\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$

Model mfd: $M=\mathrm{Sym}^2\Sigma_3$ same e & sign as $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$

$\pi_1(M)=H_1(\Sigma_3)$ $(b_1=6)$

Has disjoint Lagrangian tori carrying basis for H_1

Six Lagr. +1-surgeries give sympl mfd X with $\pi_1(X)=0$

$SW_X \neq 0 \Rightarrow X \not\cong \mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$

Nullhomologous torus $T\subset X$ &

$1/k$-surgeries give ∞ family

Baldridge-Kirk
Akhmedov-Park

Their models constructed by cut-and-paste

How to find model manifolds

- Find appropriate Kahler surface (as in last example)
- Construct via cut-and-paste.

(Akhmedov and Park do this to construct exotic $CP^2\#2\overline{CP}^2$'s)

- Santeria Surgery - Find a useful nullhomologous torus directly in a standard mfd

(Stern and I have shown how to do this in $CP^2\#n\overline{CP}^2$ for $2\leqq n\leqq 7$.)

The Canonical Class

Rational surface: $CP^2 \# n\overline{CP}^2$, $K = -3H + E_1 + \ldots + E_n$

say $0 \leqq n < 9$, so $c_1^2 > 0$.

K not holo, <u>-K is</u>. K rep by torus

Seek exotic sympl. mfd X homeo to $CP^2 \# n\overline{CP}^2$

with K_X pseudoholo.

Adj formula would $\Rightarrow$ K_X rep. by surface of genus 10-n

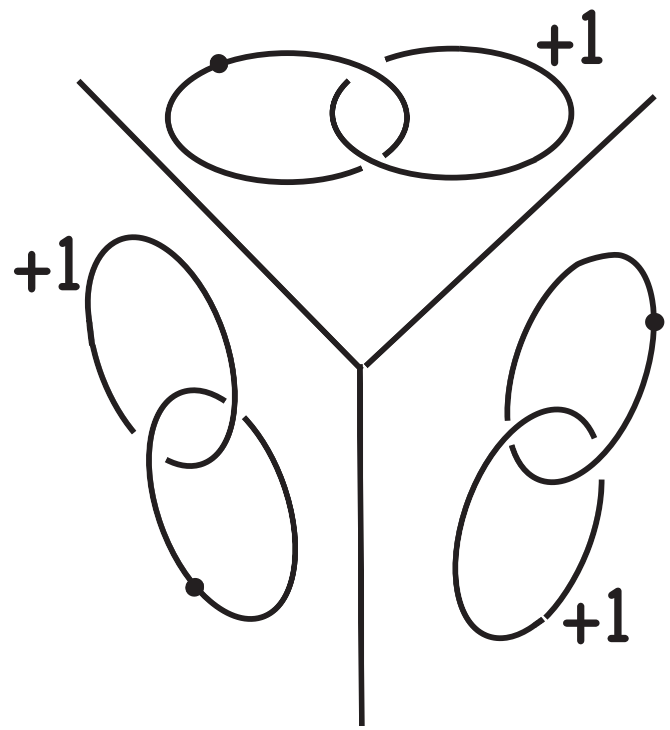
(not a torus)

Need to look for tori to surger in $CP^2 \# n\overline{CP}^2$

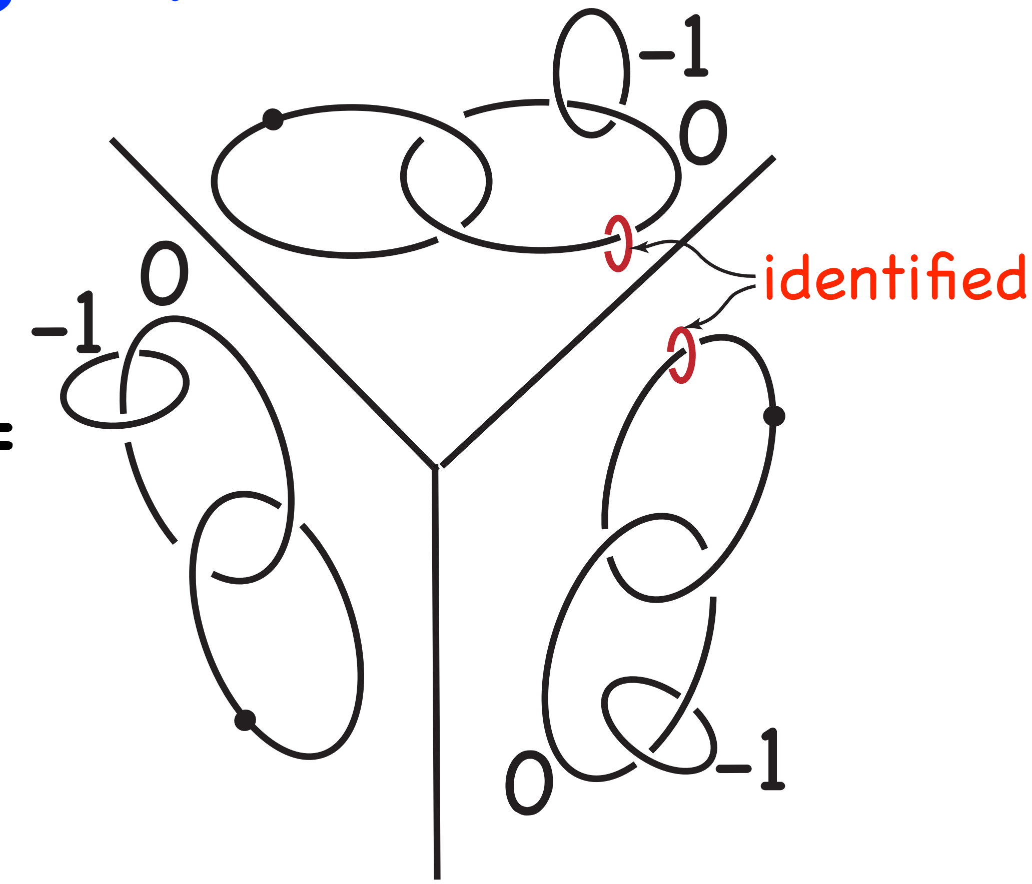
such that genus of K is "forced up"

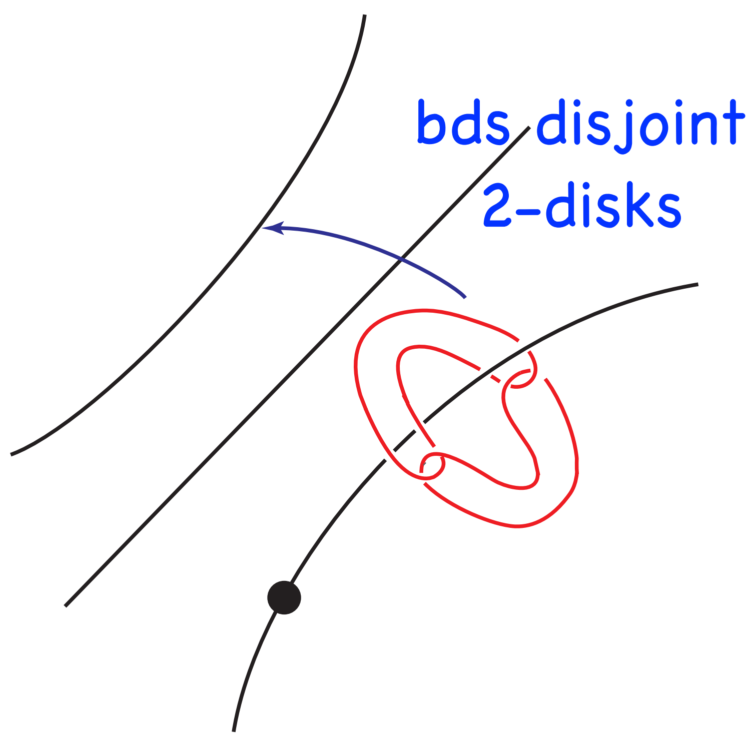
$CP^2 \# 3\overline{CP}^2$ (again)

$CP^2 =$



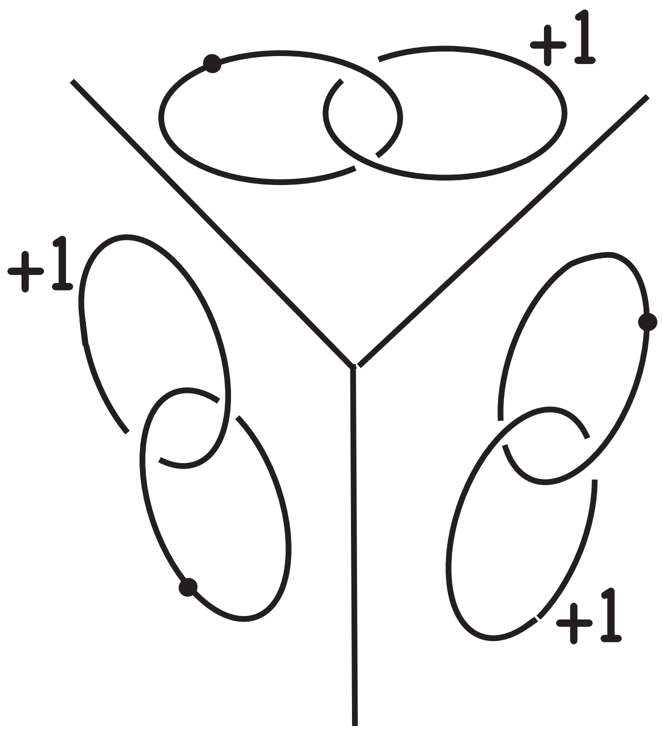
$CP^2 \#$
$3\overline{CP}^2$ $=$



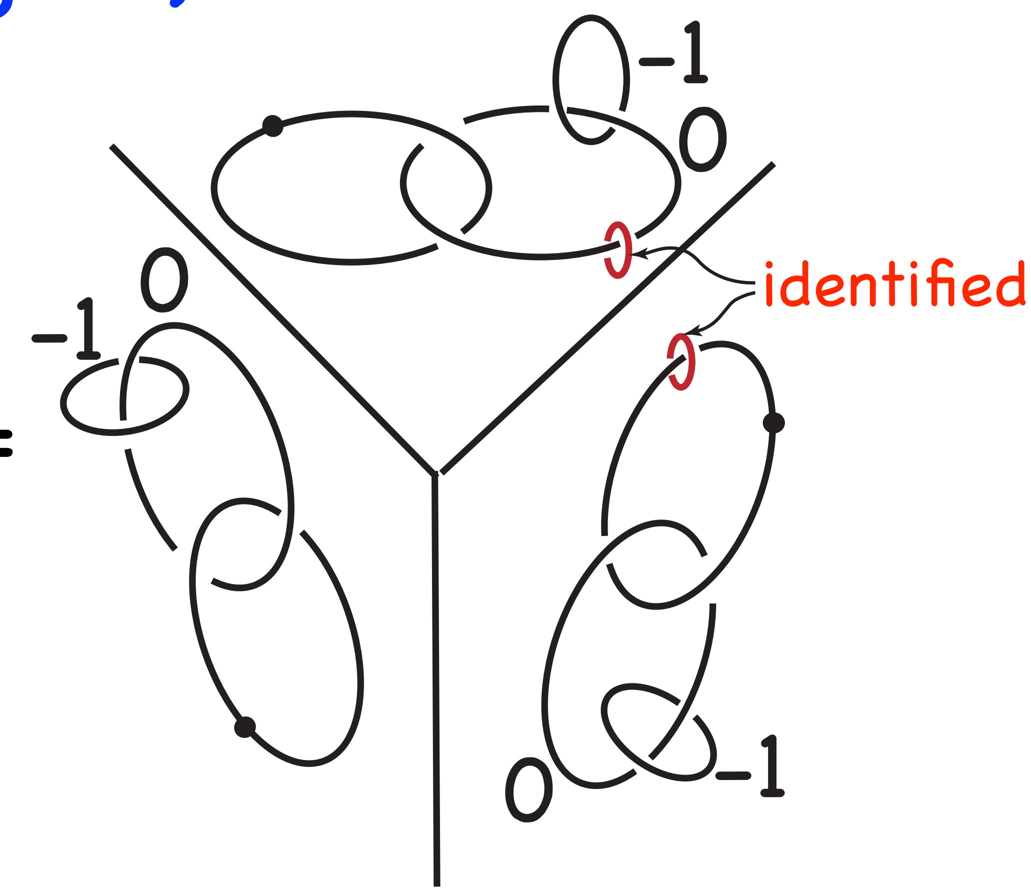


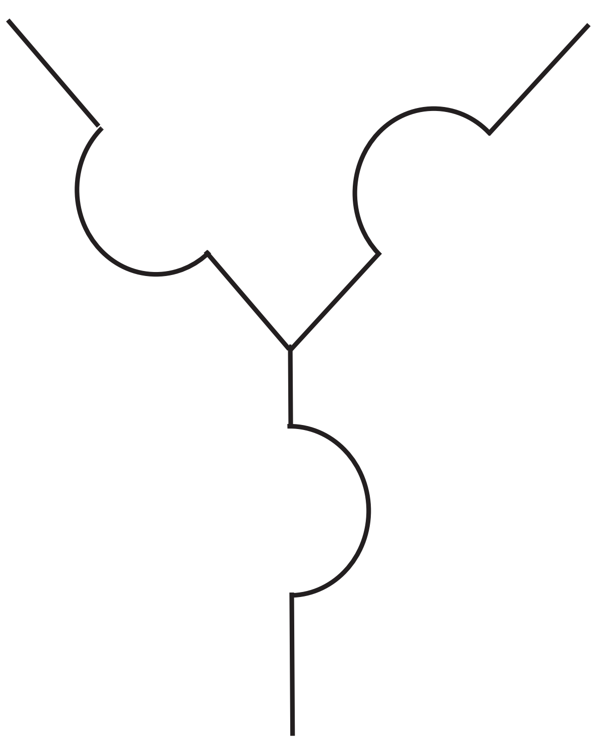
$CP^2\#3\overline{CP}^2$ (again)

CP^2 =

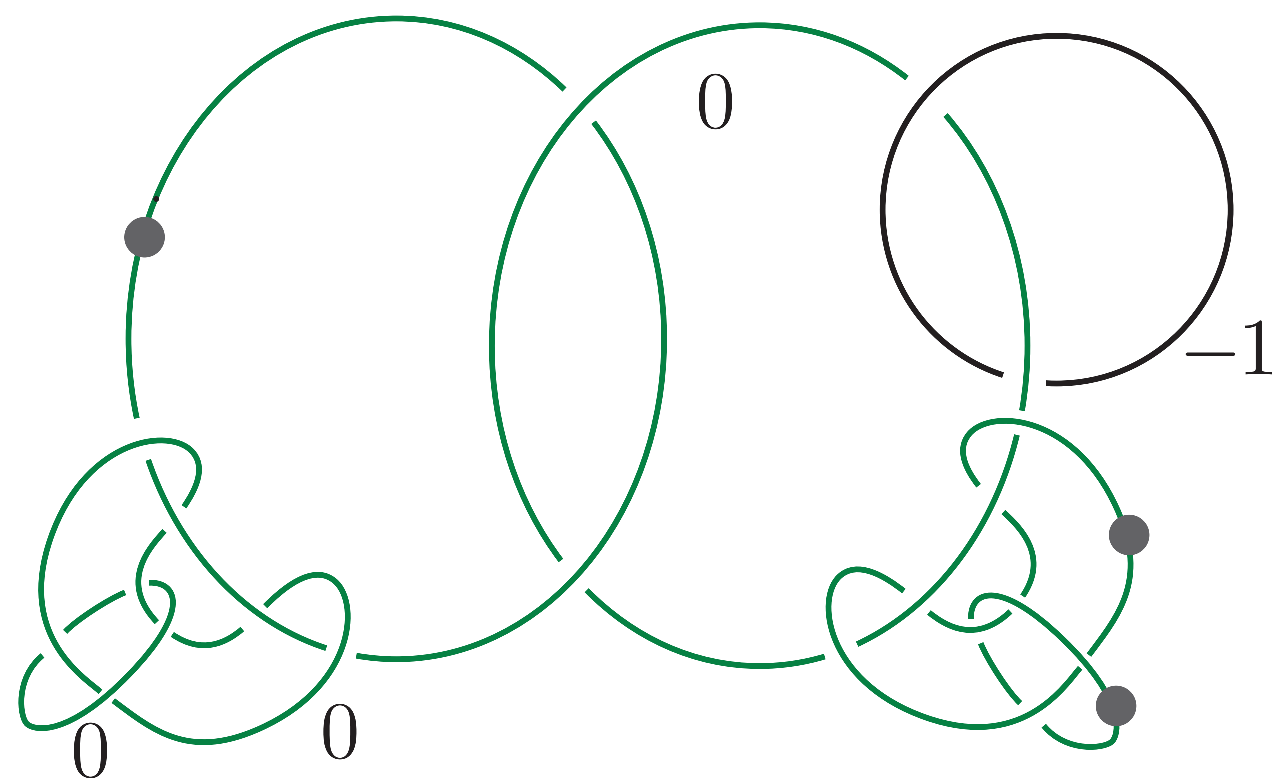


$CP^2\#$
$3\overline{CP}^2$ =



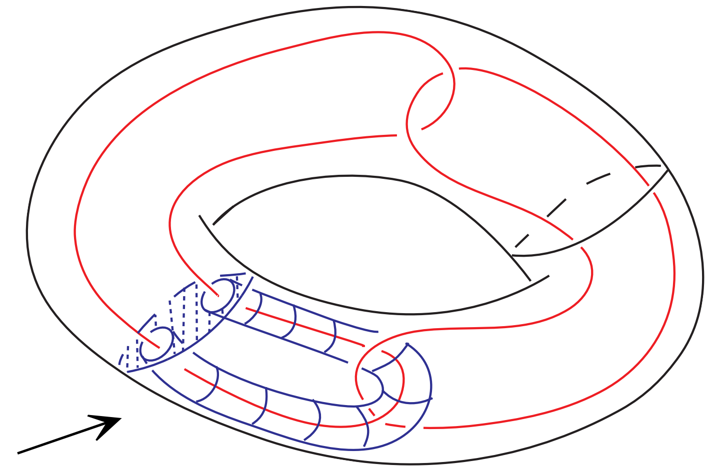


each piece



Bing Tori

A=Green part= $S^1 \times$



$= T^2 \times D^2 - pt \times T_0$

$- \, pt \times$

contains pair of "Bing tori"

$K_{CP^2 \# 3\overline{CP}^2}$ intersects this in pair of normal disks

Surgery on both Bing tori forces genus of K up by 2.

Do all 6 surgeries to get sympl mfd $\cong CP^2 \# 3\overline{CP}^2$

and SW≠0 – and genus(K)=1+6=7

- One surgery will suffice
- Similar constr of exotic $CP^2 \# 2\overline{CP}^2$

Moral: Look for useful emb's of A.

Current State of Affairs

- $\exists$ exotic smooth str's on $CP^2 \# n\overline{CP}^2$, $2 \leqq n \leqq 9$
- $\exists$ exotic smooth str's on $CP^2 \# n\overline{CP}^2$, $n \geqq 10$
  (but no examples minimal, and $c_1^2 < 0$ for these)
- Open: CP^2, $CP^2 \# \overline{CP}^2$, $S^2 \times S^2$
- $\exists$ proposed examples for $S^2 \times S^2$
  but π_1 calculation incorrect

These use reverse eng. with model $M = \Sigma_2$ bundle over Σ_2
$\Rightarrow$ M aspherical

<u>Conjecture</u>: The result of Lagrangian (i.e. Luttinger) surgery on a symplectically aspherical 4-mfd is again sympl. asph. ($\Rightarrow \pi_1$ infinite).

<u>(Very) Optimistic Conj.</u> Every s.c smooth 4-mfd can be obtained from surgery on tori in a conn. sum of copies of S^4, CP^2, $\overline{CP}^2$, and $S^2 \times S^2$.