A COMBINATION THEOREM FOR SPECIAL CUBE COMPLEXES

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Abstract. We prove that certain compact cube complexes have special finite covers. This means they have finite covers whose fundamental groups are quasiconvex subgroups of right-angled Artin groups. As a result we obtain, linearity and the separability of quasiconvex subgroups, for the groups we consider.

Our result applies in particular to compact negatively curved cube complexes whose hyperplanes don't self-intersect.

For cube complexes with word-hyperbolic fundamental group, we are able to show that they are virtually special if and only if the hyperplanes are separable.

In a final application, we show that the fundamental groups of every simple type uniform arithmetic hyperbolic manifolds are cubical and virtually special.

Contents

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1. Introduction

In this paper we give a high-dimensional generalization of the 1-dimensional work in [Wis02]. The main result there can be loosely reformulated as follows: the 2-complex built by amalgamating two graphs along a malnormal immersed graph is "virtually special". Recall that a subgroup $H \subset G$ is *malnormal* if $gHg^{-1} \cap H = \{1\}$ for each $g \notin H$, and an immersed subgraph is malnormal if its π_1 maps to a malnormal subgroup. More precisely, we consider two combinatorial graph immersions $A \leftarrow M$ and $M \rightarrow B$ and form a nonpositively curved square complex $X = A \cup_M B$ by attaching a copy of $M \times [-1, 1]$ to A and B using the maps $A \leftarrow M \times \{-1\}$ and $M \times \{1\} \rightarrow B$. The main result of [Wis02] can be reformulated as:

Proposition 1.1. *If $\pi_1 M$ is malnormal in $\pi_1 A$ and $\pi_1 B$ then $X = A \cup_M B$ is virtually special.*

From a group theoretical viewpoint, two particularly salient features of a graph Γ are that: Γ retracts onto its connected subgraphs, and that any finite immersed subgraph embeds in a finite cover of Γ. These features lead to notions of canonical completion and retraction for graphs that were studied in [Wis02]. The *special cube complexes* introduced in [HW08] are higher dimension spaces that also admit canonical completion and retraction. Simple aspects of these notions and their peculiar properties were verified and extended to higher dimensions in [HW08], and more difficult such aspects are treated in this paper. Using these, we prove the following:

Theorem 1.2. *Let A and B be compact virtually special cube complexes with word-hyperbolic π_1, and let $A \leftarrow M$ and $M \rightarrow B$ be local isometries of cube complexes such that $\pi_1 M$ is quasiconvex and malnormal in $\pi_1 A$ and $\pi_1 B$. Then the nonpositively curved cube complex $X = A \cup_M B$ is itself virtually special.*

The specificness of Proposition 1.1 is misleading, in fact, it was surprisingly widely applicable to many 2-dimensional groups appearing in combinatorial group theory. We expect Theorem 1.2 to be even more powerful and dynamic, both because it can reach higher-dimensional groups, but also because sometimes 2-dimensional groups are not fundamental groups of 2-dimensional special cube complexes, but require the greater flexibility of higher-dimensions.

A $\mathcal{VH}$*-complex* X is (up to double cover) a square 2-complex with the property that the link of each vertex is bipartite. Furthermore, X is *n*egatively curved if each link has girth ≥ 5. An immediate application of Proposition 1.1 is that negatively curved

$\mathcal{VH}$-complexes are virtually special. In higher dimensions, this naturally extends to negatively curved "foldable complexes" as defined in [BŚ99]. The cube complex X is *foldable* if there is a non degenerate combinatorial map $X \to Q$ onto some cube Q. X is *negatively curved* if the link of each vertex is a flag complex, and moreover, any 4-cycle in the link bounds the union of two 2-simplices.

Theorem 1.3. *Let C be a compact negatively curved foldable cube complex. Then C has a finite cover $\widehat{C}$ such that $\widehat{C}$ is a special cube complex.*

The work in [Wis02] led to connections between negative curvature and residual finiteness. Our paper extends this connection considerably, since it is now understood that surprisingly many of the groups studied in combinatorial group theory, actually act properly and cocompactly on CAT(0) cube complexes, and are thus approachable through these results.

Theorem 1.4. *Let C be a compact nonpositively curved cube complex, and suppose that $\pi_1 C$ is word-hyperbolic. Then C is virtually special if and only if $\pi_1 D$ is separable in $\pi_1 C$ for each immersed hyperplane $D \to C$.*

Using the already known properties of virtually special cube complexes we get the following corollary:

Theorem 1.5. *Let C be a compact nonpositively curved cube complex. Suppose that $\pi_1 C$ is word-hyperbolic and that $\pi_1 D$ is separable in $\pi_1 C$ for each immersed hyperplane $D \to C$. Then $\pi_1 C \subset GL(n, \mathbb{Z})$ for some large n, and every quasiconvex subgroup of $\pi_1 C$ is separable.*

In a final application we apply Theorem 1.4 to obtain an interesting structural result about certain arithmetic hyperbolic lattices:

Theorem 1.6. *Let G be a uniform arithmetic hyperbolic lattice of "simple type". Then G has a finite index subgroup F that is the fundamental group of a compact special cube complex.*

Hidden inside Theorem 1.6 is the claim that every such lattice acts properly and cocompactly on a CAT(0) cube complex, which was not known previously. But combining with results from [HW08], we obtain the following subgroup separability consequence:

Corollary 1.7. *Let G be a uniform arithmetic hyperbolic lattice of "simple type", then every quasiconvex subgroup of G is a virtual retract, and is hence closed in the profinite topology.*

Partial results were made towards separability of arithmetic hyperbolic lattices in low dimensions in [ALR01]. Their method is similar to ours in that they virtually embed such groups into right-angled hyperbolic Coxeter groups. The results have been substantially extended by Ian Agol to deal with various lattices in up to 11 dimensions which satisfy an orthogonality condition on certain of their hyperplanes [Ago06]. Perhaps the paucity of hyperbolic reflection groups has limited the scope of Scott's method.

The application towards uniform arithmetic lattices was not the original intention of this research, but it emerged as a consequence of our combination theorem. This application does not require the full strength of our main theorem, and in a future paper, using a method more specific to the situation, we will give an account of the virtual specialness of nonuniform simple arithmetic hyperbolic lattices.

As an application of the cubulation of uniform lattices of the real hyperbolic space we get the following result:

Theorem 1.8. *Every word-hyperbolic group is quasi-isometric to a uniformly locally finite CAT(0) cube complex.*

Proof. Let Γ be a word-hyperbolic group. Then by the work of Bonk and Schramm (see [BS00]) there exists a quasi-isometric embedding $\Gamma \to \mathbb{H}^n$ for n large enough. Consider any standard arithmetic uniform lattice G of H^n we let G act freely cocompactly on a locally finite CAT(0) cube complex X. Then $\mathbb{H}^n$ is quasi-isometric to G, which is quasi-isometric to X.

We thus get a quasi-isometric embedding of Γ into X. The image of Γ in X is a quasiconvex subset $Y \subset X$. Now X is a uniformly locally finite, Gromov-hyperbolic CAT(0) cube complex and $Y \subset X$ is quasiconvex. Thus by Theorem 4.2 the combinatorial convex hull Z of Y inside X stays at finite Hausdorff distance of Y, and we are done.

□

It is not very difficult to prove that a Gromov-hyperbolic uniformly locally finite CAT(0) cube complex embeds in a product of finitely many trees (see for example [Hag07]), and the embedding is isometric on the 1-skeleton equipped with the combinatorial distance. Thus as a corollary of Theorem 1.8 we see that every word-hyperbolic group has a quasi-isometric embedding in a product of finitely many trees, a result which was first proved by Buyalo, and Schroeder (see [BS05]).

2. Special Cube Complexes

2.A. **Nonpositively curved cube complex.**

Definition 2.1. A 0-cube is a single point. A 1-*cube* is a copy of $[-1,1]$, and has a cell structure consisting of 0-cells $\{\pm 1\}$ and a single 1-cell.

An n-*cube* is a copy of $[-1,1]^n$, and has the product cell structure, so each closed cell of $[-1,1]$ is obtained by restricting some of the coordinates to $+1$ and some to -1.

A *cube complex* is obtained from a collection of cubes of various dimensions by identifying certain subcubes. We shall often call 0-cubes *vertices* and 1-cubes *edges*. A map between cube complexes is *combinatorial* if it sends homeomorphically open cubes to open cubes.

A *flag complex* is a simplicial complex with the property that every finite set of pairwise adjacent vertices spans a simplex.

Let X be a cube complex. The *link* of a vertex v in X is a complex built from simplices corresponding to the corners of cubes adjacent to v. One can think of link(v) as being the "ϵ-sphere" about v in X.

The cube complex X is *nonpositively curved* if link(v) is a flag complex for each $v \in X^0$. A (finite dimensional) simply-connected nonpositively curved cube complex has a CAT(0) metric in which each n-cube is isometric to the subspace $[-1,1]^n \subset \mathbb{E}^n$ (see [Gro87]). We thus refer to such cube complexes as CAT(0) cube complexes. Similarly, a (finite dimensional) nonpositively curved cube complex admits a locally CAT(0) metric, and hence the choice of terminology for the combinatorial flag complex condition.

A combinatorial map $f : X \to Y$ of nonpositively curved cube complexes is a *local isometry* if $f(\text{link}(v, X))$ is a full subcomplex of $\text{link}(f(v), Y)$ for each $v \in X^0$.

2.B. **Hyperplanes.**

Definition 2.2. A *hypercube* D in a cube C is the subspace obtained by restricting exactly one coordinate to 0. For instance $[-1,1] \times [-1,1] \times \{0\} \times [-1,1]$ is a hypercube in $[-1,1]^4$.

Given a cube complex X, consider the disjoint union of all hypercubes of X, and let Y be the cube complex obtained by identifying lower dimensional hypercubes which are subspaces of higher dimensional hypercubes. The connected components of Y are *hyperplanes of* X.

It is not difficult to check that when X is nonpositively curved, then each hyperplane of X is nonpositively curved. Moreover, using the metrics of nonpositive curvature, the natural map $H \to X$ is a local-isometry. When this local-isometry is an embedding we say that H embeds.

When X is simply-connected, each hyperplane of X is simply-connected, and embeds.

2.C. **Right angled Artin groups.**

Definition 2.3. The *right-angled Artin group* presentation associated to a simplicial graph Γ is defined to be:

$$\left\langle\ a\ :\ a \in \mathrm{V}(\Gamma)\ \mid\ [a,b]\ :\ \{a,b\} \in \mathrm{E}(\Gamma)\ \right\rangle$$

The associated right-angled Artin group will be denoted by $A = A(\Gamma)$.

For each cluster of n pairwise adjacent vertices in Γ, we add an n-cube to the standard 2-complex of the presentation above, to obtain a nonpositively curved cube complex, which we shall denote by $R = R(\Gamma)$.

Specifically, for each vertex a of $\mathrm{V}(\Gamma)$ let S_a denote a graph with a single vertex and a single edge. Then R is the subcomplex of the combinatorial torus $\prod_{a \in \mathrm{V}(\Gamma)} S_a$, union of all subtori corresponding to complete subgraphs of Γ. In order to get an isomorphism $A \to \pi_1 R$ we just need to choose an orientation for the edges of the circles S_a.

We call any such complex R an *Artin cube complex.*

2.D. **Special cube complex.**

Definition 2.4. A *special cube complex* is a cube complex C such that there exists a combinatorial local-isometry $C \to R$ where $R = R(\Gamma)$ for some simplical graph Γ.

We give below an intrinsic combinatorial characterization of the special property. We first introduce the adapted notations and definitions.

We denote with an arrow the oriented edges of a cube complex. The (unoriented) edge associated with an oriented edge $\overrightarrow{a}$ will always be denoted by a. We denote by $\iota(\overrightarrow{a})$ and $\tau(\overrightarrow{a})$ the initial vertex and terminal vertex of the oriented edge $\overrightarrow{a}$.

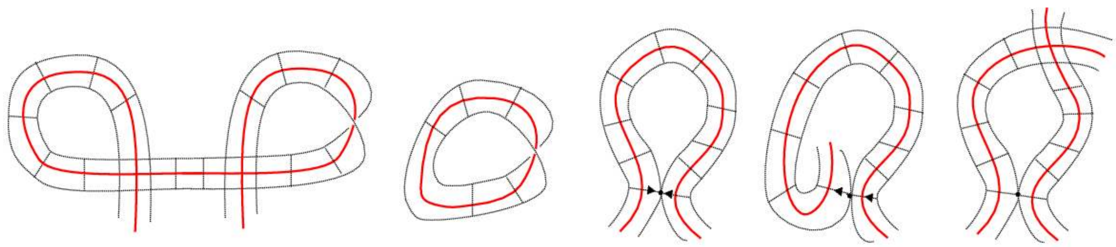

FIGURE 1. From left to right, the diagrams above correspond to the pathologies enumerated in Lemma 2.6. The third and fourth diagrams illustrate direct self-osculation and indirect self-osculation.

Definition 2.5 (parallelism)**.** Two oriented edges of a euclidean unit square are *parallel* if the associated unit vectors are the same. Then the parallelism of oriented edges in a cube complex C is the equivalence relation generated by parallelism inside a square. By forgetting orientation we also get a parallelism relation on (unoriented) edges of C. Note that two edges of C are parallel iff their midpoints belong to the same hyperplane.

An embedded hyperplane Y in X is *2-sided* if its open cubical neighborhood is isomorphic to the product $Y \times (-1,1)$, where we identify Y with $Y \times \{0\}$. For each $y \in Y^0$, the 1-cell in X whose open 1-cell corresponds to $y \times (-1,1)$ is *dual* to Y. An oriented edge $\overrightarrow{a}$ is *dual* to Y if a is dual to Y. When Y is 2-sided, the projection map $Y \times (-1,1) \to (-1,1)$ allows us to choose an orientation on each 1-cell dual to Y, so that all corresponding oriented edges are parallel. In other words no oriented edge dual to Y is parallel to its opposite edge.

The 2-sided hyperplane Y *directly self-osculates* if there are distinct oriented dual edges $\overrightarrow{a}$ and $\overrightarrow{b}$ such that $\iota(\overrightarrow{a}) = \iota(\overrightarrow{b})$. The hyperplane Y *self-osculates* if there are distinct dual edges a and b that share a vertex. Thus self-osculation of 2-sided hyperplanes consists of direct self-osculation, and also indirect self-osculation, where there are distinct oriented dual edges $\overrightarrow{a}$ and $\overrightarrow{b}$ such that $\iota(\overrightarrow{a}) = \tau(\overrightarrow{b})$.

Consider two distinct oriented edges $\overrightarrow{a}, \overrightarrow{b}$ with origin a given vertex v. Identify $\overrightarrow{a}, \overrightarrow{b}$ with vertices of $\text{link}(v)$. When $\overrightarrow{a}, \overrightarrow{b}$ span an edge of $\text{link}(v)$ we say that $\overrightarrow{a}, \overrightarrow{b}$ are *perpendicular* at v. When $\overrightarrow{a}, \overrightarrow{b}$ are not joined in $\text{link}(v)$ we say that $\overrightarrow{a}, \overrightarrow{b}$ *osculate* at v. We say that two edges a, b are perpendicular (or osculate) when there are orientations $\overrightarrow{a}, \overrightarrow{b}$ such that $\overrightarrow{a}, \overrightarrow{b}$ are perpendicular (or osculate) at some vertex.

Two hyperplanes A and B *inter-osculate* if they have perpendicular dual edges a, b that are parallel to osculating edges a', b'.

We refer the reader to Figure 1 for some simplistic diagrams suggesting these pathologies:

Lemma 2.6. *A nonpositively curved cube complex is special if and only if:*

(S_1) *Each hyperplane embeds.*
(S_2) *Each hyperplane is two-sided.*
(S_3) *No hyperplane directly self-osculates.*

(S_4) *No two hyperplanes interosculate.*

Sketch. For a nonpositively curved cube complex C satisfying conditions $(S_1),(S_2),(S_3),(S_4)$, let $V = V_C$ denote the set of hyperplanes of C. Let $\Gamma = \Gamma_C$ denote the graph on V such that vertices are adjacent if and only if the corresponding hyperplanes cross (i.e. intersect).

Let $R = R(\Gamma)$, which we shall also denote later by $R = R(C)$. Choose arbitrary orientations for the edges of R. Then there is a map $C \to R$ induced by sending an oriented edge $\overrightarrow{a}$ of C to the oriented edge of R corresponding to the hyperplane $\overrightarrow{a}$ is dual to. It is not difficult to verify that conditions $(S_1),(S_2),(S_3),(S_4)$ imply that this map is a local-isometry.

Conversely, if $C \to D$ is a local-isometry of nonpositively curved cube complexes, then it is easy to verify that if D satisfying conditions $(S_1),(S_2),(S_3),(S_4)$ then C does. The Lemma follows since it is easy to check that an Artin cube complex $R(\Gamma)$ always satisfies conditions $(S_1),(S_2),(S_3),(S_4)$. □

In the sequel, given any special cube complex C we will denote by $R = R(C)$ the Artin complex $R = R(\Gamma)$, where $\Gamma = \Gamma_C$ denotes the graph on the set $V = V_C$ of hyperplanes in C that we considered in the first part of the proof above.

3. Canonical Completion and Retraction and Wall Projections

3.A. **Canonical Completion and Retraction.** In this section we recall how to factorize some combinatorial immersions $X \to Y$ as an inclusion composed with a covering map (see [Sta83]). The key point here is to give a canonical construction, that can be used as an elementary machinery in more elaborate constructions. Due to its naturality, the construction will enjoy many nice formal properties.

The possibility of lifting an immersion $X \to Y$ to an inclusion $X \to Y'$ in a finite cover $Y' \to Y$ is related to the separability of $\pi_1 X < \pi_1 Y$. In a residually finite group, any virtual retract is separable (in a very strong way). We show below that in the context of special cube complexes it is possible to lift a local isometry $X \to Y$ to an inclusion $X \to \mathsf{C}(X,Y)$, where $\mathsf{C}(X,Y) \to Y$ is a "canonically" defined covering, *and furthermore* $\mathsf{C}(X,Y)$ *"canonically" retracts to* X. This construction has been made for graph immersions in [Wis02] and generalized to arbitrary local isometries of special cube complexes in [HW08]. We first recall the case of graphs.

Definition 3.1 (canonical completion and retraction for immersions of graphs in a bouquet)**.** Let $A \to b$ be an immersion of graphs where b consists of a single loop, and A is finite. Each component of A is either a cover of b, or can be completed to a cover of b by the addition of a single edge. We define $\mathsf{C}(A,b) \to b$ to be the resulting covering space. Note that $A \subset \mathsf{C}(A,b)$ and that there is a retraction map $\mathsf{C}(A,b) \to A$ defined by sending each new open edge to the component of A it was attached along.

Let $A \to B$ be an immersion of graphs where B is a bouquet of circles, and A is finite. For each loop b in B, let A_b denote the preimage of b in A. We define $\mathsf{C}(A,B)$ to be the union of $\mathsf{C}(A_b,b)$ amalgamated along A^0. Then the induced map $\mathsf{C}(A,B) \to B$ is a covering. Note that $A \subset \mathsf{C}(A,B)$ and that the retraction maps $\mathsf{C}(A_b,b) \to A_b$ induce a retraction map $\mathsf{C}(A,B) \to A$. In other words we have completed A to a (finite) covering of B, that furthermore retracts onto A.

We also note that the retraction map $\mathsf{C}(A,B) \to A$ is cellular (it sends cells to cells), unless there is some loop b such that A_b contains a component isomorphic to a linear graph with ≥ 3 vertices.

Definition 3.2 (canonical completion and retraction for local isometries of cube complexes in an Artin complex)**.** Let $X \to R$ be a local isometry of cube complexes where R is an Artin cube complex and X is finite. We have already defined canonical completions and retractions of graphs to obtain: $X^1 \leftarrow \mathsf{C}(X^1, R^1) \to R^1$. Using the local isometry assumption, a case by case inspection shows that the boundary of a square in R always lifts to a closed curve in $\mathsf{C}(X^1, R^1)$. Thus we can extend the previous covering $\mathsf{C}(X^1, R^1) \to R^1$ to a covering map of square complexes $\mathsf{C}(X,R)^2 \to R^2$. Now the 2-skeleton of higher dimensional cubes of R lift immediately to $\mathsf{C}(X,R)^2$, and we attach the corresponding cubes. The resulting space $\mathsf{C}(X,R)$ covers R and contains X.

Furthermore the retraction map $X^1 \leftarrow \mathsf{C}(X^1, R^1)$ extends to a (non necessarily combinatorial) retraction $X^2 \leftarrow \mathsf{C}(X,R)^2$ and thus to a retraction $X \leftarrow \mathsf{C}(X,R)$ by asphericity of X.

Note that when X and R are compact then so is $\mathsf{C}(X,R)$.

Definition 3.3 (fiber product)**.** Given a pair of combinatorial maps $X \to W$ and $Y \to W$ (between cube complexes), we define their *fiber product* $X \otimes_W Y$ to be a cube complex, whose i-cubes are pairs of i-cubes in X, Y that map to the same i-cube in W. There is a commutative diagram:

$$\begin{array}{ccc} X \otimes_W Y & \to & Y \\ \downarrow & & \downarrow \\ X & \to & W \end{array}$$

Note that $X \otimes_W Y$ is the complex in $X \times Y$ which is the preimage of the diagonal $D \subset W \times W$ under the map $X \times Y \to W \times W$. Recall that D has a natural structure of cube complex since for any cube Q, the diagonal of Q^2 is isomorphic to Q by either of the projections.

We will use several times the universal property of $X \otimes_W Y$, which is that any commutative diagram:

$$\begin{array}{ccc} C & \to & Y \\ \downarrow & & \downarrow \\ X & \to & W \end{array}$$

is the pull-back under some combinatorial map $C \to X \otimes_W Y$ of the diagram with $X \otimes_W Y$ (which is thus minimal).

Definition 3.4 (canonical completion and retraction of local isometries)**.** Let $A \to B$ be a local isometry of cube complexes, where A is finite, B is special, and let $R = R(B)$. By composition we get a local isometry $A \to R$, and we already know how to complete A to a cover $\mathsf{C}(A,R) \to R$. Since we have a local isometry $B \to R$ we just induce on B the cover $\mathsf{C}(A,R) \to R$, so that $A \to B$ will lift to an inclusion. In order to investigate the formal properties of this completion we rather use the fiber product language.

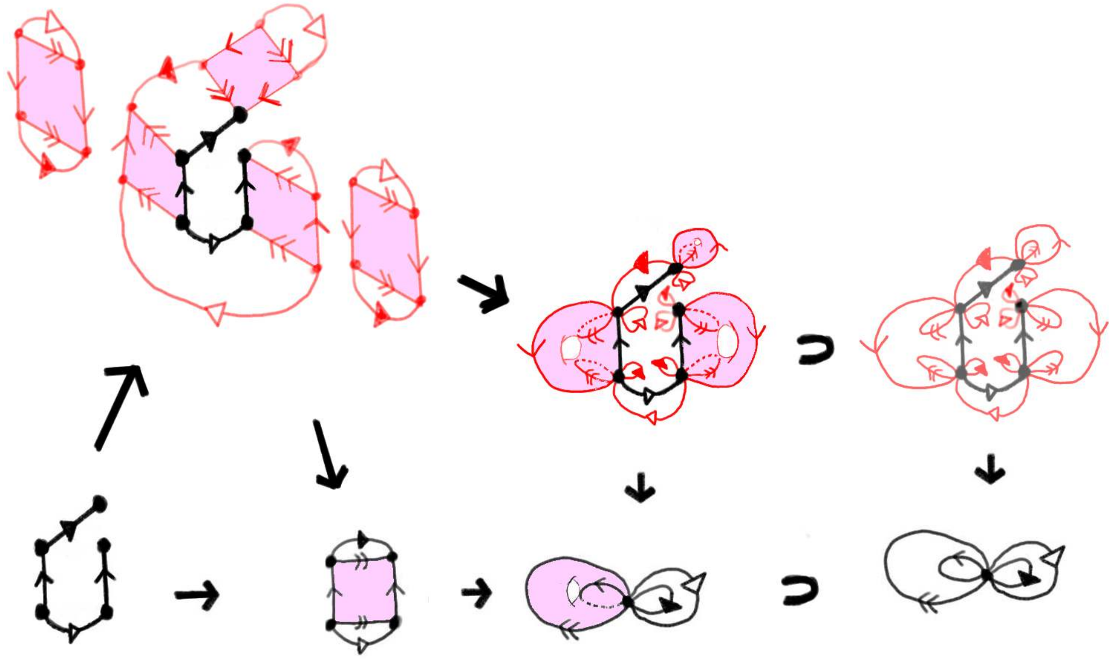

FIGURE 2. The figure corresponds to the following commutative diagram:

$$\begin{array}{ccccccc} & & \mathsf{C}(A,B) & \to & \mathsf{C}(A,R) & \supset & \mathsf{C}(A^1,R^1) \\ & \nearrow & \downarrow & & \downarrow & & \downarrow \\ A & \to & B & \to & R & \supset & R^1 \end{array}$$

We thus set $\mathsf{C}(A,B) := B \otimes_R \mathsf{C}(A,R)$. The local isometry $A \to B$ and the inclusion map coincide on R, thus define an inclusion $A \to \mathsf{C}(A,B)$. The map $\mathsf{C}(A,B) \to B$ is a covering because $\mathsf{C}(A,R) \to R$ is a covering. We define the canonical retraction map $\mathsf{C}(A,B) \to A$ to be the composition $\mathsf{C}(A,B) \to \mathsf{C}(A,R) \to A$.

We will now explain when it is possible to choose among all possible retractions a more combinatorial, uniquely defined one.

Definition 3.5 (projection-like map)**.** A map $f : X \to Y$ between cube complexes is *projection-like* provided for each cube Q of X there is a face $Q' < Q$ such that f is combinatorial on Q', and for any $x \in Q$ we have $f(x) = f(x')$ where x' is the orthogonal projection of x onto Q'.

Remark 3.6. Being projection-like is detected on the 2-skeleton. Indeed if $f : X \to Y$ is projection-like then so is $f : X^2 \to Y$. Conversely, if Y is nonpositively curved, then any projection-like map $f : X^2 \to Y$ extends to a unique projection-like map $f : X \to Y$.

Note also that projection-like maps preserve parallelism. More precisely, let $f : X \to Y$ be projection-like, and let a, b denote parallel edges of X. If $f(a)$ is an edge, then so is $f(b)$, and furthermore $f(a), f(b)$ are parallel edges of Y.

Lemma 3.7 (projection-like retraction)**.** *Let $A \to B$ be a local isometry of cube complexes, where A is finite, B is special, and let $R = R(B)$. Assume that B has no indirect self-osculation. Then the retraction map $A^1 \leftarrow \mathsf{C}(A^1, R^1)$ is cellular, and it extends to a unique projection-like retraction map $A \leftarrow \mathsf{C}(A, R)$. The induced retraction map $A \leftarrow \mathsf{C}(A, B)$ is also projection-like.*

Proof. Since B has no indirect self-osculation, the components of the preimage of a loop b in R under $A \to R$ consists of vertices or edges. It follows that the retraction map $A^1 \leftarrow \mathsf{C}(A^1, R^1)$ is cellular. A case by case inspection shows that the attaching map of squares in $\mathsf{C}(A, R)$ retract on A in a projection-like manner. Thus the retraction $A^2 \leftarrow \mathsf{C}(A, R)^2$ is projection-like, and by Remark 3.6 it extends to a unique projection-like retraction map $A \leftarrow \mathsf{C}(A, R)$. It follows that $A \leftarrow \mathsf{C}(A, B)$ is also projection-like. □

Definition 3.8 (directly special cube complexes)**.** A cube complex B is *directly special* if B is special and has no indirect self-osculation.

Note that the first cubical subdivision of a special cube complex is always directly special. And any compact special cube complex has a directly special finite cover.

Lemma 3.9 (retraction of walls)**.** *Let B be a directly special cube complex and let $A \to B$ be a local isometry. Then any edge e' of $\mathsf{C}(A, B)$ parallel with an edge e of A is retracted onto an edge e'' of A which is parallel to e.*

Proof. By Remark 3.6, projection-like maps preserve parallelism.

□

Definition 3.10. A combinatorial map $D \to C$ of (special) cube complexes always sends a hyperplane inside a well defined-hyperplane, thus induces a map $V_D \to V_C$ between the set of hyperplanes. We say that the map $D \to C$ is *wall-injective* if the map $V_D \to V_C$ is injective.

As an immediate consequence of Lemma 3.9 we get:

Corollary 3.11 (wall-injective in completion)**.** *Let C be a directly special cube complex and let $D \to C$ be a local isometry. Then D is wall-injective in $\mathsf{C}(D, C)$.*

Lemma 3.12. *Suppose $D \to C$ is a wall-injective local-isometric embedding of cube complexes with D finite and C special. Then there is a natural embedding of $\mathsf{C}(D, D)$ in $\mathsf{C}(D, C)$ which is consistent with the inclusion, retraction and covering maps, so that we have the following diagram:*

$$\begin{array}{ccc} D & = & D \\ \updownarrow & & \updownarrow \\ \mathsf{C}(D,D) & \subset & \mathsf{C}(D,C) \\ \downarrow & & \downarrow \\ D & \to & C \end{array}$$

Proof. The wall-injective local isometry $D \to C$ induces an embedding $V_D \to V_C$, and thus a combinatorial embedding $R(D) \subset R(C)$ (not necessarily a local isometry).

We first check that there is a well-defined map $\mathsf{C}(D, R(D)) \to \mathsf{C}(D, R(C))$, and start at the level of 1-skeleta.

First D^1 is a common subgraph of both $\mathsf{C}(D^1, R(D)^1)$ and $\mathsf{C}(D^1, R(C)^1)$. Let a be an edge of $\mathsf{C}(D^1, R(D)^1)$ not contained in D^1, and let b denote the image of a inside $R(D)^1$. We let D_b denote the subgraph of D^1 such that $D_b \cup a$ is a circle. Then b is also an edge of $R(C)$, and D_b is also a connected component of the preimage of b under $D \to R_C$. Thus by construction there is a unique edge a' in $\mathsf{C}(D^1, R(C)^1)$ such that $D_b \cup a'$ is a circle. We then map a to a' by the unique homeomorphism compatible with $a \to b$ and $a' \to b$.

We have now extended $D^1 \subset \mathsf{C}(D^1, R(C)^1)$ to a cellular map $\mathsf{C}(D^1, R(D)^1) \to \mathsf{C}(D^1, R(C)^1)$, which by construction is compatible with retractions onto D^1, and such that the following diagram commutes :

$$\begin{array}{ccc} \mathsf{C}(D^1, R(D)^1) & \to & \mathsf{C}(D^1, R(C)^1) \\ \downarrow & & \downarrow \\ R(D)^1 & \subset & R(C)^1 \end{array}$$

The map $\mathsf{C}(D^1, R(D)^1) \to \mathsf{C}(D^1, R(C)^1)$ is injective on the 0-skeleton and it is locally injective, thus it is injective. It sends the boundary of a square of $\mathsf{C}(D, R(D))$ onto the boundary of a square of $\mathsf{C}(D, R(C))$, by the very definition of the squares in canonical completion. The same happens for higher dimensional cubes. Thus we get an injective combinatorial map $\mathsf{C}(D, R(D)) \to \mathsf{C}(D, R(C))$, compatible with retractions onto D, and such that the following diagram commutes :

$$\begin{array}{ccc} \mathsf{C}(D, R(D)) & \subset & \mathsf{C}(D, R(C)) \\ \downarrow & & \downarrow \\ R(D) & \subset & R(C) \end{array}$$

Now the composition maps $\mathsf{C}(D, D) \to D \to C$, $\mathsf{C}(D, D) \to \mathsf{C}(D, R(D)) \to \mathsf{C}(D, R(C))$ are compatible with the projections onto C. Thus we get a map $\mathsf{C}(D, D) \to \mathsf{C}(D, C)$. Injectivity follows from the injectivity of the maps $\mathsf{C}(D, D) \to D \times \mathsf{C}(D, R(D))$, $D \to C$ and $\mathsf{C}(D, R(D)) \to \mathsf{C}(D, R(C))$. Chasing around diagrams, one easily checks that this map has the other desired properties. □

Lemma 3.13. *Let C be a special cube complex, and let $D \subset C$ be a wall-injective locally convex subcomplex.*

Then the preimage of D in $\mathsf{C}(D, C)$ is (isomorphic to) $\mathsf{C}(D, D)$.

Proof. By definition the preimage of D in $\mathsf{C}(D, C)$ is $D \otimes_{R(C)} \mathsf{C}(D, R(C))$. The image of a cube $Q \subset D$ in $R(C)$ is in fact a cube of the subcomplex $R(D)$. It follows that $D \otimes_{R(C)} \mathsf{C}(D, R(C)) = D \otimes_{R(D)} \mathsf{C}(D, R(D)) = \mathsf{C}(D, D)$. □

3.B. **Wall projections.** We now study the notion of a wall projection of one subcomplex onto another. This will play the role of the intersection between subgraphs of a graph.

Definition 3.14 (parallel cubes and wall-projection)**.** Let X denote a cube complex. Recall that 1-cubes a, b are *parallel* in X provided they are dual to the same immersed hyperplane.

Let A and B be subcomplexes of X. We define $\mathsf{WProj}_X(A \to B)$, the *wall projection of A onto B in X*, to equal the union of B^0 together with all cubes of B whose 1-cubes are all parallel to 1-cubes of A.

FIGURE 3. The above examples illustrate a length 3 interval A whose wall projection onto B is all of B. On the left B is circle and on the right it is a 3-cube. The reader can show that for any B, there is a cube complex X containing B and a subcomplex $A \cong I$, such that $\mathsf{WProj}_X(A \to B) = B$.

We say the wall projection $\mathsf{WProj}_X(A \to B)$ is *trivial* when any closed loop of $\mathsf{WProj}_X(A \to B)$ is homotopically trivial inside X.

Remark 3.15 (locally convex wall projection)**.** Assume that B is a locally convex subcomplex of a nonpositively curved cube complex X. Let A be any subcomplex. Then $\mathsf{WProj}_X(A \to B)$ is locally convex. Indeed let Q denote a cube of B, and let v be a vertex of Q, then by definition $Q \subset \mathsf{WProj}_X(A \to B)$ iff each edge of Q at v belongs to $\mathsf{WProj}_X(A \to B)$.

Lemma 3.16 (wall-projection controls retraction)**.** *Let A and D be subcomplexes of a directly special cube complex B. Assume A is locally convex. Let $\widehat{D}$ denote the preimage of D in $\mathsf{C}(A, B)$, and let $r : \mathsf{C}(A, B) \to A$ be the canonical retraction map.*

Then $r(\widehat{D}) \subset \mathsf{WProj}_B(D \to A)$.

Proof. Consider an edge $\hat{b}$ of $\widehat{D}$. By definition $\hat{b}$ consists of a pair (b, b') where b is an edge of $D \subset B$, b' is an edge of $\mathsf{C}(A, R(B))$, and these edges map to the same edge e in $R(B)$. Then $r(\hat{b})$ is the image of b' under the retraction $A \leftarrow \mathsf{C}(A, R(B))$.

Assume that $r(\hat{b})$ is not a vertex but an edge a. Then a is contained in the preimage of e under $A \to R(B)$. This means that a and b are parallel in B. Thus $r(\hat{b}) \subset \mathsf{WProj}_B(D \to A)$.

The Lemma follows, since $\mathsf{WProj}_B(D \to A)$ contains all vertices of A, and r is projection-like. □

3.C. **Elevation.** We now indicate some terminology related to covering spaces.

Definition 3.17 (elevations)**.** Let $\bar{X} \to X$ denote a covering map. Let $A \subset X$ denote a connected subspace. An *elevation of A to $\bar{X}$* is a connected component of the preimage of A under $\bar{X} \to X$.

Now let $A \to X$ denote any map with A connected. Consider two commutative diagrams

$$(\mathcal{D}_1): \begin{array}{ccc} \bar{A}_1 & \to & \bar{X} \\ \downarrow & & \downarrow \\ A & \to & X \end{array} \qquad (\mathcal{D}_2): \begin{array}{ccc} \bar{A}_2 & \to & \bar{X} \\ \downarrow & & \downarrow \\ A & \to & X \end{array}$$

where $\bar{A}_i \to A$ are connected covers, and write $(\mathcal{D}_1) \leq (\mathcal{D}_2)$ if there is a map $\bar{A}_2 \to \bar{A}_1$ whose composition with $\bar{A}_1 \to \bar{X}, \bar{A}_1 \to A$ gives $\bar{A}_2 \to \bar{X}, \bar{A}_2 \to A$. Say two diagrams $(\mathcal{D}_1), (\mathcal{D}_2)$ are *equivalent* if $(\mathcal{D}_1) \leq (\mathcal{D}_2)$ and $(\mathcal{D}_2) \leq (\mathcal{D}_1)$. Then, up to equivalence, $\leq$ is a partial order. An *elevation* of $A \to X$ to $\bar{X}$ is a minimal diagram.

Here is a more concrete description. Let $\bar{A}$ be a connected component of the fiber product $A \otimes_X \bar{X}$. The projections induce two maps $\bar{A} \to \bar{X}, \bar{A} \to A$, the latter is a covering map, so we get a diagram as above:

$$(\mathcal{E}): \begin{array}{ccc} \bar{A} & \to & \bar{X} \\ \downarrow & & \downarrow \\ A & \to & X \end{array}$$

Then any elevation is equivalent to such a diagram $(\mathcal{E})$, and any such $(\mathcal{E})$ is an elevation.

When the map $A \to X$ is injective the associated map $A \otimes_X \bar{X} \to \bar{X}$ is also injective. More generally we say that a map $A \to X$ has *embedded elevations w.r.t. a cover* $\bar{X} \to X$ if the associated map $A \otimes_X \bar{X} \to \bar{X}$ is injective. In this case each elevation is injective, and two distinct components of $A \otimes_X \bar{X}$ have disjoint images, so that connected components of $A \otimes_X \bar{X}$ are in 1-to-1 correspondence with equivalence classes of elevations. We may thus identify the elevations with their images inside $\bar{X}$, and we recover the case of connected subspaces $A \subset X$.

For example let $A \subset X$ denote a locally convex subcomplex, with A compact and X special. Then the subcomplex $A \subset \mathsf{C}(A, X)$ is an elevation of A to $\mathsf{C}(A, X)$. By Lemma 3.13, the other elevations of A are the remaining components of $\mathsf{C}(A, A) \subset \mathsf{C}(A, X)$.

4. Connected intersection theorem.

4.A. **Some geometric lemmas on cubical complexes.**

The *distance* $d(u, v)$ between two vertices in a connected cube complex X is the length of the shortest combinatorial path joining them. A *geodesic between u and v* is a combinatorial path whose length is $d(u, v)$. For subcomplexes U, V we let $d(U, V)$ denote the length of the shortest geodesic connecting points $u \in U, v \in V$. For a subset $S \subset X$ its cubical neighborhood $N(S)$ is defined to be the union of closed cubes intersecting S.

We recall the following concerning the combinatorial geometry of a CAT(0) cube complex $\widetilde{X}$ (for details see for example [Hag08]).

- A combinatorial path of $\widetilde{X}$ is a geodesic iff the sequence of hyperplanes it crosses has no repetition. A path σ of a nonpositively curved cube complex X whose lift $\widetilde{\sigma}$ is a combinatorial geodesic of the universal cover $\widetilde{X}$ will be called a *local geodesic*.

- A full subcomplex $Y \subset \widetilde{X}$ is *convex* iff it is combinatorially convex, in the sense that any combinatorial geodesic of $\widetilde{X}$ with endpoints in Y has all of its vertices inside

Y. We say that a subcomplex A in a cubical complex Z is *full* provided that a cube of Z belongs to A iff its vertices do.

- For any convex subcomplex $Y \subset \widetilde{X}$ its cubical neighborhood $N(Y)$ is again a convex subcomplex.

Let S be a subset of X. We define S^{+0} to be the smallest subcomplex of X containing S. For $R \geq 1$, we define $S^{+R} = N(S^{+(R-1)})$ to be the *cubical R-thickening of S in X*. When Y is a convex subcomplex we have $Y^{+0} = Y$, and consequently Y^{+R} is convex for each R. For a 0-cell v the subcomplex v^{+R} is called the *cubical ball with center v and radius R*. We note that each cubical ball about v is convex whereas combinatorial metric balls are often not convex. Note that $H^{+0} = N(H)$ when H is a hyperplane, and in this case, $N(H)$ is actually a convex subcomplex, and consequently H^{+R} is convex for each R.

- Every combinatorial path with initial point v and length $\leq R$ is contained in v^{+R}.

- The *convex hull* of a subcomplex $Y \subset \widetilde{X}$ is the intersection of all convex subcomplexes containing Y.

Remark 4.1. If X has dimension $\leq D$, then any vertex x in v^{+R} is joined to v by an edge-path of length $\leq DR$. In particular v^{+R} has diameter $\leq 2DR$.

A fundamental result is then:

Theorem 4.2 (convex hull)**.** *Let X denote a CAT(0) cube complex. Assume that X is uniformly locally finite, in the sense that there is a uniform bound on the number of edges containing a given vertex. Then there exists a function $L \mapsto R(L)$ with the following property.*

For any subcomplex $Y \subset X$ if Y is L-quasiconvex then the convex hull of Y is contained inside $Y^{+R(L)}$.

Here we say that Y is *L-quasiconvex* if any vertex of a combinatorial geodesic with endpoints inside Y is at combinatorial distance $\leq L$ of some vertex in Y. For a proof of Theorem 4.2, see for instance [Hag08].

We will also make use of the following fundamental fact, which the reader can find in [Ger97] and [Rol]:

Theorem 4.3. *Let $Y_1, \ldots, Y_r$ be convex subcomplexes of the CAT(0) cube complex X. Suppose that $Y_i \cap Y_j$ is nonempty for each i, j. Then $\cap_1^r Y_i$ is nonempty.*

Definition 4.4 (R-embeddings)**.** Let X be a nonpositively curved, connected cube complex. Let $Y \to X$ denote any local isometry with Y connected. There is an induced equivariant isometric embedding $\widetilde{Y} \to \widetilde{X}$. We have already defined the cubical R-thickening $\widetilde{Y}^{+R}$ of $\widetilde{Y}$ in the simply-connected $\widetilde{X}$. For any integer $R \geq 0$, the *cubical R-thickening* of Y is $Y^{+R} := \pi_1(Y)\backslash\widetilde{Y}^{+R}$.

The map $Y \to X$ induces a local isometry $Y^{+R} \to X$ which will play an important role below. When $Y^{+R} \to X$ is an embedding, we say that $Y \to X$ is an *R-embedding*, In particular, a locally convex subcomplex $Y \subset X$ is *R-embedded* when the inclusion $Y \hookrightarrow X$ is R-embedding. In this case, we identify Y^{+R} with its image in X, and refer to this locally convex subcomplex as the *R-thickening* of Y in X.

The *embedding radius of* $Y \subset X$ is the supremum of the integers $R \geq 0$ for which $Y \subset X$ is *R-embedded.* The *injectivity radius of* X is the minimum of the embedding radii of vertices, denoted by $\mathsf{InjRad}(X)$. In other words $\mathsf{InjRad}(X) \geq R$ iff $v^{+R} \to X$ is injective for any vertex v.

When $H \to X$ is a hyperplane we likewise define the *R-thickening* of $H \to X$ to be $H^{+R} := \pi_1(H)\backslash \widetilde{H}^{+R}$, there is an induced local isometry $H^{+R} \to X$, and when it embeds we identify H^{+R} with its image and call it the *R-thickening of H in X.*

The *hyperplane embedding radius of* X is the minimum of the embedding radii of hyperplane neighborhoods, denoted by $\mathsf{HEmbRad}(X)$. In other words $\mathsf{HEmbRad}(X) \geq R$ iff for any hyperplane H of X with cubical neighborhood $N(H)$, we have H^{+R} embeds in X.

Note that Y^{+R} is compact provided Y is compact and X is locally finite.

Lemma 4.5. *Let B, C be locally convex connected subcomplexes of the connected nonpositively curved cube complex X. Suppose that B and C are R-embedded and that $B \cap C$ is connected.*

1. *$B \cap C$ is R-embedded.*
2. *$(B \cap C)^{+R}$ equals the component of $B^{+R} \cap C^{+R}$ containing $B \cap C$.*

Proof. We first prove the first statement. Choose a basepoint x in $D = B \cap C$ and let $\widetilde{A}, \widetilde{B}, \widetilde{D}$ be the based elevations at $\widetilde{x} \in \widetilde{X}$, and note that $\widetilde{D} = \widetilde{A} \cap \widetilde{B}$. Let $p \in \widetilde{D}^{+R}$ and $g \in \pi_1 D$ such that $gp \in \widetilde{D}^{+R}$. We show that $g \in \pi_1 D = \text{Stabilizer}(\widetilde{D})$. Observe that $gp \in \widetilde{A}^{+R}$ and so since A is R-embedded we see that $g \in \text{Stabilizer}(\widetilde{A})$. Likewise $g \in \text{Stabilizer}(\widetilde{B})$. Thus $g \in \text{Stabilizer}(\widetilde{D})$ as claimed.

We now prove the second statement. Let σ be a path in $B^{+R} \cap C^{+R}$ from $d \in B \cap C$ to p. Let $\widetilde{B}, \widetilde{C}, \widetilde{D}, \tilde{\sigma}$ be elevations of B, C, D, σ at some point $\tilde{d}$ projecting to d. And let $\tilde{p}$ be the endpoint of $\tilde{\sigma}$.

Observe that $\tilde{\sigma}$ lies entirely in $\widetilde{B}^{+R}$, and consequently $\tilde{p} \in \widetilde{B}^{+R}$. Thus $\tilde{p}^{+R} \cap \widetilde{B}$ is nonempty.

Similarly, $\tilde{p}^{+R} \cap \widetilde{C}$ is nonempty.

Applying Helly's Theorem 4.3 we see that $\tilde{p}^{+R} \cap \widetilde{D} = \tilde{p}^{+R} \cap \widetilde{B} \cap \widetilde{C}$ is nonempty and so taking the images in X we see that $p \in (B \cap C)^{+R}$. □

Lemma 4.6. *Let X be a nonpositively curved cube complex with $\mathsf{InjRad}(X) \geq R$ Let $Y \subset X$ be any connected subcomplex where any vertex is joined by a path with $\leq R$ edges to some fixed vertex y: then Y is null-homotopic in X.*

Proof. Indeed the CAT(0) complex y^{+R} embeds in X and contains Y. □

Lemma 4.7 (short self-connections ⇒ small embedding radius)**.** *Let H denote a hyperplane of a nonpositively curved cube complex X. Let β be a local geodesic of length $\leq 2R$ between two vertices of $N(H)$. If β is not contained in $N(H)$, then the embedding radius of H is $< R$.*

Proof. Any lift of $\beta = \beta_1 \ldots \beta_k$ to the universal cover $\widetilde{X}$ is a combinatorial geodesic $\widetilde{\beta} = \widetilde{\beta}_1 \ldots \widetilde{\beta}_k$ connecting the cubical neighborhoods of hyperplanes $\widetilde{H}, \widetilde{H'}$ projecting onto H in X. Note that $\widetilde{H} \neq \widetilde{H'}$, otherwise by convexity of cubical neighborhood of hyperplanes, the curve β would stay inside $N(H)$.

By assumption there is a vertex x (on $\widetilde{\beta}$) at distance $\leq R$ of both $N(\widetilde{H})$ and $N(\widetilde{H'})$. Let $g \in \pi_1 X$ denote an element sending $\widetilde{H}$ to $\widetilde{H'}$, so that $g \notin \pi_1 H$. Then the vertex $x' = gx$ is not identified with x in $H^{+R} = \pi_1 N(H) \backslash \widetilde{H}^{+R}$, but it is in X. □

Using the fundamental group interpretation for elevations and the fact that the natural map $Y \to Y^{+R}$ is a π_1-isomorphism, we get:

Lemma 4.8 (elevations of neighborhoods)**.** *Let X denote a connected, nonpositively curved cube complex, and let $Y \to R$ denote a local isometry with Y connected. Assume $X' \to X$ is a cover and let $R \geq 0$ denote some integer. For any elevation $Y' \to X'$ of $Y \to X$, the map $Y'^{+R} \to X'$ is an elevation of $Y^{+R} \to X$.*

In particular any elevation of an R-embedding is an R-embedding. And for any cover $X' \to X$ we always have $\mathsf{InjRad}(X') \geq \mathsf{InjRad}(X)$ and $\mathsf{HEmbRad}(X') \geq \mathsf{HEmbRad}(X)$.

Lemma 4.9 (virtually high embedding radius)**.** *Let X denote a compact, connected, nonpositively curved special cube complex. Let $Y \to X$ denote any local isometry of a compact cube complex. Then for any integer $R \geq 0$, there is a finite cover $X_{Y,R} \to X$ such that for any further cover $X' \to X_{Y,R}$, any elevation $Y' \to X'$ of $Y \to X$ is an R-embedding.*

Note that for $R = 0$ the Lemma provides a finite cover where each elevation is injective.

Proof in the compact case. We canonically complete the local isometry $Y^{+R} \to X$. We thus get a finite cover $\mathsf{C}(Y^{+R}, X) \to X$, such that $Y \to \mathsf{C}(Y^{+R}, X)$ is an R-embedding. We now let $X_{Y,R} \to X$ denote any regular finite cover factoring through $\mathsf{C}(Y^{+R}, X) \to X$. By Lemma 4.8 some elevation of $Y \subset X$ to $X_{Y,R}$ is an R-embedding. But by regularity all elevations are. Thus the Lemma holds for $X' = X_{Y,R}$. Applying again Lemma 4.8 we deduce that the Lemma holds for arbitrary $X' \to X_{Y,R}$. □

Corollary 4.10 (noncompact virtually high embedding radius)**.** *Let $Y \to X$ denote a local isometry of nonpositively curved cube complexes. Suppose that $Y \to X$ factors through a local isometry $\bar{Y} \to \bar{X}$ where $\bar{Y}$ and $\bar{X}$ are connected nonpositively curved cube complexes, $\bar{Y}$ is compact and $\bar{X}$ is virtually special. And suppose that the preimage of $\pi_1 \bar{Y}$ equals $\pi_1 Y$. Then for any integer $R \geq 0$, there is a finite cover $X_{Y,R} \to X$ such that for any further cover $X' \to X_{Y,R}$, any elevation $Y' \to X'$ of $Y \to X$ is an R-embedding.*

Proof. Apply Lemma 4.9 to $\bar{Y}' \to \bar{X}'$, where $\bar{X}' \to \bar{X}$ is a special cover, and $\bar{Y}'$ is the induced finite cover of $\bar{Y}$. Let $\bar{X}'_{\bar{Y}',R} \to \bar{X}'$ be the resulting space. This induces a finite cover of X such that the based elevation of Y is R-embedded. A finite regular cover factoring through this has the desired property. □

Corollary 4.11 (virtually high (hyperplane) injectivity radius)**.** *Let X denote a compact, connected, nonpositively curved special cube complex. Then for any integer $R \geq 0$, there is a finite cover $X' \to X$ such that for any further cover $X'' \to X'$, we have $\mathsf{InjRad}(X'') \geq R$ and $\mathsf{HEmbRad}(X'') \geq R$*

Proof. For Y any singleton $\{v\}$ or any hyperplane neighborhood we consider a finite cover $X_{Y,R} \to X$ as in Lemma 4.9. We then let $X' \to X$ denote any finite cover factoring through the finitely many finite covers $X_{Y,R} \to X$. We conclude as above with Lemma 4.8. □

Lemma 4.12 (embedding radius of amalgams)**.** *Let A, B be connected, locally convex subcomplexes of a connected, nonpositively curved cube complex X, such that $A \cup B$ is connected and locally convex.*

Assume the embedding radius of A, B is $\geq R$ and each component of $A^{+R} \cap B^{+R}$ intersects a component of $A \cap B$. Then the embedding radius of $A \cup B$ is $\geq R$.

Proof. Set $Y = A \cup B$. Let $\widetilde{Y}$ be some lift of Y to the universal cover $\widetilde{X}$. Let $\gamma \in \pi_1 X$ map $\tilde{p} \in \widetilde{Y}^{+R}$ to $\tilde{p}' \in \widetilde{Y}^{+R}$, and let us show that γ stabilizes $\widetilde{Y}$.

We fix a base vertex $\tilde{x}$ in $\widetilde{Y}$ mapping inside $A \cap B$ and denote by $\widetilde{A}, \widetilde{B}$ the lifts of A, B at $\tilde{x}$. The subspace $\widetilde{Y}$ is covered by the translates $g\widetilde{A}, h\widetilde{B}$ for $g, h \in \pi_1 Y$, thus $\widetilde{Y}^{+R}$ is covered by the translates $g\widetilde{A}^{+R}, h\widetilde{B}^{+R}$. The stabilizer of $\widetilde{Y}$ is identified with $\pi_1 Y$.

We may assume that $\tilde{p} \in (g\widetilde{A})^{+R}$. There are two possibilities for $\tilde{p}'$. Either there exists $h \in \pi_1 Y$ s.t. $\tilde{p}' \in (h\widetilde{A})^{+R}$. In this case we note that $h^{-1}\gamma g$ self-intersects $\widetilde{A}^{+R}$. Since the embedding radius of A is $\geq R$ it follows that $h^{-1}\gamma g \in \pi_1 A \subset \pi_1 Y$ and thus $\gamma \in \pi_1 Y$.

Otherwise there exists $h \in \pi_1 Y$ s.t. $\tilde{p}' \in (h\widetilde{B})^{+R}$, and to show that $\gamma \in \pi_1 Y$ we may assume $g = h = 1$. Then $\tilde{p}$ maps in X to a vertex p contained in $A^{+R} \cap B^{+R}$. By the relative connectedness assumption there is a path σ inside $A^{+R} \cap B^{+R}$ connecting p to $q \in A \cap B$. Choose a path $\tilde{\alpha} \subset \widetilde{A}^{+R}$ connecting $\tilde{x}$ to $\tilde{p}$, then compose its image α inside x with σ. Since $A \subset A^{+R}$ is a π_1-isomorphism the product $\alpha\sigma$ is homotopic with fixed endpoints inside A^{+R} to a path a contained in A. Thus up to translating by an element of Stabilizer$(\widetilde{A})$ the point $\tilde{p}$ is the endpoint of the lift at $\tilde{x}$ of the path $a\sigma^{-1}$. Similarly, up to translating by the element of Stabilizer$(\widetilde{B})$ there is a path $b \subset B$ connecting x to the endpoint of σ such that $\tilde{p}'$ is the endpoint of the lift at $\tilde{x}$ of the path $b\sigma^{-1}$. It follows that $\gamma \in \pi_1 A a^{-1} b \pi_1 B$ so $\gamma \in \pi_1 Y$. □

4.B. **Quasiconvex amalgams.**

Lemma 4.13 (Quasiisometric Line of Spaces)**.** *Let X be a δ-hyperbolic CAT(0) cube complex. There exists $R_0 \geq 0$ and $L_0 > 0$ with the following property:*

Let $Y_0, \dots, Y_m$ be a sequence of convex subcomplexes of X, such that $Z_{i+1} := Y_i \cap Y_{i+1}$ is nonempty for each $0 \leq i < m$. Let $Y = Y_0 \sqcup_{Z_1} Y_1 \sqcup_{Z_2} \cdots \sqcup_{Z_m} Y_m$, and note that there is an induced map $\phi : Y \to X$.

If $d(Z_i, Z_{i+1}) > R_0$ for each $0 \leq i < m$ then for $y_0 \in Y_0,\ \ y_m \in Y_m$, we have $d_X(\phi(y_0), \phi(y_m)) \leq d_Y(y_0, y_m) \leq L_0 d_X(\phi(y_0), \phi(y_m))$.

Proof. This is implicit in the proof of the quasiconvex amalgam theorem proven in [HW08]. □

Lemma 4.14 (Quasiisometric Tree of Spaces)**.** *Let X be a δ-hyperbolic CAT(0) cube complex. There exists $R_0 \geq 0$ and $L_0 > 0$ with the following property:*

Let Γ be a graph. For each $v \in \Gamma^0$, let Y_v denote a convex subcomplex of X. For each edge $\{u, v\}$ of Γ, we let $Y_{\{u,v\}} = Y_u \cap Y_v$. Suppose that each such $Y_{\{u,v\}}$ is nonempty.

Let Y_Γ denote the abstract union of the Y_v along their pairwise intersections, and consider the map $Y_T \to X$.

If $d(Y_{\{u,v\}}, Y_{\{v,w\}}) > R_0$ for each pair of distinct adjacent edges $\{u, v\}, \{v, w\}$ then $Y_T \to X$ is an $(L_0, 0)$-quasiisometric embedding.

It follows in particular that Γ is a tree!

Proof. This follows from Lemma 4.13. □

Lemma 4.15 (immersed quasiconvex amalgam)**.** *Let X be a compact nonpositively curved cube complex. Assume the universal cover $\widetilde{X}$ is δ-hyperbolic. Then there are constants R_0, K_0 depending only on $\widetilde{X}$ such that the following holds.*

Let A, B, C be (nonempty) connected locally convex subcomplexes of X such that C is a connected component of $A \cap B$. Consider the space $S = A \cup_C B$ and the natural map $f : S \to X$. If both A and B are R_0-embedded , then $S \to X$ factors through an embedding $S \to T$ and a local isometry $T \to X$, such that $S \to T$ is a π_1-isomorphism and every point $p \in T$ is at distance $\leq K_0$ of a point of S.

Proof. The universal cover $\widetilde{S}$ consists of a collection of copies of universal covers of $\widetilde{A}$ and $\widetilde{B}$. The nerve of this covering of $\widetilde{S}$ by this collection of subspaces is a tree Γ. The map $S \to X$ induces a map $\widetilde{S} \to \widetilde{X}$, putting us in the framework of Lemma 4.14. We can thus conclude that $S \to X$ is an $(L_0, 0)$-quasiisometric embedding.

We now regard $\widetilde{S}$ as a subcomplex of $\widetilde{X}$. Since $(L_0, 0)$-quasigeodesics κ-fellowtravel geodesics for some $K_0 = K_0(L_0, \delta)$ we see that $\widetilde{S}$ is actually K_0-quasiconvex.

Apply Theorem 4.2 to $\widetilde{S} \subset \widetilde{X}$ to obtain a convex $\pi_1 S$-invariant subcomplex $\widetilde{T}$ that is contained in the combinatorial K_0-neighborhood of $\widetilde{S}$ and is thus $\pi_1 S$-cocompact. We let $T = \pi_1 S \backslash \widetilde{T}$ and note that $S \to X$ factors as $S \to T \to X$ to satisfy our claim. □

Corollary 4.16 (embedded quasiconvex amalgam)**.** *Let X be a compact nonpositively curved special cube complex. Assume the universal cover $\widetilde{X}$ is Gromov-hyperbolic. Then there are constants $R \geq K$, such that the following holds.*

Let A, B, C be (nonempty) connected locally convex subcomplexes of X such that C is a connected component of $A \cap B$. Consider the space $S = A \cup_C B$ and the natural map $f : S \to X$.

If the embedding radius of both A and B are $\geq R$, then $S \to X$ can be completed to a finite cover $S \subset X' \to X$, so that there is a connected, wall-injective locally convex subcomplex T of X' that contains S, with every point $p \in T$ at distance $\leq K$ of S, and $S \subset T$ is a π_1-isomorphism.

Furthermore T is the union of two locally convex subcomplexes $\mathcal{A}, \mathcal{B}$ such that:

1. $A \subset \mathcal{A} \subset A^{+K}$ *and* $B \subset \mathcal{B} \subset B^{+K}$
2. $\mathcal{A} \cap \mathcal{B}$ *is connected and* $\mathcal{A} \cap \mathcal{B} \subset (A \cap B)^{+K} = C^{+K} \subset X'$.

We say that $T \subset X'$ *is an embedded locally convex thickening of the amalgam* $S \to X$.

Remark 4.17. Note that $C \to X'$ is an R-embedding by Lemma 4.5.1, so in particular $C^{+K} \subset X'$ is really an embedding.

Note also that $C \to \mathcal{A} \cap \mathcal{B}$ is a π_1-isomorphism. Indeed by local convexity any path with endpoints in $\mathcal{A}$ can be homotoped to a local geodesic inside $\mathcal{A}$. It follows $\pi_1(\mathcal{A} \cap \mathcal{B}) = \pi_1\mathcal{A} \cap \pi_1\mathcal{B}$.

Proof of Corollary 4.16. Let R_0, K be the constants of Lemma 4.15. Let $R = \max(R_0, K)$. We first apply Lemma 4.15 to X, A, B, C, to extend the amalgam $S \to X$ to a local isometry that we denote by $\bar{T} \to X$.

We will now perform some computations in $\bar{X} = \mathsf{C}(\bar{T}, X)$.

Let $\mathcal{A}$ be the component of $\bar{T} \cap A^{+K}$ containing A, let $\mathcal{B}$ be the component of $\bar{T} \cap B^{+K}$ containing B, and let $T = \mathcal{A} \cup \mathcal{B}$.

We claim that T is locally convex. Indeed, for $p \in \mathcal{A} - \mathcal{B}$ we have $\text{link}_T(p) = \text{link}_{\mathcal{A}}(p)$ and thus T is nonpositively curved at p. Likewise T is nonpositively curved at each $p \in \mathcal{B} - \mathcal{A}$. Consider now $p \in \mathcal{A} \cap \mathcal{B}$. We will show that $\text{link}_T(p) = \text{link}_{\bar{T}}(p)$ which we already know is a flag complex since $\bar{T}$ is locally convex in $\bar{X}$. Indeed, let Q be a cube of $\bar{T}$ containing p, then $Q \subset A^{+K} \cup B^{+K}$. So for example if $Q \subset A^{+K}$ we have $Q \subset A^{+K} \cap \bar{T}$ and Q intersects $\mathcal{A}$ at p so $Q \subset \mathcal{A}$.

Note that T is connected as it is the union of connected sets with nonempty intersection.

We now show that $\mathcal{A} \cap \mathcal{B}$ is connected. Consider $S \to T \to \bar{T}$ and note that $S \to \bar{T}$ is a π_1-isomorphism, and so $S \to T$ is a π_1-isomorphism since T is connected and locally convex in $\bar{T}$. Let $\mathcal{C}$ denote the connected component of C inside $\mathcal{A} \cap \mathcal{B}$. Form the space $\mathcal{S} := \mathcal{A} \cup_{\mathcal{C}} \mathcal{B}$. Since $S \to T$ is π_1-surjective, the composition $S \to \mathcal{S} \to T$ shows that $\mathcal{S} \to T$ is π_1-surjective. If $\mathcal{A} \cap \mathcal{B}$ is not connected, then there are non-trivial connected covers of T to which $\mathcal{S}$ lifts isomorphically. But this implies that the image of $\pi_1\mathcal{S}$ inside $\pi_1 T$ is a proper subgroup which is impossible.

Since we have shown that $\mathcal{A} \cap \mathcal{B}$ is connected it follows from Lemma 4.5.2 that $(\mathcal{A} \cap \mathcal{B}) \subset (A \cap B)^{+K}$.

Finally, since $T \to X$ is a local isometry we let $X' = \mathsf{C}(T, X)$. Then $X' \to X$ is a finite cover and $T \subset X'$ is wall-injective by Corollary 3.11. □

4.C. **Virtually Connected Intersection.**

Lemma 4.18 (à la Helly)**.** *Suppose the nonpositively curved cube complex $\mathcal{C}$ deformation retracts to the locally convex connected subcomplex C. Let $B_1, \ldots, B_n$ be locally convex connected subcomplexes of $\mathcal{C}$ such that $B_i \cap C$ is nonempty for each i. Then:*

1. *$B_i \cap C$ is connected.*
2. *Each component of $B_1 \cap B_2$ contains a point of C.*
3. *If $\cap_i(B_i \cap C)$ is connected then $\cap_i B_i$ is connected.*

Proof. We first prove the first claim. Let γ be a local geodesic in B_i that starts and ends on $B_i \cap C$. Let $\widetilde{\gamma}$ be a lift of γ. Let $\widetilde{B}_i$ be the lift of B_i that contains $\widetilde{\gamma}$. Note that since $\mathcal{C}$ deformation retracts onto C, there is only one component $\widetilde{C}$ in the preimage of C. It follows that the geodesic path $\widetilde{\gamma}$ is contained in the convex subcomplex $\widetilde{C}$. Thus $\widetilde{\gamma} \subset \widetilde{B}_i \cap \widetilde{C}$ and so $\gamma \subset B_i \cap C$.

We now prove the second claim.

For $i \in \{1, 2\}$ let σ_i be a path in B_i from $B_i \cap C$ to $p \in B_1 \cap B_2$. Since $\mathcal{C}$ deformation retracts to C, we can let σ be a path in C that is path homotopic to $\sigma_1\sigma_2^{-1}$, and so $\sigma\sigma_2\sigma_1^{-1}$ lifts to a closed path $\widetilde{\sigma}\widetilde{\sigma}_2\widetilde{\sigma}_1^{-1}$ in $\widetilde{\mathcal{C}}$. Let $\widetilde{\Sigma}_i$ be the smallest convex subcomplex of $\mathcal{C}$ containing $\widetilde{\sigma}_i$, and note that $\widetilde{\Sigma}_i \subset \widetilde{B}_i$. Similarly, let $\widetilde{\Sigma}$ be the smallest convex subcomplex containing $\widetilde{\sigma}$ and note that $\widetilde{\Sigma} \subset \widetilde{C}$. The closed path $\widetilde{\sigma}\widetilde{\sigma}_2\widetilde{\sigma}_1^{-1}$ shows that the convex subcomplexes $\widetilde{\Sigma}_1, \widetilde{\Sigma}_2, \widetilde{\Sigma}$ have nonempty pairwise intersection, and so Helly's Theorem 4.3 shows that $\widetilde{\Sigma}_1 \cap \widetilde{\Sigma}_2 \cap \widetilde{\Sigma}$ is nonempty. Its image lies in a component of $B_1 \cap B_2 \cap C$ that contains the image of the connected set $\widetilde{\Sigma}_1 \cap \widetilde{\Sigma}_2$ which is contained in the component of $B_1 \cap B_2$ containing p.

The third claim follows from the following statement:

(3') Each component of $\cap_i B_i$ contains a point of C

This follows from (2) by induction on n. Indeed given any family of subcomplexes $B_1, \ldots, B_n, B_{n+1}$ satisfying the assumption of the Lemma, let B'_n be any component of $B_n \cap B_{n+1}$. By (2) the inductive hypothesis applies to $B_1, \ldots, B_{n-1}, B'_n$.

□

Lemma 4.19 (Separating $\mathcal{D}$ from stuff)**.** *Let X be a based nonpositively curved connected cube complex. Suppose $B_1, \ldots, B_n, \mathcal{D}$ are locally convex connected subcomplexes of X containing the basepoint. Suppose $\pi_1\mathcal{D}g\pi_1B_i$ is separable in π_1X for each $g \in \pi_1X$ and each i.*

Then there is a finite cover $\widehat{X}$ with $\widehat{\mathcal{D}} \cong D$ such that the based elevations $\widehat{B}_1, \ldots, \widehat{B}_n$ have connected intersection with $\widehat{\mathcal{D}}$ in $\widehat{X}$.

Proof. For each i we shall produce below a finite index normal subgroup $N_i \subset \pi_1X$ such that for any normal subgroup $N \subset N_i$ the cover $\widehat{X}_i$ corresponding to $N\pi_1\mathcal{D}$ has $\widehat{\mathcal{D}} \cap \widehat{B}_i$ connected. We then let $N = \cap_i N_i$, and let $\widehat{X}$ be the cover corresponding to $N\pi_1\mathcal{D}$.

Let p denote the basepoint of X, and let p_k denote a point in each other component C_k of $B_i \cap \mathcal{D}$. For each k let β_k denote a local geodesic in B_i from p to p_k, and let δ_k denote a local geodesic in $\mathcal{D}$ from p to p_k.

Observe that $1 \notin \pi_1\mathcal{D}\delta_k\beta_k^{-1}\pi_1B_i$. Indeed, suppose $b_i\beta_k$ is homotopic to $d\delta_k$ for some $b_i \in \pi_1B_i$ and $d \in \pi_1\mathcal{D}$. Their lifts to $\widetilde{X}$ are homotopic to geodesic paths δ, β. But then $\delta = \beta$ by uniqueness of geodesics, so p_k and p must lie in the same component of $B_i \cap \mathcal{D}$.

By separability, let N_i be a finite index normal subgroup of π_1X such that for each k we have $N_i\pi_1\mathcal{D}\delta_k\beta_k^{-1}\pi_1B_i$ is disjoint from $N_i\pi_1\mathcal{D}\pi_1B_i$. Observe that for any $N \subset N_i$ we consequently have $N_i\pi_1\mathcal{D}\delta_k\beta_k^{-1}\pi_1B_i$ is disjoint from $N\pi_1\mathcal{D}\pi_1B_i$ for each k. Equivalently, $(N\pi_1\mathcal{D})\pi_1D\delta_k\beta_k^{-1}\pi_1B_i$ is disjoint from $(N\pi_1\mathcal{D})\pi_1\mathcal{D}\pi_1B_i$. The cover corresponding to $N\pi_1\mathcal{D}$ has the desired property. □

Definition 4.20. Let $V, U, \mathcal{U}$ be connected based subspaces of X. We say $V, \mathcal{U}$ has *connected intersections relative* to V, U, if for each cover $\widehat{X}$ and based elevations $\widehat{V}, \widehat{U}, \widehat{\mathcal{U}}$: each component of $\widehat{V} \cap \widehat{\mathcal{U}}$ contains a component of $\widehat{V} \cap \widehat{U}$.

Lemma 4.21 (Obtaining Relative Connectivity)**.** *Let $V, U, \mathcal{U}$ be connected based subspaces of X. Suppose $U \subset \mathcal{U}$ and $\pi_1U \to \pi_1\mathcal{U}$ is surjective and that $V \cap \mathcal{U}$ is connected. Then $V, \mathcal{U}$ has connected intersection relative to V, U.*

Proof. Let $f : \widehat{X} \to X$ be a cover whose basepoint $\widehat{p}$ maps to the basepoint p of X, and $\widehat{V}, \widehat{U}, \widehat{\mathcal{U}}$ be the based elevations. Note that $\widehat{\mathcal{U}} \cap \widehat{V}$ is the disjoint union of components covering $\mathcal{U} \cap V$, and by the connectivity of $\mathcal{U} \cap V$, each of these contains a point of $f^{-1}(p) \cap \widehat{\mathcal{U}}$. But since $\pi_1 U \to \pi_1 \mathcal{U}$ is surjective, $f^{-1}(p) \cap \widehat{\mathcal{U}} = f^{-1}(p) \cap \widehat{U}$. Consequently each component of $\widehat{\mathcal{U}} \cap \widehat{V}$ contains a component of $\widehat{U} \cap \widehat{V}$. □

Lemma 4.22 (Intermediate Relative Connectivity)**.** *Let V and $U \subset \mathcal{U} \subset U^+$ be connected locally convex based subcomplexes of X, and assume that $\mathcal{U} \to U^+$ is π_1-surjective. If V, U^+ has connected intersections relative to V, U then $V, \mathcal{U}$ has connected intersections relative to V, U.*

Proof. Let $\widehat{X}$ be a connected cover and note that $\widehat{V}$ and $\widehat{U} \subset \widehat{\mathcal{U}} \subset \widehat{U}^+$ are connected and locally convex. Let $q \in \widehat{V} \cap \widehat{\mathcal{U}}$. Applying relative connectivity of V, U^+ relative to V, U we see that there is a path σ in $\widehat{U}^+$ from q to $p \in \widehat{U}$. By π_1-surjectivity σ can be homotoped to a local geodesic γ in $\widehat{\mathcal{U}}$. By local convexity γ lies in $\widehat{V}$. Thus the component of $\widehat{V} \cap \widehat{\mathcal{U}}$ containing q also contains $p \in \widehat{V} \cap \widehat{U}$. □

Theorem 4.23 (Virtually Connected Intersection)**.** *Let X be a compact special cube complex with $\widetilde{X}$ δ-hyperbolic. Let $(B_0, \ldots, B_n, A)$ be connected locally convex subcomplexes containing the basepoint of X. Suppose that $A \subset \bigcap_{j \in \{0,1,\ldots,n\}} B_j$.*

Then there is a based finite cover $\bar{X}$ and based elevations $\bar{B}_0, \ldots, \bar{B}_n$ with $\bar{A} \cong A$, such that $\cap_{j \in J} \bar{B}_j$ is connected for each $J \subset \{0, \ldots, n\}$.

The *p-component of S* denoted by $[S]_p$, is the component of S containing the point p, and we use the notation $[S] = [S]_b$ where b is the basepoint.

When indices are clear from the context we will use the notation $B_J = \cap_{j \in J} B_j$, $\ddot{B}_J = \cap_{j \in J} \ddot{B}_j$ etc.

Given a subspace $A \subset X$ containing the basepoint, and a based cover $\widehat{X}$, we will employ the notation $\widehat{A}$ to denote the based elevation of A. Likewise $\ddot{X}, \ddot{A}$, and $\bar{X}, \bar{A}$ etc.

Proof. **Step 0: Relative Connectedness Preparation.** Let $R \geq K$ be the constants in Corollary 4.16. For each $I \subset \{0, \ldots, n\}$, since each $\pi_1[B_I]$ is separable, and $[B_I]^{+R}$ is compact and embeds in the cover of X corresponding to $\pi_1[B_I]$, we can pass to a finite cover $\ddot{X}$ of X such that each $[\ddot{B}_I] \cong [B_I]$ (in particular $\ddot{A} \cong A$) and each $[\ddot{B}_I]^{+R}$ embeds.

Let $I, J \subset \{0, \ldots, n\}$, and apply Lemma 4.19 to $\ddot{B}_I, \ddot{B}_J^{+R}$ to obtain a finite cover $\overset{\bowtie}{X}$ such that $[\overset{\bowtie}{B}_J] \cong [\ddot{B}_J]$ and $[\overset{\bowtie}{B}_I]^{+R}$ have connected intersection, indeed since $\pi_1 X$ is word-hyperbolic, double (quasiconvex) cosets are separable as prove in [HW08]. In particular $\overset{\bowtie}{A} \cong A$. By Lemma 4.21, in any further cover $\dot{X} \to \overset{\bowtie}{X}$, the subspaces $[\dot{B}_I]^{+R}, [\dot{B}_J]$ have connected intersection relative to $[\dot{B}_I], [\dot{B}_J]$, and the same holds if we replace $[\dot{B}_I]^{+R}$ by a deformation retract containing $[\dot{B}_I]$.

For each $I, J \subset \{0, \ldots, n\}$ we apply the above construction to obtain a finite cover $\overset{\bowtie}{X}_{IJ} \to X$. Let $\dot{X}$ be a finite cover that factors through each $\overset{\bowtie}{X}_{IJ}$ with $\dot{A} \cong A$, for instance the based fiber product of the $\overset{\bowtie}{X}_{IJ} \to X$. We complete the proof by applying the following claim to $\dot{X}$ with $(\dot{B}_0, \ldots, \dot{B}_n, \dot{A})$. The remainder of the proof will focus on verifying this claim.

Claim. Let X be a compact special cube complex with $\widetilde{X}$ δ-hyperbolic. Let $(B_0,\dots,B_n,C)$ be locally convex connected subcomplexes containing the basepoint of X. Suppose that $C\cap B_J$ is connected for each $J\subset\{0,\dots,n\}$.

Suppose that each $[B_I]^{+R}$ embeds in X, and suppose that $[B_I],[B_J]^{+R}$ has connected intersection relative to $[B_I],[B_J]$ for each $I,J\subset\{0,\dots,n\}$.

Then there is a based cover $\bar{X}$ with $\bar{C}\cong C$, such that $\bar{B}_J=\cap_{j\in J}\bar{B}_j$ is connected for each $J\subset\{0,\dots,n\}$.

The claim holds for $n=0$ with $\bar{X}=X$, and will be proven by induction on n.

Step 1: Embedding neighborhoods of C and insuring connected intersections with each $\vec{B}_j$: We first pass to a finite cover of X such that C^{+R} embeds, and then apply Lemma 4.19 several times to obtain a finite cover $\vec{X}$ such that:

(1) $C^{+R}\cong\vec{C}^{+R}$ embeds in $\vec{X}$.
(2) $\vec{C}^{+R}\cap\vec{B}_j$ is connected for each j.
(3) $\vec{C}^{+K}\cap\vec{B}_j^{+K}$ is connected for each j.

As a consequence the following holds by Lemma 4.5 since we have just enforced that C is R-embedded, and $\vec{C}\cap\vec{B}_0$ is connected and B_0 is R-embedded by hypothesis of the claim

(4) $(\vec{C}\cap\vec{B}_0)^{+R}$ embeds

Step 2: Making intersections connected in B_0: Consider the subspaces $[\vec{B}_0\cap\vec{B}_i]$ for $i\in\{1,\dots,n\}$. We claim that $\big(\cap_{j\in J}[\vec{B}_0\cap\vec{B}_j]\big)\cap\vec{C}$ is connected for any $J\subset\{1,\dots,n\}$. Indeed, using the connectivity hypothesis we have the following inclusions, which are thus equalities:

$$\vec{B}_{J\cup\{0\}}\cap\vec{C}\ \subseteq\ \big(\cap_{j\in J}[\vec{B}_0\cap\vec{B}_j]\big)\cap\vec{C}\ \subseteq\ \cap_{j\in J}[\vec{B}_0\cap\vec{B}_j]\cap\vec{C}\ =\ \cap_{j\in J}\vec{B}_0\cap\vec{B}_j\cap\vec{C}\ =\ \vec{B}_{J\cup\{0\}}\cap\vec{C}$$

Observe that $\big[\cap_{j\in J}[\vec{B}_0\cap\vec{B}_j]\big]=[\cap_{j\in J\cup\{0\}}\vec{B}_j]$. Indeed $\big[\cap_{j\in J}[\vec{B}_0\cap\vec{B}_j]\big]\subset[\cap_{j\in J\cup\{0\}}\vec{B}_j]$ because $\cap_{j\in J}[\vec{B}_0\cap\vec{B}_j]\ \subset\ \cap_{j\in J}(\vec{B}_0\cap\vec{B}_j)\ \subset\ \cap_{j\in J\cup\{0\}}\vec{B}_j$. The reverse inclusion $[\cap_{j\in J\cup\{0\}}\vec{B}_j]\ \subset\ \big[\cap_{j\in J}[\vec{B}_0\cap\vec{B}_j]\big]$ holds because $\cap_{j\in J\cup\{0\}}\vec{B}_j\ \subset\ \vec{B}_0\cap\vec{B}_j$ for each $j\in J$ and so $[\cap_{j\in J\cup\{0\}}\vec{B}_j]\ \subset\ [\vec{B}_0\cap\vec{B}_j]$ and so $[\cap_{j\in J\cup\{0\}}\vec{B}_j]\ \subset\ \cap_{j\in J\cup\{0\}}[\vec{B}_0\cap\vec{B}_j]$.

The above two observations show that the hypotheses of our claim hold for the family $\big([\vec{B}_0\cap\vec{B}_1],\dots,[\vec{B}_0\cap\vec{B}_n],\vec{C}\big)$. Thus, by induction there is a finite covering space $\widehat{X}\to\vec{X}$ and an isomorphic based elevation $\widehat{C}\to\vec{C}$ such that the collection of subspaces $\widehat{[\vec{B}_0\cap\vec{B}_j]}=[\widehat{B}_0\cap\widehat{B}_j]$ have the connected multiple intersection property: $\cap_{j\in J}[\widehat{B}_0\cap\widehat{B}_j]$ is connected for each $J\subset\{1,\dots,n\}$.

Step 3: Formation of $\mathcal{D}$: Let $D=\widehat{B}_0\cup\widehat{C}$, and since $\widehat{B}_0$ and $\widehat{C}$ are R-embedded, let $\mathcal{D}$ be the locally convex thickening of D provided by Corollary 4.16. Furthermore $\mathcal{D}$ decomposes as $\mathcal{D}=\mathcal{B}_0\cup\mathcal{C}$, where $\widehat{B}_0\subset\mathcal{B}_0\subset\widehat{B}_0^{+K}$ and $\widehat{C}\subset\mathcal{C}\subset\widehat{C}^{+K}$ (Corollary 4.16.1). The inclusions $\widehat{B}_0\subset\mathcal{B}_0\subset\widehat{B}_0^{+K}$ and $\widehat{C}\subset\mathcal{C}\subset\widehat{C}^{+K}$ are homotopy equivalence by local convexity, and we shall soon use this. Moreover $(\widehat{B}_0\cap\widehat{C})\subset\mathcal{B}_0\cap\mathcal{C}\subset(\widehat{B}_0\cap\widehat{C})^{+K}$ where the locally convex intersection $\mathcal{B}_0\cap\mathcal{C}$ is connected. Since the intermediate subcomplex $\mathcal{B}_0\cap\mathcal{C}$ is locally convex and connected it deformation retracts to $\widehat{B}_0\cap\widehat{C}$, indeed the inclusion is a π_1-isomorphism by Remark 4.17.

Step 4: Making $\mathcal{D}$ have connected intersection with each of $\widehat{B}_1, \ldots, \widehat{B}_n$: Apply Lemma 4.19 to pass to a finite cover $\check{X}$ such that $\check{\mathcal{D}} \cong \mathcal{D}$ and such that each $\check{B}_j \cap \check{\mathcal{D}}$ is connected.

Step 5: Multiple intersections are connected in $\check{\mathcal{D}}$: Recall that $\check{B}_J$ denotes $\cap_{j \in J} \check{B}_j$. Our goal now is to show that $\check{\mathcal{D}} \cap \check{B}_J$ is connected for each $J \subset \{1, \ldots, n\}$. This intersection can be expressed as the union of two sets containing the basepoint p:

$$\check{\mathcal{D}} \cap \check{B}_J = (\check{\mathcal{C}} \cup \check{\mathcal{B}}_0) \cap \check{B}_J = (\check{\mathcal{C}} \cap \check{B}_J) \cup (\check{\mathcal{B}}_0 \cap \check{B}_J)$$

and it therefore suffices to verify the connectivity of both $(\check{\mathcal{C}} \cap \check{B}_J) = (\check{\mathcal{C}} \cap (\cap_{j \in J} \check{B}_j))$ and $(\check{\mathcal{B}}_0 \cap \check{B}_J) = (\check{\mathcal{B}}_0 \cap (\cap_{j \in J} \check{B}_j))$.

To reach this goal we aim to apply Lemma 4.18.3 to the families $(\check{\mathcal{C}} \cap \check{\mathcal{B}}_j)_{j \in J}$ and $(\check{\mathcal{B}}_0 \cap \check{\mathcal{B}}_j)_{j \in J}$, respectively contained in $\check{\mathcal{C}}, \check{\mathcal{B}}_0$ where $\check{\mathcal{C}}$ deformation retracts to $\check{C}$ and $\check{B}_0$ deformation retracts to $\check{B}_0$ (as we have seen in Step 3).

We will be done after verifying the connectedness of each of the following:

(a) $\check{\mathcal{C}} \cap \check{\mathcal{B}}_j$,
(b) $\check{C} \cap \check{\mathcal{B}}_J$,
(c) $\check{\mathcal{B}}_0 \cap \check{B}_j$,
(d) $\check{B}_0 \cap \check{B}_J$.

(a) We now show that $\check{\mathcal{C}} \cap \check{B}_j$ is connected. By choice of $\vec{X}$ we know that $\vec{C}^{+K} \cap \vec{B}_j$ is connected. Since $\vec{\mathcal{C}} \subset \vec{C}^{+K}$ is a homotopy equivalence it follows by Lemma 4.18.1 that $\vec{\mathcal{C}} \cap \vec{B}_j$ is connected and we are done.

(b) By assumption $\check{C} \cap \check{B}_J$ is connected. At this point we already have that each $\check{\mathcal{C}} \cap \check{B}_J$ is connected.

(c) We now show that $\check{\mathcal{B}}_0 \cap \check{B}_j$ is connected for each j. Let $q \in \check{\mathcal{B}}_0 \cap \check{B}_j$. By assumption $\check{B}_j, \check{B}_0^{+R}$ has connected intersection relative to $\check{B}_j, \check{B}_0$. Thus by Lemma 4.22, $[\check{\mathcal{B}}_0 \cap \check{B}_j]_q$ intersects $\check{B}_0$. Since $\check{B}_j \cap \check{\mathcal{D}}$ is connected, we see that $[\check{\mathcal{B}}_0 \cap \check{B}_j]_q$ intersects $\check{\mathcal{B}}_0 \cap \check{\mathcal{C}}$ at some point r. Recall that $(\check{\mathcal{B}}_0 \cap \check{\mathcal{C}}) \cong (\mathcal{B}_0 \cap \mathcal{C}) \subset (B_0^{+K} \cap C^{+K})$ is connected: Apply Lemma 4.18.2 to see that the r-component of $(\check{\mathcal{B}}_0 \cap \check{\mathcal{C}}) \cap [\check{\mathcal{B}}_0 \cap \check{B}_j]_q$ contains a point s of $(\check{B}_0) \cap (\check{\mathcal{B}}_0 \cap \check{\mathcal{C}}) \cap [\check{\mathcal{B}}_0 \cap \check{B}_j]_q$. But $s \in \check{B}_0 \cap \check{B}_j \cap \check{\mathcal{C}}$ which we proved is connected and which contains p.

(d) We now show that $\cap_{j \in J}(\check{B}_0 \cap \check{B}_j)$ is connected. Since $\check{\mathcal{B}}_0 \cap \check{B}_j$ is connected it follows from Lemma 4.18.1 that $\check{B}_0 \cap \check{B}_j$ is connected, and so $[\check{B}_0 \cap \check{B}_j] = \check{B}_0 \cap \check{B}_j$. We thus see that $\cap_{j \in J}(\check{B}_0 \cap \check{B}_j) = \cap_{j \in J}[\check{B}_0 \cap \check{B}_j]$, but the latter is connected since $\cap_{j \in J}[\widehat{B}_0 \cap \widehat{B}_j]$ is connected by choice of $\widehat{X}$.

Step 6: Applying induction again: We now apply the inductive assumption to the family $(\check{B}_1, \ldots, \check{B}_n, \check{\mathcal{D}})$. There is a finite covering space $\bar{X} \to X$ and an isomorphic based elevation $\bar{\mathcal{D}} \cong \check{\mathcal{D}}$ such that: letting $\bar{B}_j$ denote the based elevation of $\check{B}_j$, then for any nonempty subset $J \subset \{1, \ldots, n\}$ the intersection $\bar{B}_J$ is connected.

For any subset $J = \{0\} \cup I$ with $I \subset \{1, \ldots, n\}$, we have $\bar{B}_J = \cap_{i \in I}(\bar{B}_0 \cap \bar{B}_i)$. Since $\bar{B}_0 \subset \bar{\mathcal{D}}$, the map $\bar{B}_0 \to \check{B}_0$ is an isomorphism and we have already shown in step (5d) that $\cap_{i \in I}(\check{B}_0 \cap \check{B}_i)$ is connected. □

Corollary 4.24 (Virtually Connected Intersection for Local Isometries)**.** *Let X be a compact special cube complex with $\widetilde{X}$ δ-hyperbolic. Let $(B_0 \to X, \dots, B_n \to X, A \to X)$ be local isometries of connected complexes. Suppose that $A \to X$ is injective and factors as $A \to B_j \to X$ for each j.*

Then there is a finite cover $\bar{X}$ with an elevation $\bar{A} \cong A$ such that the elevations $\bar{B}_0 \to \bar{X}, \dots, \bar{B}_n \to \bar{X}$ at $\bar{A} \to \bar{X}$ are injective, and the intersection $\cap_{j \in J} \bar{B}_j$ of each subcollection of their images is connected.

Proof. For each i, let $X_i = \mathsf{C}(B_i, X)$, and let $\hat{X}$ denote the component of the fiber-product of $X_0 \to X, \dots, X_n \to X$ that contains $\bar{A} \cong A$. Each elevation $\hat{B}_i$ embeds in $\hat{X}$ and contains A. We may thus apply Theorem 4.23 to $(\widehat{B}_0, \dots, \widehat{B}_n, A)$. □

5. Trivial Wall Projections

5.A. **Introduction.** This section is devoted to proving Corollary 5.8, which is a higher-dimensional generalization of the following:

Proposition 5.1. *Let $A \to X, B \to X$ be immersions of finite connected graphs to the connected graph X. Suppose that all conjugates of $\pi_1 A$ and $\pi_1 B$ have trivial intersection in $\pi_1 X$. Then there is a finite cover $\widehat{X} \to X$, such that each pair of distinct elevations $\widehat{A}, \widehat{B}$ embed and intersect in a forest.*

In Corollary 5.8 we have a similar statement for local isometries of complexes of (virtually) special cube complexes (with word hyperbolic fundamental groups), by substituting "trivial wall projection" for "forest intersection". In the case of graphs the wall projection of A onto B equals $B^0 \cup (A \cap B)$, so that the statement in Proposition 5.1 is really about trivial wall projections. While the proof of Proposition 5.1 is rather simple (see for example [Wis02]), we found the extension to special cube complexes to be a very challenging part of the proof. The reader might choose to skip this lengthy section at first reading, after becoming familiar with Corollary 5.8 which of course uses the language of elevations from Definition 3.17.

The conclusion of Proposition 5.1 still holds in any further cover $X' \to \ddot{X}$. Trivial wall projection is also preserved under covering:

Lemma 5.2. *Suppose $X' \to \dot{X}$ is a covering map of connected cube complexes. Let $\dot{A}, \dot{B} \subset \dot{X}$ denote connected subcomplexes, and let $A', B' \subset X'$ denote elevations of those. If $\mathsf{WProj}_{\dot{X}}(\dot{B} \to \dot{A})$ is trivial then $\mathsf{WProj}_{X'}(B' \to A')$ is trivial.*

Proof. This holds since $\mathsf{WProj}_{X'}(B' \to A')$ maps to $\mathsf{WProj}_{\dot{X}}(\dot{B} \to \dot{A})$ under the map $X' \to \dot{X}$. □

The main work in proving Corollary 5.8 will be to prove the following theorem which focuses on a single local isometry. In fact this is a special case of Corollary 5.8 when B consist of single 0-cell. Observe that the following result is immediate when $A \to X$ is an injection of graph. The difficulty in higher dimensions comes from the reach of wall-projections, even when $A \to X$ is injective.

Theorem 5.3 (Trivial Wall Projections)**.** *Let X be a compact virtually special cube complex, and let $A \to X$ be a compact local isometry with $\pi_1 A \subset \pi_1 X$ malnormal. Assume $\pi_1 X$ is word hyperbolic.*

Then there exists a finite cover $A_0 \to A$ such that any further finite cover $\bar{A} \to A_0$ can be completed to a finite special cover $\bar{X} \to X$ with the following properties:

1. *all elevations of $A \to X$ to $\bar{X}$ are injective*
2. *$\bar{A}$ is wall-injective in $\bar{X}$*
3. *every elevation of A distinct from $\bar{A}$ has trivial wall-projection onto $\bar{A}$.*

One of the difficulties in proving Theorem 5.3 is that the triviality of $\text{WProj}_{\widehat{X}}(B \to \widehat{A})$ for elevations $B, \widehat{A}$ of A is certainly not stable under taking further covers. Lemma 5.2 gives no control in the case that $\dot{A} = \dot{B}$. For then $\mathsf{WProj}_{\dot{X}}(\dot{B} \to \dot{A}) = \dot{A}$ can be nontrivial, and indeed, there can be elevations $B' \neq A'$ with nontrivial wall projection onto A'. This behavior is exhibited in the following example which illustrates the delicacy of Theorem 5.3. This example shows that in general there does not exist a finite cover $\widehat{X} \to X$ with the property that we have trivial wall projections in any further cover.

Example 5.4. Let X denote the standard 2-complex of $\langle a, b, c \mid a^{-1}b^{-1}ac \rangle$ so X is obtained from a cylinder by identifying two points on distinct bounding circles. Let A denote the subcomplex consisting of the 0-cell and the 1-cell labeled by a.

Consider any based finite cover $\widehat{X} \to X$. Let $\widehat{A}$ denote the based elevation of A, and let d denote the degree of $\widehat{A} \to A$. Observe that for some n there is an immersion $D \to X$ where D is formed from $\widehat{A}$ by attaching a distinct strip $I_n \times I$ to each 1-cell of $\widehat{A}$ along both $\{0\} \times I$ and $\{n\} \times I$. For instance, we could let n denote the order of the image of $\pi_1 X$ in the left coset representation on $\pi_1 \widehat{X}$.

Let $\widehat{D}$ denote any double cover of D in which the preimage of $\widehat{A}$ consists of two isomorphic components $\widehat{A}_1, \widehat{A}_2$. It is easy to see that $\text{WProj}_{\widehat{D}}(\widehat{A}_1 \to \widehat{A}_2)$ consists of all of $\widehat{A}_2$. Now let $\bar{X}$ denote a cover of $\widehat{X}$ which contains $\widehat{D}$, and we see that $\widehat{A}_2 = \text{WProj}_{\widehat{D}}(\widehat{A}_1 \to \widehat{A}_2) \subset \text{WProj}_{\bar{X}}(\widehat{A}_1 \to \widehat{A}_2) \subset \widehat{A}_2$.

5.B. **Narrow wall-projection.** We say hyperplanes H, K of X are *M-close* if $d(N(H), N(K)) \leq M$ where we recall that $d(U, V)$ is the length of the shortest combinatorial path with endpoints on U, V.

Lemma 5.5 (narrow implies trivial)**.** *Suppose X is virtually special with finitely many hyperplanes. Let $A \to X$ be a compact local isometry and let $M > 0$ be a positive number. There exists a finite cover $X_0 \to X$ with a based elevation A_0 such that all elevations of A to X_0 are injective and for any further cover $(\bar{X}, \bar{A}) \to (X_0, A_0)$ we have the following:*

If $\bar{A}' \neq \bar{A}$ is another elevation of A to $\bar{X}$, and any two hyperplanes from $\bar{A}$ to $\bar{A}'$ are M-close then $\bar{A}'$ has trivial wall-projection onto $\bar{A}$.

Proof. By Lemma 4.9 and Corollary 4.11 there is a finite special cover $X' \to X$ such that all elevations of A to X' are injective and all hyperplanes of X' have embedding radius $> M$. Let N denote the number of hyperplanes in X'. Let $A_0 \to A$ be a finite cover factoring through X' such that the finite CAT(0) complex $v^{+(N+1)} \to A_0$ embeds as a subcomplex for each vertex $v \in A_0$, and such that $v^{+(N+1)}$ is wall-injective in A_0. Let $X_0 = \mathsf{C}(A_0, X')$. Note that each $v^{+(N+1)}$ is then wall-injective in X_0.

Remark 5.6. The hypothesis that A is compact can be relaxed to the following hypothesis: $A \to X$ factors through a local isometry $\bar{A} \to \bar{X}$ such that the preimage of $\pi_1 \bar{A}$ equals $\pi_1 A$, and $\bar{A}$ is compact, and $\bar{X}$ is virtually special.

Let $\hat{\bar{X}}$ be a finite special cover of $\bar{X}$. Let $\bar{A}_0 \to \bar{A}$ be a finite cover factoring through the base elevation of $\bar{A}$ to $\hat{\bar{X}}$, such that each $v^{+(N+1)}$ is injective and wall-injective. Now let $\bar{X}_0 = \mathsf{C}(\bar{A}_0, \hat{\bar{X}})$, and observe that $\bar{A}_0$ is wall-injective in $\bar{X}_0$ and thus each subspace $v^{+(N+1)}$ of $\bar{A}_0$ is wall-injective in $\bar{X}_0$.

Note that the $v^{+(N+1)}$ is contained and is wall-injective in $\bar{v}^{+(N+1)}$ for each v, since it is a convex subcomplex of a CAT(0) cube complex.

We now let X_0 be the fiber product of X' and $\bar{X}_0$, and we let A_0 be the fiber product of A' and $\bar{A}_0$, where A' is the based elevation of A to X'.

Let $(\bar{X}, \bar{A}) \to (X_0, A_0)$ be any further cover. For each $\bar{v}$ of $\bar{A}$ the CAT(0) cubical ball $\bar{v}^{+(N+1)}$ is still a CAT(0) and wall-injective subcomplex of $\bar{X}$.

Consider any edge-path $\bar{\sigma}$ in $\bar{A}$ of length $N+1$, and assume that any two dual hyperplanes are M-close. Then $\bar{\sigma}$ has two distinct edges dual to hyperplanes $\bar{H}_1, \bar{H}_2$ which are M-close but project to the same hyperplane H' of X'. Since $\bar{\sigma}$ projects to a short path σ' from $N(H')$ to itself, we see by Lemma 4.7 that σ' is homotopic into $N(H')$ and thus $\bar{H}_1 = \bar{H}_2$. The cubical $(N+1)$-ball $\bar{B}$ of $\bar{A}$ centered at the origin of $\bar{\sigma}$ is wall-injective in $\bar{X}$, and thus there is a $\bar{B}$-hyperplane dual to two distinct edges of $\bar{\sigma}$. Since $\bar{B}$ is CAT(0) it follows that $\bar{\sigma}$ is not a local geodesic.

By the local convexity of wall projections (Remark 3.15), the above argument shows that for any distinct elevation $\bar{A}'$ whose common hyperplanes with $\bar{A}$ are M-close, each connected component of $\mathrm{WProj}_{\bar{X}}(\bar{A}' \to \bar{A})$ is contained in a cubical N-ball, thus it is trivial. □

Lemma 5.7 (nearby elevations). *Let $A \to X$ be a compact local isometry of special cube complexes. Assume $\pi_1 A \to \pi_1 X$ is malnormal and the universal cover $\widetilde{X}$ is Gromov-hyperbolic.*

Then for any positive number $D > 0$ there is a finite cover $X_1 \to X$ with a based elevation A_1 with the following properties.

All elevations of $A \to X$ to X_1 are injective, and for any further cover $(\widehat{X}, \widehat{A}) \to (X_1, A_1)$ and any elevation $\widehat{A}' \neq \widehat{A}$, if $d(\widehat{A}', \widehat{A}) \leq D$ then $\widehat{A}'$ has trivial wall-projection onto $\widehat{A}$.

Proof. By Lemma 4.9, we may pass to an initial finite cover $\dot{X} \to X$ we may assume that all elevations of $A \to X$ are embedded, and have embedding radius $> D + R$, where R is the constant of Corollary 4.16. Let $\dot{A}$ be a fixed elevation of A to $\dot{X}$. Since the embedding radius of $\dot{A}$ is $> D + R$ we see that $\dot{A}^{+D}$ embeds in $\dot{X}$. Let $\dot{A}_i$ be an elevation intersecting $\dot{A}^{+D}$, and let C_{ij} be the various components of this intersection. Let $M = \max_{ij} \big(\mathrm{diameter}(C_{ij}^{+K})\big)$ which is obviously finite.

We apply Lemma 5.5 to $\dot{A} \subset \dot{X}$ with the constant M in order to obtain the finite cover $\bar{X}$ and elevation $\bar{A}$, such if $\bar{A}' \neq \bar{A}$ is an elevation whose common hyperplanes with $\bar{A}$

are M-close, then $\bar{A}'$ has trivial wall projection onto $\bar{A}$ in $\bar{X}$, and this persists in further covers.

Since the embedding radius of $\bar{A}$ is $> D+R$ the space $(\bar{A})^{+D}$ embeds in $\bar{X}$. For any distinct elevation $\bar{A}_k$ within distance D of $\bar{A}$ we choose a connected component $\bar{C}_{k\ell}$ of the nonempty intersection $\bar{A}^{+D} \cap \bar{A}_k$, and we form a space $S_{k\ell}$ by attaching $\bar{A}^{+D}$ with $\bar{A}_k$ along $\bar{C}_{k\ell}$. By malnormality of $\pi_1 A$, each C_{ij} is simply-connected and thus each $\bar{C}_{k\ell}$ factors isomorphically through some C_{ij}. Consequently diameter$(\bar{C}_{k\ell}^{+K}) \leq M$ always holds.

The embedding radius of $\bar{A}_k$ inside $\bar{X}$ is $> D+R \geq R$. The embedding radius of $\bar{A}^{+D}$ inside $\bar{X}$ is $> (D+R)-D = R$. Thus by Corollary 4.16 the natural map $S \to \bar{X}$ factors as $S_{k\ell} \to T_{k\ell} \hookrightarrow X_{k\ell} \to \bar{X}$, where $X_{k\ell} \to \bar{X}$ is a finite cover, $S_{k\ell} \to T_{k\ell}$ is an injective π_1-isomorphism, $T_{k\ell} \subset X_{k\ell}$ is a connected, wall-injective locally convex subcomplex, and any path of $T_{k\ell}$ connecting $\bar{A}_k$ to $\bar{A}^{+K}$ enters $\bar{C}_{k\ell}^{+K}$.

Since $T_{k\ell} \subset X_{k\ell}$ is wall-injective it follows that any hyperplane of $X_{k\ell}$ from $\bar{A}_k$ to $\bar{A}$ enters $\bar{C}_{k\ell}^{+K}$, and thus any two such hyperplanes are M-close. Consequently the wall projection from $\bar{A}_k$ to $\bar{A}$ is trivial.

We consider the various covers $X_{k\ell} \to \bar{X}$ associated to the finitely many choices of $C_{k\ell}$. Each finite cover $X_{k\ell}$ contains an isomorphic elevation of $\bar{A}$ and we denote by $(X_1, A_1) \to (\bar{X}, \bar{A})$ the $\bar{A}$-component of the fiber-product of the various covers $\{X_{k\ell} \to \bar{X}\}$. Note that $A_1 \cong \bar{A}$.

The cover $X_1 \to X$ has the desired properties. Indeed consider any further cover $(\widehat{X}, \widehat{A}) \to (X_1, A_1)$ and any elevation $\widehat{A}' \neq \widehat{A}$ with $d(\widehat{A}', \widehat{A}) \leq D$.

In $\bar{X}$ the embedding radius of $\bar{A}$ is $> D$. Thus since $\widehat{A}' \neq \widehat{A}$ and $d(\widehat{A}', \widehat{A}) \leq D$ the image of $\widehat{A}'$ inside $\bar{X}$ is distinct from $\bar{A}$ and thus equals one of the $\bar{A}_k$ discussed above. Let A_1' be the image of $\widehat{A}'$ in X_1, and again note that $A_1' \neq A'$ - indeed it maps to $\bar{A}_k \neq \bar{A}$ in $\bar{X}$.

Observe that $X_1 \to \bar{X}$ factors through $X_{k\ell}$ and A_1' maps onto $\bar{A}_k \subset T_{k\ell} \subset X_{k\ell}$. Since the wall-projection of $\bar{A}_k$ onto $\bar{A}$ inside $X_{k\ell}$ is trivial, it follows that $\text{WProj}_{X_1}(A_1' \to A_1)$ is trivial, and hence so is $\text{WProj}_{\widehat{X}}(\widehat{A}' \to \widehat{A})$. □

5.C. **Proof of Theorem 5.3.**

Step 1: Preparation. Let $R \geq K$ be the constants of Corollary 4.16. Choose constants $D > (3K+1)\dim(X)$, and $M \geq 4K\dim(X)$, and $R_1 \geq \max(K+R, 4K+2)$. We first pass to a finite cover $(X_0, A_0) \to (X, A)$ such that:

(1) X_0 is special
(2) each elevation of A to X_0 is injective
(3) each elevation of A to X_0, each hyperplane of X_0 has embedding radius $> R_1$
(4) each elevation $A_0' \neq A_0$ with $d(A_0, A_0') \leq D$ has trivial wall projection
(5) Let $(\bar{X}, \bar{A}) \to (X_0, A_0)$ be any further cover, then for any other elevation $\bar{A}'$ of A, if the hyperplanes between $\bar{A}, \bar{A}'$ are M-close then $\mathsf{WProj}_{\bar{X}}(\bar{A}' \to \bar{A})$ is trivial.

(The adequate values of the constants D, M, R_1 will be adjusted later.)

These properties are stable under covers, so we obtain them consecutively. First we choose a finite special cover $X_1 \to X$. Let A_1 be an elevation of $A \to X$ to X_1. We form the canonical completion $\mathsf{C}(A_1, X_1) \to X$ and note that its based elevation is injective.

Let $X_2 \to X$ be a finite regular cover factoring through $\mathsf{C}(A_1, X_1) \to X$, and note that all elevations of A to X_2 are injective. Then we apply Lemma 4.9 and Corollary 4.11 to get a finite cover $X_3 \to X_2$ with arbitrarily high embedding radii of the desired subcomplexes. To get the two last properties we consecutively apply Lemmas 5.7 and 5.5.

Let $\bar{A} \to A_0$ be any finite cover. Using separability we complete $\bar{A} \to X_0$ to a finite cover $\dot{X} \to X_0$.

Step 2: Connected intersection of thickened elevations and hyperplanes. We claim that there is a further finite cover $\ddot{X} \to \dot{X}$ with an isomorphic elevation of $\bar{A}$, such that the following holds: for any hyperplane $\ddot{H}$ of $\ddot{X}$ dual to an edge of $\bar{A}$ and for any pair of connected locally convex subcomplexes $\ddot{Y}, \ddot{Z} \subset \ddot{X}$ satisfying $\bar{A} \subset \ddot{Y} \subset (\bar{A})^{+R_1},\ N(\ddot{H}) \subset \ddot{Z} \subset N(\ddot{H})^{+R_1}$, the intersection $\ddot{Y} \cap \ddot{Z}$ is connected.

Indeed for any hyperplane $\dot{H}$ of $\dot{X}$ dual to an edge e of $\bar{A}$ and for any pair of connected locally convex subcomplexes $\dot{Y}, \dot{Z} \subset \dot{X}$ satisfying $\bar{A} \subset \dot{Y} \subset (\bar{A})^{+R_1},\ N(\dot{H}) \subset \dot{Z} \subset N(\dot{H})^{+R_1}$, let $C_{\dot{Y}\dot{Z}}$ denote the component of $\dot{Y} \cap \dot{Z}$ containing e. By Lemma 4.19, there is a finite cover $\dot{X}_{\dot{H},\dot{Y},\dot{Z}} \to \dot{X}$ with an isomorphic elevation of $\bar{A}$ such that the elevations of $\dot{Y}, \dot{Z}$ at e intersect connectedly. The fiber product of the various covers $\dot{X}_{\dot{H},\dot{Y},\dot{Z}} \to \dot{X}$ contains a natural isomorphic elevation of $\bar{A}$ that is contained in a connected component $\ddot{X}$ of the fiber product.

We claim that $\ddot{X}$ has the required connectedness property. This is clear for $\ddot{Y}$ arbitrary and $\ddot{Z} = N(\ddot{H})$ or $\ddot{Z} = N(\ddot{H})^{+R_1}$, because in that case $\ddot{Y}, \ddot{Z}$ cover subcomplexes $\dot{Y}, \dot{Z}$. The result follows for an intermediate $N(\ddot{H}) \subset \ddot{Z} \subset N(\ddot{H})^{+R_1}$ using Lemma 4.18.

Step 3: Geometric properties of the union of $\bar{A}$ and a hyperplane.

We now consider the collection of spaces $\ddot{U}_1, \ldots, \ddot{U}_k, \ddot{V}_1, \ldots, \ddot{V}_\ell$ arising in the following two ways:

- the union $\ddot{U}_i$ of $\bar{A}$ and a hyperplane H passing through it;
- the union $\ddot{V}_j$ of $\bar{A}$, a distant elevation $\ddot{A}'$, and a hyperplane H passing through both.

Each $\ddot{U}_i, \ddot{V}_j$ is quasiconvex and we consider their locally convex thickenings $\ddot{B}_i \to \ddot{X}, \ddot{C}_j \to \ddot{X}$. Below we describe precisely these constructions.

Observe that $\bar{A}$, and more generally any intermediate subcomplex $\bar{A} \subset \ddot{Y} \subset (\bar{A})^{+R_1}$, is wall-injective in $\ddot{X}$. Indeed, let e_1, e_2 be 1-cells of $\ddot{Y}$ dual to a hyperplane $\ddot{H}$. Since $\ddot{Y} \cap N(\ddot{H})$ is connected by Step 2, there is a path p in $\ddot{Y} \cap N(\ddot{H})$ from e_1 to e_2. By local convexity, $e_1 p e_2$ travels on a sequence of squares of $\ddot{Y}$ dual to $\ddot{H}$. Thus any hyperplane $\ddot{H}$ that intersects $\bar{A}$ actually meets $\bar{A}$ along a single hyperplane of $\bar{A}$, which we denote by $\ddot{H}_{\bar{A}}$.

We then form the space $\ddot{U}$ by gluing $\bar{A}$ and $N(\ddot{H})$ along their connected intersection. By construction the embedding radii of both $\bar{A}$ and H in $\ddot{X}$ are $> R_1$. In the sequel we denote by R, K the constant of Corollary 4.16, and we assume from now on that $R_1 \geq R$. By Corollary 4.16 the map $\ddot{U} \to \ddot{X}$ factors through a local isometry $\ddot{B} \to \ddot{X}$, so that $\ddot{U} \to \ddot{B}$ is an injective π_1-isomorphism. Furthermore there are connected locally convex subcomplexes $\mathcal{A}_{\ddot{H}}, \ddot{\mathcal{H}}$ of $\ddot{B}$ such that $\bar{A} \subset \mathcal{A}_{\ddot{H}} \subset (\bar{A})^{+K},\ N(\ddot{H}) \subset \ddot{\mathcal{H}} \subset (N(\ddot{H}))^{+K}$, $\ddot{B} = \mathcal{A}_{\ddot{H}} \cup \ddot{\mathcal{H}}$, the intersection $\mathcal{A}_{\ddot{H}} \cap \ddot{\mathcal{H}}$ is connected and contained inside $\big(\bar{A} \cap N(\ddot{H})\big)^{+K}$.

We note that since $R_1 \geq R \geq K$ the map $\ddot{B} \to \ddot{X}$ is injective on both $\mathcal{A}_{\ddot{H}}, \ddot{\mathcal{H}}$. We claim that in fact $\ddot{B} \to \ddot{X}$ is injective. We prove this by checking that $\mathcal{A}_{\ddot{H}} \cap \ddot{\mathcal{H}}$ is connected. This follows from Step 2 where we established the connectedness of $\mathcal{A}_{\ddot{H}} \cap \ddot{\mathcal{H}}$ provided $\bar{A} \subset \mathcal{A}_{\ddot{H}} \subset (\bar{A})^{+R_1}$, $N(\ddot{H}) \subset \ddot{\mathcal{H}} \subset N(\ddot{H})^{+R_1}$ (recall that $K \leq R_1$).

Since $\mathcal{A}_{\ddot{H}} \subset \bar{A}^{+K}$ we see that the embedding radius of $\mathcal{A}_{\ddot{H}}$ is $\geq R_1 - K \geq R$, and likewise $\ddot{H}$ has embedding radius $\geq R$. Since $\bar{A} \subset (\mathcal{A}_{\ddot{H}})^{+R} \subset (\bar{A})^{+R_1}$ and $N(\ddot{H}) \subset \ddot{\mathcal{H}}^{+R} \subset (N(\ddot{H}))^{+R_1}$, step 2 implies that $(\mathcal{A}_{\ddot{H}})^{+R} \cap \ddot{\mathcal{H}}^{+R}$ is connected. We can now apply Lemma 4.12 to see that the embedding radius of $\ddot{B}$ is $\geq R$.

Assume now that $\ddot{H}$ cuts an elevation $\ddot{A}'$ with $d(\ddot{A}', \bar{A}) > D$. Let γ be a connected component of $\ddot{B} \cap \ddot{A}'$ that contains an edge dual to $\ddot{H}$. Let $\ddot{W}$ be the space obtained by attaching $\ddot{B}$ to $\ddot{A}'$ along γ. We denote by $\ddot{V} \subset \ddot{W}$ the subspace $\ddot{U} \cup \ddot{A}'$, which supports all of the fundamental group. By Corollary 4.16 the map $\ddot{W} \to \ddot{X}$ factors through a local isometry $\ddot{C} \to \ddot{X}$, so that $\ddot{W} \to \ddot{C}$ is an injective π_1-isomorphism. Furthermore there are connected locally convex subcomplexes $\mathcal{B}, \mathcal{A}'_\gamma$ of $\ddot{C}$ such that $\ddot{B} \subset \mathcal{B} \subset \ddot{B}^{+K}$, $\ddot{A}' \subset \mathcal{A}'_\gamma \subset (\ddot{A}')^{+K}$, $\ddot{C} = \mathcal{B} \cup \mathcal{A}'_\gamma$, the intersection $\mathcal{B} \cap \mathcal{A}'_\gamma$ is connected and contained inside γ^{+K}.

Step 4: Construction of $\bar{X} \to X$. The inclusion $\bar{A} \to \ddot{X}$ factors through the various local isometries $\ddot{B}_i, \ddot{C}_j \to \ddot{X}$. We apply Corollary 4.24 to the collection of local isometries $\{\ddot{B}_i \to \ddot{X}\}, \{\ddot{C}_j \to \ddot{X}\}$ relative to $\bar{A}$. This yields a cover $(\bar{X}, \bar{A})$ in which their elevations $\{\bar{B}_i, \bar{C}_j\}$ are injective and have pairwise connected intersection.

Step 5: Verifying that wall projections are trivial. Let $\bar{A}' \neq \bar{A}$ be an elevation of A, and let $\ddot{A}'$ denote the image of $\bar{A}'$ inside $\ddot{X}$.

If $d(\ddot{A}', \bar{A}) \leq D$ and $\ddot{A}' \neq \bar{A}$ then $\mathrm{WProj}_{\ddot{X}}(\ddot{A}' \to \bar{A})$ is already trivial and we are done by Lemma 5.2.

Otherwise either $\ddot{A}' = \bar{A}$ or $\ddot{A}'$ is distant from $\bar{A}$. In each of these cases we will deduce the triviality of the wall-projection by showing M-closeness of the hyperplanes cutting through both $\bar{A}$ and $\bar{A}'$.

In the first case consider any two hyperplanes $\bar{H}_1, \bar{H}_2$ that are common to $\bar{A}$ and $\bar{A}'$. Their images in $\ddot{X}$ are $\ddot{H}_1, \ddot{H}_2$. For each i, let $\ddot{U}_i$ and $\ddot{B}_i$ be the spaces from step 3 associated to $(\bar{A}, \ddot{H}_i)$.

Since $\bar{A}'$ maps to $\bar{A}$ and $\bar{H}_1, \bar{H}_2$ map to $\ddot{H}_1, \ddot{H}_2$ we have $\bar{A}' \subset \bar{U}_1 \cap \bar{U}_2 \subset \bar{B}_1 \cap \bar{B}_2$ where $\bar{U}_i \subset \bar{B}_i$ denotes the elevation of $\ddot{U}_i$ that contains $\bar{A}$.

There is an edge-path $\bar{\sigma}$ inside the connected subcomplex $\bar{B}_1 \cap \bar{B}_2$ that starts at $\bar{A}$ and ends in $\bar{A}'$. Since $\ddot{B}_1 = (\mathcal{A}_{\ddot{H}_1} \cup \ddot{\mathcal{H}}_1) \subset (\bar{A}^{+K} \cup \ddot{H}_1^{+K})$, the locally convex subcomplex $\bar{B}_1$ is contained in a union of elevations of $\bar{A}^{+K}$ and $\ddot{H}_1^{+K}$. Since $R_1 \geq K + 1$ we have $\bar{A}^{+K} \cap \bar{A}' = \emptyset$, thus there is a first vertex $\bar{p}$ on $\bar{\sigma}$ which is not in $\bar{A}^{+K}$. Note that $\bar{p} \in \bar{A}^{+(K+1)}$. Since $R_1 \geq 2K + 1$ this vertex $\bar{p}$ does not belong to any elevation of $\bar{A}^{+K} \subset \ddot{X}$ to $\bar{X}$. Thus $\bar{p} \in (\bar{e}'_1)^{+K}$ where $\bar{e}'_1$ is an edge dual to a hyperplane $\bar{H}'_1$ mapping to the hyperplane $\ddot{H}_1$ and contained inside $\bar{B}_1$. Given an edge $\bar{e}_1$ dual to $\bar{H}_1$ and contained in $\bar{A}$, we have $\bar{e}'_1, \bar{e}_1 \subset \bar{A}^{+(2K+2)}$. The images of these edges under $\bar{X} \to \ddot{X}$ are both dual to $\ddot{H}_1$. Since $R_1 \geq 2K + 2$ we have already seen that $\bar{A}^{+(2K+2)}$ is wall-injective in $\ddot{X}$. Since $\bar{X} \to \ddot{X}$ induces an isomorphism $\bar{A}^{+(2K+2)} \to \bar{A}^{+(2K+2)}$ it follows that $\bar{e}_1', \bar{e}_1$ are dual to the same hyperplane of $\bar{A}^{+(2K+2)}$, and so $\bar{H}_1' = \bar{H}_1$. Similarly $\bar{p} \in \bar{H}_2^{+K}$. It follows that there is a path of length $\leq 2K\dim(X)$ from $N(\bar{H}_1)$ to $N(\bar{H}_2)$. See Remark 4.1 for the relationship between combinatorial neighborhoods and cubical thickenings.

The second case (when $\ddot{A}'$ is a distant elevation) is similar except that we use locally convex thickenings $\ddot{C}_1, \ddot{C}_2$ of the spaces built from $\bar{A}$, $\ddot{A}'$, together with $\ddot{H}_1, \ddot{H}_2$ respectively. There is no difference from the previous explanation at the vertex of $\bar{\sigma}$ leaving the K-thickening of $\bar{A}$: it comes within a uniform distance of both $\bar{H}_1$ and $\bar{H}_2$. And such a vertex exists provided D is large enough. We shall now provide the details:

Consider any two hyperplanes $\bar{H}_1, \bar{H}_2$ cutting both $\bar{A}$ and $\bar{A}'$. Choose an edge $\bar{a}_i'$ of $\bar{A}'$ dual to $\bar{H}_i$, and denote by a_i' the image of $\bar{a}_i'$ inside $\ddot{X}$. The images of $\bar{H}_1, \bar{H}_2$ in $\ddot{X}$ are $\ddot{H}_1, \ddot{H}_2$. For each i, let $\ddot{B}_i$ be the locally convex thickening of $\bar{A} \cup N(\ddot{H}_i)$. We let γ_i denote the connected component of a_i' inside $\ddot{B}_i \cap \ddot{A}'$. Let $\ddot{V}_i, \ddot{W}_i, \ddot{C}_i$ be the immersed spaces associated to $\ddot{B}_i, \ddot{A}', \gamma_i$ that we constructed in Step 3.

Since $\bar{A}'$ maps to $\ddot{A}'$ and $\bar{H}_1, \bar{H}_2$ map to $\ddot{H}_1, \ddot{H}_2$ we have $\bar{A}' \subset \bar{W}_1 \cap \bar{W}_2 \subset \bar{C}_1 \cap \bar{C}_2$.

There is an edge-path $\bar{\sigma}$ inside the connected subcomplex $\bar{C}_1 \cap \bar{C}_2$ that starts at $\bar{A}$ and ends in $\bar{A}'$.

The following inclusion shows that the locally convex thickening $\bar{C}_1$ is contained in the union of elevations of $\ddot{A}'^{+K}$, elevations of $\bar{A}^{+2k} \subset \ddot{X}$, and elevations of $\ddot{H}_1^{+2K}$:

$$\ddot{C}_1 \;=\; \mathcal{B}_1 \cup \mathcal{A}_1' \;\subset\; (\mathcal{A}_{\ddot{H}_1} \cup \ddot{\mathcal{H}})^{+K} \cup (\ddot{A}')^{+K} \;\subset\; \bar{A}^{+2K} \cup \ddot{H}_1^{+2K} \cup (\ddot{A}')^{+K}.$$

Since $D > (2K)\dim(X)$ we have $\bar{A}^{+2K} \cap \bar{A}' = \emptyset$, thus there is a first vertex $\bar{p}$ on $\bar{\sigma}$ which is not in $(\bar{A})^{+2K}$. Since $R_1 \geq 4K+1$ this vertex $\bar{p}$ does not belong to the $2K$-thickening of any elevation of $\bar{A} \subset \ddot{X}$. Since $D > (3K+1)\dim(X)$ the point $\bar{p}$ does not belong to the K-thickening of any elevation of $\ddot{A}'$ contained in $\bar{C}_1$. It follows that $\bar{p}$ belongs to the $2K$-thickening of an elevation of $N(\ddot{H}_1)$. Equivalently $\bar{p} \in (\bar{e}_1')^{+2K}$ for some edge $\bar{e}_1'$ dual to a hyperplane $\bar{H}_1'$ mapping to the hyperplane $\ddot{H}_1$ and contained inside $\bar{B}_1$. The argument ends in the same manner as in the first case: if $R_1 \geq 4K+2$ then $\bar{H}_1' = \bar{H}_1$, and so any two hyperplanes between $\bar{A}$ and $\bar{A}'$ are within a distance $4K\dim(X)$. □

5.D. **A variation on the theme.** We will need later the following result, which is a consequence of Theorem 5.3.

Corollary 5.8. *Let X be a compact virtually special cube complex, and let $A, B \to X$ be compact local isometries with $\pi_1 A \subset \pi_1 X$ malnormal, $\pi_1 B \subset \pi_1 X$ malnormal, and each conjugate of $\pi_1 A$ has trivial intersection with $\pi_1 B$. More precisely if $a \to A$ and $b \to B$ are immersed circles that are homotopic to each other in X, then they are null-homotopic.*

Assume $\pi_1 X$ is word hyperbolic.

Then there exists a finite cover $A_0 \to A$ such that any further finite cover $\bar{A} \to A_0$ can be completed to a finite special cover $\bar{X} \to X$ with the following properties:

1. *all elevations of $A \to X, B \to X$ to $\bar{X}$ are injective.*
2. *$\bar{A}$ is wall-injective.*
3. *every elevation of A distinct from $\bar{A}$ has trivial wall-projection onto $\bar{A}$.*

(4) *every elevation of B has trivial wall-projection onto $\bar{A}$.*

Proof. **Step 1: The auxiliary pair $C \to Y$.** We choose base points $\bar{a}, \bar{b}$ in A, B, and let a, b be their images inside X. We then consider the space Y obtained by adding a single 1-cube e to X with origin at a and with endpoint at b. We also form a connected cube complex C by setting $C = A \sqcup [0,1] \sqcup B/_{\bar{a}=0, \bar{b}=1}$. We denote by $\bar{e}$ the image of $[0,1]$ inside C. Mapping $\bar{e}$ to e we get a natural map $C \to Y$.

Step 2: Geometric properties of Y. Observe that $\widetilde{Y}$ is a hyperbolic CAT(0) cube complex and $C \to Y$ is a local isometry. Indeed the universal cover $\widetilde{Y}$ is a tree-like space, where the vertex spaces are disjoint copies of $\widetilde{X}$, connected by the edges mapping to e.

Step 3: $\pi_1 C \subset \pi_1 Y$ is malnormal. This can be proven by either simple disc diagram arguments, combinatorial group theory arguments involving normal forms, or geometric considerations in the universal cover. We leave the details to the reader.

Step 4: Y is virtually special. Let $\widehat{X} \to X$ be a special cover of finite degree d. The preimage of a consists of d points, and the preimage of b consists of d points. We then choose a one-to-one correspondence between preimages of a and preimages of b, and glue d edges accordingly. The resulting cube complex $\widehat{Y}$ covers Y, and it is special. Indeed the union of a special special cube complexes meeting along vertices is itself special since it straightforwardly satisfies the condition of Lemma 2.6.

Step 5: Constructing A_0. We can now apply Theorem 5.3 to the local isometry $C \to Y$. We thus obtain a finite cover $C_0 \to C$ such that any further finite cover $\bar{C} \to C_0$ extends to a finite special cover $\bar{Y} \to Y$ where all elevations of $C \to Y$ are injective and have trivial wall-projection onto $\bar{C}$, provided they are distinct from $\bar{C}$, and moreover $\bar{C} \subset \bar{Y}$ is wall-injective. We choose $A_0 \subset C_0$ to be a fixed elevation of $A \subset C$ to the covering space $C_0 \to C$.

Step 6: Conclusion. We now verify that $A_0 \to A$ has the desired property. Let $\bar{A} \to A_0$ be any finite cover. C_0 is special since it is a locally convex subcomplex of the special complex Y_0, and thus $A \to A_0 \subset C_0$ extends to a finite cover $\bar{C} \to C_0$.

We next further complete $\bar{C} \to C_0$ to a finite special cover $\bar{Y} \to Y$ with the properties of Theorem 5.3. We denote by $\bar{X}$ the elevation of $X \subset Y$ that contains $\bar{A}$, and we claim that $\bar{X}$ has the desired properties.

$\bar{X}$ is special since it is a locally convex subcomplex of the special cube complex $\bar{Y}$.

Note that $\bar{A}$ is wall-injective in $\bar{C}$, thus also in $\bar{Y}$ and a fortiori in $\bar{X}$.

Each elevation of $A \to X$ or $B \to X$ to $\bar{X}$ extends to an elevation of $C \to Y$, and is thus injective.

Consider an elevation $E \neq \bar{A}$ of either A or B to $\bar{X}$. Let $\bar{C}'$ be the elevation of C containing E. We first treat the case that $\bar{C}' = \bar{C}$. Observe that E and $\bar{A}$ have no common hyperplane in $\bar{C}$ since $\bar{C}$ is the disjoint union of covers of A and B attached together along isolated 1-cells. The wall injectivity of $\bar{C} \subset \bar{X}$ implies that E and $\bar{A}$ have no common hyperplane in $\bar{X}$ either. Thus $\mathsf{WProj}_{\bar{X}}(E \to \bar{A}) = \bar{A}^0$ and is thus trivial.

In the other case where $\bar{C}' \neq \bar{C}$, we see that $\mathsf{WProj}_{\bar{X}}(E \to \bar{A})$ is contained in $\mathsf{WProj}_{\bar{Y}}(\bar{C}' \to \bar{C})$ and is thus trivial. □

6. The Main Technical Result: A Symmetric Covering Property

Let P be an embedded 2-sided hyperplane in the cube complex Q. We define $Q - P$ to be the complex obtained by first subdividing along P, then cutting along P to obtain one or two components in $Q - P$, and then attaching exactly two copies of P in the obvious manner.

Theorem 6.1. *Let Q be a compact connected nonpositively curved cube complex, and let P be a hyperplane in Q such that the following hold:*

(1) *$\pi_1 Q$ word-hyperbolic.*
(2) *P is an embedded, nonseparating, 2-sided hyperplane in Q.*
(3) *$\pi_1 P$ is malnormal in $\pi_1 Q$.*
(4) *$Q - P$ is virtually special.*

Let $X = Q - P$. For any finite cover $\widehat{X}$ of X, there is a finite regular cover $\overset{\bowtie}{X}$ factoring through $\widehat{X}$, such that $\overset{\bowtie}{X} \to X$ induces the same cover on each side A, B of P.

Proof. Let $\alpha : P \to X$ and $\beta : P \to X$ denote the two maps corresponding to the two sides of P in Q, so Q is the quotient of X obtained by identifying $\alpha(p) = \beta(p)$ for $p \in P$. Let $A = \alpha(P)$ and $B = \beta(P)$.

We will show that for each finite cover $\widehat{X}$ there is a finite regular cover $\overset{\bowtie}{X}$ such that the isomorphism $\gamma = \beta\alpha^{-1}$ from A to B, lifts to an isomorphism $\overset{\bowtie}{\gamma}$ from some elevation $\overset{\bowtie}{A}$ to some elevation $\overset{\bowtie}{B}$.

Step 1: (Covers with trivial wall projection onto $\widehat{A}_a$ and $\widehat{B}_b$)

There exists finite special connected covers $\widehat{X}_a$ and $\widehat{X}_b$ factoring through $\widehat{X}$ such that:

(1) The base elevation $\widehat{A}_a$ of A to $\widehat{X}_a$ is wall injective in $\widehat{X}_a$.
(2) The base elevation $\widehat{B}_b$ of B to $\widehat{X}_b$ is wall injective in $\widehat{X}_b$.
(3) The isomorphism $\gamma : A \to B$ lifts to an isomorphism $\widehat{\gamma} : \widehat{A}_a \to \widehat{B}_b$.
(4) $\mathsf{WProj}_{\widehat{X}_a}(\breve{B} \to \widehat{A}_a)$ is trivial for each elevation $\breve{B}$ of B to $\widehat{X}_a$.
(5) $\mathsf{WProj}_{\widehat{X}_a}(\breve{A} \to \widehat{A}_a)$ is trivial for each elevation $\breve{A}$ of A to $\widehat{X}_a$ with $\breve{A} \neq \widehat{A}_a$.
(6) $\mathsf{WProj}_{\widehat{X}_b}(\breve{A} \to \widehat{B}_b)$ is trivial for each elevation $\breve{A}$ of A to $\widehat{X}_b$.
(7) $\mathsf{WProj}_{\widehat{X}_b}(\breve{B} \to \widehat{B}_b)$ is trivial for each elevation $\breve{B}$ of B to $\widehat{X}_b$ with $\breve{B} \neq \widehat{B}_b$.

This follows by Corollary 5.8.

Step 2: We form the following canonical completions and note the inclusion maps hold by Lemma 3.12 and the isomorphism is obtained from the isomorphism $\widehat{A}_a \cong \widehat{B}_b$.

$$\mathsf{C}(\widehat{A}_a, \widehat{X}_a) \hookleftarrow \mathsf{C}(\widehat{A}_a, \widehat{A}_a) \cong \mathsf{C}(\widehat{B}_b, \widehat{B}_b) \hookrightarrow \mathsf{C}(\widehat{B}_b, \widehat{X}_b)$$

Step 3: There exist based covers $\bar{A} \to A$ and $\bar{B} \to B$ such that:

(1) the isomorphism $\gamma : A \to B$ lifts to an isomorphism $\bar{\gamma} : \bar{A} \to \bar{B}$ so we have the following commutative diagram:

$$\begin{array}{ccc} \bar{A} & \to & \bar{B} \\ \downarrow & & \downarrow \\ \widehat{A}_a & \to & \widehat{B}_b \end{array}$$

(2) $\bar{A}$ factors through each elevation of A to $\mathsf{C}(\widehat{A}_a, \widehat{X}_a)$ and to $\mathsf{C}(\widehat{B}_b, \widehat{X}_b)$.
(3) $\bar{B}$ factors through each elevation of B to $\mathsf{C}(\widehat{A}_a, \widehat{X}_a)$ and to $\mathsf{C}(\widehat{B}_b, \widehat{X}_b)$.

Indeed, we simply choose based covers of A and B that factor through all elevations, and then a common cover using the isomorphism $\gamma : A \to B$.

Step 4: The canonical retraction map $\mathsf{C}(\widehat{A}_a, \widehat{X}_a)$ together with the cover $\bar{A} \to A_a$ induces the covers $\mathsf{C}(\widehat{\widehat{A}_a, \widehat{X}_a})$ and $\mathsf{C}(\widehat{\widehat{A}_a, \widehat{A}_a})$. Similarly, we obtain $\mathsf{C}(\widehat{\widehat{B}_b, \widehat{X}_b})$ and $\mathsf{C}(\widehat{\widehat{B}_b, \widehat{B}_b})$ so we have the following commutative diagrams:

$$\begin{array}{ccc} \mathsf{C}(\widehat{\widehat{A}_a, \widehat{X}_a}) & \to & \bar{A} \\ \downarrow & & \downarrow \\ \mathsf{C}(\widehat{A}_a, \widehat{X}_a) & \to & \widehat{A}_a \end{array} \quad \begin{array}{ccc} \mathsf{C}(\widehat{\widehat{A}_a, \widehat{A}_a}) & \to & \bar{A} \\ \downarrow & & \downarrow \\ \mathsf{C}(\widehat{A}_a, \widehat{A}_a) & \to & \widehat{A}_a \end{array} \quad \begin{array}{ccc} \mathsf{C}(\widehat{\widehat{B}_b, \widehat{B}_b}) & \to & \bar{B} \\ \downarrow & & \downarrow \\ \mathsf{C}(\widehat{B}_b, \widehat{B}_b) & \to & \widehat{B}_b \end{array} \quad \begin{array}{ccc} \mathsf{C}(\widehat{\widehat{B}_b, \widehat{X}_b}) & \to & \bar{B} \\ \downarrow & & \downarrow \\ \mathsf{C}(\widehat{B}_b, \widehat{X}_b) & \to & \widehat{B}_b \end{array}$$

Because of the isomorphism between $\bar{A} \to \widehat{A}_a$ and $\bar{B} \to \widehat{B}_b$ above, we have the following commutative diagrams:

$$\begin{array}{ccccccc} \mathsf{C}(\widehat{\widehat{A}_a, \widehat{X}_a}) & \hookleftarrow & \mathsf{C}(\widehat{\widehat{A}_a, \widehat{A}_a}) & \cong & \mathsf{C}(\widehat{\widehat{B}_b, \widehat{B}_b}) & \hookrightarrow & \mathsf{C}(\widehat{\widehat{B}_b, \widehat{X}_b}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathsf{C}(\widehat{A}_a, \widehat{X}_a) & \hookleftarrow & \mathsf{C}(\widehat{A}_a, \widehat{A}_a) & \cong & \mathsf{C}(\widehat{B}_b, \widehat{B}_b) & \hookrightarrow & \mathsf{C}(\widehat{B}_b, \widehat{X}_b) \end{array}$$

Step 5:
Let $\overset{\bowtie}{X}_a$ denote the smallest regular cover factoring through each component of $\bar{X}_a$.
Let $\overset{\bowtie}{X}_b$ denote the smallest regular cover factoring through each component of $\bar{X}_b$.
Let $\overset{\bowtie}{X}$ denote the smallest regular cover factoring through $\overset{\bowtie}{X}_a$ and $\overset{\bowtie}{X}_b$.
Let $\overset{\bowtie}{A}$ denote the smallest regular cover factoring through each component of $\bar{A}_a$.
Let $\overset{\bowtie}{B}$ denote the smallest regular cover factoring through each component of $\bar{B}_b$.
Let $\overset{\bowtie}{A}_a$, and $\overset{\bowtie}{A}_b$ denote the elevations of A to $\overset{\bowtie}{X}_a$ and $\overset{\bowtie}{X}_b$.
Let $\overset{\bowtie}{B}_a$, and $\overset{\bowtie}{B}_b$ denote the elevations of B to $\overset{\bowtie}{X}_a$ and $\overset{\bowtie}{X}_b$.
It is clear that the isomorphism $\gamma : A \to B$ lifts to an isomorphism $\overset{\bowtie}{\gamma}:\overset{\bowtie}{A}\to\overset{\bowtie}{B}$.

We will show that $\overset{\bowtie}{A}\cong\overset{\bowtie}{A}_a$ since they factor through each other, and that $\overset{\bowtie}{A}$ factors through $\overset{\bowtie}{A}_b$. It will follow that each elevation of A to $\overset{\bowtie}{X}$ is isomorphic to $\overset{\bowtie}{A}$. An analogous argument shows that $\overset{\bowtie}{B}\cong\overset{\bowtie}{B}_b$ and that $\overset{\bowtie}{B}$ factors through $\overset{\bowtie}{B}_a$. Consequently, each elevation of B to $\overset{\bowtie}{X}$ is isomorphic to $\overset{\bowtie}{B}$.

Since $\mathsf{C}(\widehat{A_a, A_a}) \subset \mathsf{C}(\widehat{A_a, X_a})$, it is obvious that $\overset{\bowtie}{A}_a$ factors through $\overset{\bowtie}{A}$.

Since $\overset{\bowtie}{X}_a$ is the smallest regular cover induced by $\mathsf{C}(\widehat{X_a, A_a})$, to see that $\overset{\bowtie}{A}$ factors through $\overset{\bowtie}{A}_a$, it is enough to check that $\overset{\bowtie}{A}$ factors through each elevation of A to $\mathsf{C}(\widehat{X_a, A_a})$.

There are two cases to consider, according to the history of A as it follows a sequence of elevations indicated below:

$$\begin{array}{ccccccc} A_3 & \to & A_2 & \to & A_1 & \to & A \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathsf{C}(\widehat{\widehat{A}_a, \widehat{X}_a}) & \to & \mathsf{C}(\widehat{A}_a, \widehat{X}_a) & \to & \widehat{X}_a & \to & X \end{array}$$

If $A_1 = \widehat{A}_a$ is the base elevation of A, then by Lemma 3.13 $A_2 \subset \mathsf{C}(\widehat{A}_a, \widehat{A}_a) \subset \mathsf{C}(\widehat{A}_a, \widehat{X}_a)$. Since A_3 is contained in $\mathsf{C}(\widehat{\widehat{A}_a, \widehat{A}_a})$ we see that $\overset{\bowtie}{A}$ factors through A_3.

If $A_1 \neq \widehat{A}_a$, then $\mathsf{WProj}_{\widehat{X}_a}(A_2 \to \widehat{A}_a)$ is trivial, and so by Lemma 3.16 A_2 is nullhomotopic in the retraction map $\mathsf{C}(\widehat{A}_a, \widehat{X}_a) \to \widehat{A}_a$. Thus $A_3 \cong A_2$. But $\overset{\bowtie}{A}$ factors through $\bar{A}$ which factors through A_2 which is isomorphic to A_3.

To see that $\overset{\bowtie}{A}$ factors through $\overset{\bowtie}{A}_b$, we show that $\bar{A}$ factors through $\overset{\bowtie}{A}_b$ by showing $\bar{A}$ factors through each elevation of A to $\mathsf{C}(\widehat{\widehat{B}_b, \widehat{X}_b})$. Again consider the history of elevations of A to $\mathsf{C}(\widehat{\widehat{B}_b, \widehat{X}_b})$

$$\begin{array}{ccccccc} A_3 & \to & A_2 & \to & A_1 & \to & A \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathsf{C}(\widehat{\widehat{B}_b, \widehat{X}_b}) & \to & \mathsf{C}(\widehat{B}_b, \widehat{X}_b) & \to & \widehat{X}_b & \to & X \end{array}$$

Observe that A_1 has trivial wall projection onto B_b, and so by Lemma 3.16, A_2 is nullhomotopic in the retraction map $\mathsf{C}(\widehat{B}_b, \widehat{X}_b) \to \widehat{X}_b$. Thus $A_3 \cong A_2$. But $\bar{A}$ factors through A_2 and hence $\overset{\bowtie}{A}$ factors through $\bar{A}$ which factors through $A_3 \cong A_2$. □

7. Subgroup Separability of Quasiconvex Subgroups

Without hyperbolicity, the hypothesis of Theorem 6.1 permits a proof of residual finiteness of $\pi_1 Q$ using the normal form theorem for graphs of groups. In this section we prove the following stronger property from this hypothesis together with hyperbolicity:

Theorem 7.1. *Let Q be a compact nonpositively curved cube complex. Let P be an embedded nonseparating 2-sided hyperplane in Q. Let X be the cube complex obtained from Q by deleting an open regular neighborhood of P, and assume X is connected. Suppose that Q has the cocompact convex core property. Suppose that for each finite cover $\widehat{X} \to X$, there is finite regular cover $\overset{\bowtie}{X} \to X$ factoring through $\widehat{X}$, such that $\overset{\bowtie}{X}$ induces the same cover on each side A, B of P. Assume that $\pi_1 X$ has separable double cosets of quasiconvex subgroups.*

Then every quasiconvex subgroup of $\pi_1 Q$ is separable.

Definition 7.2. The space Y has the *cocompact convex core property* if for each quasiconvex subgroup H of $\pi_1 Y$, and each compact subset $C \subset \widetilde{Y}$, there is a convex subset S of $\widetilde{Y}$ such that $C \subset S$ and S is H-stable and H-cocompact.

By Lemma 7.5, the compact nonpositively curved cube complex X has this property if $\pi_1 X$ is δ-hyperbolic.

Remark 7.3. It is likely that we can relax the hypothesis to assume that X has the compact core property for CAT(0) quasiconvex subgroups.

Double quasiconvex coset separability always for word-hyperbolic groups provided that quasiconvex subgroups are separable. Double coset separability implies virtual specialness for finite type cube complexes.

The proof follows the scheme for proving subgroup separability given in [Wis00].

Proof. We will consider on Q a structure of graph of spaces whose open edge space is the open cubical neighborhood of P, and whose vertex space is its complement X.

Let $\dot{Q}$ denote the based cover corresponding to a quasiconvex subgroup of $\pi_1 Q$. Let $\sigma \in \pi_1 Q - \pi_1 \dot{Q}$, and let S be a compact subspace of $\dot{Q}$ containing the based lift of σ. By the cocompact convex core property for Q, let Y be a compact core of $\dot{Q}$ containing S.

Y has an induced graph of space structure whose vertex spaces are components of the preimage of X in Y, and whose open edge spaces are components of the preimage of the open edge space of Q. Since vertex and edge spaces of $\dot{Q}$ are connected and locally convex, the intersection of Y with each vertex and edge space of $\dot{Q}$ is a connected and locally convex subcomplex by Lemma 4.18.1. These intersections correspond precisely to the induced vertex and edge spaces of Y. Let $Y_i = Y \cap X_i$ denote the vertex spaces of Y where $X_1, \ldots, X_n$ denote the vertex spaces of $\dot{Q}$ that have nonempty intersection with Y. We emphasize that $Y_i \hookrightarrow X_i$ induces a π_1-isomorphism.

For each i, we shall now use the separability of certain double cosets to produce a finite cover $\widehat{X}_i \to X$ with the following injectivity properties:

(1) $\dot{X}_i \to X$ factors as $\dot{X}_i \to \widehat{X}_i \to X$ and $Y_i \subset \dot{X}_i$ embeds in $\widehat{X}_i$ under this map.
(2) The distinct elevations $\dot{A}_{ij} \to \dot{X}_i$ of $A \to X$ whose images intersect $Y_i \subset \dot{X}_i$ factor through distinct elevations in $\widehat{X}_i$.
(3) The distinct elevations $\dot{B}_{ik} \to \dot{X}_i$ of $B \to X$ whose images intersect $Y_i \subset \dot{X}_i$ factor through distinct elevations in $\widehat{X}_i$.

For each i, the finitely many intersecting elevations correspond to double cosets of the form $\pi_1 Y_i \alpha_{ij} \pi_1 A$ and $\pi_1 Y_i \beta_{ik} \pi_1 B$. Without loss of generality, we can add a tail to B so that A, B, X are based spaces, and the maps are basepoint preserving. Consider the A-type cosets, each α_{ij} is a closed based path, and the elevation $\dot{A}_{ij} \to \dot{X}_i$ is the based elevation at the endpoint of the lift $\dot{\alpha}_{ij}$ to $\dot{X}_i$. We note that right-concatenating α_{ij} with an element of $\pi_1 A$ doesn't change the choice of elevation. Left-multiplication by elements of $\pi_1 Y_i = \pi_1 X_i$ lead to no change either.

Enumerate the finitely many pairs p_ℓ, q_ℓ of distinct vertices of Y_i that map to the same vertex of X, and for each ℓ let ω_ℓ be a path from the basepoint to p_ℓ and let σ_ℓ be a path from p_ℓ to q_ℓ. Note that the projections of the paths $\omega_\ell, \sigma_\ell, \omega_\ell^{-1}$ to X are concatenable to a closed path $\gamma_\ell = \omega_\ell \sigma_\ell \omega_\ell^{-1}$. A based lift of Y_i embeds in a based cover of $\ddot{X}$ of X precisely if each path $\gamma_\ell \notin \pi_1 \ddot{X}$.

By double coset separability (note that these compact local isometries have quasiconvex fundamental groups), for each i we are able to choose a finite index normal subgroup N_i such that the $N_i \pi_1 Y_i \alpha_{ij} \pi_1 A$ are all disjoint from each other, and the $N_i \pi_1 Y_i \beta_{ik} \pi_1 A$ are all disjoint from each other, and finally each $N_i \pi_1 Y_i \gamma_{i\ell}$ is disjoint from $N_i \pi_1 Y_i$. It follows that the based covering space $\widehat{X}_i$ with $\pi_1\widehat{X}_i = N_i\pi_1 Y_i$ has the properties enumerated above.

Let Z denote the space obtained from Y by extending each Y_i to $\widehat{X}_i$, so Z is the quotient of the disjoint union $Y \cup (\cup_i \widehat{X}_i)$ obtained by identifying each Y_i with its embedded image in $\widehat{X}_i$. Note that there is an induced map $Z \to Q$.

Let $\widehat{X}$ denote a finite cover factoring through each $\widehat{X}_i$. By hypothesis, let $\overset{\bowtie}{X} \to \widehat{X}$ be a finite connected regular cover, such that $\overset{\bowtie}{X}$ induces the same cover $\overset{\bowtie}{A}, \overset{\bowtie}{B}$ on each side A, B of P. Choose a one-to-one correspondence between the elevations of A and the elevations of B and attach a copy of $\overset{\bowtie}{P} \times [-1, 1]$ to form a finite cover $\overset{\bowtie}{Q} \to Q$ whose vertex space is $\overset{\bowtie}{X}$.

Let $\overset{\bowtie}{Z} = Z \otimes_Q \overset{\bowtie}{Q}$ denote the fiber product of $\overset{\bowtie}{Q} \to Q$ and $Z \to Q$.

Note that each vertex space of $\overset{\bowtie}{Z}$ is isomorphic to $\overset{\bowtie}{X}$, since it is a component of the fiber product of $\widehat{X}_i$ and $\overset{\bowtie}{X}$. Moreover, note that for each edge space of $\overset{\bowtie}{Z}$, its two ends are contained in copies of $\overset{\bowtie}{A}$ and $\overset{\bowtie}{B}$. We extend this edge space to a copy of $\overset{\bowtie}{P} \times I$.

Finally, we choose a one-to-one correspondence between remaining copies of $\overset{\bowtie}{A}$ and $\overset{\bowtie}{B}$ that do not have incident edge spaces. A simple count shows that there are the same number of each. And we attach edge spaces according to this one-to-one correspondence.

The result is a finite cover $\bar{Q}$ of Q. Indeed, the construction gives a natural combinatorial map $\bar{Q} \to Q$ that is a local isomorphism. This related to the enumerated properties that kept the edge space attachments disjoint from each other.

Finally, observe that the element σ is separated from $\pi_1\dot{Q}$ in the right representation on cosets of $\pi_1\bar{Q}$. Indeed, the endpoint of σ is not in the preimage in $\overset{\bowtie}{Z} \subset \bar{Q}$ of the basepoint of Y in $\overset{\bowtie}{Z} \subset \bar{Q}$. Thus $\pi_1 Y$ is separated from σ in the right coset representation, since they act differently on the base coset. □

Combining this and Theorem 6.1 we obtain the following:

Corollary 7.4. *Let Q be a compact connected nonpositively curved cube complex, and let P be a hyperplane in Q such that the following hold:*

1. *$\pi_1 Q$ word-hyperbolic.*
2. *P is an embedded 2-sided hyperplane in Q.*
3. *$\pi_1 P$ is malnormal in $\pi_1 Q$.*
4. *$Q - P$ is virtually special.*

Then every quasiconvex subgroup of $\pi_1 Q$ is separable.

Proof. The Corollary follows in case when P is not separating by combining Theorem 7.1 and Theorem 6.1. Note that the hypotheses of these Theorems hold as in Remark 7.3.

Assume now P is separating. Let Q' be obtained from Q by adding a new 1-cube which connects the components of X. Observe that $Q \to Q'$ is a local isometry. Obviously $\pi_1 Q'$ is word-hyperbolic. It is easy to verify that $\pi_1 P$ is still malnormal in $\pi_1 Q'$. And $Q' - P$ is virtually special: the disjoint union of connected special finite covers of the two connected components with a collection of 1-cubes provides a special cover. We conclude using the non-separating case since any quasiconvex subgroup of $\pi_1 Q$ maps to a quasiconvex subgroup of $\pi_1 Q'$. □

Lemma 7.5. *Let $\widehat{X} \to X$ be a covering space of a nonpositively curved cube complex X with $\pi_1 X$ word-hyperbolic. Suppose $\pi_1 \widehat{X}$ is quasiconvex. Then each compact subspace $K \subset \widehat{X}$ is contained in a compact locally convex subcomplex $Y \subset \widehat{X}$.*

Proof. This follows from Theorem 4.2. □

8. Main Theorem: Virtual Specialness of Malnormal Cubical Amalgams

In [HW08] we proved the following criterion:

Proposition 8.1. *Let C be a compact nonpositively curved cube complex, and suppose that $\pi_1 C$ is word-hyperbolic. Then C is virtually special if and only if every quasiconvex subgroup of $\pi_1 C$ is separable.*

Combining Corollary 7.4 and Proposition 8.1 we obtain the following result:

Theorem 8.2. *Let Q be a compact connected nonpositively curved cube complex, and let P be a hyperplane in Q such that the following hold:*

1. *$\pi_1 Q$ word-hyperbolic.*
2. *P is an embedded 2-sided hyperplane in Q.*
3. *$\pi_1 P$ is malnormal in $\pi_1 Q$.*
4. *$Q - P$ is virtually special.*

Then Q is virtually special.

Theorem 8.3. *Let C be a compact nonpositively curved cube complex such that:*

1. *$\pi_1 C$ is word-hyperbolic*
2. *each hyperplane of C is 2-sided and embeds*
3. *$\pi_1 D$ is malnormal in $\pi_1 C$ for each hyperplane D of C.*

Proof. We repeatedly cut along hyperplanes and apply Theorem 8.2. Here it is convenient to remove open regular neighborhoods of hyperplanes when cutting. The base case where Q consists of a single vertex is reached after finitely many cuts. □

Remark 8.4. We note that a "malnormal hyperplane hierarchy" gives a more general formulation using Theorem 8.2.

9. Virtually special ⇔ separable hyperplanes

A subgroup H of K is *almost malnormal* if $H^k \cap H$ is finite for each $k \in K - H$.

Lemma 9.1. *Let H be a separable quasiconvex subgroup of a word-hyperbolic group G. Then G has a finite index subgroup K containing H such that H is almost malnormal in K.*

Proof. As proven in [GMRS98], (see also [HWb]) there are finitely many cosets $g_i H$ such that $g_i H g_i^{-1} \cap H$ is infinite. By separability, we can choose K containing H but not containing any g_i. Thus H is almost malnormal in K. □

We are now able to obtain the following characterization of virtual specialness in the word-hyperbolic case. It remains an open problem whether such a characterization holds in general. See [HW08] for a characterization using double hyperplane cosets.

Theorem 9.2. *Let C be a compact nonpositively curved cube complex such that $\pi_1 C$ is word-hyperbolic. Then C is virtually special if and only $\pi_1 D$ is separable in $\pi_1 C$ for each immersed hyperplane D of C.*

Proof. For each hyperplane D_i apply Lemma 9.1 to obtain a finite cover $C_i \to C$ such that $\pi_1 D_i$ is malnormal in $\pi_1 C_i$. Then let C' be a regular cover factoring through all the C_i's, and observe that each hyperplane of C_i has malnormal fundamental group.

We can then pass to a finite cover $\bar{C}$ such that each hyperplane is embedded.

Indeed for each immersed hyperplane $D \to C$, let $N \to C$ denote its immersed cubical regular neighborhood. By convexity of $\widetilde{N} \subset \widetilde{C}$, we see that N embeds in the cover $\widehat{C}_N$ with $\pi_1 \widehat{C} \cong \pi_1 N$. By separability, we see that N embeds in a finite cover $\bar{C}_N$ of C.

We now let $\bar{C}$ be a finite regular cover factoring through $\bar{C}_N$ as N varies over all regular neighborhoods of immersed hyperplanes.

At this stage producing a further cover in which hyperplanes are 2-sided is easily done [HW08]. We conclude by applying Theorem 8.3. □

10. Uniform arithmetic hyperbolic manifolds of simple type

The main goal of this section is to show that certain arithmetic lattices in $\mathcal{H}^n = \text{Isom}(\mathbb{H}^n)$ are virtually special. In the first subsection, we describe conditions on a hyperbolic lattice which imply virtual specialness. Assuming that there are sufficiently many codimension-1 immersed closed geodesic submanifolds, we can cubulate the group and then apply Theorem 9.2 to obtain virtual specialness. In the second subsection, we verify that uniform arithmetic hyperbolic lattices of simple type satisfy this geodesic submanifold criterion and are thus virtually special.

Our application to subgroup separability of quasiconvex subgroups generalizes earlier results in [ALR01] as well as more recent work of Agol [Ago06] who has remarkably pushed Scott's original reflection group idea to handle many arithmetic examples up to dimension 11.

10.A. Criterion for virtual specialness of closed hyperbolic manifolds.

Theorem 10.1. *Let G be a uniform lattice in $\mathcal{H}^n$. Let $H_1, \ldots, H_k$ be isometric copies of $\mathbb{H}^{n-1}$ in $\mathbb{H}^n$. Suppose that $\text{Stabilizer}_G(H_i)$ acts cocompactly on H_i for each i. Suppose there exists D such that any length D geodesic intersects gH_i for some $g \in G$ and $1 \leq i \leq k$.*

Then G acts properly and cocompactly on a CAT(0) cube complex C. Moreover, G contains a finite index subgroup F such that $F \backslash C$ is a special cube complex.

The cubulation utilizes Sageev's construction, and can be deduced from the following formulation which we quote from [HWc]:

Proposition 10.2. *Let G act cocompactly on a δ-hyperbolic CAT(0) space X. Let $H_1, \ldots H_k$ in X be a set of convex hyperplanes in X. Suppose that the union of their* *translates $T = \{gH_i : g \in G, 1 \leq i \leq k\}$ is locally finite in X. Suppose there exists D such that any geodesic segment of length D, crosses some hyperplane gH_i. Then G acts properly and cocompactly on a CAT(0) cube complex C. Moreover, the distinct hyperplanes Y of C are in one-to-one correspondence with distinct hyperplanes gH_i, and* $\text{Stabilizer}(Y) = \text{Stabilizer}(gH_i)$.

Lemma 10.3. *Let G be a finitely generated subgroup of $\mathcal{H}^n$. Let H be an isometric copy of $\mathbb{H}^{n-1}$ in $\mathbb{H}^n$. Then* $\text{Stabilizer}(H)$ *is a separable subgroup of G.*

Proof. Let r be the reflection along H. Let $G' = \langle G, r\rangle$, and observe that G' is finitely generated, and hence residually finite since it is linear. Observe that the centralizer $\text{Cent}_{G'}(r)$ is a separable subgroup of G'. Indeed, if $k \notin \text{Cent}_{G'}(r)$ then $[r,k] \neq 1$. Let $G' \to \bar{G}'$ be a finite quotient in which $[\bar{r}, \bar{k}] \neq \bar{1}$. Then the preimage of $\text{Cent}_{\bar{G}'}(\bar{r})$ in G' separates k from $\text{Cent}_{G'}(r)$. Finally, observe that $\text{Stabilizer}_G(H) = \text{Cent}_{G'}(r) \cap G$ is separable in G. Indeed, $\text{Stabilizer}_{\mathcal{H}^n}(H) = \text{Cent}_{\mathcal{H}^n}(r)$. □

Proof of Theorem 10.1. The proper and cocompact action of G on a CAT(0) cube complex C follows immediately from Proposition 10.2. The stabilizer of each hyperplane of C equals the stabilizer of a hyperplane gH_i in $\mathbb{H}^n$, and is therefore separable by Lemma 10.3. Since G is residually finite, and there are finitely many torsion elements, we can pass to a finite index subgroup G' which is torsion-free. We can then apply Theorem 9.2 to $G'\backslash C$ to obtain a finite index subgroup F of G' such that $F\backslash C$ is special. □

10.B. **Uniform Arithmetic Hyperbolic Lattices of "Simple" or "Standard" Type.** The results in this subsection were motivated by a lecture of Alex Lubotzky, who mentioned that arithmetic hyperbolic manifolds of simple type have many totally geodesic lattices. We are very grateful to Nicolas Bergeron for pointing out to us that the density of the commensurator was an easy way to see that there are sufficiently many such sublattices to apply our criterion. After we developed this point of view on the virtual specialness of these lattices, an alternate treatment has been developed in [BHW] which uses the double coset separability criterion very much along the lines of the proof of virtual specialness of Coxeter groups [HWa].

Theorem 10.4. *Let G be a uniform arithmetic lattice in $\mathcal{H}^n$ of simple type. Then:*

1. *G acts properly and cocompactly on a CAT(0) cube complex C.*
2. *G contains a finite index subgroup F such that $F\backslash C$ is special.*

Proof. This follows from Lemma 10.10 where we verify the criterion of Theorem 10.1. □

Combining with [HW08], we obtain the following consequence:

Corollary 10.5. *Every quasiconvex subgroup of G is a virtual retract, and is hence separable.*

To prove Theorem 10.4, we will show that G contains sufficiently many subgroups acting on codimension-1 hyperplanes. Before embarking on the proof, it will be helpful to state an explicit characterization of a simple arithmetic lattice, and to note two of their elementary properties.

Remark 10.6 (Simple Arithmetic Lattices in $SO(1,n)$)**.** Up to commensurability and conjugation, the "simple lattices" are described precisely in the following formulation which we quote from [Wit06]. Aside from some exceptional families that appear for $n = 3, 7$, these simple lattices are the only arithmetic lattices in $\mathcal{H}^n$ for odd n. For even n, there is an additional families of uniform lattices arising from quaternion algebras.

Let $\mathbb{F}$ be a totally real algebraic number field. Let $\mathcal{O}$ be the ring of integers in $\mathbb{F}$, and let $a_1, \ldots, a_n \in \mathcal{O}$ be such that:

(1) each a_j is positive.
(2) each $\sigma(a_j)$ is negative for every place $\sigma \neq 1$.

Let $G = SO(a_1x_1^2 + \cdots + a_nx_n^2 - x_{n+1}^2; \mathbb{R}) \cong SO(n,1)$. Then:

(1) $G_{\mathcal{O}}$ is an arithmetic lattice in G.
(2) $G_{\mathcal{O}}$ is uniform if and only if $(0, \ldots, 0)$ is the only solution in $\mathcal{O}^{n+1}$ of the equation $a_1x_1^2 + \cdots + a_nx_n^2 - x_{n+1}^2 = 0$

By imposing the restriction $x_j = 0$ for some $1 \leq j \leq n$, and noting that the corresponding equation $x_{n+1} = \sum_{i \neq j} a_i x_i^2$ still admits only the trivial solution, we obtain the following immediate corollary of the result specified in Remark 10.6.

Corollary 10.7. *Let $G_{\mathcal{O}}$ be a uniform arithmetic lattice of simple type in $SO(1,n)$. Then $G_{\mathcal{O}}$ contains a subgroup K' which stabilizes an isometric copy of $\mathbb{H}^{n-1}$ in $\mathbb{H}^n$, and is itself a uniform arithmetic lattice (of simple type).*

Definition 10.8. The *commensurator* of a subgroup $\mathcal{H}$ in a group $\mathcal{G}$ is the following subgroup of $\mathcal{G}$:

$$\mathrm{Comm}(\mathcal{H}, \mathcal{G}) = \{g \in \mathcal{G} : [\mathcal{H} : \mathcal{H} \cap g\mathcal{H}g^{-1}] < \infty \text{ and } [\mathrm{g}\mathcal{H}\mathrm{g}^{-1} : \mathcal{H} \cap \mathrm{g}\mathcal{H}\mathrm{g}^{-1}]\}$$

Since $\mathrm{Comm}(G_{\mathcal{O}}, \mathcal{H}^n)$ obviously contains $SO(1, n; \mathbb{Q})$, the following result is obvious for arithmetic lattices of simple type:

Proposition 10.9. *Let G be an arithmetic lattice in $\mathcal{H}^n$. Then* $\mathrm{Comm}(G, \mathcal{H}^n)$ *is dense in $\mathcal{H}^n$.*

Dense means that $\mathcal{H}^n$ is the closure of the subspace $\mathrm{Comm}(G, \mathcal{H}^n)$, where we view $\mathcal{H}^n$ as a topological space in the ordinary way as a lie group. In fact, the converse to Proposition 10.9 holds and is a deeper result of Margulis which we do not need.

Lemma 10.10. *Let G be a uniform arithmetic hyperbolic lattice in $\mathcal{H}^n$ of simple type. There are totally geodesic codimension one submanifolds $H_1, \ldots, H_k$ (which we call hyperplanes) such that:*

(1) $\mathrm{Stabilizer}_G(H_i)$ *acts cocompactly on H_i.*
(2) *The set of hyperplanes $\{gH_i : g \in G, 1 \leq i \leq k\}$ is locally finite.*
(3) *There exists D such that any length D geodesic crosses some gH_i.*

Proof. By Corollary 10.7, there is an $(n-1)$-dimensional hyperplane $H \subset \mathbb{H}^n$, such that $\mathrm{Stabilizer}_G(H)$ acts properly and cocompactly on H.

Let $c \in \mathrm{Comm}(G)$. We claim that $\mathrm{Stabilizer}_G(cH)$ acts cocompactly on cH. Indeed let S denote the stabilizer of H in $\mathcal{H}^n$. Since $[G : G \cap G^c] < \infty$ we have $[\mathrm{Stabilizer}_G(cH) :$

$\mathrm{Stabilizer}_G(cH) \cap G^c] < \infty$, and it is enough to prove that $\mathrm{Stabilizer}_G(cH) \cap G^c$ is cocompact on cH. Now $\mathrm{Stabilizer}_G(cH) \cap G^c = S^c \cap G \cap G^c$. By assumption $G \cap G^c$ is of finite index in G^c, thus $S^c \cap G \cap G^c$ is of finite index in S^c. Since S is cocompact on H it follows that S^c - the stabilizer of cH in G^c - is also cocompact on cH, which ends the argument.

By choosing various elements $c_i \in \mathrm{Comm}(G)$, we are thus able to produce a collection of hyperplanes $H_i = c_i H$ each of which has cocompact stabilizer in G.

Let us check that the cocompactness of $\mathrm{Stabilizer}_G(H)$ on H implies the local finiteness of the family of subsets $\{gH\}_{g \in G}$. By cocompactness there is a ball A centered at some point of H such that for any $p \in H$ there exists $h \in \mathrm{Stabilizer}_G(H)$ with $h^{-1}p \in A$. For any ball B of $\mathbb{H}^n$ and any g such that $gH \cap B \neq \emptyset$ there exists an $h \in \mathrm{Stabilizer}_G(H)$ such that $ghA \cap B = \emptyset$. Since G acts properly it follows that the set $\{g \in G, gH \cap B \neq \emptyset\}$ is the union of finitely many cosets $g_1 \mathrm{Stabilizer}_G(H) \cup \cdots \cup g_k \mathrm{Stabilizer}_G(H)$. Thus only finitely many translates of H meet B.

It thus remains to verify that the third property holds for an appropriate choice of $c_1, \ldots, c_k$.

Since G is uniform, we can choose a closed radius r ball A such that $GA = \mathbb{H}^n$. Let B, C denote the balls with same center as A and with radius $r+1, r+2$.

As any closed convex subset of $\mathbb{H}^n$, B is the intersection of the closed half-spaces containing B. In this intersection we may restrict to the family of half-spaces whose complement meets the sphere ∂C in a nonempty open subset. By compactness of ∂C there is a finite collection of half-spaces $K_1, \ldots, K_m$ containing B and such that the union of the complements of K_j covers ∂C. In other words the polytope $\Pi_0 = \cap_i K_i$ is contained in the interior of C.

We now approximate the polyhedron Π (containing B) by a polyhedron Π' whose faces span hyperplanes which are translates of H by elements of the commensurator, and such that $A \subset \Pi' \subset C$. The complements of the K_i provide an open covering of the sphere ∂C. By density of the commensurator there exists $c_1, \ldots, c_m \in \mathrm{Comm}(G)$ such that $H_i := c_i H$ is so near to ∂K_i that each H_i is disjoint of A, and the complements of the half-spaces K_i' of $\mathbb{H}^n$ bounded by H_i and containing A provide a covering of ∂C. This exactly means that we have $A \subset \Pi' \subset C$.

We now show that any geodesic γ of length $D = 2(r+2)$ intersects some gH_i. Since D is the diameter of C, any geodesic with initial point in $A \subset \Pi' \subset C$ and length $> D$ has its terminal point outside C, thus crosses the boundary of Π', which means it intersects some bounding hyperplane H_i. Now by the choice of A the initial point of any geodesic may be translated into A by an element of G, thus the geodesic intersects some translate of some H_i. □

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