NOTES ON SIMPLE LIE ALGEBRAS AND LIE GROUPS. MATH 261A

These notes are intended to clarify some aspects of simple Lie groups and Lie algebras.

A Lie algebra $\mathfrak{g}$ over the field $\mathbb{K}$ is *simple* if there is no non-trivial $\mathbb{K}$-ideal and $\dim\mathfrak{g} > 1$. Equivalently, the adjoint representation $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is irreducible (and non-zero).

Lemma 0.1. *Let G be a connected matrix Lie group, with (real) Lie algebra $\mathfrak{g}$, and $H < G$ a connected analytic subgroup with Lie algebra $\mathfrak{h} < \mathfrak{g}$. Then $H \lhd G \Leftrightarrow \mathfrak{h}$ is an ideal of $\mathfrak{g}$.*

Proof. Suppose $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Then we may restrict $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$. So for $X \in \mathfrak{g}, Y \in \mathfrak{h}$, $ad_X^n(Y) \in \mathfrak{h}$. Thus, $e^{ad_X}(Y) \in \mathfrak{h}$. Then $e^X e^Y e^{-X} = exp(Ad_{e^X}(Y)) = exp(e^{ad_X}(Y)) \in exp(\mathfrak{h})$. So $exp(\mathfrak{g})$ normalizes $exp(\mathfrak{h})$. Since $G = \bigcup_{n\geq 0} exp(\mathfrak{g})^n$, $H = \bigcup_{n\geq 0} exp(\mathfrak{h})^n$, we see that G normalizes H.

Conversely, suppose that G normalizes H. Then G acts on H by conjugation. For $g \in G$, the derivative of this map is $Ad_g : T_eH \to T_eH = \mathfrak{h}$. Then for $X \in \mathfrak{g}$, $Ad_{e^{tX}} : \mathfrak{h} \to \mathfrak{h}$. Taking the derivative at $t = 0$, we see that $ad_X : \mathfrak{h} \to \mathfrak{h}$, so $\mathfrak{h}$ is an ideal (see Proposition 2.24). □

Lemma 0.2. $\mathfrak{sl}(2;\mathbb{C})$ *is simple.*

Proof. As mentioned in class, one can follow the classification of dimension 3, rank 3 Lie algebras to prove this. One may also identify ad with the irreducible 3-dimensional representation of $\mathfrak{sl}(2;\mathbb{C})$ following the proof of Theorem 4.9. Here's a direct argument.

We've seen that $\mathfrak{sl}(2;\mathbb{C})$ has structure

$$[H, X] = 2X,\ [H, Y] = -2Y,\ [X, Y] = H.$$

Suppose $0 \neq \mathfrak{h} < \mathfrak{sl}(2;\mathbb{C})$ is an ideal. Let $0 \neq Z \in \mathfrak{h}$. Let $Z = \alpha X + \beta Y + \gamma H$. We compute $ad_H^2 Z = [H, [H, Z]] = [H, 2\alpha X - 2\beta Y] = 4\alpha X + 4\beta Y$. Then $Z - \frac{1}{4} ad_H^2 Z = \gamma H \in \mathfrak{h}$.

If $\gamma \neq 0$, then $H \in \mathfrak{H}$. Then $[X, H] = -2X \in \mathfrak{h}, [Y, H] = 2Y \in \mathfrak{h}$, and we see $\mathfrak{h} = \mathfrak{sl}(2; \mathbb{C})$.

If $\gamma = 0$, then $[X, Z] = \beta H, [Y, Z] = -\alpha H$. So in some case (since $Z \neq 0$), we see that $H \in \mathfrak{h}$, and thus $\mathfrak{h} = \mathfrak{sl}(2; \mathbb{C})$ $\square$

Definition 0.3. *A simple Lie group is a connected non-abelian Lie group G which does not have nontrivial connected (analytic) normal Lie subgroups.*

Note: Under this definition, the one-dimensional Lie group is not considered to be simple.

The above Lemma implies that G is a simple Lie group if and only if $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{R}$.

The following theorem may be proven by first classifying simple Lie algebras, and then proving that each associated Lie group is a simple group (modulo its center). This theorem does not usually appear in courses in Lie theory, but we include it here.

Theorem 0.4. *If G is a connected Lie group, such that $\mathfrak{g}$ is a simple (real) Lie algebra, then any normal subgroup $K \lhd G$ must be discrete.*

Remark: In particular, $G/Z(G)$ is a *simple group.*

Proof. Let $K < G$ be an arbitrary subgroup. We want to weaken the notion of a Lie algebra for a subgroup, as defined in Definition 3.11. Suppose $e \in M \subset K$ is (a germ of) a smooth submanifold. Let $T_eM \subset \mathfrak{g} = T_eG$. Choose M to be maximal dimensional.

Claim: T_eM is independent of M. If $M_1, M_2 \subset K$ are two maximal dimensional submanifolds near e such that $T_eM_1 \neq T_eM_2$, then there is a vector $X \in T_eM_1 \backslash T_eM_2$. Let $C \subset M_1$ be a one-dimensional curve tangent to X. Then $m : X \times M_2 \to K$, where m is the multiplication on G gives a submanifold near e of dimension $1 + dim M_2$, since $m_* : T_eG \times T_eG \to T_eG$ is given by $m_*(X, Y) = X + Y$ (this follows from Baker-Campbell-Hausdorff). This contradicts that M_2 was maximal.

Remark: One may show that T_eM is a Lie subalgebra of $\mathfrak{g}$.

Let $M \subset K$ be such a maximal germ of a submanifold near e. Since $K \lhd G$, for $g \in G$, $gMg^{-1} \subset K$ is a maximal dimensional germ of a submanifold near e, and therefore $T_eM = T_e(gMg^{-1})$. Thus,

$Ad_g(T_eM) = T_eM$. For $X \in \mathfrak{g}$, $Ad_{e^{tX}}(T_eM) = T_eM$. Taking derivatives at $t = 0$, we see that $ad_X : T_eM \to T_eM$. Thus, T_eM is an ideal of $\mathfrak{g}$, so $T_eM = 0$ or $\mathfrak{g}$. If $T_eM = \mathfrak{g}$, then M must contain a neighborhood of $e \in G$, and thus $K = G$. We want to show that $T_eM \neq 0$.

The closure $\overline{K}$ is a closed normal subgroup of G, and is therefore a Lie subgroup. Either $\overline{K}$ is discrete, and therefore $\overline{K} = K$, so K was discrete, or $\overline{K} = G$, since by the above Lemma G has no non-trivial connected normal Lie subgroups. For $k \in K$, consider the map $\psi_k : G \to K$, given by $\psi_k(g) = k^{-1}g^{-1}kg$. Since G is non-abelian, and K is dense in G, we may find $k \in K$ such that $Ad_{k^{-1}} : \mathfrak{g} \to \mathfrak{g}$ is not the identity. Choose $X \in \mathfrak{g}$ such that $Ad_{k^{-1}}X \neq X$. We have

$$\frac{d\psi_k}{dt}|_{t=0} = (\frac{d}{dt}k^{-1}e^{-tX}ke^{tX})|_{t=0}$$
$$= (k^{-1}(-X)e^{-tX}ke^{tX} + k^{-1}e^{-tX}ke^{tX}X)|_{t=0} = -Ad_{k^{-1}}X + X \neq 0.$$

Thus, $\psi_k(e^{tX})$ gives a submanifold of K near e, and therefore $\dim T_eM > 0$. □

Exercise: Let $\mathfrak{g}$ be a complex Lie algebra. Then $\mathfrak{g}$ is simple over $\mathbb{C} \Leftrightarrow$ it is simple over $\mathbb{R}$.

Proposition 0.5. *If $\mathfrak{h}$ is a simple algebra over $\mathbb{R}$, then either $\mathfrak{h}_{\mathbb{C}}$ is simple, or $\mathfrak{h}$ admits a complex vector space structure, and $\mathfrak{h}_{\mathbb{C}} \cong \mathfrak{h} \oplus \overline{\mathfrak{h}}$.*

Proof. As a vector space $\mathfrak{h}_{\mathbb{C}} \cong \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{h} + i\mathfrak{h}$. Suppose $\mathfrak{k} \subset \mathfrak{h}_{\mathbb{C}}$ is a complex ideal. For $0 \neq k \in \mathfrak{k}$, let $k = h_1 + ih_2$, $h_i \in \mathfrak{h}$. If $h_2 = 0$, we see that $\mathfrak{h} \subset \mathfrak{k}$, and therefore $\mathfrak{k} = \mathfrak{h}_{\mathbb{C}}$. Similarly if $h_1 = 0$. Otherwise, for all $0 \neq h_1 + ih_2 \in \mathfrak{k}$, $h_1 \neq 0 \neq h_2$. Since $\mathfrak{h}$ is simple, for any $h_1 \in \mathfrak{h}$, there exists $k = h_1 + ih_2$, such that $h_2 = 0 \Leftrightarrow h_1 = 0$. Thus, $\mathfrak{k}$ is the graph of a real vector space isomorphism $J : \mathfrak{h} \to \mathfrak{h}$. Moreover, $-i(h + iJ(h)) = J(h) - ih$, so we see that $J^2 = -I$. The map J also preserves the bracket on $\mathfrak{h}$, since $[h_1 + iJ(h_1), h_2] = [h_1, h_2] + i[J(h_1), h_2] = [h_1, h_2] + iJ[h_1, h_2] \in \mathfrak{k}$. Thus, J induces a complex structure on $\mathfrak{h}$, and $\mathfrak{h}_{\mathbb{C}} \cong \mathfrak{k} \oplus \overline{\mathfrak{k}} \cong \mathfrak{h} \oplus \overline{\mathfrak{h}}$. □

Thus, in order to classify simple Lie algebras, it suffices to classify simple complex Lie algebras and their real forms.