# Final Solutions, Math 1A, section 1

Wednesday, December 17, 2008, 12:30 pm - 3:30 pm

1. Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?

   **Solution:** Choose the starting point to be at the origin. The position of the first car at time $t$ $hr.$ is $-60t$ $mi.$ The position of the second car at time $t$ $hr.$ is $-25t$ $mi.$. The distance between them at time $t$ therefore is $d(t) = \sqrt{(-25t)^2 + (60t)^2} = 65t$ (since $t$ is positive). Thus, the rate that the distance is increasing is $d'(t)$ $mph = 65$ $mph$, regardless of the hour.

2. Show that $\tan(x) > x$ for $0 < x < \pi/2$.

   **Solution:** Consider the function $g(x) = \tan(x) - x$. Then $g(0) = 0$. Also, we have $g$ is differentiable on its domain and $g'(x) = \sec^2(x) - 1 > 0$ for $0 < x < \pi/2$. Thus, $g(x)$ is increasing for $0 < x < \pi/2$ by the ID test, so we have $g(x) > 0$ for $0 < x < \pi/2$, so $\tan(x) > x$.

3. A box with a square base and open top must have a volume of $32,000$ $cm^3$. Find the dimensions of the box that minimizes the amount of material used.

   **Solution:** Let $x$ $cm$ be the length of the sides of the square bottom, and $hcm$ be the height of the box. The volume of the box is given by $hx^2 = 32,000$, so $x > 0$, and the area of the box is given by $A = x^2 + 4xh = x^2 + 128000/x$. We compute $A'(x) = 2x - 128000/x^2$ and $A''(x) = 2 + 256000/x^3$. We see that $A''(x) > 0$ for all $x > 0$. We compute $A'(x) = 0$ for $x > 0$ if $x^3 = 64000$, so $x = 40$. This is the global minimum by the second derivative test.

4. Find $\frac{d}{dx}\int_{\sin x}^{\cos x} \frac{1}{\sqrt{1-t^2}}\,dt$ for $0 < x < \pi/2$, justifying your answer.

   **Solution:** Let $F(u) = \int_{\pi/4}^{u} \frac{1}{\sqrt{1-t^2}}\,dt$, then by rules for definite integrals, we have $\int_{\sin x}^{\cos x} \frac{1}{\sqrt{1-t^2}}\,dt = F(\sin(x)) - F(\cos(x))$ for $0 < x < \pi/2$. We want to find $\frac{d}{dx}(F(\sin(x)) - F(\cos(x))) = F'(\sin(x))\cos(x) - F'(\cos(x))(-\sin(x))$ by the chain rule. By FTC1, $F'(u) = \frac{1}{\sqrt{1-u^2}}$, so we get $\frac{\cos(x)}{\sqrt{1-\sin^2(x)}} + \frac{\sin(x)}{\sqrt{1-\cos^2(x)}} = \frac{\cos(x)}{\cos(x)} + \frac{\sin(x)}{\sin(x)}$ since $0 < x < \pi/2$ and using trigonometric identities. Thus, $\frac{d}{dx}\int_{\sin x}^{\cos x} \frac{1}{\sqrt{1-t^2}}\,dt = 2$ for $0 < x < \pi/2$.

5. Show that the tangent line to the ellipse

   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

   at the point $(x_0, y_0)$ is

   $$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1.$$

   **Solution:** We implicitly differentiate:

   $$\frac{\partial}{\partial x}\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2x/a^2 + 2y/b^2\frac{\partial y}{\partial x} = \frac{\partial}{\partial x}1 = 0.$$

   Plugging in $(x_0, y_0)$ and solving we get $\frac{\partial y}{\partial x}(x_0, y_0) = -\frac{x_0 b^2}{y_0 a^2}$. We plug into the point-slope equation of the tangent line to get $y - y_0 = -\frac{x_0 b^2}{y_0 a^2}(x - x_0)$. Multiplying both sides by $y_0/b^2$ and rearranging, we get

   $$yy_0/b^2 + xx_0/a^2 = x_0^2/a^2 + y_0^2/b^2 = 1.$$

6. Show that the tangent lines to the curves $x = y^3$ and $y^2 + 3x^2 = 5$ are perpendicular when the curves intersect. Justify your answer.

   **Solution:** We use implicit differentiation $\frac{d}{dx}x = 1 = \frac{d}{dx}y^3 = 3y^2\frac{dy}{dx}$ using the power and chain rules. Let $m$ be the slope of the tangent line to the curve $x = y^3$, and let $k$ be the slope of the tangent line to the curve $y^2 + 3x^2 = 5$ at a point $(x, y)$ of intersection of the two curves. Also, $\frac{d}{dx}y^2 + 3x^2 = 2y\frac{dy}{dx} + 6x = \frac{d}{dx}5 = 0$. Let $(x, y)$ be in the intersection of the two curves. If $y = 0$, then $x = y^3 = 0^3 = 0$, and $y^2 + 3x^2 = 0 = 5$, a contradiction. Thus, $y > 0$, and similarly $x > 0$. Thus, we may divide the derivatives by $y$ to obtain $m = \frac{dy}{dx} = 1/3y^2$ and $k = -6x/2y = -3x/y = -3y^3/y = -3y^2 = -1/m$. Thus, $k$ and $m$ represent complementary slopes, and thus the two curves intersect perpendicularly.

7. Evaluate the following integrals, justifying your answers:

   (a) $\int_0^1 x - \frac{\tan^{-1} x}{1+x^2} dx$

   **Solution:** By linearity, $\int_0^1 x - \frac{\tan^{-1} x}{1+x^2} dx = \int_0^1 x dx - \int_0^1 \frac{\tan^{-1} x}{1+x^2} dx$. Since the integrands are continuous, we may use FTC2. $\int_0^1 x dx = \left[x^2/2\right]_0^1 = 1/2$ by the power rule.
   Let $u = \tan^{-1} x, du = 1/(1 + x^2)dx$. Then $u(0) = \tan^{-1}(0) = 0, u(1) = \tan^{-1}(1) = \pi/4$. $\int_0^1 \frac{\tan^{-1} x}{1+x^2} dx = \int_0^{\pi/4} u du = \left[u^2/2\right]_0^{\pi/4} = \pi^2/32$, using the substitution method, the power rule, and FTC2.

   (b) $\int_0^2 \sqrt{4 - x^2} dx$ This integral represents a quarter disk in the first quadrant of radius 2. Since the area of a disk of radius 2 is $\pi 2^2$, the area is $\pi$.

   (c) $\int e^x \sqrt{1 + e^x} dx$
   We use the substitution method. Let $u = 1 + e^x, du = e^x dx$. Then $\int e^x \sqrt{1 + e^x} = \int u^{1/2} du = 2/3u^{3/2} = 2/3(1 + e^x)^{3/2}$ by the power rule.

8. For the function $f(x) = e^x/x$, find with justification

   (a) the domain
   (b) intercepts
   (c) symmetry
   (d) asymptotes
   (e) intervals of increase or decrease
   (f) local maximum and minimum values
   (g) concavity and points of inflection

   Then sketch the graph $y = f(x)$, marking on your graph all of the information you have found.

   **Solution:**

   (a) The domain of $f(x)$ is a ratio of functions defined for all $x$, so the ratio is defined as long as the denominator is non-zero, which occurs for $x \neq 0$.

   (b) We set $f(x) = 0$ to find the $x$-intercept. This occurs only when the numerator $= 0$. Since $e^x > 0$ for all $x$, $f(x) \neq 0$ for all $x$, and there is no $x$-intercept.
   $f$ is undefined at $x = 0$, so there is no $y$-intercept.

(c) Let $x \neq 0$. Suppose $f(-x) = e^{-x}/(-x) = -1/(xe^x) = \pm e^x/x$, then $1/e^x = \pm e^x$, so $\pm 1 = e^{2x}$, which is only possible for $x = 0$, a contradiction. Therefore $f$ is not even or odd.

$f(x)$ is not periodic, since its domain is not periodic.

(d) The only possible vertical asymptote is at $x = 0$, since $f$ is undefined there. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^x \lim_{x\to 0^+} 1/x = 1\cdot\infty = \infty$. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^x \lim_{x\to 0^-} 1/x = 1\cdot(-\infty) = -\infty$.
$\lim_{x\to-\infty} \frac{e^x}{x} = \lim_{x\to-\infty} e^x \lim_{x\to-\infty} 1/x = 0\cdot 0 = 0$ $\lim_{x\to\infty} \frac{e^x}{x}$ is indeterminate of type $\infty/\infty$. $\lim_{x\to\infty} \frac{(e^x)'}{x'} = \lim_{x\to\infty} \frac{e^x}{1} = \infty$. Thus, by l'Hospital's rule, $\lim_{x\to\infty} \frac{e^x}{x} = \infty$. So there is a horizontal asymptote $y = 0$ as $x \to -\infty$.

(e) $f'(x) = (e^x/x)' = \frac{(e^x)'x - e^x x'}{x^2} = \frac{e^x x - e^x}{x^2} = e^x \frac{x-1}{x^2}$ by the quotient rule. For $x > 1$, $x - 1 > 1$, and $e^x/x^2 > 0$ for all $x \neq 0$, so $f'(x) > 1$. By the increasing test, $f(x)$ is increasing on this interval. For $x < 1$ and $x \neq 0$ , $x - 1 < 1$, so $f'(x) < 0$. Thus, on the intervals $x < 0$ and $0 < x < 1$, $f(x)$ is decreasing by the decreasing test.

(f) Since $f'(1) = 0$, and $f''(1) < 0$, $f$ has a local maximum at $x = 1$ by the second derivative test. By the first derivative test, since 1 is the only zero of $f'(x)$, $f(x)$ has no local extremum at any other point in its domain.

(g) $f''(x) = \frac{-e^x(x-1)(x^2)' + (e^x(x-1))'(x^2)}{(x^2)^2} = \frac{-e^x(x-1)2x + (e^x(x-1)' + (e^x)'(x-1))(x^2)}{x^4} = \frac{-e^x(2x^2-2x) + e^x(x^2+x^3-x^2)}{x^4} = e^x \frac{x^2-2x+2}{x^3} = e^x \frac{(x-1)^2+1}{x^3}$. Then $e^x((x-1)^2+1) > 0$ for all $x$, and $x^3 > 0$ for $x > 0$, and $x^3 < 0$ for $x < 0$, so $f''(x) > 0$ for $x > 0$, and $f''(x) < 0$ for $x < 0$. By the concavity test, $f(x)$ is concave up for $x > 0$, and $f(x)$ is concave down for $x > 0$. $f(x)$ has no inflection points, since it changes sign only at $x = 0$ which is not in the domain of $f(x)$.

(h) Sketch the graph.

9. Let $f(x) = x - 2\sqrt{x}$.

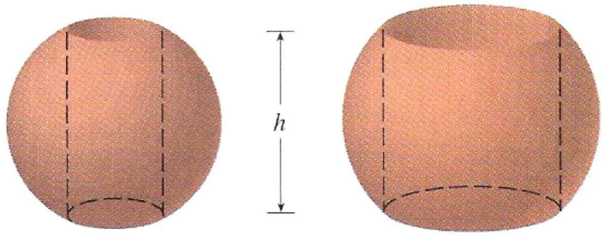
(a) Prove that $f$ is increasing for $x > 1$.
**Solution:** For $x > 1$, $x = \sqrt{x}^2$, so $f(x) = \sqrt{x}^2 - 2\sqrt{x} + 1 - 1 = (\sqrt{x} - 1)^2 - 1$. For $x_1 > x_2 > 1$, $\sqrt{x_1} > \sqrt{x_2} > 1$, so $\sqrt{x_1} - 1 > \sqrt{x_2} - 1 > 0$, and $(\sqrt{x_1} - 1)^2 - 1 > (\sqrt{x_2} - 1)^2 - 1$. Thus, $f(x)$ is increasing for $x > 1$. One can also take the derivative and apply the ID test.

(b) Find an inverse function for $f(x)$ on the interval $x > 1$.
**Solution:** We solve for $x$: $y = (\sqrt{x} - 1)^2 - 1$, so $(\sqrt{x} - 1)^2 = y + 1$, and $(\sqrt{x} - 1) = \sqrt{y+1}$ for $y > -1$. Then $x = (1 + \sqrt{y+1})^2$, so $g(y) = (1 + \sqrt{y+1})^2$ is the inverse function, where $y > -1$.

(c) Prove rigorously the following limit, using the precise definition of an infinite limit:

$$\lim_{x\to\infty} x - 2\sqrt{x} = \infty$$

**Solution:** Let $M > -1$, and let $N = g(M) = (1 + \sqrt{M+1})^2$. Then $f(g(M)) = M$. Since $f$ is increasing for $x > 1$, then if $x > N = g(M)$, we have $f(x) > M$. If $M \leq -1$, then take $N = 1$, so $f(x) > -1 \geq M$ for $x > 1$. Thus $\lim_{x\to\infty} f(x) = \infty$ by the precise definition of a limit (Definition 9, p. 140).




10. Suppose you make napkin rings by drilling holes through the centers of balls with different diameters and different sized holes. Suppose that the napkin rings have the same height $h$. Show that the volumes of the napkin rings are the same. Justify your answer.

    **Solution:** Let $r$ be the radius of the cylindrical hole, and $R$ the radius of the sphere. Then they satisfy the relation $R^2 = r^2 + (h/2)^2$. Let $r \leq x \leq R$ be the radius of a cylindrical shell (or let $-h/2 \leq y \leq h/2$ be the height of a washer cross-section). Using the cylindrical shells method to compute the volume, we have $Volume = \int_r^R (2\pi x) 2\sqrt{R^2 - x^2}dx$. Making the substitution $u = R^2 - x^2, du = -2xdx$, and using FTC2 since the integrand is continuous, the power rule, and the constant multiple rule, we get

$$Volume = \int_{(h/2)^2}^{0} -2\pi u^{1/2} du = \left[-2\pi \frac{2}{3} u^{3/2}\right]_{(h/2)^2}^{0} = 2\pi \frac{2}{3}((h/2)^2)^{3/2} = \pi h^3/6.$$

    Using the washers method, we have

$$Volume = \int_{-h/2}^{h/2} \pi(\sqrt{R^2 - y^2}^2 - r^2)dy = 2\pi \int_0^{h/2} R^2 - r^2 - y^2 dy$$

$$= 2\pi \left[(h/2)^2 y - \frac{1}{3}y^3\right]_0^{h/2} = \pi h^3/6.$$

    Here, we have used the fact that the integrand is even and integrated over a symmetric interval, and $R^2 - r^2 = (h/2)^2$, and the FTC2 since the integrand is continuous, the sum, constant multiple, and power rules to evaluate the anti-derivative.