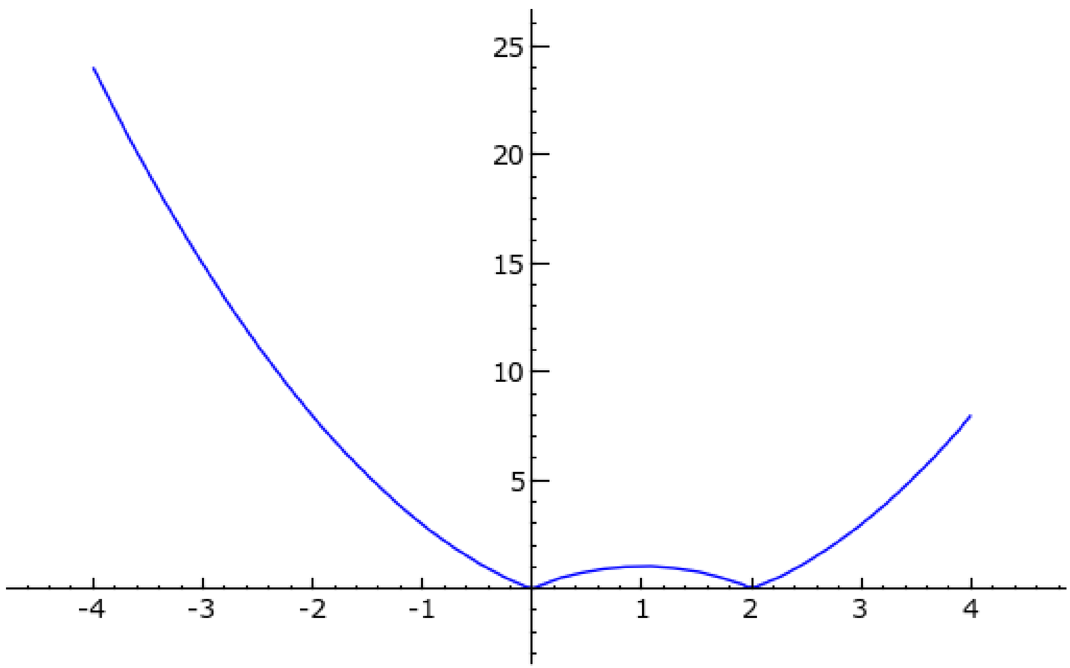
Math 1A, practice Midterm 1 from fall 2009, solutions

1. Sketch the graph of $y = |x^2 - 2x|$ for $-4 \le x \le 4$.

   Solution: The plot looks like that of a usual parabola, except when $0 < x < 2$, the part underneath the x-axis gets reflected above the x-axis to be positive, since in this interval $x^2 - 2x = x(x-2) < 0$ since $x > 0, x - 2 < 0$. We compute the points $y(-4) = (-4)^2 - 2(-4) = 24, y(0) = 0, y(1) = |1-2| = 1, y(2) = 0, y(4) = 4^2 - 2(4) = 8$.

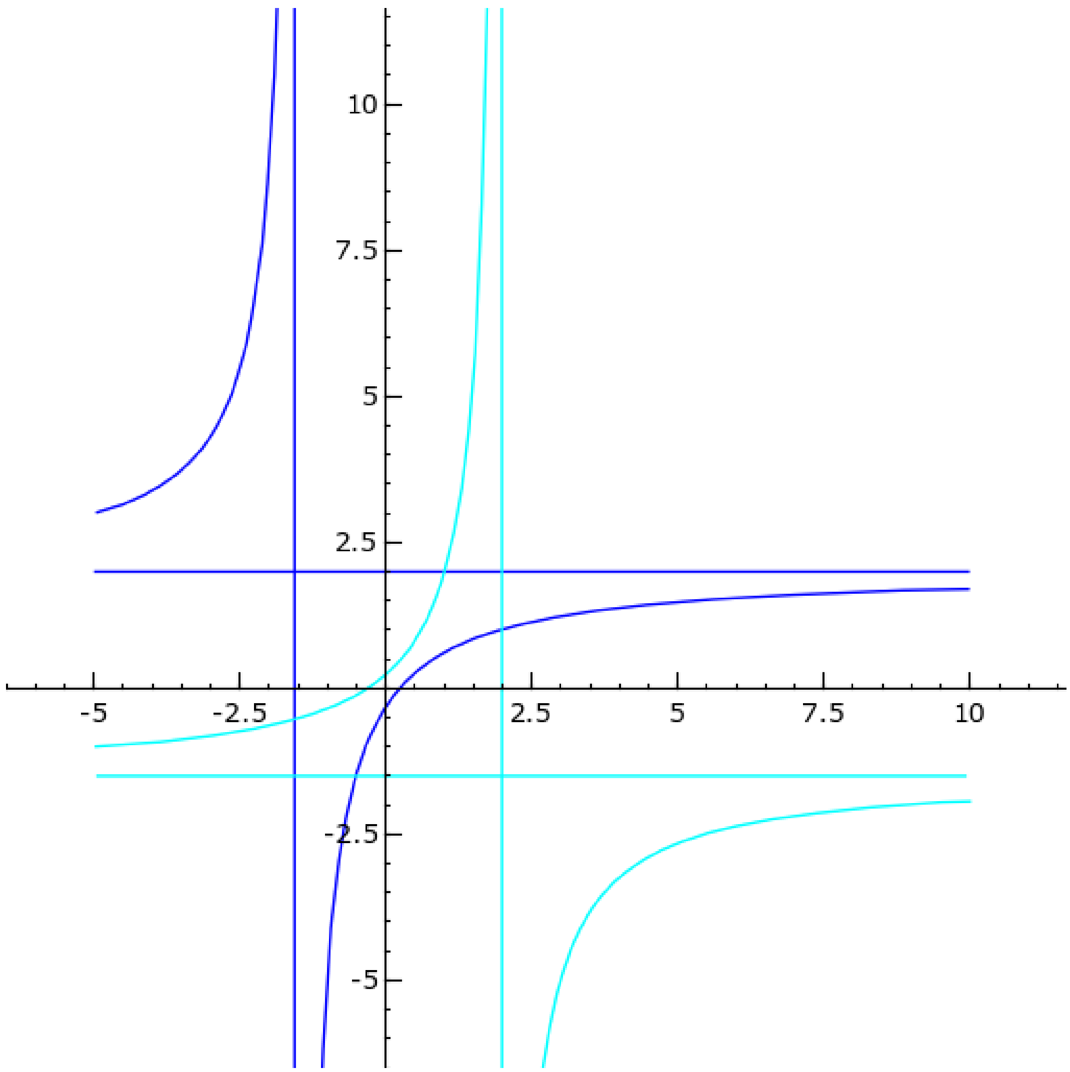


2. Sketch the graph of the function $f(x) = (4x-1)/(2x+3)$. Find a formula for its inverse f^{-1} and sketch the graph of f^{-1} on the same plot.

   Solution: We compute $f(0) = -1/3$. Also, dividing numerator and denominator by $x > 0$, we have $\lim\limits_{x\to\infty} \frac{4x-1}{2x+3} = \lim\limits_{x\to\infty} \frac{4-1/x}{2+3/x} = \frac{4 - \lim\limits_{x\to\infty} 1/x}{2+3\lim\limits_{x\to\infty} 1/x} = 4/2 = 2$ using the fact that $\lim\limits_{x\to\infty} 1/x = 0$ and the direct substitution property for rational functions (and the fact that the denominator is non-zero). Similarly, $\lim\limits_{x\to-\infty} \frac{4x-1}{2x+3} = 2$. Thus the graph will have a horizontal asymptote the line $y = 2$, which we include in the plot.

   The y-intercept is $\frac{4x-1}{2x+3} = 0$, so $4x - 1 = 0$, and we see that $x = \frac{1}{4}$. There is also vertical asymptotes when the denominator $2x + 3 = 0$, so $x = -3/2$. We have $\lim\limits_{x\to-3/2^+} \frac{4x+1}{2x+3} = -\infty$ since the numerator approaches -5, and the denominator is a small positive number. Similarly, $\lim\limits_{x\to-3/2^-} \frac{4x+1}{2x+3} = \infty$, since now the denominator is a small negative number. We incorporate all of this information into a graph.

   The domain of the function is $x \ne -3/2$. We compute the inverse function by setting $\frac{4y-1}{2y+3} = x$, and solving for y in terms of x. Since $2y + 3 \ne 0$, we may multiply through to get $4y - 1 = x(2y+3) = 2xy + 3x$. Gather the terms involving y on one side of the equation and the rest on the other to obtain $y(4-2x) = 3x + 1$. Now, x cannot equal 2 since this would give $y(4 - 2\cdot 2) = 0 = 3 \cdot 2 + 1 = 7$, which is impossible. So $4 - 2x \ne 0$, and we may divide both sides of the equation by $4-2x$ to obtain $y=\frac{3x+1}{4-2x}$. From before, this has horizontal asymptote $y=-\frac{3}{2}$ and vertical asymptote at 2 since we've exchanged the roles of x and y, and the x and y intercepts get interchanged.



3. Evaluate the limit

$$\lim_{x\to 4}\frac{2-\sqrt{x}}{4x-x^2}.$$

Solution: This is an indeterminate limit, since both the numerator and denominator approach 0 as $x\to 4$. The domain of the function is $x>0$ and $x\neq 4$.

We compute $\frac{2-\sqrt{x}}{4x-x^2}=\frac{2-\sqrt{x}}{x(2-\sqrt{x})(2+\sqrt{x})}=\frac{1}{x(2+\sqrt{x})}$, which holds for $x>0, x\neq 4$. Now the denominator does not approach zero, so we may plug in by Theorem 2.5.7, to get

$$\lim_{x\to 4}\frac{2-\sqrt{x}}{4x-x^2}=\frac{1}{4(2+\sqrt{4})}=\frac{1}{16}.$$

4. Show that there is a number x such that $e^x+\sin(x)=5$.

Solution: Consider the function $f(x) = e^x + \sin(x)$. This is a continuous function by Theorem 2.5.7 and the sum rule 2.5.4. We compute $f(0) = e^0 + \sin(0) = 1 + 0 = 1$. Also, we compute $f(\ln(7)) = e^{\ln(7)} - \sin(\ln(7)) \geq 7 - 1 = 6$. Thus, by the intermediate value theorem, since $f(0) < 5 < f(\ln(7))$ and f is continuous, there must exist c with $0 < c < \ln(7)$ such that $f(c) = 5$. Then we see that $f(c) = e^c + sin(c) = 5$.

5. What is $\lim_{x \to +\infty} \sqrt{x^2 + 3x} - \sqrt{x^2 + 2x}$?

   Solution: This is a special case of problem 2.6 #27 from the book.

   Multiply by the conjugate:

$$\sqrt{x^2 + 3x} - \sqrt{x^2 + 2x} = (\sqrt{x^2 + 3x} - \sqrt{x^2 + 2x})(\sqrt{x^2 + 3x} + \sqrt{x^2 + 2x})/(\sqrt{x^2 + 3x} + \sqrt{x^2 + 2x})$$

   which holds for all x in the domain of the function. We obtain

$$(x^2 + 3x - (x^2 + 2x))/(\sqrt{x^2 + 3x} + \sqrt{x^2 + 2x}) = x/(\sqrt{x^2 + 3x} + \sqrt{x^2 + 2x}).$$

   Since $x \to \infty$, x is positive, so we may divide out by x in the denominator:

$$x/(x(\sqrt{1 + 3/x} + \sqrt{1 + 2/x})) = 1/(\sqrt{1 + 3/x} + \sqrt{1 + 2/x}).$$

   Now $\lim_{x \to \infty} 1/x = 0$, so we may plug into the limit since it is an algebraic function by Theorem 2.5.7, and the denominator does not approach zero, to get $1/(\sqrt{1} + \sqrt{1}) = 1/2$.

6. Find the equation of the tangent line to the curve $y = 2x^3 - 5x$ at the point where $x = -1$.

   Solution: We compute $\frac{dy}{dx} = (2x^3 - 5x)' = 2(x^3)' - 5x'$ by the sum and constant multiple rules. Then using the power rule, we get $\frac{dy}{dx} = 2 \cdot 3x^2 - 5 \cdot 1 = 6x^2 - 5$. When $x = -1$, we get $y'(-1) = 6(-1)^2 - 5 = 1$. We also have $y(-1) = 2 \cdot (-1)^3 - 5 \cdot (-1) = -2 + 5 = 3$. We plug into the point-slope formula to obtain the tangent line:

$$y - y(1) = y'(-1)(x - (-1)) = y - 3 = x + 1,$$

   so $y = x + 4$.

7. State the definition of the derivative of a function, and fine the derivative of the function $f(x) = x^2 - 1$ using the definition of the derivative.

   Solution: The derivative is defined as

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

   when the limit exists.

   First we compute the difference quotient: $\frac{x^2 - 1 - (a^2 - 1)}{x - a} = \frac{x^2 - a^2}{x - a} = \frac{(x - a)(x + a)}{x - a} = x + a$, where the last equality holds for all $x \neq a$. The we plug this into the limit definition of the derivative:

$$f'(a) = \lim_{x \to a} \frac{x^2 - 1 - (a^2 - 1)}{x - a} = \lim_{x \to a} x + a = a + a = 2a.$$

   The substitution is valid since the two functions are equal for $x \neq a$. Also, $x + a$ is continuous since it is a polynomial by Theorem 2.5.7, so we may plug in the limit.

8. (Skip this problem, since we haven't covered $\arctan(x)$ yet).

9. Differentiate the function $y = e^{x+1} + x^{-10}$.

   Solution: The domain of this function is $x \neq 0$, since $e^{x+1} = e \cdot e^x$ is defined for all x, and x^{-10} is defined for $x \neq 0$. We differentiate using the sum, exponential, constant multiple, and power rules:

   $\frac{dy}{dx} = (e^{x+1} + x^{-10})' = (ee^x)' + (x^{-10})' = ee^x + (-10)x^{-10-1} = e^{x+1} - 10x^{-11}$.

10. Differentiate $e^x\sqrt{x}$

    Solution: The domain of the function is $x \geq 0$. The function $(e^x)' = e^x$ by the exponential law, and $(\sqrt{x})' = (x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}}$ by the power law for $x > 0$. We then apply the product rule (for $x > 0$):

    $$(e^x x^{\frac{1}{2}})' = (e^x)' x^{\frac{1}{2}} + e^x (x^{\frac{1}{2}})' = e^x x^{\frac{1}{2}} + e^x \frac{1}{2} x^{-\frac{1}{2}} = e^x(\sqrt{x} + \frac{1}{2}/\sqrt{x}).$$

11. Differentiate $\frac{e^x}{x^2+1}$.

    Solution:

    We apply the quotient rule. Since $x^2+1 \geq 1$, the denominator is never 0. Also, the numerator and denominator are differentiable functions (by the exponential, sum, and power rules). So we may apply the quotient rule:

    $$\frac{d}{dx}\frac{e^x}{x^2+1} = \frac{(e^x)'(x^2+1) - (e^x)(x^2+1)'}{(x^2+1)^2} = \frac{e^x(x^2+1) - e^x(2x)}{(x^2+1)^2} = e^x \frac{(x-1)^2}{(x^2+1)^2}.$$