Midterm 1, Math 1A, section 1 solutions

1. Let $F(x) = \sqrt{2+x}$, $G(x) = \sqrt{2-x}$. Find $F - G$, FG, F/G, and $G \circ F$, and find their domains. Determine which of these functions is even, odd, or neither.

   Solution:

   First, we find the domain of F and G. If $F(x) = \sqrt{2+x}$, then we must have $2 + x \geq 0$, so $x \geq -2$. If $G(x) = \sqrt{2-x}$, then $2 - x \geq 0$, so $x \leq 2$.

   We have $F(x) - G(x) = \sqrt{2+x} - \sqrt{2-x}$. The domain is $-2 \leq x \leq 2$.

   $FG(x) = \sqrt{2+x}\sqrt{2-x} = \sqrt{(2+x)(2-x)} = \sqrt{4-x^2}$. Then FG has domain $-2 \leq x \leq 2$, and FG is even since $FG(-x) = \sqrt{4-(-x)^2} = \sqrt{4-x^2} = FG(x)$.

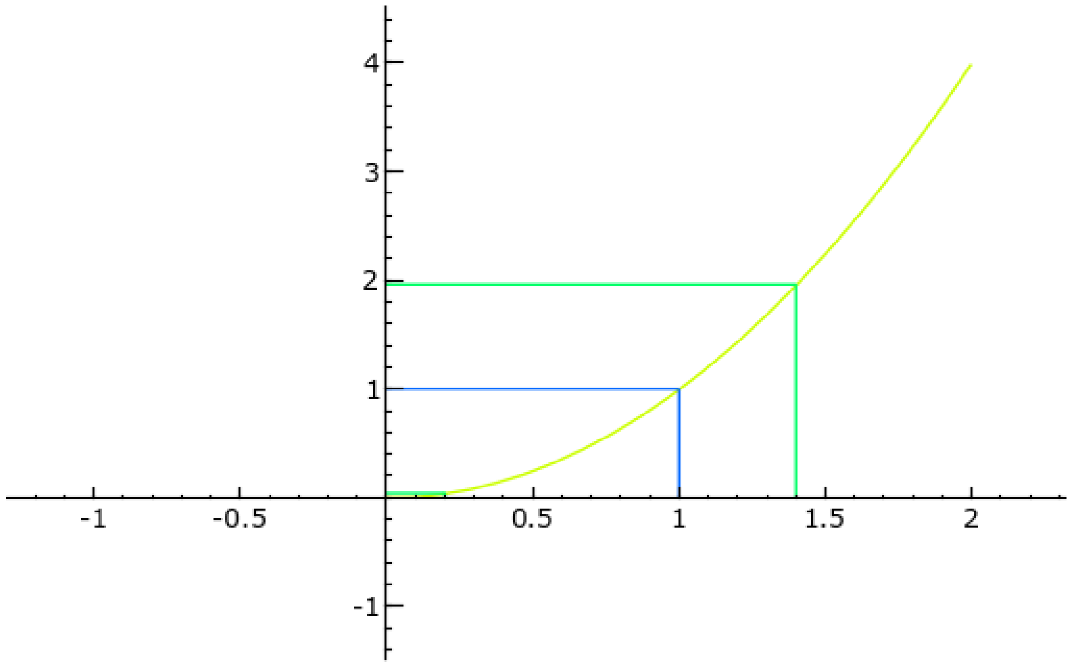
   $(F/G)(x) = \frac{\sqrt{2+x}}{\sqrt{2-x}}$. This has domain $-2 \leq x < 2$, since the denominator cannot $= 0$. $F/G(x)$ is neither even nor odd since its domain is not symmetric about $x = 0$.

   $G \circ F(x) = \sqrt{2 - \sqrt{2+x}}$. For $G \circ F$ to be defined, we must have $F(x)$ lies in the domain of $G(x)$, so $F(x) \leq 2$. Then $\sqrt{2+x} \leq 2$, so we have $0 \leq 2 + x \leq 4$, and thus $-2 \leq x \leq 2$. $G \circ F$ is neither odd nor even, since since $G \circ F(2) = \sqrt{2 - \sqrt{2+2}} = 0$, while $G \circ F(-2) = \sqrt{2 - \sqrt{2-2}} = \sqrt{2}$, so $0 = G \circ F(2) \neq \pm G \circ F(-2) = \pm\sqrt{2}$.

2. Draw the graph of $y = x^2$. Use the graph to find a number δ such that if $|x - 1| < \delta$, then $|x^2 - 1| < .96 = \frac{24}{25}$. Label the corresponding intervals on your graph.

   Solution:

   The inequality $|x^2 - 1| < \frac{24}{25}$ is equivalent to $-\frac{24}{25} < x^2 - 1 < \frac{24}{25}$. Adding 1 to each part of the inequality, we obtain $\frac{1}{25} = 1 - \frac{1}{25} < x^2 < 1 + \frac{24}{25} = \frac{49}{25}$. Since the positive square root preserves inequalities, this is equivalent to $\sqrt{\frac{1}{25}} = \frac{1}{5} < x < \frac{7}{5} = \sqrt{\frac{49}{25}}$ for $x > 0$. Now, we subtract 1 from both sides, obtaining $-\frac{4}{5} = \frac{1}{5} - 1 < x - 1 < \frac{2}{5} = \frac{7}{5} - 1$. Thus, we see that if we let $\delta < \frac{2}{5}$, then if $|x - 1| < \delta$, we have $-\frac{4}{5} < -\frac{2}{5} < x - 1 < \frac{2}{5}$, and therefore from the above reversible derivations, we get $|x^2 - 1| < \frac{24}{25}$.



3. Let $f(x) = \frac{x+8}{x^2-4}$

   (a) What is the domain of f?

   (b) Find $f(1)$, $f(-3)$, and the x- and y-intercepts of f.

   (c) Is f even, odd, or neither? Give an explanation.

   (d) Find $\lim_{x\to\infty} f(x)$, $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$. What are the asymptotes of $y = f(x)$?

   (e) Sketch all of the points and asymptotes you have found from the previous parts on a graph. Then sketch the graph of $y = f(x)$ on the same graph.

Solution:

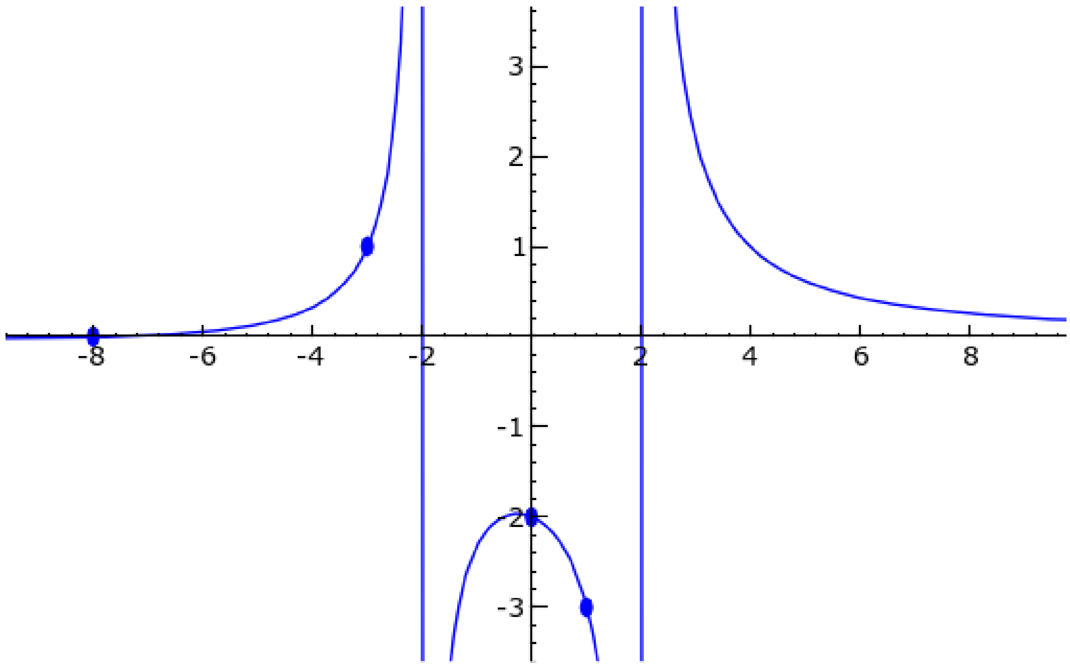
(a) $f(x)$ is defined when the denominator is non-zero, so when $x^2 - 4 = (x-2)(x+2) \neq 0$, which is equivalent to $x - 2 \neq 0$ and $x + 2 \neq 0$. Thus, the domain of $f(x)$ is $x \neq \pm 2$.

(b) $f(1) = \frac{1+8}{1^2-4} = \frac{9}{-3} = -3$. $f(-3) = \frac{-3+8}{(-3)^2-4} = \frac{5}{5} = 1$. $f(0) = \frac{8}{-4} = -2$.
The y-intercept is obtained by setting $f(x) = 0$, which happens when the numerator is zero, and therefore $x = -8$.

(c) f is neither even nor odd, since the denominator is even, but the numerator is neither odd nor even. Alternatively, one may use that $f(0) = -2 \neq 0$, so f is not odd, and $f(8) = \frac{4}{15} > 0 = f(-8)$, so f is not even.

(d) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x+8}{x^2-4} = \lim_{x\to\infty} \frac{1/x+8(1/x)^2}{1-4(1/x)^2} = \frac{0}{1} = 0$, where we are using the fact that $\lim_{x\to\infty} 1/x = 0$, and we may plug this into a limit of a continuous function.
When $-2 < x < 2$, we have $x^2 - 4 < 0, x + 8 > 6$, and $\lim_{x\to 2^-} x^2 - 4 = 0$. Thus, $f(x) < 0$.
So we have $\lim_{x\to 2^-} \frac{x+8}{x^2-4} = -\infty$.

When $x > 2$, we have $x+8 > 10, x^2-4 > 0$, $\lim_{x\to 2^+} x^2 - 4 = 0$, and thus we have $\lim_{x\to 2^+} \frac{x+8}{x^2-4} = \infty$.



(e)

4. Let

$$h(x) = \begin{cases} x^2 + 2a, & x \le 1 \\ ax + 3, & x > 1 \end{cases} \tag{1}$$

Determine a so that h is continuous for all real numbers. Explain with upper and lower limits.

Solution:

On the intervals $(-\infty, 1]$ and $(1, \infty)$, $h(x)$ is equivalent to a polynomial function restricted to that interval, and thus $h(x)$ is continuous on both of these intervals. Thus, we need only choose a so that $h(x)$ is continuous at $x = 1$. Since $\lim_{x\to 1^-} h(x) = \lim_{x\to 1^-} x^2 + 2a = 1 + 2a = h(1)$, we need only choose a so that $\lim_{x\to 1^+} h(x) = \lim_{x\to 1^+} ax + 3 = a + 3 = h(1) = 1 + 2a$. Solving for a, we see that $a = 2$, so $h(x)$ is continuous if $a = 2$.

5. Prove rigorously the following limit, using the M-N definition of an infinite limit:

$$\lim_{x\to\infty} x - 100\cos x = \infty$$

Solution:

Let $M > 0$, and let $N = M + 100$. Suppose $x > N$. We have $\cos x \le 1$, so $-\cos x \ge -1$, and $-100\cos x \ge -100$. Then $x - 100\cos x > N - 100 = M$. From the definition of infinite limits (Definition 8, p. 66 Stewart), we conclude that

$$\lim_{x\to\infty} x - 100\cos x = \infty$$

6. Find the tangent line to the graph of $y = 2x^3 - 5x$ at the point $(-1, 3)$.

Solution:

We compute $\frac{dy}{dx}(-1) = \lim_{x \to -1} \frac{2x^3-5x-3}{x+1} = \lim_{x \to -1} \frac{(x+1)(2x^2-2x-3)}{x+1} = \lim_{x \to -1} 2x^2 - 2x - 3 = 1$ is the slope of the tangent line.

Then we use the formula for a line through $(-1, 3)$ of slope 1 to be $y - 3 = x + 1$, or $y = x + 4$.

7. Find $g'(x)$, where $g(x) = \sqrt{x-2}$ using the limit definition of the derivative and the methods for finding limits that we have developed so far. What are the domains of $g(x)$ and $g'(x)$?

Solution: The domain of $g(x)$ is $x \geq 2$.

First, some algebra to reduce the difference quotient. Let $x > 2$.

$$\frac{\sqrt{x+h-2} - \sqrt{x}}{h} = \frac{(\sqrt{x+h-2} - \sqrt{x-2})(\sqrt{x+h-2} + \sqrt{x-2})}{h(\sqrt{x+h-2} + \sqrt{x-2})}$$
$$= \frac{x+h-2-(x-2)}{h \cdot (\sqrt{x+h-2} + \sqrt{x-2})} = \frac{1}{(\sqrt{x+h-2} + \sqrt{x-2})},$$

assuming $h \neq 0$ and $h > 2 - x$.

Now we may compute

$$g'(x) = \lim_{h \to 0} \frac{\sqrt{x+h-2} - \sqrt{x-2}}{h} = \lim_{h \to 0} \frac{1}{(\sqrt{x+h-2} + \sqrt{x-2})}$$
$$= \frac{1}{2\sqrt{x-2}} = \frac{1}{2}(x-2)^{-1/2}.$$

The second to last inequality, we are using that the function is continuous at $h = 0$, so we may plug in $h = 0$ to obtain the limit. The domain of $g'(x)$ is $x > 2$, since these give the values where the denominator $(x-2)^{\frac{1}{2}}$ is defined and non-zero.