Midterm 1, Math 1A, section 1 solutions

1. Let \( F(x) = \sqrt{2 + x} \), \( G(x) = \sqrt{2 - x} \). Find \( F - G \), \( FG \), \( F/G \), and \( G \circ F \), and find their domains. Determine which of these functions is even, odd, or neither.

Solution:

First, we find the domain of \( F \) and \( G \). If \( F(x) = \sqrt{2 + x} \), then we must have \( 2 + x \geq 0 \), so \( x \geq -2 \). If \( G(x) = \sqrt{2 - x} \), then \( 2 - x \geq 0 \), so \( x \leq 2 \).

We have \( F(x) - G(x) = \sqrt{2 + x} - \sqrt{2 - x} \). The domain is \(-2 \leq x \leq 2\).

\( FG(x) = \sqrt{2 + x} \sqrt{2 - x} = \sqrt{(2 + x)(2 - x)} = \sqrt{4 - x^2} \). Then \( FG \) has domain \(-2 \leq x \leq 2\), and \( FG \) is even since \( FG(-x) = \sqrt{4 - (-x)^2} = \sqrt{4 - x^2} = FG(x) \).

\( (F/G)(x) = \frac{\sqrt{2 + x}}{\sqrt{2 - x}} \). This has domain \(-2 \leq x < 2\), since the denominator cannot = 0. \( F/G \) is neither even nor odd since its domain is not symmetric about \( x = 0 \).

\( G \circ F(x) = \sqrt{2 - \sqrt{2 + x}} \). For \( G \circ F \) to be defined, we must have \( F(x) \) lies in the domain of \( G(x) \), so \( F(x) \leq 2 \). Then \( \sqrt{2 + x} \leq 2 \), so we have \( 0 \leq 2 + x \leq 4 \), and thus \(-2 \leq x \leq 2 \). \( G \circ F \) is neither odd nor even, since since \( G \circ F(2) = \sqrt{2 - \sqrt{2 + 2}} = 0 \), while \( G \circ F(-2) = \sqrt{2 - \sqrt{2 - 2}} = \sqrt{2} \), so \( 0 = G \circ F(2) \neq \pm G \circ F(-2) = \pm \sqrt{2} \).

2. Draw the graph of \( y = x^2 \). Use the graph to find a number \( \delta \) such that if \( |x - 1| < \delta \), then \( |x^2 - 1| < .96 = \frac{24}{25} \). Label the corresponding intervals on your graph.

Solution:

The inequality \( |x^2 - 1| < \frac{24}{25} \) is equivalent to \( -\frac{24}{25} < x^2 - 1 < \frac{24}{25} \). Adding 1 to each part of the inequality, we obtain \( 1 - \frac{24}{25} < x^2 < 1 + \frac{24}{25} = \frac{49}{25} \). Since the positive square root preserves inequalities, this is equivalent to \( \sqrt{\frac{1}{25}} = \frac{1}{5} < x < \frac{7}{5} = \sqrt{\frac{49}{25}} \) for \( x > 0 \). Now, we subtract 1 from both sides, obtaining \( -\frac{4}{5} = \frac{1}{5} - 1 < x - 1 < \frac{2}{5} = \frac{7}{5} - 1 \). Thus, we see that if we let \( \delta = \frac{2}{5} \), then if \( |x - 1| < \delta \), we have \(-\frac{1}{5} < x - 1 < \frac{2}{5} \), and therefore from the above reversible derivations, we get \( |x^2 - 1| < \frac{24}{25} \).
3. Let \( f(x) = \frac{x+8}{x^2-4} \)

(a) What is the domain of \( f \)?

(b) Find \( f(1) \), \( f(-3) \), and the \( x \)- and \( y \)-intercepts of \( f \).

(c) Is \( f \) even, odd, or neither? Give an explanation.

(d) Find \( \lim_{x \to \infty} f(x) \), \( \lim_{x \to -2^+} f(x) \), \( \lim_{x \to -2^-} f(x) \). What are the asymptotes of \( y = f(x) \)?

(e) Sketch all of the points and asymptotes you have found from the previous parts on a graph. Then sketch the graph of \( y = f(x) \) on the same graph.

Solution:

(a) \( f(x) \) is defined when the denominator is non-zero, so when \( x^2 - 4 = (x - 2)(x + 2) \neq 0 \), which is equivalent to \( x - 2 \neq 0 \) and \( x + 2 \neq 0 \). Thus, the domain of \( f(x) \) is \( x \neq \pm 2 \).

(b) \( f(1) = \frac{1+8}{1^2-4} = \frac{9}{-3} = -3 \). \( f(-3) = \frac{-3+8}{(-3)^2-4} = \frac{5}{5} = 1 \). \( f(0) = \frac{8}{-4} = -2 \).

The \( y \)-intercept is obtained by setting \( f(x) = 0 \), which happens when the numerator is zero, and therefore \( x = -8 \).

(c) \( f \) is neither even nor odd, since the denominator is even, but the numerator is neither odd nor even. Alternatively, one may use that \( f(0) = -2 \neq 0 \), so \( f \) is not odd, and \( f(8) = \frac{4}{15} > 0 = f(-8) \), so \( f \) is not even.

(d) \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x+8}{x^2-1} = \lim_{x \to \infty} \frac{1/x+8(1/x)^2}{1-4(1/x)^2} = \frac{0}{1} = 0 \), where we are using the fact that \( \lim_{x \to 0} \frac{1}{x} = 0 \), and we may plug this into a limit of a continuous function.

When \(-2 < x < 2\), we have \( x^2 - 4 < 0 \), \( x + 8 > 6 \), and \( \lim_{x \to 2^-} x^2 - 4 = 0 \). Thus, \( f(x) < 0 \).

So we have \( \lim_{x \to 2^-} \frac{x+8}{x^2-4} = -\infty \).
When \( x > 2 \), we have \( x + 8 > 10, x^2 - 4 > 0 \), \( \lim_{x \to 2^+} x^2 - 4 = 0 \), and thus we have \( \lim_{x \to 2^+} \frac{x + 8}{x^2 - 4} = \infty \).

(e)

4. Let

\[
h(x) = \begin{cases} 
    x^2 + 2a, & x \leq 1 \\
    ax + 3, & x > 1 
\end{cases}
\]  

(1)

Determine \( a \) so that \( h \) is continuous for all real numbers. Explain with upper and lower limits.

Solution:

On the intervals \(( -\infty, 1 \) and \((1, \infty)\), \( h(x) \) is equivalent to a polynomial function restricted to that interval, and thus \( h(x) \) is continuous on both of these intervals. Thus, we need only choose \( a \) so that \( h(x) \) is continuous at \( x = 1 \). Since \( \lim_{x \to 1^-} h(x) = \lim_{x \to 1^-} x^2 + 2a = 1 + 2a = h(1) \), we need only choose \( a \) so that \( \lim_{x \to 1^+} h(x) = \lim_{x \to 1^+} ax + 3 = a + 3 = h(1) = 1 + 2a \). Solving for \( a \), we see that \( a = 2 \), so \( h(x) \) is continuous if \( a = 2 \).

5. Prove rigorously the following limit, using the \( M-N \) definition of an infinite limit:

\[
\lim_{x \to \infty} x - 100 \cos x = \infty
\]

Solution:

Let \( M > 0 \), and let \( N = M + 100 \). Suppose \( x > N \). We have \( \cos x \leq 1 \), so \( -\cos x \geq -1 \), and \(-100 \cos x \geq -100 \). Then \( x - 100 \cos x > N - 100 = M \). From the definition of infinite limits (Definition 8, p. 66 Stewart), we conclude that

\[
\lim_{x \to \infty} x - 100 \cos x = \infty
\]
6. Find the tangent line to the graph of \( y = 2x^3 - 5x \) at the point \((-1, 3)\).

**Solution:**

We compute \( \frac{dy}{dx}(-1) = \lim_{x \to -1} \frac{2x^3 - 5x - 3}{x + 1} = \lim_{x \to -1} \frac{(x+1)(2x^2 - 2x - 3)}{x+1} = \lim_{x \to -1} 2x^2 - 2x - 3 = 1 \) is the slope of the tangent line.

Then we use the formula for a line through \((-1, 3)\) of slope 1 to be \( y - 3 = x + 1 \), or \( y = x + 4 \).

7. Find \( g'(x) \), where \( g(x) = \sqrt{x - 2} \) using the limit definition of the derivative and the methods for finding limits that we have developed so far. What are the domains of \( g(x) \) and \( g'(x) \)?

**Solution:** The domain of \( g(x) \) is \( x \geq 2 \).

First, some algebra to reduce the difference quotient. Let \( x > 2 \).

\[
\frac{\sqrt{x + h - 2} - \sqrt{x}}{h} = \frac{(\sqrt{x + h - 2} - \sqrt{x - 2})(\sqrt{x + h - 2} + \sqrt{x - 2})}{h(\sqrt{x + h - 2} + \sqrt{x - 2})} = \frac{1}{h(\sqrt{x + h - 2} + \sqrt{x - 2})},
\]

assuming \( h \neq 0 \) and \( h > 2 - x \).

Now we may compute

\[
g'(x) = \lim_{h \to 0} \frac{\sqrt{x + h - 2} - \sqrt{x - 2}}{h} = \lim_{h \to 0} \frac{1}{\sqrt{x + h - 2} + \sqrt{x - 2}} = \frac{1}{2\sqrt{x - 2}} = \frac{1}{2} (x - 2)^{-1/2}.
\]

The second to last inequality, we are using that the function is continuous at \( h = 0 \), so we may plug in \( h = 0 \) to obtain the limit. The domain of \( g'(x) \) is \( x > 2 \), since these give the values where the denominator \( (x - 2)^{\frac{1}{2}} \) is defined and non-zero.