Math 113 Homework # 6, due 2/23/01 at 5:00 PM

Feel free to use MATLAB for questions 1, 2, and 3.

1. Find the matrix of $d/dx : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ with respect to the basis $1+x, x+x^2, x^2+2$ of $P_2(\mathbb{R})$.

2. Let $W = \operatorname{span}((1,2,3),(4,5,6)) \subset \mathbb{R}^3$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection across W.

   (a) Find a nonzero vector $v \in W^\perp$.

   (b) Consider the basis $\beta = ((1,2,3),(4,5,6),v)$ of $\mathbb{R}^3$. What is the matrix $[T]_\beta^\beta$?

   (c) Find the matrix A of T with respect to the standard basis of $\mathbb{R}^3$. Your answer should satisfy $A^t = A^{-1} = A$. (Why?)

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a 60 degree rotation around $(1,1,-1)$.

   (a) Find an orthonormal basis v_1, v_2 of $(1,1,-1)^\perp$.

   (b) Consider the basis $\beta = (v_1, v_2, (1,1,-1))$ of $\mathbb{R}^3$. What should the matrix $[T]_\beta^\beta$ be?

   (c) Find the matrix A of T with respect to the standard basis of $\mathbb{R}^3$. Your answer should satisfy $A^6 = I$, $A^{-1} = A^t$, and $A(1,1,-1) = (1,1,-1)$. (Why?)

   (For parts (b) and (c) there are two possible answers.)

4. Let V, W be FDIPS's and let $T : V \to W$ be linear.

   (a) For $x \in V$, show that $x = 0$ if and only if $\langle x, y\rangle = 0$ for all $y \in V$.

   (b) Show that $\operatorname{Im}(T)^\perp = \operatorname{Ker}(T^*)$.

   (c) Show that $(T^*)^* = T$.

5. Let W be a subspace of $\mathbb{R}^n$. Let $w_1, \ldots, w_m$ be any basis for W. Let A be the $n \times m$ matrix whose columns are $w_1, \ldots, w_m$. We can regard A as a linear map $\mathbb{R}^m \to \mathbb{R}^n$.

   (a) Show that A^tA is injective. Conclude that A^tA is invertible.

   (b) Show that $P_W = A(A^tA)^{-1}A^t$. Hint: for $x \in \mathbb{R}^n$, show that $P_W(x) = Ay$ for some $y \in \mathbb{R}^m$ and $x - P_W(x) \in \operatorname{Im}(A)^\perp$, and use problem 4(b).