Math 113 Homework # 5, due 2/16/01 at 5:00 PM

1. Show that if $T : V \to W$ is linear, then $\mathrm{Ker}(T)$ is a subspace of V, and $\mathrm{Im}(T)$ is a subspace of W.

2. Show that if $T : V \to W$ is an injective linear map and $v_1, \dots, v_n$ are linearly independent vectors in V, then $T(v_1), \dots, T(v_n)$ are linearly independent in W.

3. Section 2.4 problem 2.

4. Show that if $T : V \to W$ is linear and bijective, then $\dim(V) = \dim(W)$.

5. (a) Show that if $f : X \to Y$ is bijective, then the inverse function $g : Y \to X$, satisfying (i) $g \circ f = \mathrm{id}_X$ and (ii) $f \circ g = \mathrm{id}_Y$, is unique.
   (b) Show that if $f : X \to Y$ is bijective and $g : Y \to X$ is a function, then each of the equations (i), (ii) implies the other.
   (c) Show that if A is an $n \times m$ matrix and B is an $m \times n$ matrix, with $m > n$, then it is possible that $AB = I_n$, but impossible that $BA = I_m$.
   (d) Let A and B be $n \times n$ matrices with $AB = I_n$. Show that $BA = I_n$. (Hint: first show that A is surjective and B is injective, then use linear algebra to deduce that A and B are invertible.)

6. (a) Show that if $f : W \to X$, $g : X \to Y$, and $h : Y \to Z$ are functions, then $(h \circ g) \circ f = h \circ (g \circ f) : W \to Z$.
   (b) Show that if A is an $a \times b$ matrix, B is a $b \times c$ matrix, and C is a $c \times d$ matrix, then $(AB)C = A(BC)$.

7. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$. Compute A^{-1}, and check that your answer satisfies $AA^{-1} = A^{-1}A = I$. (No MATLAB this time.)

8. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation which rotates by 120 degrees around the line $\{x \in \mathbb{R}^3 \mid x_1 = x_2 = x_3\}$. Find the matrix A such that $T = T_A$. (There are two possible answers depending on which direction you rotate.) Check that your answer satisfies $A^3 = I$.