Math 113 Final, due Wednesday 3/21/01 at 11:30 AM

Name ________________________ **Signature** ________________________

Instructions. You may use your textbook, lecture notes, and homework. You may not use other sources, computing devices, or the assistance of other people. Please sign above to acknowledge that you accept the honor code.

Please give a random 5-digit number here. ____________ This will be used to post your homework, final exam, and course grades.

Please try to give *clear* and *detailed* explanations of your answers to *each* question. You may cite results proved in class; when in doubt, add details.

Please slide your exam all the way under the door of room 383M. Do not leave your exam in a public place where it could be seen or misplaced.

Each of the 10 problems is worth 10 points. The lowest of your scores on the 10 problems will be dropped.

It was a pleasure having you all in class. Good luck on this exam, and in your future studies!

1. Let $v_1, \ldots, v_n$ be linearly independent vectors in $\mathbb{R}^n$, and let $\lambda_1, \ldots, \lambda_n$ be real numbers. Show that there exists an $n \times n$ real matrix A, such that for each $i = 1, \ldots, n$, the vector v_i is an eigenvector of A with eigenvalue λ_i.

2. Find a 3×3 matrix whose null space (kernel) is the span of the vector $(1, 2, 3)$, and whose column space (image) is the span of the vectors $(4, 5, 6)$ and $(7, 8, 9)$.

3. Find the eigenvalues of the following matrix:
$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 10 & 11 & 12 \\ 0 & 0 & 0 & 13 & 14 \\ 0 & 0 & 0 & 0 & 15 \end{pmatrix}.$$

4. Let A be an $m \times m$ matrix and let B be an $n \times n$ matrix. Consider the $(m+n) \times (m+n)$ matrix
$$C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Show that $\det(C) = \det(A)\det(B)$.

5. Let A and B be $n \times n$ matrices[1].

   (a) Show that $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$.

   (b) Show that $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$.

6. Let $P_n(\mathbb{R})$ denote the space of polynomials $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with $a_0, \ldots, a_n \in \mathbb{R}$. Find all eigenvalues and eigenvectors of the linear transformation

$$\frac{d}{dx} : P_n(\mathbb{R}) \to P_n(\mathbb{R}).$$

7. Let A be an $n \times n$ real matrix satisfying $A^t = -A$. (Such a matrix is called "skew-symmetric".) Show that if n is odd, then A is not invertible.

8. Let V be an inner product space and $T : V \to V$ a linear map. Assume $T^2 = T = T^*$. Show that T is the orthogonal projection[2] onto $\operatorname{Im}(T)$.

9. True or false:

   (a) If A is a symmetric matrix and v and w are eigenvectors of A with different eigenvalues, then $v \perp w$.

   (b) If x is a vector in an inner product space V, and if $\langle x, y\rangle = 0$ for all $y \in V$, then $x = 0$.

10. True or false:

   (a) Let $(x_1, y_1), (x_2, y_2), \ldots, (x_5, y_5)$ be five points in $\mathbb{R}^2$. Then there exist real numbers a, b, c, d, e, f, not all zero, such that all five of the points (x_i, y_i) lie on the quadratic curve

$$\{(x, y) \in \mathbb{R}^2 \mid ax^2 + bxy + cy^2 + dx + ey + f = 0\}.$$

   (b) If V is a vector space and $T : V \to V$ is an injective linear transformation, then T is surjective. (Be careful.)

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[1] Recall that the rank of a linear transformation is defined to be the dimension of its image.

[2] Recall that if W is a subspace of V, then "T is the orthogonal projection onto W" means that for each $x \in V$, we have $Tx \in W$ and $x - Tx \in W^\perp$.