1. (a) Taking the complex conjugate of both sides of the equation

\[ A(v + iw) = (a + bi)(v + iw), \]

we obtain \( \overline{A}(v - iw) = (a - bi)(v - iw). \) (Recall that if \( z \) and \( w \) are complex numbers, then \( z + w = \overline{z} + \overline{w} \) and \( z\overline{w} = \overline{zw} \), as is easily checked. It follows from the definition of matrix multiplication that these rules also hold when \( z \) and \( w \) are matrices and vectors.) Since \( A \) is real, \( \overline{A} = A \), so we have

\[ A(v - iw) = (a - bi)(v - iw). \]

This means that \( v - iw \) is an eigenvector of \( A \) with eigenvalue \( a - bi \). (Since \( v + iw \neq 0 \), the vector \( v - iw \) is also nonzero.)

(b) Suppose \( v, w \) are dependent. Since \( v + iw \) is an eigenvector, \( v \) and \( w \) are not both zero. Suppose first that \( v \neq 0 \). Then \( w = \theta v \) for some real number \( \theta \). So \( (1 + i\theta)v \) is an eigenvector of \( A \) with eigenvalue \( a + bi \). Dividing by \( 1 + i\theta \), we find that \( v \) is an eigenvector of \( A \) with eigenvalue \( a + bi \). But this is impossible because \( A \) and \( v \) are real. (If \( Av = (a + bi)v \), then the left side of the equation is real, but the right side is not real, since \( v \neq 0 \) and \( b \neq 0 \).) If \( w \neq 0 \), we get a similar contradiction.

(c) We begin with two observations:

i. \( f \) and \( g \) are solutions to the differential equation. (Try it and see.) Hence any linear combination of \( f \) and \( g \) is a solution, since the differential equation is linear and homogeneous.

ii. \( f(0) \) and \( g(0) \) span \( \mathbb{R}^2 \), by part (b).

Now if \( x \) is any solution, write \( x(0) = af(0) + bg(0) \). Then the function \( af + bg \) is a solution to the differential equation with the same initial condition (value at \( t = 0 \)) as \( x \). By the uniqueness theorem from analysis, \( x(t) = af(t) + bg(t) \) for all \( t \).

2. We have

\[ \det(A - \lambda I) = (-1)^n(\lambda - \lambda_1) \cdots (\lambda - \lambda_n). \]  

(1)
Let’s evaluate \( \det(A - \lambda I) \) using the sum over permutations formula. First, there is the product of the diagonal entries:

\[
(A_{11} - \lambda)(A_{22} - \lambda) \cdots (A_{nn} - \lambda).
\]

Then there are \((n! - 1)\) other terms. In each of these other terms, at most \(n - 2\) diagonal entries appear, and hence no \(\lambda^{n-1}\) terms appear. Reason: if a permutation \(\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}\) has \(\sigma(i) = i\) for \(n - 1\) numbers \(i\), then \(\sigma\) must be the identity permutation, because if \(\sigma(i) = i\) for all \(i\) except \(i = i_0\), then \(i_0\) is not in the image of \(\sigma\), contradicting the fact that all permutations are surjective. So the coefficient of \(\lambda^{n-1}\) in \(\det(A - \lambda I)\) is the coefficient of \(\lambda^{n-1}\) in the product of the diagonal entries (2), namely

\[
(-1)^{n-1}(A_{11} + A_{22} + \cdots + A_{nn}) = (-1)^{n-1} \text{tr}(A).
\]

Now the coefficient of \(\lambda^{n-1}\) in \((-1)^n(\lambda - \lambda_1) \cdots (\lambda - \lambda_n)\) is

\[
(-1)^{n+1}(\lambda_1 + \cdots + \lambda_n).
\]

So by (1), we have \(\text{tr}(A) = \lambda_1 + \cdots + \lambda_n\).

3. Let \(m\) denote the maximum number of linearly independent eigenvectors of \(A\) one can find. I claim that

\[
\sum_{\lambda \in \mathbb{C}} \dim(V_\lambda) = m.
\]

This claim will imply what we want, since by definition \(A\) is diagonalizable if and only if one can find \(n\) linearly independent eigenvectors. It is obvious that \(\sum_{\lambda \in \mathbb{C}} \dim(V_\lambda) \geq m\), because if we have a bunch of linearly independent eigenvectors, then each of them must live in some \(V_\lambda\), but each \(V_\lambda\) can contain no more than \(\dim(V_\lambda)\) of them.

To prove that

\[
\sum_{\lambda \in \mathbb{C}} \dim(V_\lambda) \leq m,
\]

let \(\lambda_1, \ldots, \lambda_k\) denote the different eigenvalues of \(A\) (throw away repeats), and write \(V_i = V_{\lambda_i}\). Let \(d_i = \dim(V_i)\), and for each \(i\), let
$v_{i,1}, \ldots, v_{i,d_i}$ be a basis for $V_i$. I claim that the vectors $v_{i,j}$ are linearly independent. Since these vectors are all eigenvectors of $A$, this will prove (3). To prove independence, suppose

$$\sum_{i=1}^{k} \sum_{j=1}^{d_j} a_{ij} v_{ij} = 0. \quad (4)$$

(We want to show that $a_{ij} = 0$ for all $i,j$.) For $i = 1, \ldots, k$, define

$$v_i = \sum_{j=1}^{d_j} a_{ij} v_{ij}.$$

Then $v_i$ is an eigenvector of $A$ with eigenvalue $\lambda_i$, and equation (4) says that $v_1 + \cdots + v_k = 0$. We know that eigenvectors with different eigenvalues are independent, so $v_i = 0$ for each $i$. But for a fixed $i$, the vectors $v_{ij}$ are independent, because we chose them to be a basis for $V_i$. So $a_{ij} = 0$ for all $i,j$. This completes the proof.

4. (a) Since $A^*A$ maps $\mathbb{R}^m$ to itself, to show that $A^*A$ is invertible, it is enough to show that $A^*A$ is injective. Suppose $x \in \mathbb{R}^m$ and $A^*Ax = 0$. (We need to show that $x = 0$.) Taking the inner product with $x$, we obtain

$$0 = \langle A^*Ax, x \rangle = \langle Ax, Ax \rangle = |Ax|^2.$$

It follows that $Ax = 0$. But we assumed that the columns of the matrix $A$ are independent, so $A$ is injective. Hence $x = 0$.

(b) Let $v \in \mathbb{R}^n$; we need to show that $Pv = A(A^*A)^{-1}A^*v$. Since $Pv \in \text{Im}(A)$, we can write $Pv = Ax$ for some $x \in \mathbb{R}^m$. Now $v - Pv$ is orthogonal to any vector in $\text{Im}(A)$, that is to any vector of the form $Ay$ with $y \in \mathbb{R}^m$. So

$$\langle v - Ax, Ay \rangle = 0$$

for all $y \in \mathbb{R}^m$. Equivalently,

$$\langle A^*v - A^*Ax, y \rangle = 0$$

for all $y \in \mathbb{R}^m$. It follows that $A^*v - A^*Ax = 0$. Since $A^*A$ is invertible, we can apply $(A^*A)^{-1}$ to this equation to obtain

$$x = (A^*A)^{-1}A^*v.$$

Since $Pv = Ax$, this proves what we wanted.