
2. Let $A$ be an $m \times n$ real matrix, and let $R(A) \subset \mathbb{R}^n$ denote the span of the rows of $A$. Show that $\text{Ker}(A) = R(A)^\perp$.

3. Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix} : \mathbb{R}^4 \to \mathbb{R}^4$. Find bases for $\text{Ker}(A)$, $\text{Ker}(A)^\perp$, $\text{Im}(A)$, and $\text{Im}(A)^\perp$.

4. Let $V = \{ x \in \mathbb{R}^4 \mid x_1 = x_2 \text{ and } x_3 = x_4 \}$. Find a basis for $V$, and find the orthogonal projection (with respect to the standard inner product on $\mathbb{R}^4$) of $(1, 2, 3, 4)$ onto $V$.

5. Let $v_1, \ldots, v_n$ be nonzero vectors in an inner product space. Show that if $v_i \perp v_j$ for all $i \neq j$, then $v_1, \ldots, v_n$ are independent.

6. A subset $S$ of a vector space $V$ is called an affine subspace if there exists a subspace $W \subset V$ and a vector $v \in V$ such that $x \in S \iff x - v \in W$. (That is, $S$ is “parallel” to the subspace $W$, but might not contain 0.)

   (a) Let $A$ be an $m \times n$ real matrix and $b \in \mathbb{R}^m$. Show that the set of solutions $x$ to the equation $Ax = b$ is an affine subspace of $\mathbb{R}^n$, if it is nonempty.

   (b) Show that a nonempty subset $S \subset V$ is an affine subspace if and only if the following condition holds: if $x, y \in S$ and $\lambda$ is a scalar, then $\lambda x + (1 - \lambda)y \in S$. (What does this condition mean geometrically?)

7. Let $V, W$ be subspaces of $\mathbb{R}^n$ of dimension $k, l$ respectively. Show that if $k + l \geq n$, then $\dim(V \cap W) \geq n - k - l$.

8. (Extra credit) Let $v_1, \ldots, v_k$ be independent vectors in $\mathbb{R}^n$, and let $w_1, \ldots, w_{n-k}$ be independent vectors in $\mathbb{R}^m$. Does there necessarily exist an $m \times n$ matrix whose kernel is spanned by the $v_i$'s and whose image is spanned by the $w_j$'s? If so, how do you find it? Try an example.