

Math 53 Final, 5/18/07, 12:30 PM – 3:30 PM

No calculators or notes. Each question is worth 10 points. Please write your solution to each of the 10 questions on a separate sheet with your name, SID#, and GSI on it. (If you are removing an incomplete for professor X, write “Incomplete/X” on each page. For Math 49, do questions 6-10 only.) To get full credit for a question, you must obtain the correct answer, put a box around it, and show correct work/justification. Please do not leave the exam between 3:00 and 3:30. Good luck!

1. Find the point on the sphere $x^2 + y^2 + z^2 = 1$ that minimizes the function

$$f(x, y, z) = (x - 2)^2 + (y - 2)^2 + (z - 1)^2.$$

2. Find the volume of the region consisting of all points that are inside the sphere $x^2 + y^2 + z^2 = 4$, above the plane $z = 0$, and below the plane $z = x$.

3. Let R be the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1/2, 1/2)$. Evaluate the integral

$$\iint_R \frac{e^{x+y}}{x+y} dA.$$

Hint: Use the change of variables $u = x + y$, $v = x - y$.

4. A particle moves along the intersection of the surfaces

$$x^2 + y^2 + 2z^2 = 4, \quad z = xy.$$

Let $\langle x(t), y(t), z(t) \rangle$ denote the location of the particle at time t . Suppose that $\langle x(0), y(0), z(0) \rangle = \langle 1, 1, 1 \rangle$ and $x'(0) = 1$. Calculate $y'(0)$ and $z'(0)$.

5. Suppose f is a function on \mathbb{R}^2 satisfying the following conditions on its directional derivatives:

$$D_{\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle} f(x, y) = \sqrt{2}x, \quad D_{\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle} f(x, y) = \sqrt{2}y.$$

- (a) Find $f_x(x, y)$ and $f_y(x, y)$.
- (b) Assuming also that $f(0, 0) = 0$, find $f(x, y)$.

6. Let S be the triangle with vertices $(0, 0, 0)$, $(1, 0, 1)$, and $(1, 1, 2)$, oriented upward. Calculate the surface integral

$$\iint_S \langle 3, 4, 5 \rangle \cdot d\mathbf{S}.$$

7. Consider the vector field

$$\mathbf{F} = \sqrt{x^2 + y^2 + z^2} \langle x, y, z \rangle.$$

- (a) There is a constant c such that $\operatorname{div} \mathbf{F} = c\sqrt{x^2 + y^2 + z^2}$. Find c .
 (b) Compute the outward flux of the vector field \mathbf{F} through the boundary of the solid region $x^2 + y^2 + z^2 \leq 1$, $z \geq \sqrt{x^2 + y^2}$.
8. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the space curve $\mathbf{r}(t) = \langle \cos t, 0, \sin t \rangle$ for $0 \leq t \leq 2\pi$, and

$$\mathbf{F} = \langle \sin(x^3) + z^3, \sin(y^3), \sin(z^3) - x^3 \rangle.$$

Hint: Use Stokes' Theorem.

9. Let S_1 be the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, oriented upward. Let

$$\mathbf{F} = \langle x + y^2 + z^2, x^2 - y + z^2, x^2 + y^2 \rangle.$$

- (a) Use the Divergence Theorem to show that there is an oriented surface S_2 in the x, y plane such that $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$.
 (b) Use part (a) to calculate $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$.
10. Let C be the semicircle $x^2 + y^2 = 1$, $y \geq 0$, oriented counterclockwise. Calculate the line integral

$$\int_C (-y + \cos x) dx + (x + \sin y) dy.$$

Hint: Part of this integral can be evaluated directly from the definition, and the rest using the Fundamental Theorem of Line Integrals.