Math 53 Final, 5/18/07, 12:30 PM – 3:30 PM

No calculators or notes. Each question is worth 10 points. Please write your solution to each of the 10 questions on a separate sheet with your name, SID#, and GSI on it. (If you are removing an incomplete for professor X, write "Incomplete/X" on each page. For Math 49, do questions 6-10 only.) To get full credit for a question, you must obtain the correct answer, put a box around it, and show correct work/justification. Please do not leave the exam between 3:00 and 3:30. Good luck!

1. Find the point on the sphere $x^2 + y^2 + z^2 = 1$ that minimizes the function
$$f(x, y, z) = (x - 2)^2 + (y - 2)^2 + (z - 1)^2.$$

2. Find the volume of the region consisting of all points that are inside the sphere $x^2 + y^2 + z^2 = 4$, above the plane $z = 0$, and below the plane $z = x$.

3. Let R be the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1/2, 1/2)$. Evaluate the integral
$$\iint_R \frac{e^{x+y}}{x + y}\, dA.$$
Hint: Use the change of variables $u = x + y$, $v = x - y$.

4. A particle moves along the intersection of the surfaces
$$x^2 + y^2 + 2z^2 = 4, \qquad z = xy.$$
Let $\langle(x(t), y(t), z(t)\rangle$ denote the location of the particle at time t. Suppose that $\langle x(0), y(0), z(0)\rangle = \langle 1, 1, 1\rangle$ and $x'(0) = 1$. Calculate $y'(0)$ and $z'(0)$.

5. Suppose f is a function on $\mathbb{R}^2$ satisfying the following conditions on its directional derivatives:
$$D_{\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle} f(x, y) = \sqrt{2}\, x, \qquad D_{\left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle} f(x, y) = \sqrt{2}\, y.$$
   (a) Find $f_x(x, y)$ and $f_y(x, y)$.
   (b) Assuming also that $f(0, 0) = 0$, find $f(x, y)$.

6. Let S be the triangle with vertices $(0,0,0)$, $(1,0,1)$, and $(1,1,2)$, oriented upward. Calculate the surface integral

$$\iint_S \langle 3,4,5\rangle \cdot d\mathbf{S}.$$

7. Consider the vector field

$$\mathbf{F} = \sqrt{x^2+y^2+z^2}\,\langle x,y,z\rangle\,.$$

   (a) There is a constant c such that $\operatorname{div}\mathbf{F} = c\sqrt{x^2+y^2+z^2}$. Find c.

   (b) Compute the outward flux of the vector field $\mathbf{F}$ through the boundary of the solid region $x^2+y^2+z^2 \leq 1$, $z \geq \sqrt{x^2+y^2}$.

8. Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the space curve $\mathbf{r}(t) = \langle \cos t, 0, \sin t\rangle$ for $0 \leq t \leq 2\pi$, and

$$\mathbf{F} = \left\langle \sin(x^3)+z^3, \sin(y^3), \sin(z^3)-x^3\right\rangle.$$

   Hint: Use Stokes' Theorem.

9. Let S_1 be the hemisphere $x^2+y^2+z^2=1$, $z \geq 0$, oriented upward. Let

$$\mathbf{F} = \left\langle x+y^2+z^2, x^2-y+z^2, x^2+y^2\right\rangle.$$

   (a) Use the Divergence Theorem to show that there is an oriented surface S_2 in the x,y plane such that $\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_2}\mathbf{F}\cdot d\mathbf{S}$.

   (b) Use part (a) to calculate $\iint_{S_1}\mathbf{F}\cdot d\mathbf{S}$.

10. Let C be the semicircle $x^2+y^2=1$, $y \geq 0$, oriented counterclockwise. Calculate the line integral

$$\int_C (-y+\cos x)\,dx + (x+\sin y)\,dy.$$

   Hint: Part of this integral can be evaluated directly from the definition, and the rest using the Fundamental Theorem of Line Integrals.