Math 242 Homework #3 from 9/30/24

The following exercises are suggested to help you understand the material. This homework will not be collected or graded.

1. (a) Let λ be a contact form on Y^{2n-1}. Check that

$$\omega = d(e^s\lambda)$$

is a symplectic form on $\mathbb{R} \times Y$, where s denotes the $\mathbb{R}$ coordinate. The symplectic manifold $(\mathbb{R}\times Y, \omega)$ is called the **symplectization** of (Y, λ).

   (b) Let (M^{2n}, ω) be a symplectic manifold, and let $\imath : Y^{2n-1} \hookrightarrow M$ be a compact, contact type hypersurface. Let λ be a contact form on Y such that $d\lambda = \imath^*\omega$. Show that there exists a symplectomorphism between a neighborhood of Y in M and a neighborhood of $\{0\}\times Y$ in the symplectization of (Y, λ). *Hint:* Use the Moser trick.

2. Let $\imath : Y^{2n-1} \hookrightarrow \mathbb{R}^{2n}$ be a compact, connected, contact type hypersurface, and let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function such that 0 is a regular value of H and $Y = H^{-1}(0)$ and $H > 0$ in the region bounded by Y.

   (a) Show that if λ is a contact form on Y with $d\lambda = \imath^*\omega$, then the associated Reeb vector field is a *positive* multiple of X_H on Y.

   (b) Suppose that $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$ is a periodic orbit of X_H. Consider a disk in $\mathbb{R}^{2n}$ bounded by γ, i.e. a map $u : D^2 \to \mathbb{R}^{2n}$ such that $u(e^{i\theta}) = \gamma(T\theta/(2\pi))$. Show that $\int_{D^2} u^*\omega > 0$.

   (c) *Extra credit:* Use part (b) to find a compact connected hypersurface in $\mathbb{R}^4$ which is not contact type.

3. Prove the Darboux theorem for contact forms: If λ is a contact form on Y^{2n-1}, then for each point $p \in Y$, there exists a coordinate chart containing p with local coordinates $x_1, y_1, \ldots, x_{n-1}, y_{n-1}, z$ in which

$$\lambda = dz - \sum_{i=1}^{n-1} y_i \, dx_i.$$

Hint: Imitate the proof of the Darboux theorem for symplectic forms, and see Geiges's book if you get stuck.

4. Let ω_{FS} denote the symplectic form on $\mathbb{C}P^1$, whose pullback to $S^3 = \{z \in \mathbb{C}^2 \mid |z| = 1\}$ via the Hopf fibration is the restriction of the standard symplectic form on $\mathbb{C}^2$. Calculate $\int_{\mathbb{C}P^1} \omega_{FS}$.