Math 242 Homework #2 from 9/9/24

The following exercises are suggested to help you understand the material. This homework will not be collected or graded.

1. Let (M, ω) be a symplectic manifold and let $\{\phi_s\}|_{s\in[0,1]}$ be a symplectic isotopy, i.e. ϕ_s is a symplectomorphism of (M, ω) and $\phi_0 = \mathrm{id}_M$. If $\gamma : S^1 \to M$ is a loop, define $u : [0,1] \times S^1 \to M$ by $u(s,t) = \phi_s(\gamma(t))$, and define
$$\mathrm{Flux}(\{\phi_s\})(\gamma) = \int_{[0,1]\times S^1} u^*\omega.$$
   (a) Show that this gives a well-defined map
$$\mathrm{Flux}(\{\phi_s\}) : H_1(M) \longrightarrow \mathbb{R}.$$
   (b) Show that $\mathrm{Flux}(\{\phi_s\})$ depends only on the homotopy class of path in $\mathrm{Symp}(M, \omega)$ from id_M to ϕ_1, so that Flux is a function on the universal cover $\widetilde{\mathrm{Symp}}(M, \omega)$ of $\mathrm{Symp}(M, \omega)$.
   (c) Show that $\mathrm{Flux} : \widetilde{\mathrm{Symp}}(M, \omega) \to \mathbb{R}$ is a homomorphism.

2. Let (M, ω) be a symplectic manifold, and suppose that ω is exact. (Note that M is necessarily noncompact.) Fix a 1-form λ with $d\lambda = \omega$. Let $L \subset M$ be a Lagrangian submanifold and let $\imath : L \to M$ denote the inclusion. Say that L is *exact* if $\imath^*\lambda$ is exact. Show that if a Lagrangian L' is Hamiltonian isotopic to L, then L' is also exact.

3. Consider the two-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the standard symplectic form $\omega = dx\,dy$. Consider the Lagrangian $L = S^1 \times \{0\} \subset T^2$. Suppose that L' is Hamiltonian isotopic to L and that L' is transverse to L. Give an elementary proof that $|L \cap L'| \geq 2$.

4. Let M be a smooth manifold and let $\phi : M \to M$ be a diffeomorphism. A fixed point x of ϕ is said to be *nondegenerate* if 1 is not an eigenvalue of $d\phi_x : T_xM \to T_xM$. Check that all fixed points of ϕ are nondegenerate if and only if the graph of ϕ is transverse to the diagonal in $M \times M$.