Math 242 Homework #1 from 8/30/24

The following exercises are suggested to help you understand the material. This homework will not be collected or graded.

1. Let V be a finite dimensional vector space and let
$$\omega : V \otimes V \longrightarrow \mathbb{R}$$
be an antisymmetric bilinear form. Define
$$\operatorname{Ker}(\omega) = \{x \in V \mid \omega(x,y) = 0 \quad \forall y \in V\}.$$
Say that ω is *nondegenerate* if $\operatorname{Ker}(\omega) = \{0\}$.
   (a) Show that if ω is nondegenerate then $\dim(V)$ is even.
   (b) Show that ω is nondegenerate if and only if $\underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ times}} \neq 0$ where $n = \frac{1}{2}\dim(V)$.
   (c) Show that if W is a codimension 1 subspace of V, then $\dim(\operatorname{Ker}(\omega|_W)) = 1$.
   (d) If W is a codimension 2 subspace of V, what can you say about $\dim(\operatorname{Ker}(\omega|_W))$?

2. Let M be the unit sphere $x^2 + y^2 + z^2 = 1$ and let ω be the Euclidean area form on M. Define a Hamiltonian $H : M \to \mathbb{R}$ by $H(x,y,z) = z$. Show that the associated Hamiltonian vector field is
$$X_H = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$
Conclude that the flow of X_H for time 2π is the identity on M.

3. If (M,ω) is a symplectic manifold and $f, g : M \to \mathbb{R}$, define the *Poisson bracket*
$$\{f,g\} = \omega(X_f, X_g).$$
   (a) Show that the vector space of smooth functions $M \to \mathbb{R}$, together with the Poisson bracket, is a Lie algebra.
   (b) If M is the unit sphere as in the previous exercise, compute $\{x,y\}$.

4. In $\mathbb{R}^4$ with the standard symplectic form, let Y be the unit sphere $x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1$. Show that every point in Y is contained in a closed characteristic of Y.

5. Let M be the 4-torus $T^4 = (\mathbb{R}^4/2\pi\mathbb{Z})^4$ with coordinates $x_1, \ldots, x_4$. Let ε be an irrational constant, and define a symplectic form on M by

$$\omega = dx_1\, dx_2 + \varepsilon dx_2\, dx_3 + dx_3\, dx_4.$$

Define $H : M \to \mathbb{R}$ by $H(x_1, \ldots, x_4) = \sin x_4$. Show that the associated Hamiltonian vector field X_H does not have a periodic orbit on any regular level set of H. (This example is due to Zehnder, 1987.)