Applications of Modularity in Topology

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ABSTRACT. Now Modular invariance is one of the most fundamental principle in mathematical physics. In this expository paper, we will discuss what they are, and how to use them to get some formulas of characteristic forms, which play an important role in proving Ochanine congruence.

§0 Introduction

In the recent thirty years, the famous theorem of Rokhlin [R1] has been extended to various versions. Rokhlin's theorem states that the signature of an closed oriented smooth spin 4-manifold is divisible by 16. In 1987, Ochanine [O] generalized this result to manifolds of $8k + 4$ dimensions, while another generalization was given by Atiyah and Hirzebruch [AH], which states that $\hat{A}$-genus of a closed oriented smooth spin manifold is an even integer. Landweber [La] shows that we can use the elliptic genus to get the Ochanine result directly from the divisibility results of Atiyah and Hizebruch [AH].

In 1972, Rokhlin [R2] established a congruence formula of the type $\phi(B) \equiv \frac{\mathrm{Sign}(M)-\mathrm{Sign}(B\cdot B)}{8} \bmod 2Z$, where $B \cdot B$ is the self-intersection of B in M, and B is an orientable characteristic submanifold of M, which is the Poincaré dual of the scond class of tangent bundle. Ochanine [O] generalized this congruence formula to the case of $8k + 4$ dimensional closed spin^c manifolds. Guillon and Marin [GM] generalizes the result when B might be non-orientable.

On the physics aspect, when Alvarez-Gaumé and Witten [AW] directly computed gravitational anomaly, they discoverd a so called "Miraculous Cancellation" formula, which is actually a formula of $\hat{L}$-class, $\hat{A}$-class and a twisted $\hat{A}$-class of 12-dimension manifold. By using this formula, we can briefly deal with the case of 12 dimensional spin manifolds to build a bridge between divisibility in [AH] and [O]. Liu [L] established a higher dimensional "miraculous cancellation" formula by developing modular invariance properties of characteristic forms. Liu and Zhang [LZh] thus got an intrinsic analytic interpretation of Ochanine invariant $\phi(B)$ for any $8k + 2$ dimesional closed spin manifold B, which leads to an analytic version of Ochanine congruence formula.

The purpose of this paper is to introduce the various characteristic forms and modular invariance, then give a proof of the Han-Zhang formula in [HZ], i.e. a twisted "miraculous cancellation" formula. Finally we use Han-Zhang formula to give a topological proof [HZ] of Ochanine congruence formula, where we do not use too much analytic arguments.

§1 Characteristic Forms

In this section, we introduce some fundamental knowledge for this paper and at the end of this section, one can get a simpliest cancellation formular. Thus one can get some sense for the following section.

Considering an m-dimensional complex vector bundle E over a smooth manifold M, we denote the curvature of a connection ∇^E by Ω^E.

Definition 1.1 The (total) Chern form denoted by $c(E, \nabla^E)$ associated to ∇^E is defined by $c(E, \nabla^E) = \det(I + \frac{\sqrt{-1}}{2\pi}\Omega^E)$

Property 1.2 We have the following identity for matrix

(1.1) $\det(1+A) = \exp tr(\ln(1+A)) = \exp tr(A - \frac{A^2}{2} + \cdots + \frac{(-1)^{n+1}A^n}{n} + \cdots) = 1 + tr(A - \frac{A^2}{2} + \cdots) + \frac{1}{2}[tr(A - \frac{A^2}{2} + \cdots)]^2 + \cdots = 1 + trA + \frac{1}{2}[(trA)^2 - trA^2]^2 + \cdots$

Definition 1.3 The cohomology class $[c(E, \nabla^E)]$ denote (total) Chern class of E. Write $c(E, \nabla^E)$ as $1 + c_1(E, \nabla^E) + c_2(E, \nabla^E) + \cdots c_k(E, \nabla^E)$ with each $c_i(E, \nabla^E) \in \Omega^{2i}(M)$. We call $c_i(E, \nabla^E)$ the i-th Chern form associated to ∇^E and its cohomology class $[c_i(E, \nabla^E)]$ denoted by $c_i(E)$, the i-th Chern class of E.

Remark:1) every $c_i(E, \nabla^E) \in \Omega^{2i}(M)$ is a closed differential form.

2) $c_i(E, \nabla^E)$ determines a cohomology class $[c_i(E, \nabla^E)] \in H^*_{dR}(M; C)$.

3) This class, i.e. $c_i(E)$ does not depend on choice of ∇^E.

For property 1.2 and Definition 1.3, we can write down the first two term of Chern class

$c_1(E) = \frac{\sqrt{-1}}{2\pi} tr\Omega^E$ and $c_2(E) = \frac{1}{8\pi^2} tr(\Omega^E \wedge \Omega^E - (tr\Omega^E) \wedge (tr\Omega^E)]$

Definition 1.4 We can also introduce the Chern character as

(1.2) $ch(E) = tr(exp\frac{\sqrt{-1}}{2\pi}\Omega^E)$

Property 1.5 1) $ch(E) = tr(\sum_{k=0}^{\infty} \frac{1}{k!}(\frac{\sqrt{-1}}{2\pi}\Omega^E)^k \in H^*(M; R)$

2) $ch(E) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=1}^{r} x_j^k = \sum_{j=1}^{r} e^{x_j} = r + \sum_{j=1}^{r} x_j + \frac{1}{2}\sum_{j=1}^{r} x_j^2 + \cdots = r + c_1(E) + \frac{1}{2}[(c_1(E))^2 - 2c_2(E)] + \cdots = \sum_{l=0}^{\infty} ch_l(E)$, where x_j's are the Chern root corresponding to the complex vector bundle E.

3) i) $ch(E \oplus F) = tr(exp\frac{\sqrt{-1}}{2\pi}\Omega^{E\oplus F}) = \sum_{i=1} e^{x_i} + \sum_{j=1} e^{y_j} = ch(E) + ch(F)$

ii) $ch(E \otimes F) = tr(exp\frac{\sqrt{-1}}{2\pi}\Omega^{E\otimes F}) = \sum_{i,j} e^{x_i + y_j} = \sum_i e^{x_j} \sum_j e^{y_j} = ch(E)ch(F)$

Let $\Lambda_t(E)$ and $S_t(E)$ be the total exterior and symmetric powers of E, i.e.

(1.3) $\Lambda_t(E) = C\mid_M + tE + t^2\Lambda^2(E) + \cdots$ and $S_t(E) = C\mid_M + tE + t^2 S^2(E) + \cdots$, where $C\mid_M$ denotes a trivial line bundle on M.

From Atiyah's book, we know

(1.4) $S_t(E) = \frac{1}{\Lambda_{-t}(E)}$ and $\Lambda_t(E - F) = \frac{\Lambda_t(E)}{\Lambda_t(F)}$

By Preoperty 1.5/3)/ii), we can get the formulas for Chern Character forms

(1.5) $ch(S_t(E)) = \frac{1}{ch(\Lambda_{-t}(E))}$ and $ch(\Lambda_t(E - F)) = \frac{ch(\Lambda_t(E))}{ch(\Lambda_t(F))}$

From the Hirzebruch and roof polynomial, we can define $\hat{L}$-genus and $\hat{A}$-genus of a (real) vector bundle E as follows.

(1.6) $\hat{L}(E) = \prod_j \frac{x_j}{\tanh x_j} = \prod_j (1 - \frac{1}{3}x_j^2 + \frac{2}{15}x_j^4 - \frac{17}{315}x_j^6 + \cdots)$, where x_j's are the Chern roots of bundle E.

(1.7) $\hat{A}(E)=\prod_j \frac{\frac{1}{2}x_j}{\sinh\frac{x_j}{2}}=\prod_j (1+\frac{1}{3!}(\frac{x_j}{2})^2+\frac{1}{5!}(\frac{x_j}{2})^4+\cdots)^{-1}$

Definition 1.6 In a similar way of Definition 1.1&1.3, we can also define (total) Pontrjagin form associated to ∇^E by

(1.8) $p(E,\nabla^E)=\det(I-\frac{\Omega^E}{2\pi})$, where E is a real vector bundle over M.

We still have following decompsition for the same reason

(1.9) $p(E,\nabla^E)=1+p_1(E,\nabla^E)+p_2(E,\nabla^E)+\cdots p_k(E,\nabla^E)$, with $p_i(E,\nabla^E)\in\Omega^{4i}(M)$

We call $p_i(E,\nabla^E)$ the i-th Pontrjagin form associated to ∇^E and call the corresponding cohomology class $p_i(E)$, the i-th Pontrjagin class of E.

Let $E\otimes C$ be the complaexification bundle of the real vector bundle E. By comparing the total pontrjagin classes and the Chern classes of $E\otimes C$, we have the following relation

(1.10) $p_i(E)=(-1)^i c_{2i}(E\otimes C)$

For the above definitions and results, one can refer to [Hou] and [Zh3].

Using a differential form view introduced by Zhang, the $\hat{L}$-form and $\hat{A}$-form associated to ∇^E, denoted by

(1.11) $\hat{L}(E,\nabla^E)=\det^{\frac{1}{2}}(\frac{\frac{\sqrt{-1}}{2\pi}\Omega^E}{\tanh(\frac{\sqrt{-1}}{4\pi}\Omega^E)})$ and $\hat{A}(E,\nabla^E)=\det^{\frac{1}{2}}(\frac{\frac{\sqrt{-1}}{4\pi}\Omega^E}{\sinh(\frac{\sqrt{-1}}{4\pi}\Omega^E)})$, which are just another expressions of the above terms.(The former one is actually in the De Rham cohomology ring and here is a differential form).

In the fashion of Pontrjagin class, we thus can rewrite

(1.12) $\hat{L}(E,\nabla^E)=1+\frac{1}{3}p_1+(-\frac{1}{45}p_1^2+\frac{7}{45}p_2)+\cdots$

Anyway, for the same reason as stated in Remark, we can write $\hat{L}(E)$ for convenience.

In a similar way, we can also rewrite $\hat{A}(E)$ as

(1.12) $\hat{A}(E,\nabla^E)=1+\frac{1}{2^2}(-\frac{1}{6}p_1)+\frac{1}{2^4}(\frac{7}{360}p_1^2-\frac{1}{90}p_2)+\cdots$

Remark: (1) In the following section, we can see the prototype of all the cancellation formulas is $\hat{L}=-8\,\hat{A}$ (only compare the top form of them) for the case of $dimM=4$, which can be directly derived from the above discussion. Actually Alvarez-Gaumé and Witten [AW] directly compute these identities (they have other appearence in physics) and then got the so called "miraculous cancellation" formula for dimension 12.

(2) $\hat{A}$ and $\hat{L}$ do have geometric meaning. Integration of $\hat{A}$ is the index of a spin complex, while Integration of $\hat{L}$ is the index of a signature complex, which are the direct application of the Atiyah-Singer Index Theorem.

§2 The main result-A twisted "miraculous cancellation" formula

Notations: Let V be a rank $2l$ real Euclidean vector bundle over M and ξ be a rank two real oriented Euclidean vector bundle over M, which respectively have connection ∇^V and ∇^ξ.

Denote $E-C^{rank(E)}$ by $\widetilde{E}$, where E is a complex vector bundle.

Set

(2.1) $\Theta_1(T_CM, V_C, \xi_C) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_CM}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(\widetilde{V_C} - 2\widetilde{\xi_C}) \otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-\frac{1}{2}}}(\widetilde{\xi_C}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{-q^{s-\frac{1}{2}}}(\widetilde{\xi_C})$

$$\Theta_2(T_CM, V_C, \xi_C) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_CM}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{-q^{m-\frac{1}{2}}}(\widetilde{V_C} - 2\widetilde{\xi_C}) \otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-\frac{1}{2}}}(\widetilde{\xi_C}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{q^s}(\widetilde{\xi_C})$$

where $q = e^{2\pi\sqrt{-1}\tau}$ with $\tau \in H$, the upper half complex plane, and V_C denotes the complexification of the real bundle V.

We can expand $\Theta_1(T_CM, V_C, \xi_C)$ and $\Theta_2(T_CM, V_C, \xi_C)$ as formal Fourier series with respect to $q^{\frac{1}{2}}$

(2.2) $\Theta_1(T_CM, V_C, \xi_C) = A_1(T_CM, V_C, \xi_C) + A_1(T_CM, V_C, \xi_C)q^{\frac{1}{2}} + \cdots$

$\Theta_2(T_CM, V_C, \xi_C) = B_0(T_CM, V_C, \xi_C) + B_1(T_CM, V_C, \xi_C)q^{\frac{1}{2}} + \cdots$

where the A_j's and B_j's are elements in the semi-group formally generated by Hermitian vector bundles over M.

Let $c = e(\xi, \nabla^\xi)$ be the Euler form of ξ canonically associated to ∇^ξ.

Now we can state the main theorem of this article as follows.

Main Theorem 2.1(Han-Zhang) Assume M is a manifold of dimension $8k+4$. If the equality of first Pontrjagin class of tangent bundle and real vector bundle V holds, i.e. $p_1(TM, \nabla^{TM}) = p_1(V, \nabla^V)$, then we have

(2.3) $\left(\frac{1}{2}\right)^{l+2k+1} \left\{ \frac{\widehat{A}(TM, \nabla^{TM}) \det^{\frac{1}{2}}(2\cosh(\frac{\sqrt{-1}}{4\pi}\Omega^V))}{\cosh^2(\frac{c}{2})} \right\}^{(8k+4)}$

$$= \sum_{r=0}^{k} 2^{-6r} \left\{ \widehat{A}(TM, \nabla^{TM}) ch(b_r(T_CM, V_C, \xi_C)) \cosh(\tfrac{c}{2}) \right\}^{(8k+4)}$$

with each $b_r(T_CM, V_C, \xi_C)$, $0 \le r \le k$, is a canonical integral linear combination of $B_j(T_CM, V_C, \xi_C)$, $0 \le j \le r$.

Remark:

1) In (2.3), we compare two top-dimensional differential forms on M.

2) If we let ξ be the trivial bundle, i.e. $\xi = \mathbf{R}^2$, thus we have $c = 0$ and recover Liu's main result.

3) If we consider a 12-dimensional manifold M with $V = TM$ and $\nabla^V = \nabla^{TM}$, which means that the conditional equality automatically holds, we just recover the "miraculous cancellation" formula of Alvarez-Gaumé and Witten.

§3 Modular forms and its property

Let Γ be a subgroup of $SL_2(Z) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in Z, ad - bc = 1 \right\}$

Defnition 3.1 A modular form over Γ is a holomorphic function $f(\tau)$ on $H \cup \{\infty\}$ which, for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, satisfies the transformation formula

(3.1) $f(g\tau) = f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(g)(c\tau+d)^k f(\tau)$

where $\chi : \Gamma \to C^*$ is a character of Γ and k is called the weight of f.

Defnition 3.2 Four Jacobi theta-functions are defined as follows

(3.2) $$\theta(\upsilon,\tau) = 2q^{\frac{1}{8}}\sin(\pi\upsilon)\prod_{j=1}^{\infty}\left[(1-q^j)(1-e^{2\pi\sqrt{-1}\upsilon}q^j)(1-e^{-2\pi\sqrt{-1}\upsilon}q^j)\right]$$

$$\theta_1(\upsilon,\tau) = 2q^{\frac{1}{8}}\cos(\pi\upsilon)\prod_{j=1}^{\infty}\left[(1-q^j)(1+e^{2\pi\sqrt{-1}\upsilon}q^j)(1+e^{-2\pi\sqrt{-1}\upsilon}q^j)\right]$$

$$\theta_2(\upsilon,\tau) = \prod_{j=1}^{\infty}\left[(1-q^j)(1-e^{2\pi\sqrt{-1}\upsilon}q^{j-\frac{1}{2}})(1-e^{-2\pi\sqrt{-1}\upsilon}q^{j-\frac{1}{2}})\right]$$

$$\theta_3(\upsilon,\tau) = \prod_{j=1}^{\infty}\left[(1-q^j)(1+e^{2\pi\sqrt{-1}\upsilon}q^{j-\frac{1}{2}})(1+e^{-2\pi\sqrt{-1}\upsilon}q^{j-\frac{1}{2}})\right]$$

where $q = e^{2\pi\sqrt{-1}\tau}$ with $\tau \in H$.

Remark: All these Jacobi theta-functions are holomorphic functions for $(\upsilon,\tau) \in C \times H$.

Proposition 3.3 $\theta'(0,\tau) := \frac{\partial\theta(\upsilon,\tau)}{\partial\upsilon}\mid_{\upsilon=0} = \pi\theta_1(0,\tau)\theta_2(0,\tau)\theta_3(0,\tau)$

Let S, T be the two generators of $SL_2(Z)$ defined by $S\tau = -\frac{1}{\tau}$, $T\tau = \tau+1$.

Property 3.4 We have the following transformation laws of Jacobi theta-functions with respect to two generators S, T.

(3.3) $\theta(\upsilon,T\tau) = e^{\frac{\pi\sqrt{-1}}{4}}\theta(\upsilon,\tau), \quad \theta(\upsilon,S\tau) = \frac{1}{\sqrt{-1}}\left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}} e^{-\tau\upsilon^2}\theta(\tau\upsilon,\tau)$

(3.4) $\theta_1(\upsilon,T\tau) = e^{\frac{\pi\sqrt{-1}}{4}}\theta_1(\upsilon,\tau)$, $\theta_1(\upsilon,S\tau) = \left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}} e^{-\tau\upsilon^2}\theta_2(\tau\upsilon,\tau)$

(3.5) $\theta_2(\upsilon,T\tau) = \theta_3(\upsilon,\tau), \qquad \theta_2(\upsilon,S\tau) = \left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}} e^{-\tau\upsilon^2}\theta_1(\tau\upsilon,\tau)$

(3.6) $\theta_3(\upsilon,T\tau) = e^{\frac{\pi\sqrt{-1}}{4}}\theta_2(\upsilon,\tau)$, $\theta_3(\upsilon,S\tau) = \left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}} e^{-\tau\upsilon^2}\theta_3(\tau\upsilon,\tau)$

§4 Proof of the Main Theorem

Lemma 4.1 If $\{w_i\}$ are the formal Chern roots of a Hermitian vector bundle E carrying a Hermitian connection ∇^E, then we have the following formula for the Chern character form of $\Lambda_t(E)$

(4.1) $ch(\Lambda_t(E)) = \prod_i (1+e^{w_i}t)$

Proof. From the Property 1.5/2), we know $ch(E) = \sum_{j=1}^{r} e^{w_j}$. Because $\{w_i\}$ are the formal Chern roots of E, $\{w_i + w_j\}_{i\neq j}$ are the formal Chern roots of $\Lambda^2 E$. Thus we get $ch(\Lambda^2 E) = \sum_{\substack{i,j=1\\ i\neq j}}^{r} e^{w_i+w_j}$.

Thus we have the following formula,

(4.2) $ch(\Lambda_t(E)) = ch(C\mid_M) + tch(E) + t^2 ch(\Lambda^2 E) + \cdots$

$$= 1 + t\sum_{j=1}^{r} e^{w_j} + t^2 \sum_{\substack{i,j=1\\ i\neq j}}^{r} e^{w_i+w_j} + \cdots = \prod_i (1+e^{w_i}t)$$

Set

(4.3) $P_1(\tau) = \left\{ \frac{\hat{A}(TM,\nabla^{TM}) \det^{\frac{1}{2}}(2\cosh(\frac{\sqrt{-1}}{4\pi}\Omega^V))}{\cosh^2(\frac{c}{2})} ch\left(\Theta_1(T_CM, V_C, \xi_C), \nabla^{\Theta_1(T_CM,V_C,\xi_C)}\right)\right\}^{(8k+4)}$

$P_2(\tau) = \left\{ \hat{A}(TM,\nabla^{TM}) ch\left(\Theta_2(T_CM, V_C, \xi_C), \nabla^{\Theta_2(T_CM,V_C,\xi_C)}\right)\cosh(\frac{c}{2})\right\}^{(8k+4)}$

where $\nabla^{\Theta_1(T_CM,V_C,\xi_C)}$, $i = 1, 2$, are the Hermitian connections with $q^{\frac{j}{2}}$-coefficients on $\Theta_1(T_CM, V_C, \xi_C)$ induced from those on the A_j's and B_j's.

By using the terminology of formal Chern roots (c.f (1.7)), we can rewrite $P_1(\tau)$ as follows,

(4.4) $P_1(\tau) = 2^l \left\{ \left(\prod_{j=1}^{4k+2} \frac{\pi x_j}{\sin(\pi x_j)} \right) \left(\prod_{v=1}^{l} \cos(\pi y_v) \right) \frac{ch\Theta_1((T_CM, V_C, \xi_C))}{\cos^2(\pi u)} \right\}^{(8k+4)}$

where we denote the formal Chern roots of $(V_C, \nabla^{V_C})(resp.(T_CM, \nabla^{T_CM}))$ by $\{\pm 2\pi\sqrt{-1}y_v\}(resp.\{\pm 2\pi\sqrt{-1}x_j\}$ and let $c = 2\pi\sqrt{-1}u$.

From (1.5) (2.1), we can write $ch(\Theta_1(T_CM, V_C, \xi_C))$ explicitly,

(4.5) $ch(\Theta_1(T_CM, V_C, \xi_C)) = \prod_{n=1}^{\infty} \frac{ch(\Lambda_{-q^n}(C^{8k+4}))}{ch(\Lambda_{-q^n}(T_CM))} \prod_{m=1}^{\infty} \frac{ch(\Lambda_{q^m}(V_C))}{ch(\Lambda_{q^m}(C^{2l})} \prod_{t=1}^{\infty} \left(\frac{ch(\Lambda_{-q^n}(C^2))}{ch(\Lambda_{-q^n}(\xi_C))} \right)^2$

$\prod_{r=1}^{\infty} \frac{ch(\Lambda_{q^{r-\frac{1}{2}}}(\xi_C))}{ch(\Lambda_{q^{r-\frac{1}{2}}}(C^2))} \prod_{n=1}^{\infty} \frac{ch(\Lambda_{-q^{s-\frac{1}{2}}}(\xi_C))}{ch(\Lambda_{-q^{s-\frac{1}{2}}}(C^2))}$

By (3.2) (4.1) (4.4) (4.5) and Proposition 3.3, we can get following identities.

Proposition 4.2

(4.6) $P_1(\tau) = 2^l \left\{ \left(\prod_{j=1}^{4k+2} x_j \frac{\theta'(0,\tau)}{\theta(x_j,\tau)} \right) \left(\prod_{v=1}^{l} \frac{\theta_1(y_v,\tau)}{\theta_1(0,\tau)} \right) \frac{\theta_1^2(0,\tau)\theta_3(u,\tau)\theta_2(u,\tau)}{\theta_1^2(u,\tau)\theta_3(0,\tau)\theta_2(0,\tau)} \right\}^{(8k+4)}$

In a similar way, we can also compute $P_2(\tau)$ and thus get

Proposition 4.2'

(4.6)' $P_2(\tau) = 2^l \left\{ \left(\prod_{j=1}^{4k+2} x_j \frac{\theta'(0,\tau)}{\theta(x_j,\tau)} \right) \left(\prod_{v=1}^{l} \frac{\theta_2(y_v,\tau)}{\theta_2(0,\tau)} \right) \frac{\theta_2^2(0,\tau)\theta_3(u,\tau)\theta_1(u,\tau)}{\theta_2^2(u,\tau)\theta_3(0,\tau)\theta_1(0,\tau)} \right\}^{(8k+4)}$

Now we'd like to introduce some terminology.

Let $M_R(\Gamma)$ denote the ring of modular forms over Γ with real Fourier coefficients.

Definition 4.3 We defined δ and ϵ functions as follows

(4.7) $\delta_1(\tau) = \frac{1}{8}(\theta_2^4(0,\tau) + \theta_3^4(0,\tau))$, $\epsilon_1(\tau) = \frac{1}{16}\theta_2^4(0,\tau)\theta_3^4(0,\tau)$

$\delta_2(\tau) = -\frac{1}{8}(\theta_1^4(0,\tau) + \theta_3^4(0,\tau))$, $\epsilon_2(\tau) = \frac{1}{16}\theta_1^4(0,\tau)\theta_3^4(0,\tau)$

where θ_i's are Jacobi theta-functions.

We can expand them as formal Fourier series

(4.8) $\delta_1(\tau) = \frac{1}{4} + 6q + \cdots$, $\epsilon_1(\tau) = \frac{1}{16} - q + \cdots$

$\delta_2(\tau) = -\frac{1}{8} - 3q^{\frac{1}{2}} + \cdots$, $\epsilon_2(\tau) = q^{\frac{1}{2}} + \cdots$

Let $\Gamma_0(2)$, $\Gamma^0(2)$ be two subgroups pf $SL_2(Z)$ defined as

(4.9) $\Gamma_0(2)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(Z) \mid c\equiv 0 \mod 2Z\right\}$

$\Gamma^0(2)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(Z) \mid b\equiv 0 \mod 2Z\right\}$

Property 4.4 1) T, ST^2ST are two generators of $\Gamma_0(2)$; while STS, T^2STS are the two generators of $\Gamma^0(2)$.

2) $\delta_2(-\frac{1}{\tau})=\tau^2\ \delta_1(\tau)$, $\delta_2(\tau+2)=\delta_2(\tau)$, $\delta_1(-\frac{1}{\tau})=\tau^2\ \delta_2(\tau)$, $\delta_1(\tau+1)=\delta_1(\tau)$
$\epsilon_2(-\frac{1}{\tau})=\tau^4\ \epsilon_1(\tau)$, $\epsilon_2(\tau+2)=\epsilon_2(\tau)$, $\epsilon_1(-\frac{1}{\tau})=\tau^4\ \epsilon_2(\tau)$, $\epsilon_1(\tau+1)=\epsilon_1(\tau)$

3) δ_2 and ϵ_2 are modular forms of weight 2 and 4 respectively over $\Gamma^0(2)$; while δ_1 and ϵ_1 are modular forms of weight 2 and 4 respectively over $\Gamma_0(2)$. Moreover, we have $M_R(\Gamma^0(2))=R[\delta_2(\tau),\epsilon_2(\tau)]$.

Proof. 1) 2) trvial

3) We can check directly on denerators of $\Gamma^0(2)$ and $\Gamma_0(2)$, where we will use 1) and 2)

The last result need to refer lemma 2 in [L]

Proposition 4.5 If $p_1(TM,\nabla^{TM})=p_1(V,\nabla^V)$ holds, then $P_1(\tau)$ is a modular form of weight $4k+2$ over $\Gamma_0(2)$; while $P_2(\tau)$ is a modular form of weight $4k+2$ over $\Gamma^0(2)$. Futhermore, we also have the following formula

(4.10) $P_1(-\frac{1}{\tau})=2^l\tau^{4k+2}P_2(\tau)$

Proof. For the first part, we only need to check

$P_1(T\tau)=P_1(\tau)$ and $P_1(ST^2ST\tau)=P_1(\frac{-\tau-1}{2\tau+1})=(2\tau+1)^{4k+2}P_1(\tau)$

$P_2(STS\tau)=P_2(-\frac{\tau}{\tau-1})=(\tau-1)^{4k+2}P_1(\tau)$ and $P_2(T^2STS\tau)=P_2(\frac{\tau-2}{\tau-1})=(\tau-1)^{4k+2}P_2(\tau)$, which are trivial. (by Prop 4.2 and Property 3.4)

For the second part, it is still a routine to check.

At any point $x\in M$, up to the volumn form determined by the metric of T_xM, $P_2(\tau)$ can be considered as a formal power serise of $q^{\frac{1}{2}}$ with real Fourier coefficients.

By Property 4.4 and Proposition 4.5, we can expand $P_2(\tau)$ in such a way

(4.11) $P_2(\tau)=h_0(8\delta_2)^{2k+1}+h_1(8\delta_2)^{2k-1}\epsilon_2+\cdot\cdot+h_k(8\delta_2)\epsilon_2^k$

where each h_j, $0\le j\le k$, is a real multiple of the volumn form at x.

By (4.10) and (4.11), we can rewrite $P_1(\tau)$ as follows

(4.12) $P_1(\tau)=\frac{2^l}{\tau^{4k+2}}P_2(-\frac{1}{\tau})$

$=\frac{2^l}{\tau^{4k+2}}[h_0(8\delta_2(-\frac{1}{\tau}))^{2k+1}+h_1(8\delta_2(-\frac{1}{\tau}))^{2k-1}\epsilon_2(-\frac{1}{\tau})+\cdots+h_k(8\delta_2(-\frac{1}{\tau}))(\epsilon_2(-\frac{1}{\tau}))^k]$

By using Property 4.4/2), we get the similar expression as (4.11)

(4.13) $P_1(\tau)=2^l[h_0(8\delta_1)^{2k+1}+h_1(8\delta_1)^{2k-1}\epsilon_1+\cdot\cdot+h_k(8\delta_1)\epsilon_1^k]$

Setting $q=0$ in both (4.3) and (4.13) and by using (4.8), we get the following identity

(4.14) $\left\{\frac{\hat{A}(TM,\nabla^{TM})\det^{\frac{1}{2}}(2\cosh(\frac{\sqrt{-1}}{4\pi}\Omega^V))}{\cosh^2(\frac{c}{2})}\right\}^{(8k+4)}$

$$=2^l[h_0(2)^{2k+1}+h_1(2)^{2k-1}\frac{1}{16}+\cdot\cdot+h_k(2)(\frac{1}{16})^k]=2^{l+2k+1}\sum_{r=0}^{k}2^{-6r}h_r$$

Then the remain thing is to prove that the above h_r can be expressed as a canonical integral linear combination of $\left\{\hat{A}(TM,\nabla^{TM})ch(B_j,\nabla^{B_j})\cosh(\frac{c}{2})\right\}^{(8k+4)}$

$, 0 \leq j \leq r$.

By (4.8) (4.11), we have the following Fourier expansion for $P_2(\tau)$ with respect to $q^{\frac{1}{2}}$

(4.15)

$P_2(\tau) = -[h_0(1+24q^{\frac{1}{2}}+\cdots)^{2k+1} + h_1(1+24q^{\frac{1}{2}}+\cdots)^{2k-1}(q^{\frac{1}{2}}+\cdots) + h_2(1+24q^{\frac{1}{2}}+\cdots)^{2k-3}(q^{\frac{1}{2}}+\cdots)^2 + \cdots + h_k(1+24q^{\frac{1}{2}}+\cdots)(q^{\frac{1}{2}}+\cdots)^k]$

$= -[h_0(1+24(2k+1)q^{\frac{1}{2}} + 288(2k+1)(2k-1)+\cdots) + h_1(q^{\frac{1}{2}}+24(2k-1)q+\cdots) + h_2(q+\cdots)+\cdots]$

$= -h_0 - [24(2k+1)h_0 + h_1]q^{\frac{1}{2}} - [576k(2k+1)h_0 + 24(2k-1)h_1 + h_2]q + \cdots$

By (2.2) (4.3) and Property (1.5)/3), we have another Fourier expansion for $P_2(\tau)$ with respect to $q^{\frac{1}{2}}$

(4.16)

$P_2(\tau) = \{\widehat{A}(TM,\nabla^{TM})ch(B_0(T_CM,V_C,\xi_C) + B_1(T_CM,V_C,\xi_C)q^{\frac{1}{2}} + B_2(T_CM,V_C,\xi_C)q + \cdots)\cosh(\frac{c}{2})\}^{(8k+4)}$

$= \{\widehat{A}(TM,\nabla^{TM})ch(B_0(T_CM,V_C,\xi_C))\cosh(\frac{c}{2}) + \widehat{A}(TM,\nabla^{TM})ch(B_1(T_CM,V_C,\xi_C))\cosh(\frac{c}{2})q^{\frac{1}{2}} + \widehat{A}(TM,\nabla^{TM})ch(B_2(T_CM,V_C,\xi_C))\cosh(\frac{c}{2})q + \cdots\}^{(8k+4)}$

By comparing (4.15) with (4.16), we can get the following equations

(4.17) $-h_0 = \left\{\widehat{A}(TM,\nabla^{TM})ch(B_0(T_CM,V_C,\xi_C))\cosh(\frac{c}{2})\right\}^{(8k+4)}$

$-[24(2k+1)h_0+h_1] = \left\{\widehat{A}(TM,\nabla^{TM})ch(B_1(T_CM,V_C,\xi_C))\cosh(\frac{c}{2})\right\}^{(8k+4)}$

$-[576k(2k+1)h_0+24(2k-1)h_1+h_2] = \left\{\widehat{A}(TM,\nabla^{TM})ch(B_2(T_CM,V_C,\xi_C))\cosh(\frac{c}{2})\right\}^{(8k+4)}$

$\cdots$

Then we can find all the h_r can be determined in this way by indection method. For example, we have

(4.18) $h_0 = -\left\{\widehat{A}(TM,\nabla^{TM})ch(B_0(T_CM,V_C,\xi_C))\cosh(\frac{c}{2})\right\}^{(8k+4)}$

$h_1 = \left\{\widehat{A}(TM,\nabla^{TM})[24(2k+1) - ch(B_1(T_CM,V_C,\xi_C))]\cosh(\frac{c}{2})\right\}^{(8k+4)}$

$h_2 = \left\{\widehat{A}(TM,\nabla^{TM})[-576k(2k+1) + 24(2k-1) - (ch(B_2(T_CM,V_C,\xi_C))\cosh(\frac{c}{2})\right\}^{(8k+4)}$

Remark: From the above identity and assuming $V = TM$, $\nabla^V = \nabla^{TM}$, $k=1$ and $c=0$, we can get original "miraculous cancellation" formula which stated in [Al, W].

§5 Application in Topology

Now we use this twisted "miraculous cancellation" formula to Spin^c manifolds to give a direct "topological" proof of the analytic version of the Ochanne congruence stated in Theorem 4.2 of [Liu, Zh].

Let M be an oriented and closed $8k+4$ dimensional Riemannian manifold and B is an $8k+2$ dimensional closed oriented submanifold whose Poincaré dual is the second Stiefel-Whitney class of TM, i.e. $\widetilde{c} \in H^2(M,Z)$ is the Poincaré dual of B and

(5.1) $\widetilde{c} \equiv w_2(TM) \bmod 2Z$

Then we can find an oriented real rank two Euclidean vector bundle ξ over M having a Euclidean connection ∇^{ξ} s.t. $\widetilde{c} = [c] = [e(\xi, \nabla^{\xi})] \in H^2(M,Z)$.

Let $V = TM$ and $\nabla^V = \nabla^{TM}$, then (2.3) becomes

(5.2) $\left\{\frac{\widehat{L}(TM,\nabla^{TM})}{\cosh^2(\frac{c}{2})}\right\}^{(8k+4)} = 8\sum_{r=0}^{k} 2^{6k-6r}\left\{\widehat{A}(TM,\nabla^{TM})ch(b_r(T_CM,\xi_C))\cosh(\frac{c}{2})\right\}^{(8k+4)}$

In this section, we briefly use $b_r(T_CM,\xi_C)$ for $b_r(T_CM,T_CM,\xi_C)$ and same for B_j and Θ_j.

There is an important identity in [O]

(5.3) $\int_M \frac{\widehat{L}(TM,\nabla^{TM})}{\cosh^2(\frac{c}{2})} = \mathrm{Sign}(M) - \mathrm{Sign}(B\cdot B)$

By (5.2) (5.3), because only the top form can contribute to the integral, we have the following formula

(5.4) $\frac{\mathrm{Sign}(M)-\mathrm{Sign}(B\cdot B)}{8} = \int_M \sum_{r=0}^{k} 2^{6k-6r}\left\{\widehat{A}(TM,\nabla^{TM})ch(b_r(T_CM,\xi_C))\cosh(\frac{c}{2})\right\}$

Another important result we need to use is given by [AH]

$\int_M \widehat{A}(TM,\nabla^{TM})ch(b_r(T_CM,\xi_C))\cosh(\frac{c}{2}) \in Z,\ 0 \le r \le k$

In (5.4), thus all the term $r < k \bmod 64Z$ are zero and the only remain term is $r = k$.

Then we have the following congruence formula

Theorem 5.1

(5.5) $\frac{\mathrm{Sign}(M)-\mathrm{Sign}(B\cdot B)}{8} \equiv \int_M \widehat{A}(TM,\nabla^{TM})ch(b_k(T_CM,\xi_C))\cosh(\frac{c}{2}) \bmod 64Z$

where $b_k(T_CM,\xi_C)$ is just the same term as in our Main Theorem and can be can be canonically expressed as an integral linear combination of $B_j(T_CM,\xi_C), 0 \le j \le k$.

Finally we prove our analytic version of the Ochanine congruence stated in Theorem 4.2 of [LZh], where we translate it to our notataion.

Theorem 5.2 (Theorem 4.2 of [LZh])

$\frac{\mathrm{Sign}(M)-\mathrm{Sign}(B\cdot B)}{8} \equiv \int_M \widehat{A}(TM,\nabla^{TM})ch(b_k(T_CM + C^2 - \xi_C, C^2))\cosh(\frac{c}{2})$ mod $2Z$

Proof. By (2.1), we can get the following identity through direct computation

(5.6) $\Theta_2(T_CM,\xi_C) = \Theta_2(T_CM + C^2 - \xi_C, C^2) \otimes \frac{\bigotimes_{r=1}^{\infty}\Lambda_{q^{r-\frac{1}{2}}}(\widetilde{\xi_C}) \otimes \bigotimes_{s=1}^{\infty}\Lambda_{q^s}(\widetilde{\xi_C})}{\bigotimes_{m=1}^{\infty}\Lambda_{-q^{m-\frac{1}{2}}}(\widetilde{\xi_C}) \otimes \bigotimes_{n=1}^{\infty}\Lambda_{-q^n}(\widetilde{\xi_C})}$

By (1.4) and $\mathrm{rk}(\xi) = 2$, we can verifies the following identity

(5.7) $\Lambda_{q^r}(\widetilde{\xi_C}) \equiv \Lambda_{-q^r}(\widetilde{\xi_C}) \bmod 2q^r\widetilde{\xi_C}Z[[q^r]]$ and $\Lambda_{q^{r-\frac{1}{2}}}(\widetilde{\xi_C}) \equiv \Lambda_{-q^{r-\frac{1}{2}}}(\widetilde{\xi_C})$ mod $2q^{r-\frac{1}{2}}\widetilde{\xi_C}Z[[q^{r-\frac{1}{2}}]]$

From (5.6) and (5.7), we have

(5.8) $\Theta_2(T_CM,\xi_C)\equiv\Theta_2(T_CM+C^2-\xi_C,C^2)$ mod $2q^{\frac{1}{2}}\widetilde{\xi_C}Z[T_CM,\xi_C][[q^{\frac{1}{2}}]]$

By (2.2) and (5.8), we get the relation between B_j's

(5.9) $B_j(T_CM,\xi_C)\equiv B_j(T_CM+C^2-\xi_C,C^2)$ mod $2\widetilde{\xi_C}Z[T_CM,\xi_C]$ for $j\geq 1$

Because the b_r's are defined by induction as those in section 4, we have the same realtion for b_r's as follow

(5.10) $b_r(T_CM,\xi_C)\equiv b_r(T_CM+C^2-\xi_C,C^2)$ mod $2\widetilde{\xi_C}Z[T_CM,\xi_C]$ for $j\geq 1$

Then we can have the expression as

(5.11) $b_r(T_CM,\xi_C)=b_r(T_CM+C^2-\xi_C,C^2)+2\widetilde{\xi_C}C_r$ for some $C_r\in Z[T_CM,\xi_C]$

On another hand, [AH] states that

(5.12) $\int_M \hat{A}(TM,\nabla^{TM})ch(\widetilde{\xi_C}C_r)\cosh(\frac{c}{2})\in Z$, $0\leq r\leq k$

Combining (5.5), (5.11) and (5.12), we fufill the proof of Theorem 5.2.

Finally, we want to point out that

(1) By using this "miraculous cancellation" formula, we also can deal with the Finashin congruence, which need some knowledge of $\eta-invariants$.

(2) We also have the such kind of twisted cancellation formula for $8k$ dimension.

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