COHOMOLOGY OF THE COMPLEX GRASSMANNIAN

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Abstract. The Grassmannian is a generalization of projective spaces–instead of looking at the set of lines of some vector space, we look at the set of all n-planes. It can be given a manifold structure, and we study the cohomology ring of the Grassmannian manifold in the case that the vector space is complex. The multiplicative structure of the ring is rather complicated and can be computed using the fact that for smooth oriented manifolds, cup product is Poincaré dual to intersection. There is some nice combinatorial machinery for describing the intersection numbers. This includes the symmetric Schur polynomials, Young tableaux, and the Littlewood-Richardson rule. Sections 1, 2, and 3 introduce notation and the necessary topological tools. Section 4 uses linear algebra to prove Pieri's formula, which describes the cup product of the cohomology ring in a special case. Section 5 describes the combinatorics and algebra that allow us to deduce all the multiplicative structure of the cohomology ring from Pieri's formula.

1. Basic properties of the Grassmannian

The Grassmannian can be defined for a vector space over any field; the cohomology of the Grassmannian is the best understood for the complex case, and this is our focus. Following [MS], the complex *Grassmannian* $G_n(\mathbb{C}^{m+n})$ is the set of n-dimensional complex linear spaces, or *n-planes* for brevity, in the complex vector space $\mathbb{C}^{m+n}$ topologized as follows: Let $V_n(\mathbb{C}^{n+m})$ denote the subspace of the n-fold direct sum $\mathbb{C}^{m+n} \oplus \ldots \oplus \mathbb{C}^{m+n}$ that consists of all n-tuples of linearly independent vectors in $\mathbb{C}^{m+n}$. Two points in $V_n(\mathbb{C}^{n+m})$ are equivalent if they span the same n-plane. $G_n(\mathbb{C}^{m+n})$ is the quotient space induced by this equivalence relation.

Proposition 1.1. *Each point $W \in G_n(\mathbb{C}^{n+m})$ has a neighborhood U homeomorphic to $\mathbb{C}^{nm}$. $G_n(\mathbb{C}^{n+m})$ is a compact complex manifold of dimension nm. Its tangent bundle is isomorphic to $Hom(\gamma^n(\mathbb{C}^{n+m}), \gamma^{\perp})$, where γ^n is the canonical complex n-plane bundle over $G_n(\mathbb{C}^{n+m})$.*

To prove this, we look at a neighborhood U of W_0 homeomorphic to $\mathbb{C}^{nm}$ for each $W_0 \in G_n(\mathbb{C}^{n+m})$. Let U be the set of all n-planes W such that $W \cap W_0^{\perp} = 0$. Consider the map that takes $W \in U$ to $p : W_0 \to W \to W_0^{\perp}$, where both maps are the orthogonal projections and the first is an isomorphism. This defines a homeomorphism between U and $\mathrm{Hom}(W_0, W_0^{\perp})$, which is homeomorphic to $\mathbb{C}^{nm}$. This also describes the tangent space because the tangent space at a point in $\mathbb{C}^{nm}$ is canonically isomorphic to $\mathbb{C}^{nm}$. See lemma 5.1 [MS] for more details.

It is not clear that the proof in [MS] can be adapted to show the Grassmannian is a complex manifold. This is proved in [GH] using a different approach. Recall that any complex manifold has a canonical preferred orientation. We will need this in section 3.

2. A CW-complex structure and additive cohomology

A partition λ of r is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $r = \sum_{i=1}^{n} \lambda_i$. We also define $|\lambda| = r$. Typically it is required that $\lambda_n > 0$, but it is convenient to not make this restriction. Let i be the largest integer such that $\lambda_i > 0$; we say λ is a partition of r into i *parts*.

We define a CW-complex structure for $G_n(\mathbb{C}^{n+m})$. Choose a basis $v_1, \ldots, v_{n+m}$ for $\mathbb{C}^{n+m}$ and let F_i be the span of $v_1, \ldots, v_i$. The chain of nested subspaces

$$0 = F_0 \subset F_1 \subset \ldots \subset F_{n+m} = \mathbb{C}^{n+m}$$

is a *complete flag*, which we denote by F. For λ with at most n parts and $\lambda_1 \leq m$ define $e(\lambda, F) \subset G_n(\mathbb{C}^{n+m})$ to be the set of all n-planes W such that

$$\dim(W \cap F_{m+i-\lambda_i}) = i, \ \dim(W \cap F_{m+i-\lambda_i-1}) = i - 1$$

for $i = 1, 2, \ldots n$. We'll use $e(\lambda)$ instead of $e(\lambda, F)$ when there is no confusion. The restriction $\lambda_1 \leq m$ is made because if $\lambda_1 > m$, $F_{m+1-\lambda_1}$ is either 0 or undefined. Therefore, we define $e(\lambda)$ to be empty if λ has more than n parts or $\lambda_1 > m$.

Example 2.1. Let $n = 3$, $m = 4$, and $\mu = (3, 3, 1)$. Points in $e(\mu)$ can be thought of as the row space of a 3 by 7 matrix. $W \in e(\mu)$ implies $\dim(W \cap F_1) = 0$, $\dim(W \cap F_2) = 1$, $\dim(W \cap F_3) = 2$, $\dim(W \cap F_5) = 2$, and $\dim(W \cap F_6) = 3$. $\dim(W \cap F_1) = 0$ and $\dim(W \cap F_2) = 1$ imply that $c_1 v_1 + v_2 \in W$ for some complex number c_1. We can make the first row of the matrix correspond to the vector $c_1 v_1 + v_2$. Similar arguments show that such a matrix represents a point in $e(\mu)$ if and only if it is row equivalent to a matrix of the form (columns correspond to the basis

$v_1, \ldots, v_{m+n}$)

$$\begin{bmatrix} * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 1 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & * & * & 1 & 0 \end{bmatrix},$$

where * denotes an arbitrary element of $\mathbb{C}$.

The closure $\overline{e}(\lambda)$ of $e(\lambda)$ is a *Schubert variety.* [MS] shows that $\overline{e}(\lambda)$ is the set of all n-planes W such that

$$\dim(W \cap F_{m+i-\lambda_i}) \geq i.$$

It is not hard to see that each $W \in G_n(\mathbb{C}^{n+m})$ is in exactly one of the open sets $e(\lambda)$. Each $e(\lambda)$ is homeomorphic to an open disk of real dimension $2(nm - |\lambda|)$ [MS]. It is shown in detail in [MS] that the $e(\lambda)$ give the real Grassmannian a CW-complex structure and the same proof works for the complex case.

CW-cohomology determines the additive structure of the cohomology ring because all cells are even dimensional so the boundary maps are zero [GH]. CW-cohomology shows that to each $e(\lambda)$ there corresponds a generator of the cohomology, and these generators have no relations. The *Schubert cycles* are the cohomology classes $\sigma_\lambda = [\overline{e}(\lambda)]^* \in H^{2|\lambda|}(G_n(\mathbb{C}^{n+m}), \mathbb{Z})$, where * denotes Poincaré dual. In this paper all the homology and cohomology will be computed over the integers so from now on we omit this from the notation. Note that σ_λ is 0 if λ has more than n parts or $\lambda_1 > m$. To summarize,

Proposition 2.2. *The set of Schubert cycles σ_λ such that $|\lambda| = r$, λ has at most n parts, and $\lambda_1 \leq m$ is a basis for $H^{2r}(G_n(\mathbb{C}^{n+m}))$ over $\mathbb{Z}$.*

3. Intersection and cohomology

Following [Hu], we introduce machinery we will need to compute cup products in the cohomology ring of the Grassmannian in terms of intersections of the Schubert varieties. Let X be a closed oriented smooth manifold of dimension d. Let A and B be closed oriented smooth submanifolds of X of dimensions $d-a$ and $d-b$. Let $[A] \in H_{d-a}(X)$, $[B] \in H_{d-b}(X)$ be the images of the fundamental classes of A and B under the inclusions $A \hookrightarrow X$, $B \hookrightarrow X$. Denote the Poincaré duals of these classes by $[A]^* \in H^a(X)$ and $[B]^* \in H^b(X, \mathbb{Z})$.

Since there are many submanifolds that represent the same homology class, it is not surprising that we need to put restrictions on how the submanifolds A and B intersect to say something useful about how intersection relates to cup product. For example, in $S^1 \times \mathbb{R}$, the submanifolds $S^1 \times 0$ and $S^1 \times 1$ represent the same homology class and have empty intersection whereas if $A = B = S^1 \times 0$, $A \cap B = A$; clearly, the cup product can't be the homology class of the empty set *and* the homology class of A. We want to rule out situations like $A = B$ in the example. Intuitively, we want A and B to be "randomly chosen" and their intersection to look like what happens "most of the time." The notion of intersecting transversely captures this intuition. A and B *intersect transversely* means for every $p \in A \cap B$, the map $T_p(A) \oplus T_p(B) \to T_p(X)$ induced by the inclusions is surjective, where T_p denotes the tangent space at p. This implies $A \cap B$ is a submanifold of dimension $d - (a + b)$.

Recall that the *tangent bundle* of a manifold, τ_X, of the smooth manifold X has as its total space the tangent manifold, and X as its base space. By lemma 11.6 of [MS] an orientation of X gives rise to an orientation of the tangent bundle τ_X and vice-versa. The fundamental classes $[A]$, $[B]$, and $[X]$ determine orientations for A, B, and X and therefore also for their tangent bundles. We need a convention for determining which fundamental class of $A \cap B$ to take, or equivalently, a convention to give an orientation to the tangent space of $A \cap B$. In this paper we only need the convention for the case when $A \cap B$ is finite set of points: $p \in A \cap B$ is positively oriented if and only if the isomorphism $T_p(A) \oplus T_p(B) \simeq T_p(X)$ induced by the inclusions is orientation preserving. See [Hu] for the general case.

Now it makes sense to talk about $[A \cap B] \in H_{d-(a+b)}(X)$, the image of the fundamental class of $A \cap B$ induced by the inclusion $A \cap B \hookrightarrow X$. We can now state the main theorem; a proof can be found in [Hu].

Theorem 3.1. *Cup product is Poincaré dual to intersection. If A and B intersect transversely, then*

$$[A]^* \smile [B]^* = [A \cap B]^* \in H^{a+b}(X)$$

Example 3.2. Let X be a torus described explicitly as $\mathbb{R}^2/\mathbb{Z}^2$. Let A and B be circles in the x and y directions respectively and let p be their point of intersection. We must choose the classes $[A]$, $[B]$ and $[X]$, or equivalently orientations for $T_p(A)$, $T_p(B)$, and $T_p(X)$. Let v_x, v_y be vectors in the positive x and y directions respectively. Declare that v_x is an orientated basis for $T_p(A)$, v_y is an orientated basis for $T_p(B)$, and v_x, v_y is an orientated basis for $T_p(X)$. $T_p(A) \oplus T_p(B) \simeq T_p(X)$ is orientation preserving, while $T_p(B) \oplus T_p(A) \simeq T_p(X)$ is orientation reversing; $A \cap B$ is a positively oriented point, while $B \cap A$ is a negatively oriented point. Theorem 3.1 implies $[A]^* \smile [B]^* = [p]^* = [X]$ and $[B]^* \smile [A]^* = -[p]^* = -[X]$.

The situation above cannot happen with complex manifolds and submanifolds. For suppose X, A and B are complex manifolds and A and

B intersect transversely at the point p. Let $u_1, \dots, u_{d-a}, v_1, \dots, v_{d-b}$ be a complex basis for $T_p(X)$, $u_1, \dots, u_{d-a}$ a complex basis for $T_p(A)$, and $v_1, \dots, v_{d-b}$ an complex basis for $T_p(B)$. Let $w_1, \dots, w_d$ be some reordering of the v_i and u_i. Then the isomorphism (induced by the inclusions of A and B into X) from the ordered basis $w_1, iw_1, \dots, w_d, iw_d$ to the ordered basis $u_1, iu_1 \dots, u_{d-a}, iu_{d-a}, v_1, iv_1 \dots, v_{d-b}, iv_{d-b}$ is orientation preserving. $A \cap B$ and $B \cap A$ are positively oriented points.

Unfortunately it is not completely correct to apply this theorem to our situation. We need a more general theorem that handles subvarieties rather than just closed smooth submanifolds. For this to work the intersections must occur away from singularities of the subvarieties. This is certainly true in our case as we are looking at Schubert varieties that intersect in their interiors and these interiors are homeomorphic to open disks. Appendix B of [Fu] gives a sketch of the topology and algebraic geometry needed for this.

4. Pieri's formula

Given a partition λ of r, a *Young diagram* with shape λ is a collection of r top-left-justified boxes with λ_i boxes in the ith row. We also talk about the columns of a Young diagram or its corresponding partition; λ has λ_1 columns. This is a convenient way to picture partitions and prepares us to define Young tableaux in the next section.

Pieri's formula is

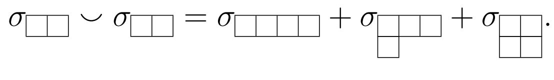
(1) $$\sigma_\lambda \smile \sigma_{(k)} = \sum \sigma_{\lambda'}$$

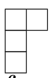
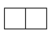
where the sum is over λ' obtained from λ by adding k boxes, no two in a column.

Example 4.1. This simple example of Pieri's formula will be continued in more detail after the necessary theory has been developed. Let $n = 2$, $m = 4$, $\lambda = (2)$, and $k = 2$. Pieri's formula implies

$$\sigma_{(2)} \smile \sigma_{(2)} = \sigma_{(4)} + \sigma_{(3,1)} + \sigma_{(2,2)}.$$

Replacing each partition with its Young diagram makes this easier to see:



 does not appear in the right hand side because this is obtained from  by adding two boxes to the same column.

We follow the proof of Pieri's formula given in section 9.4 of [Fu]. First we show that $\overline{e}(\lambda, F)$ and $\overline{e}(\lambda, F')$ represent the same cohomology class for any complete flags F and F'. $GL(\mathbb{C}^{n+m})$ acts transitively on the set of flags so there is a $g \in GL(\mathbb{C}^{n+m})$ such that $g(F_k) = g(F'_k)$ for all k. $GL(\mathbb{C}^{n+m})$ also acts on $G_n(\mathbb{C}^{n+m})$ continuously. That is, there is a continuous map $GL(\mathbb{C}^{n+m}) \times G_n(\mathbb{C}^{n+m}) \to G_n(\mathbb{C}^{n+m})$ taking $f \times W \to f(W)$. Since $GL(\mathbb{C}^{n+m})$ is connected, there is a path $G : I \to GL(\mathbb{C}^{n+m})$ from $G(0) = g$ to the identity $G(1) = I_{n+m}$. This path induces a homotopy from $g : G_n(\mathbb{C}^{n+m}) \to G_n(\mathbb{C}^{n+m})$ (the function taking W to $g(W)$) to the identity on map on $G_n(\mathbb{C}^{n+m})$. Therefore the induced maps on cohomology, g^* and I^*_{n+m}, are the same. Thus $\overline{e}(\lambda, F)$ and $g(\overline{e}(\lambda, F)) = \overline{e}(\lambda, F')$ represent the same cohomology class. Equivalently, $[\overline{e}(\lambda, F)]^* = \sigma_\lambda = [\overline{e}(\lambda, \widetilde{F})]^*$.

Let $\widetilde{F}_k$ be the subspace of $\mathbb{C}^{n+m}$ spanned by $v_{n+m}, v_{n+m-1}, \ldots, v_{n+m-k+1}$ and let $\widetilde{F}$ be the flag

$$\widetilde{F}_0 \subset \widetilde{F}_1 \subset \ldots \subset \widetilde{F}_{n+m}.$$

We need this flag because we want to look at Schubert varieties that intersect transversely. $\overline{e}(\mu, F)$ and $\overline{e}(\lambda, F)$ do not intersect transversely in general. (Probably most flags distinct from F would work, but this one makes computations particularly easy.)

Example 4.2. Let $n = 3$, $m = 4$, $\mu = (3, 3, 1)$, and $\lambda = (3, 1, 1)$. Recall from example 2.1 that $W \in e(\mu, F)$ if and only if it is the row space of a matrix of the form

$$\begin{bmatrix} * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 1 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & * & * & 1 & 0 \end{bmatrix}.$$

Points in $\overline{e}(\mu, F) - e(\mu, F)$ are represented by matrices like the above except with 1's appearing further to the left. Similarly, an element of $e(\lambda, \widetilde{F})$ is the row space of a matrix of the form

$$\begin{bmatrix} 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}.$$

The intersection $\overline{e}(\mu, F) \cap \overline{e}(\lambda, \widetilde{F})$ clearly contains the point in $G_n(\mathbb{C}^{n+m})$ that is the row space of

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Call this point W_0. It is not hard to see that W_0 is the only point of intersection and we will prove this in general. To show that $\overline{e}(\mu, F)$ and $\overline{e}(\lambda, \widetilde{F})$ intersect transversely at W_0 we look at the neighborhood U of W_0 homeomorphic to $\mathbb{C}^{nm}$ described in section 1.

Given any matrix M such that row space$(M) = W \in U$, the orthogonal projection of W onto $W_0 = \text{span}(v_2, v_3, v_6)$ is an isomorphism. Therefore the submatrix consisting of columns 2,3, and 6 of M is invertible, which implies M is row equivalent to a unique matrix of the form

$$\begin{bmatrix} * & 1 & 0 & * & * & 0 & * \\ * & 0 & 1 & * & * & 0 & * \\ * & 0 & 0 & * & * & 1 & * \end{bmatrix}.$$

We see that $e(\mu, F)$ and $e(\lambda, \widetilde{F})$ are contained in U. U is homeomorphic to $\mathbb{C}^{12}$ and it is clear from the matrices above that $T_{W_0}(e(\mu, F))$ and $T_{W_0}(e(\lambda, \widetilde{F}))$ are orthogonal to each other and of dimensions 5 and 7 respectively. We conclude that the inclusion $T_{W_0}(e(\mu, F)) \oplus T_{W_0}(e(\lambda, \widetilde{F})) \to T_{W_0}(U)$ is surjective and the intersection is transverse.

This is an example of the duality theorem. The following subspaces are convenient to work with: $A_0 = B_0 = 0$; $A_i = F_{m+i-\mu_i}$, $B_i = \widetilde{F}_{m+i-\lambda_i}$, and $C_i = A_i \cap B_{n+1-i}$ for $i = 1, \ldots, n$. Let C be the span of the C_i.

Theorem 4.3. *Suppose* $|\mu| + |\lambda| = nm$. *Then*

$$\sigma_\mu \smile \sigma_\lambda = \begin{cases} \sigma_{(m,\ldots,m)} & \text{if } \mu_i + \lambda_{n+1-i} = m \text{ for } i = 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

PROOF. If $\mu_i + \lambda_{n+1-i} > m$, then $(m+i-\mu_i)+(m+n+1-i-\lambda_{n+1-i}) < m + n + 1$. Therefore A_i and B_{n+1-i} intersect only at 0. So for any n-plane W, $\dim(W \cap A_i) + \dim(W \cap B_{n+1-i}) \leq n$. On the other hand, $W \in \overline{e}(\mu, F) \cap \overline{e}(\lambda, \widetilde{F})$ implies $\dim(W \cap A_i) + \dim(W \cap B_{n+1-i}) \geq i + (n + 1 - i) = n + 1$, so $\overline{e}(\mu, F) \cap \overline{e}(\lambda, \widetilde{F})$ is empty.

If $\mu_i + \lambda_{n+1-i} \leq m$ for $i = 1, \ldots, n$, then since $|\mu| + |\lambda| = nm$, $\mu_i + \lambda_{n+1-i} = m$ for all i. In this case, $(m + i - \mu_i) + (m + n + 1 - i - \lambda_{n+1-i}) = m + n + 1$ for each i and $C_i = \mathbb{C}v_{m+i-\mu_i}$. The exact sequence

$$0 \longrightarrow W \cap A_i \cap B_{n+1-i} \longrightarrow (W \cap A_i) \oplus (W \cap B_{n+1-i}) \longrightarrow W \longrightarrow 0$$

and dimension counting shows that $\dim(W \cap A_i \cap B_{n+1-i}) = \dim(W \cap C_i) \geq 1$. This is an equality since $\dim(C_i) = 1$, and thus $W = \bigoplus_i \mathbb{C}v_{m+i-\mu_i}$ is the point of intersection of $\overline{e}(\mu, F)$ and $\overline{e}(\lambda, \widetilde{F})$. The proof given in example 4.2 that the intersection is transverse easily generalizes so by theorem 3.1

$$\sigma_\mu \smile \sigma_\lambda = [\overline{e}(\mu, F) \cap \overline{e}(\lambda, \widetilde{F})]^* = \sigma_{(m,\dots,m)},$$

the cohomology class of a point. This point is positively oriented by the discussion in section 3. □

The unique partition μ such that $\sigma_\mu \smile \sigma_\lambda$ is nonzero is called the *dual* to λ, which we denote $\widetilde{\lambda}$.

$\sigma_\lambda \smile \sigma_{(k)}$ has cohomological dimension $2(|\lambda| + k)$ and therefore

$$\sigma_\lambda \smile \sigma_{(k)} = \sum c_{\lambda'} \sigma_{\lambda'} \tag{2}$$

for some integers $c_{\lambda'}$, where the sum is over all λ' with $|\lambda'| = |\lambda| + k$. Now by the duality theorem, taking a cup product of (2) with any σ_μ so that $|\mu| + |\lambda| + k = nm$ yields

$$\sigma_\mu \smile \sigma_\lambda \smile \sigma_{(k)} = c_{\widetilde{\mu}}(\sigma_\mu \smile \sigma_{\widetilde{\mu}}) = c_{\widetilde{\mu}}\, \sigma_{(m,\dots,m)}. \tag{3}$$

To prove Pieri's formula we must determine the $c_{\widetilde{\mu}}$ and this is done by computing the left hand side of (3).

Given an $(m + 1 - k)$-plane L, define $\overline{e}((k), L)$ (a slight abuse of notation) to be the set of n-planes that intersect L nontrivially. For each μ such that

$$|\mu| + |\lambda| + k = nm \tag{4}$$

we compute $\sigma_\mu \smile \sigma_\lambda \smile \sigma_{(k)}$ by determining $\overline{e}(\mu, F) \cap \overline{e}(\lambda, \widetilde{F}) \cap \overline{e}((k), L)$. We compute this for a small example before doing the general case.

Example 4.4. Let $n = 3$, $m = 4$, $\lambda = (2)$, and $k = 2$. Points in $\overline{e}((2), \widetilde{F})$ are 3-planes in $G_3(\mathbb{C}^6)$ that intersect $\widetilde{F}_3$ nontrivially. Let L be a generic 3-plane. $\overline{e}((2), L)$ is the set of 3-planes that intersect L nontrivially. According to Pieri's formula, $\sigma_{(2)} \smile \sigma_{(2)} = \sigma_{(4)} + \sigma_{(3,1)} + \sigma_{(2,2)}$. To check that $\sigma_{(4)}$ should appear on the right, cup product both sides with $\sigma_{\widetilde{(4)}} = \sigma_{(4,4)}$, which yields

$$[\overline{e}((4,4), F) \cap \overline{e}((2), \widetilde{F}) \cap \overline{e}((2), L)]^* = \sigma_{(4,4,4)}$$

by the duality theorem. $\overline{e}((4,4), F) \cap \overline{e}((2), \widetilde{F})$ is the set of 3-planes containing v_1 and v_2 and intersecting $\operatorname{span}(v_5, v_6, v_7)$ nontrivially. L intersects the $\operatorname{span}(v_1, v_2, v_5, v_6, v_7)$ in a 1-plane; there are complex numbers c_1, c_2, c_5, c_6, c_7 such that this 1-plane is $\operatorname{span}(c_1 v_1 + c_2 v_2 + c_5 v_5 + c_6 v_6 + c_7 v_7)$. Thus $\overline{e}((4,4), F) \cap \overline{e}((2), \widetilde{F}) \cap \overline{e}((2), L)$ is the single 3-plane $\operatorname{span}(v_1, v_2, c_1 v_1 + c_2 v_2 + c_5 v_5 + c_6 v_6 + c_7 v_7)$, as required.

Lemma 4.5. *(a)* $C = A_0 + B_0 + \sum_{i=1}^{n} A_i \cap B_{n+1-i} = \cap_{i=0}^{n}(A_i + B_{n-i})$. *(b)* $W \in \overline{e}(\mu, F) \cap \overline{e}(\lambda, \widetilde{F})$ *implies* $W \subset C$. *(c) If* $\overline{e}(\mu, F) \cap \overline{e}(\lambda, \widetilde{F})$ *is nonempty, then* $\sum_{i=1}^{n} \dim(C_i) = k + n$. *(d)* $C_1, \ldots, C_n$ *linearly independent and* $W \in \overline{e}(\mu, F) \cap \overline{e}(\lambda, \widetilde{F})$ *implies* $W = \bigoplus_{i=1}^{n}(W \cap C_i)$.

PROOF. To prove (a) all we need to use is that $A_0 \subset A_1 \subset \ldots \subset A_n$ and $B_0 \subset B_1 \subset \ldots \subset B_n$. In general given $W, X, Y \subset \mathbb{C}^{n+m}$ with $W \subset Y$, $(W + X) \cap Y = W + X \cap Y$. If $w \in W$, $x \in X$, and $w + x \in Y$, then $x = (w + x) - w \in Y$ since $w \in W \subset Y$. Therefore $w + x \in W + X \cap Y$. The other direction is easier. By applying this twice we obtain the following fact: If $W \subset Y$ and $Z \subset X$, then $(W + X) \cap (Y + Z) = W + Z + Y \cap X$. Applying this n times yields (a):

$$(A_0+B_n)\cap(A_1+B_{n-1})\cap\ldots\cap(A_n+B_0) = (A_0+A_1\cap B_n+B_{n-1})\cap(A_2+B_{n-2})\cap\ldots$$

$$= (A_0+A_1\cap B_n+A_2\cap B_{n-1}+B_{n-2})\cap\ldots = \ldots = A_0+A_1\cap B_n+A_2\cap B_{n-1}+\ldots A_n\cap B_1+B_0$$

The second equality uses that $A_0 + A_1 \cap B_n \subset A_2$ and $B_{n-2} \subset B_{n-1}$.

By (a) it suffices to show $W \subset A_i + B_{n-i}$ for all i. If $A_i \cap B_{n-i} \neq 0$, then $W \subset \mathbb{C}^{n+m} = A_i + B_{n-i}$ as needed. Otherwise $A_i \cap B_{n-i} = 0$ and $(W\cap A_i)\oplus(W\cap B_{n-i}) \subset W$. $\dim(W\cap A_i) \geq i$ and $\dim(W\cap B_{n-i}) \geq n-i$ so $(W \cap A_i) \oplus (W \cap B_{n-i})$ has dimension n, the same dimension as W. Therefore $(W \cap A_i) \oplus (W \cap B_{n-i})$ must be all of W so $W \subset A_i + B_{n-i}$.

$\dim(C_i) = (m + i - \mu_i) + (m + n + 1 - i - \lambda_{n+1-i}) - (n + m) = m + 1 - \mu_i - \lambda_{n+1-i}$ if this is nonnegative, and 0 otherwise. $W \in \overline{e}(\mu, F) \cap \overline{e}(\lambda, \widetilde{F})$ implies (by dimension counting as in the proof of Pieri's formula) $\dim(W \cap C_i) \geq 1$. Therefore $\sum_{i=1}^{n} \dim(C_i) = nm + n - |\mu| - |\lambda| = k + n$; the last equality is by (4).

The C_i linearly independent implies the $W \cap C_i$ are linearly independent and thus $\bigoplus_i W \cap C_i \subset W$. We just mentioned that $W \in \overline{e}(\mu, F)\cap\overline{e}(\lambda, \widetilde{F})$ implies $\dim(W\cap C_i) \geq 1$. The dimension of $\bigoplus_i W\cap C_i$ is at least the dimension of W so $\bigoplus_i W \cap C_i = W$. □

The condition that the C_i are independent corresponds to the right hand side of Pieri's formula. If the C_i are independent, this means they are nonzero so $C_i = \text{span}(v_{m+n+1-(m+n+1-i-\lambda_{n+1-i})}, \ldots, v_{m+i-\mu_i})$. The C_i are independent if and only if

the intervals $[m+n+1-(m+n+1-i-\lambda_{n+1-i}), m+i-\mu_i]$ are disjoint

if and only if

$$0 < m+n+1-(m+n-\lambda_n) \leq m+1-\mu_1 < m+n+1-(m+n-1-\lambda_{n-1}) \leq$$

$$m + 2 - \mu_2 < \ldots \leq m + n - \mu_n$$

if and only if

$$0 < 1 + \lambda_n \leq m + 1 - \mu_1 < 2 + \lambda_{n-1} \leq m + 2 - \mu_2 < \ldots < m + n - \mu_n$$

Because these are integers, we can get rid of the strict inequalities to obtain the equivalent statement

$$0 \leq \lambda_n \leq m - \mu_1 \leq \lambda_{n-1} \leq m - \mu_2 \leq \ldots \leq m - \mu_n, \text{ or}$$

$$0 \leq \lambda_n \leq \widetilde{\mu}_n \leq \lambda_{n-1} \leq \widetilde{\mu}_{n-1} \leq \ldots \leq \widetilde{\mu}_1 \tag{5}$$

This precisely means that $\widetilde{\mu}$ is obtained from λ by adding k boxes, no two in a column.

By lemma 4.5 (c), if $\overline{e}(\mu, F) \cap \overline{e}(\lambda, \widetilde{F}) \neq \emptyset$ and the C_i are linearly dependent, then $\dim(C) < k + n$. A generic $(m - k + 1)$-plane L intersects C only at 0. $W \in \overline{e}(\mu, F) \cap \overline{e}(\lambda, \widetilde{F})$ implies $W \subset C$ by lemma 4.5 (b) and this implies $W \cap L = 0$. Thus $\overline{e}(\mu, F) \cap \overline{e}(\lambda, \widetilde{F}) \cap \overline{e}((k), L)$ is empty. (3) implies

$$0 = \sigma_\mu \smile \sigma_\lambda \smile \sigma_{(k)} = c_{\widetilde{\mu}}\ \sigma_{(m,\ldots,m)}$$

and therefore $c_{\widetilde{\mu}} = 0$. Combining this with the previous paragraph, we conclude that if $\widetilde{\mu}$ is not obtained from λ by adding k boxes, no two in a column, then $c_{\widetilde{\mu}} = 0$.

Conversely, if (5) holds, then the C_i are linearly independent and $\dim(C) = k + n$. Let L be a generic $(m - k + 1)$-plane that intersects C in a 1-plane of the form $\mathbb{C} \cdot u$, with $u = u_1 \oplus \ldots \oplus u_n$, u_i a nonzero vector in C_i. $\overline{e}((k), L)$ intersects $\overline{e}(\mu, F) \cap \overline{e}(\lambda, \widetilde{F})$ transversely. Given $W \in \overline{e}(\mu, F) \cap \overline{e}(\lambda, \widetilde{F})$ such that W intersects L nontrivially, W must be $\operatorname{span}(u_1, \ldots, u_n)$. This is because $\operatorname{span}(u_1, \ldots, u_n) \subset \oplus(W \cap C_i) = W$ and equality in dimensions show the inclusion is an equality. Thus the triple intersection is a single point and (3) implies $c_{\widetilde{\mu}} = 1$. This completes the proof of Pieri's formula.

5. Littlewood-Richardson numbers

In this section we sketch a nice combinatorial description of the cohomology ring of the Grassmannian that is implied by Pieri's formula. Notation and proofs follow various parts of [Fu]. We give more specific Let λ be a partition of r. A Young tableaux or, more briefly, a *tableaux* of shape λ is Young diagram of shape λ filled with positive integers, one in each box, such that entries weakly increase along each row and strongly increase along each column. The following is a tableaux with shape $(4, 3, 2)$.

1	1	3	4
2	4	4	
5	6		

Given a tableaux T define $x^T = \prod_{i=1}^{\infty} x_i^{b_i(T)}$, where $b_i(T)$ is the number of boxes of T filled with an i. For the tableaux above, $x^T = x_1^2 x_2 x_3 x_4^3 x_5 x_6$. The *Schur polynomial* corresponding to partition λ on variables $x_1, \dots, x_n$ is $s_\lambda(x_1, \dots, x_n) = \sum x^T$, where the sum is over all tableaux T of shape λ with entries in $[n] = \{1, 2, \dots, n\}$.

Theorem 5.1. *The Schur polynomials satisfy Pieri's formula (we omit the dependence of the Schur polynomials on $x_1, \dots, x_n$):*

$$s_\lambda s_{(k)} = \sum s_{\lambda'}$$

where the sum is over all λ' obtained from λ by adding k boxes, no two in a column.

This formula is naturally the "projection" of a similar formula that holds in the tableaux ring. The *tableaux ring* $R_{[n]}$ is the free $\mathbb{Z}$-module with a basis element for each tableaux with entries in $[n]$. Multiplication is determined by a certain product on tableaux, which we now describe.

Given a tableaux T and positive integer a_1, the result of *row bumping* a_1 into T is a tableaux, T', with one more box than T. To obtain T', first find the left-most entry in the first row of T that is larger than a_1 and replace this by a_1. Let a_2 be the entry that a_1 bumped out. If there is no entry larger than a_1 in the first row, add a new box to the end of first row with entry a_1. If a new box wasn't added, continue by finding the left-most entry in the second row that is larger than a_2 and replace this with a_2; let a_3 be the deleted entry. Continue this process until a new box is added. If an entry is bumped out from the bottom row, add a box below T to make a new row with one entry. The result is T'.

For example, the following sequence of tableaux show the process of row bumping 2 into the first tableaux in the list.

$$\begin{array}{lllll} 1\,1\,2\,2\,3 \\ 2\,4\,5 \\ 5\,6 \end{array}, \quad \begin{array}{lllll} 1\,1\,2\,2\,\mathbf{2} \\ 2\,4\,5 \\ 5\,6 \end{array}, \quad \begin{array}{lllll} 1\,1\,2\,2\,\mathbf{2} \\ 2\,\mathbf{3}\,5 \\ 5\,6 \end{array}, \quad \begin{array}{lllll} 1\,1\,2\,2\,\mathbf{2} \\ 2\,\mathbf{3}\,5 \\ \mathbf{4}\,6 \end{array}, \quad \begin{array}{lllll} 1\,1\,2\,2\,\mathbf{2} \\ 2\,\mathbf{3}\,5 \\ \mathbf{4}\,6 \\ \mathbf{5} \end{array}$$

In this example $a_1 = 2$, $a_2 = 3$, $a_3 = 4$, and $a_4 = 5$; the entries bumped in are in bold.

The product, $T \cdot U$, of the tableaux T and U is the result of row bumping the entries of U into T, in the following order: begin with the bottom-left entry of U and continue from left to right along the bottom row; proceed with the next to the last row, from left to right, and so on. For example,

$$\begin{array}{|c|c|c|}\hline 1&1&3\\\hline 4&4\\\cline{1-2} 5\\\cline{1-1}\end{array} \cdot \begin{array}{|c|c|}\hline 1&2\\\hline 4\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline 1&1&3&4\\\hline 4&4\\\cline{1-2} 5\\\cline{1-1}\end{array} \cdot \begin{array}{|c|c|}\hline 1&2\\\hline\end{array} = \begin{array}{|c|c|c|c|}\hline 1&1&1&4\\\hline 3&4\\\cline{1-2} 4\\\cline{1-1} 5\\\cline{1-1}\end{array} \cdot \begin{array}{|c|}\hline 2\\\hline\end{array} = \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 3&4&4\\\cline{1-3} 4\\\cline{1-1} 5\\\cline{1-1}\end{array}$$

This product is associative, although this is not obvious from this definition (see chapters 1 and 2 of [Fu]). It is not commutative. We now have the machinery to prove theorem 5.1.

Proof of theorem 5.1. There is a canonical map from $R_{[n]}$ onto $\mathbb{Z}[x_1, \ldots, x_n]$ that sends the tableaux T to the monomial x^T. Let $S_\lambda[n] \in R_{[n]}$ be the sum of all tableaux of shape λ with entries in $[n]$. The image of $S_\lambda[n]$ is the Schur polynomial $s_\lambda(x_1, \ldots, x_n)$. It therefore suffices to show

$$S_\lambda[n] S_{(k)}[n] = \sum S_{\lambda'}[n] \tag{6}$$

holds in $R_{[n]}$ where the sum is over all λ' obtained from λ by adding k boxes, no two in a column. We show that there is a bijection between terms in the left hand side and terms in the right hand side of (6). More precisely, given T of shape λ and U of shape (k), we must show that $T \cdot U$ is a tableaux with shape obtained from λ by adding k boxes, no two in a column. We must also show that if λ' is a shape obtained from λ by adding boxes with no two in a column, then a tableaux V on λ' determines tableaus of shape λ and shape (k). Furthermore, we must show that these two processes are inverses of each other.

Suppose that a_1 is row bumped into T and $a_1, \ldots, a_r$ is the sequence of entries that are bumped, and let $B(a_i)$ be the box a_i is bumped into. Note that $a_1 < a_2 < \ldots < a_r$ and that $B(a_{i+1})$ is below and weakly to the left of $B(a_i)$.

Consider the product of tableaux $T \cdot \begin{array}{|c|c|c|c|}\hline c_1&c_2&\ldots&c_k\\\hline\end{array}$, for some c_i such that $1 \leq c_1 \leq \ldots \leq c_k \leq n$. Let $c_1 = a_1 < a_2 < \ldots < a_r$ be the sequence of entries obtained from row bumping c_1 and let $c_2 = b_1 < b_2 < \ldots < b_s$ be the sequence of entries obtained from row bumping c_2. Put $T' = T \cdot \begin{array}{|c|c|}\hline c_1&c_2\\\hline\end{array}$. As above, use the notation $B()$ to denote the box a given entry was bumped into.

For example if $T = \begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&3\\\hline 2&4&5\\\cline{1-3} 5&6\\\cline{1-2}\end{array}$, $c_1 = 1$, and $c_2 = 2$, we have

$$\begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&3\\\hline 2&4&5\\\cline{1-3} 5&6\\\cline{1-2}\end{array} \cdot \begin{array}{|c|c|}\hline 1&2\\\hline\end{array} = \begin{array}{|c|c|c|c|c|}\hline 1&1&\mathbf{1}&2&3\\\hline 2&\mathbf{2}&5\\\cline{1-3} \mathbf{4}&6\\\cline{1-2} \mathbf{5}\\\cline{1-1}\end{array} \cdot \begin{array}{|c|}\hline 2\\\hline\end{array} = \begin{array}{|c|c|c|c|c|}\hline 1&1&\mathbf{1}&2&\mathit{2}\\\hline 2&\mathbf{2}&\mathit{3}\\\cline{1-3} \mathbf{4}&\mathit{5}\\\cline{1-2} \mathbf{5}&\mathit{6}\\\cline{1-2}\end{array}$$

The sequences $a_1 = 1, a_2 = 2, a_3 = 4, a_4 = 5$ and $b_1 = 2, b_2 = 3, b_3 = 5, b_4 = 6$ are in bold and italics respectively. Note that $B(a_i)$ is to the left of $B(b_i)$ for $i \in [4]$. We will prove that this happens in general.

$a_1 \leq b_1$ implies a_1 and b_1 both appear in the first row of T' with $B(a_1)$ to the left of $B(b_1)$. Therefore the entries bumped out by a_1 and b_1 came from two entries in the first row of T; T a tableaux implies $a_2 \leq b_2$. Since b_2 was bumped into the second row after a_2, $B(a_2)$ is to the left of $B(b_2)$. Repeating this reasoning we see $B(a_i)$ is to the left of $B(b_i)$ for $i = 1, \ldots, s$ (and $s \leq r$). As noted in the previous paragraph, $B(a_r)$ is below and weakly to the left of $B(a_s)$. This implies $B(a_r)$ is strictly left of $B(b_s)$ and these are the new boxes in T' and not in T. They are in different columns, as desired. More generally, this shows that the new box after row bumping c_{i-1} is strictly to the left of the new box after row bumping c_i, and therefore no two of the new boxes is in the same column.

To go the other direction, the proof is similar (section 1.1 [Fu] has more details). We note that row bumping is reversible if the position of the new box is known. Given V of shape λ', reverse bump the boxes that are not contained in the Young diagram λ. Do this in order from right to left. We obtain a tableaux T of shape λ and the entries reverse bumped out form a tableaux U of shape (k). $T \cdot U = V$, as required. □

Define Λ_i^n to be the ring of homogeneous symmetric polynomials over $\mathbb{Z}$ of degree i in n variables. Using theorem 5.1, it takes a little work to show

Proposition 5.2. *The Schur polynomials $s_\lambda(x_1, \ldots, x_n)$ are symmetric in the variables $x_1, \ldots, x_n$. The set $\{s_\lambda(x_1, \ldots, x_n) : |\lambda| = i\}$ is a basis over $\mathbb{Z}$ for Λ_i^n.*

See sections 2.2 and 6.1 of [Fu] for a proof; we omit it here.

The $S_\lambda \in R_{[n]}$ and the Schur polynomials $s_\lambda(x_1, \ldots, x_n)$ satisfy the *Littlewood-Richardson rule*:

$$S_\lambda[n] \cdot S_\mu[n] = \sum_\nu c_{\lambda\mu}^\nu S_\nu[n]. \tag{7}$$

The $c_{\lambda\mu}^\nu$ are positive integers called the *Littlewood-Richardson numbers.* These numbers have several combinatorial interpretations. $c_{\lambda\mu}^\nu$ is the number of ways a tableaux V of shape ν can be written as a product of a tableaux T of shape λ times a tableaux U of shape μ. Remarkably, this number is the same for every tableaux V of shape ν. See chapter 5 of [Fu] for many more facts about these numbers.

The fact that $c^\nu_{\lambda\mu}$ does not depend on n and the rest of formula (7) does may seem strange at first. Here's an example with $n = 2$:

$$S_{(1,1)}[2]\cdot S_{(2)}[2] = \begin{smallmatrix}1\\2\end{smallmatrix} \cdot \left(\begin{smallmatrix}1&1\end{smallmatrix} + \begin{smallmatrix}1&2\end{smallmatrix} + \begin{smallmatrix}2&2\end{smallmatrix}\right) = \begin{smallmatrix}1&1&1\\2\end{smallmatrix} + \begin{smallmatrix}1&1&2\\2\end{smallmatrix} + \begin{smallmatrix}1&2&2\\2\end{smallmatrix} = S_{(3,1)}[2]$$

implying $c^{(3,1)}_{(1,1)(2)} = 1$. However this does not imply all the other Littlewood-Richardson numbers $c^\nu_{(1,1)(2)}$ are 0 because, for instance, $S_{(2,1,1)}[2] = 0$. The same computation for $n = 3$ follows.

$$S_{(1,1)}[3]\cdot S_{(2)}[3] = \left(\begin{smallmatrix}1\\2\end{smallmatrix} + \begin{smallmatrix}1\\3\end{smallmatrix} + \begin{smallmatrix}2\\3\end{smallmatrix}\right) \cdot \left(\begin{smallmatrix}1&1\end{smallmatrix} + \begin{smallmatrix}1&2\end{smallmatrix} + \begin{smallmatrix}1&3\end{smallmatrix} + \begin{smallmatrix}2&2\end{smallmatrix} + \begin{smallmatrix}2&3\end{smallmatrix} + \begin{smallmatrix}3&3\end{smallmatrix}\right) =$$

$$\begin{smallmatrix}1&1&1\\2\end{smallmatrix} + \begin{smallmatrix}1&1&2\\2\end{smallmatrix} + \begin{smallmatrix}1&1&3\\2\end{smallmatrix} + \begin{smallmatrix}1&2&2\\2\end{smallmatrix} + \begin{smallmatrix}1&2&3\\2\end{smallmatrix} + \begin{smallmatrix}1&3&3\\2\end{smallmatrix} + \begin{smallmatrix}1&1&1\\3\end{smallmatrix} + \begin{smallmatrix}1&1&2\\3\end{smallmatrix} + \begin{smallmatrix}1&1&3\\3\end{smallmatrix} + \begin{smallmatrix}1&2&2\\3\end{smallmatrix} +$$

$$\begin{smallmatrix}1&2&3\\3\end{smallmatrix} + \begin{smallmatrix}1&3&3\\3\end{smallmatrix} + \begin{smallmatrix}1&1\\2\\3\end{smallmatrix} + \begin{smallmatrix}1&2\\2\\3\end{smallmatrix} + \begin{smallmatrix}1&3\\2\\3\end{smallmatrix} + \begin{smallmatrix}2&2&2\\3\end{smallmatrix} + \begin{smallmatrix}2&2&3\\3\end{smallmatrix} + \begin{smallmatrix}2&3&3\\3\end{smallmatrix} = S_{(3,1)}[3] + S_{(2,1,1)}[3].$$

Therefore $c^{(2,1,1)}_{(1,1)(2)} = 1$. We could have deduced this much more quickly from theorem 5.1 since there are only two ways to add two boxes to $(1,1)$ with no two in the same column. This also tells us that we have found all nonzero Littlewood-Richardson numbers of the form $c^\nu_{(1,1)(2)}$.

We conclude with the main theorem.

Theorem 5.3. *The Schubert cycles σ_λ in $G_n(\mathbb{C}^{m+n})$ satisfy*

$$\sigma_\lambda \smile \sigma_\mu = \sum_\nu c^\nu_{\lambda\mu}\sigma_\nu,$$

where $c^\nu_{\lambda\mu}$ are the Littlewood-Richardson numbers.

Proof. There is an additive surjective homomorphism Θ from $\bigoplus_{i=0}^\infty \Lambda^n_i$ to $H^*(G_n(\mathbb{C}^{m+n}))$ that sends $s_\lambda(x_1,\ldots,x_n)$ to σ_λ for each λ (recall that σ_λ is 0 if λ has more than n parts or more than m columns). This determines Θ completely by proposition 5.2. Now $\bigoplus_{i=0}^\infty \Lambda^n_i$ is generated as a *ring* by the Schur polynomials $s_{(k)}$ for $k = 0,1,\ldots,n$. (It is well known that the elementary symmetric polynomials generate the ring of symmetric polynomials as a $\mathbb{Z}$-algebra. These Schur polynomials are the complete symmetric polynomials and it is not surprising that they generate as well. See section 6.1 [Fu] for more details.) Because of Pieri's formula and the corresponding formula for the Schur polynomials (theorem 5.1), Θ is a homomorphism of rings.

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$$

holds in $\bigoplus_{i=0}^\infty \Lambda^n_i$, so the image of this formula holds in $H^*(G_n(\mathbb{C}^{m+n}))$, as desired. □

This method of proof shows indirectly that Pieri's formula is enough to compute any cup product in the cohomology ring of the complex Grassmannian. Although we have not described many properties of the Littlewood-Richardson numbers, we have shown that multiplication in the cohomology ring of the complex Grassmannian can be given an explicit combinatorial interpretation. We conclude with a small example where we can do all the combinatorial calculations and visualize the geometry (at least a little).

Example 5.4. Shown below is a multiplication table for the cohomology ring of $G_2(\mathbb{C}^5)$, where σ_λ has been replaced by the Young diagram of λ. The Schubert cycles $\sigma_{(0,0)}$ and $\sigma_{(3,3)}$ have been omitted because they are less interesting. The remainder of the table can be filled in using the fact that cup product is commutative in this case and that $\sigma_\mu \smile \sigma_\lambda = 0$ when $|\mu| + |\lambda| > 6$.

	(1)	(2)	(3)	$(1,1)$	$(2,1)$	$(3,1)$	$(2,2)$	$(3,2)$
(1)	$(2)+(1,1)$	$(3)+(2,1)$	$(3,1)$	$(2,1)$	$(3,1)+(2,2)$	$(3,2)$	$(3,2)$	$(3,3)$
(2)	$(3)+(2,1)$	$(3,1)+(2,2)$	$(3,2)$	$(3,1)$	$(3,2)$	$(3,3)$	0	0
(3)	$(3,1)$	$(3,2)$	$(3,3)$	0	0	0	0	0
$(1,1)$	$(2,1)$	$(3,1)$	0	$(2,2)$	$(3,2)$	0	$(3,3)$	0
$(2,1)$	$(3,1)+(2,2)$	$(3,2)$	0	$(3,2)$	$(3,3)$	0	0	0

The first three rows and columns were computed using Pieri's formula. The duality theorem applies when $|\mu| + |\lambda| = 6$. There are several good ways to compute the other products. For example, we can use (7) to compute $(1,1) \smile (2,1) = (3,2)$:

$$S_{(1,1)}[2] \cdot S_{(2,1)}[2] = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \cdot \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & \\ \cline{1-2}\end{array}.$$

Alternatively, we can use Pieri's formula in a tricky way:

$$(2,1) \smile (1,1) = (2,1) \smile ((1) \smile (1) - (2)) = = 2(3,2) - (3,2) = (3,2)$$

Proposition 5.2 and theorem 5.3 imply that we can always use Pieri's formula for such computations.

We can use this multiplication table to compute the number of 2-planes in $\mathbb{C}^5$ intersecting 6 2-planes in general position, as is done in [GH]. This is the 6-fold intersection of the Schubert variety $\overline{e}((1))$, which we can compute from $\sigma_{(1)}^6$.

$$((1))^6 = ((1) \smile ((1,1) + (2)))^2 = (2(2,1) + (3))^2 = 4(3,3) + (3,3) = 5(3,3),$$

the class of 5 distinct points in $G_2(\mathbb{C}^5)$. So there are 5 2-planes intersecting 6 2-planes in general position.

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