ALEXANDER POLYNOMIAL OF KNOTS

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ABSTRACT. The Alexander polynomial is the very first polynomial knot invariant discovered. In this expository paper, we will discuss what they are, how to compute them, and their properties.

1. Introduction

First discovered by J.W. Alexander in 1928, the Alexander polynomial was the only known polynomial invariant of knot types for over 50 years, until Jones polynomial was discovered by Vaughan Jones in 1984.

We will first give the classical definition of Alexander polynomials. We will then show how to compute them using skein relations, and finally we will discuss the properties of the Alexander polynomial and its meaning in algebraic topology. We will omit most proofs for interest of time and space.

This paper is organized as followed. In section 2, we will give a classical definition of the Alexander polynomial as in [3, 4]. In section 3, we will see how to compute them combinatorially using skein diagrams. In section 4, we will discuss how the Alexander polynomial relates to homology, and finally in section 5, we will state some properties of the Alexander polynomials.

2. Classical Definition

There are several different (but equivalent) ways to define the Alexander polynomial. In this section, we will give one such definition used in [3, 4]. Alexander's original definition in [2] is similar.

Definition 2.1. A subset K in a space A is called a *knot* if it is homeomorphic to a sphere S^n.

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Let K_1 and K_2 be knots in A. If there is a homeomorphism f from A into itself such that $f(K_1) = K_2$, then the knots K_1 and K_2 are said to be of the same type. The main problem is to classify such types.

In this paper, we are only concerned with 1-dimensional knots in 3-space, i.e., subsets of $\mathbb{R}^3$ or S^3 that are homeomorphic to S^1. A knot is called *tame* if it has a polygonal representative. The existence of Alexander polynomials is guaranteed only for tame knots, so we will assume that all knots we discuss in this paper are tame.

The knot complement $R^3\backslash K$ or $S^3\backslash K$ is obviously a knot invariant. By the Alexander duality, $\tilde{H}_*(S^3\backslash K) \cong \tilde{H}^*(S^1)$ for any knot K, so the homology of a knot complement is useless as a knot invariant. However, the homotopy groups of the knot complements are useful knot invariants.

2.1. **Wirtinger Presentation of a Knot Group.** We will describe a method for finding a presentation of the knot group $\pi_1(\mathbb{R}^3\backslash K)$ of a knot K from a knot diagram. Without loss of generality, we may assume that the knot diagram we see is the projection of K to $\mathbb{R}^2$, and there are only finitely many double points (where two points are mapped to the same point under the projection), and no more than two points are mapped to any single point. Let $(0, 0, z_0)$ be a point lying "above" the knot K, i.e., the third coordinates of the points in K are all smaller than z_0. For any arc a in the diagram, define a loop x_a in $\mathbb{R}^3$ with base-point $(0, 0, z_0)$ which goes around the arc a and back. We can put an orientation on K, so we get a compatible orientation on the arcs, and we can choose the loops x_a to have compatible orientation. See figure 1 .

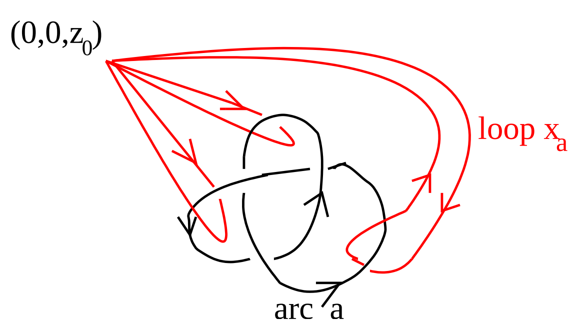

FIGURE 1. Loops around every arc

At every crossing, we must have the relations among x_a's as shown in figure 2.

Let n be the number of arcs in a diagram of K. Then that diagram also contains exactly n crossings. The n loops $x_1, \ldots, x_n$ represent some

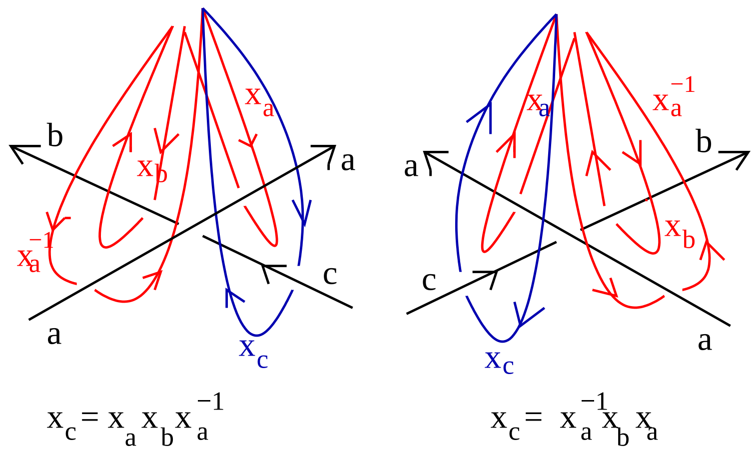

FIGURE 2. Relations in the Wirtinger Presentation

homotopy classes in $\pi_1(X)$, where $X = \mathbb{R}^3 \setminus K$, and we have n relations $r_1, \ldots, r_n$, one at every crossing, as in the figure 2. It turns out that these are the only generators and relations needed for a presentation of the knot group.

Proposition 2.2.

$$\pi_1(X) = (x_1, \ldots, x_n; r_1, \ldots, r_n).$$

In fact, the set of relations $r_1, \cdots, r_n$ contains redundancy. We can remove a single r_i and the above statement still holds.

A proof that uses Van Kampen's Theorem can be found in [7, pages 57–60]. Next, we will show how this presentation can be used to compute the Alexander polynomial.

2.2. **Derivatives and Jacobian.** This is the method used in [3, 4].

Definition 2.3. The *free derivative* $\frac{\partial}{\partial x_i}$ on a free group $F = (x_1, \ldots, x_n)$ is defined recursively by:

$$\frac{\partial}{\partial x_i} 1 = 0, \ \frac{\partial}{\partial x_i} x_j = \delta_{ij}, \ \frac{\partial}{\partial x_i} x_j^{-1} = -\delta_{ij} x_j^{-1},$$

and for any word $w = ux_j \in F$,

$$\frac{\partial}{\partial x_i} w = \frac{\partial}{\partial x_i} u + u \frac{\partial}{\partial x_i} x_j$$

The free derivative is a map from a free group F to the corresponding group ring $\mathbb{Z}[F]$.

Let $G = \pi_1(X)$. There is a group homomorphism

$$\phi : F = (x_1, \ldots, x_n) \longrightarrow G = (x_1, \ldots, x_n; r_1, \ldots, r_{n-1}).$$

The map ϕ can be extended to a map $\phi : \mathbb{Z}[F] \to \mathbb{Z}[G]$ whose kernel is generated by $r_1, \ldots, r_{n-1}$ in F. Define

$$\psi : \mathbb{Z}[G] \to \Lambda = \mathbb{Z}[t, t^{-1}]$$
$$x_i \mapsto t \text{ for each } i = 1, \cdots, n$$

In fact, ψ is the abelianization of $\mathbb{Z}[G]$, and $\psi(G)$ is an infinite cyclic group. We can then associate to any presentation $(x_1, \ldots, x_n; r_1, \ldots, r_{n-1})$ of G an $(n-1) \times n$ matrix J (called *Jacobian* in [3, 4]) whose ij^{th} entry is $\psi\phi\left(\frac{\partial r_i}{\partial x_j}\right)$.

Then the ideal in Λ generated by $(n-1) \times (n-1)$ minors of J is called the *Alexander ideal*. It can be shown using the Tietze theorem ([3, pages 43–46] and [4, pages 126–127]) that the Alexander ideal is a knot invariant and that it does not depend on the presentation of the knot group. As noted earlier, by the Alexander duality, the first homology group of the knot complement of a tame knot is the infinite cyclic group Z, so is the abelianization of the knot group. Therefore, the corresponding Alexander ideal is a principal ideal, and any generator is called an *Alexander polynomial* of the knot, denoted Δ. It is defined up to multiplication by units of Λ, i.e., elements of the form $\pm t^i$ for some integer i.

Example:

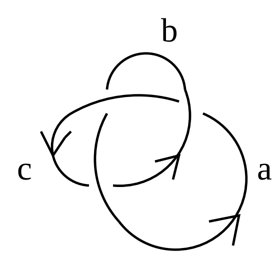

FIGURE 3. A Trefoil

A presentation for the knot group of a trefoil knot is: (compare Figure 3 and Figure 2)

$$(a, b, c;\ aba^{-1}c^{-1}, bcb^{-1}a^{-1}, cacb^{-1}),$$

so the Jacobian matrix is:

$$\begin{pmatrix} \psi(1 - aba^{-1}) & \psi(a) & \psi(-aba^{-1}c^{-1}) \\ \psi(-bcb^{-1}a^{-1}) & \psi(1 - bcb^{-1}) & \psi(b) \\ \psi(c) & \psi(-cac^{-1}b^{-1}) & \psi(1 - cac^{-1}) \end{pmatrix} = \begin{pmatrix} 1-t & t & -1 \\ -1 & 1-t & t \\ t & -1 & 1-t \end{pmatrix}.$$

Thus, the 2×2 minors of the matrix generate the ideal $(1 - t + t^2)$, and an Alexander polynomial is $\Delta = 1 - t + t^2$.

3. A Combinatorial Definition

In 1969, Jonh Conway found a way to calculate the Alexander polynomial of a knot or a link using *skein relation*, a relation between the polynomial of a link and the polynomial of other links obtained by changing the crossings in a projection of the original link.

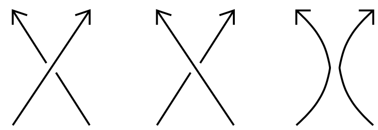

Figure 4. A skein triple

An Alexander polynomial Δ_K of a tame knot K can be defined by the following two equations:
(1) $\Delta_{\text{trivial knot}} = 1$.
(2) $\Delta_{L_+} - \Delta_{L_-} + (t^{1/2} - t^{-1/2})\Delta_{L_0} = 0$, where L_+, L_-, L_0 are three links which differ only at one crossing, as shown in figure 4.

The fact that this polynomial is a knot/link invariant can be shown easily by verifying that it is invariant under Reidemeister moves.

Example: The Alexander polynomial of a splittable link (a link whose components lie in the interiors of disjoint 3-balls) is 0. We can consider a splittable link with two components as L_0 in figure 4. Then we have two knots K_+ and K_- corresponding to L_+ and L_-. These two knots K_+ and K_- are of the same type, hence have the same Alexander polynomial. Then

$$\Delta_{K_+} - \Delta_{K_-} + (t^{1/2} - t^{-1/2})\Delta_{\text{splittable}} = 0 \implies \Delta_{\text{splittable}} = 0.$$

Similarly, we can see inductively that a splittable link with any number of components have Alexander polynomial 0.

Example: Trefoil knot. From figure 5 and the example above, we get:

$$\begin{aligned}\Delta_{\text{trefoil}} &= \Delta_{K_1} - (t^{1/2} - t^{-1/2})\Delta_{K_2} \\ &= 1 - (t^{1/2} - t^{-1/2})(\Delta_{K_3} + (t^{1/2} - t^{-1/2})\Delta_{K_4}) \\ &= 1 - (t^{1/2} - t^{-1/2})(0 + (t^{1/2} - t^{-1/2}))1 \\ &= t^{-1} - 1 + t.\end{aligned}$$

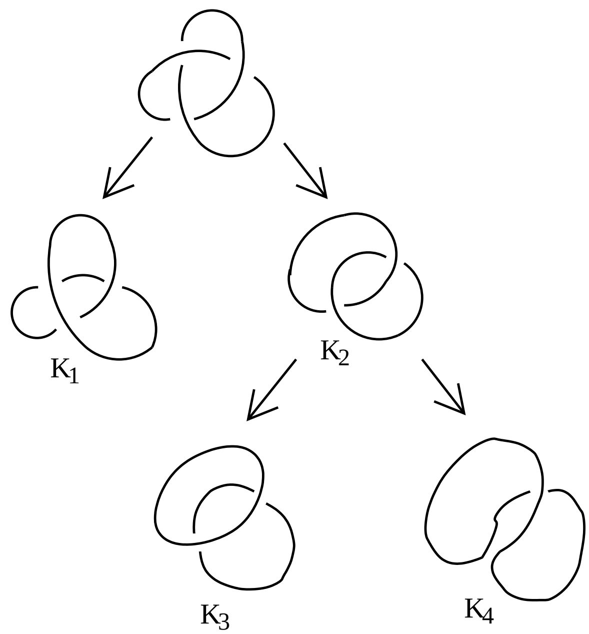

Figure 5. Trefoil knot

4. Alexander Polynomial and Homology

In this section, we will discuss about a covering space $\tilde{X}$ of the knot complement $X = \mathbb{R}^3 \backslash K$ whose homology $H_*(\tilde{X})$ turns out to be related to the Alexander polynomial.

Lemma 4.1. *Every knot complement X has an infinite cyclic cover $p : \tilde{X} \longrightarrow X$. Moreover, $\tilde{X}$ is also the universal abelian cover of X.*

That is, the group of covering automorphisms $Aut(\tilde{X}, p)$ of $(\tilde{X}, p)$ is isomorphic to the infinite cyclic group $\mathbb{Z}$. Moreover, the fundamental group $\pi_1(\tilde{X})$ is isomorphic to commutator subgroup $[\pi_1(X), \pi_1(X)]$. One can find a construction involving Seifert surfaces in [7, pages 128–131].

4.1. Alexander Invariant and Alexander Matrix.

Definition 4.2. The *Alexander invariant* of a knot is the homology $H_*(\tilde{X})$ of the infinite cyclic cover $\tilde{X}$ of the knot complement X.

Let $\Lambda = \mathbb{Z}[t, t^{-1}]$ as before. Then the homology group $H_i(\tilde{X})$ has a Λ-module structure as follows. Choose a generator $\tau : \tilde{X} \longrightarrow \tilde{X}$ of the group of covering translations. There are two choices here since there are two generators for $\mathbb{Z}$. Then τ induces an isomorphism

$$\tau_* : H_i(\tilde{X}) \longrightarrow H_i(\tilde{X})$$

on the homology. Let $f(t) = a_{-m}t^{-m} + \cdots + a_{-1}t^{-1} + a_0 + a_1 t + \cdots + a_n t^{-n}$ be an element of Λ. Then f acts on an element α of $H_i(\tilde{X})$ by

$$f(t)\alpha = a_{-m}\tau_*^{-m}(\alpha) + \cdots + a_{-1}\tau_*^{-1}(\alpha) + a_0\alpha + a_1\tau_*(\alpha) + \cdots + a_n\tau_*^{-n}(\alpha).$$

The projection $p : \tilde{X} \to X$ induces an injective homomorphism

$$p_* : \pi_1(\tilde{X}) \longrightarrow \pi_1(X),$$

and $p_*\pi(\tilde{X}) = [\pi(X), \pi(X)]$.

Denote the commutator subgroup $[\pi(X), \pi(X)]$ by C, so

$$C = [\pi(X), \pi(X)] = p_*\pi(\tilde{X}) \cong \pi(\tilde{X}).$$

Then p_* induces an isomorphism

$$\bar{p_*} : \frac{\pi(\tilde{X})}{[\pi(\tilde{X}), \pi(\tilde{X})]} \xrightarrow{\sim} \frac{C}{[C, C]},$$

so

$$\bar{p_*} : H_1(\tilde{X}) \xrightarrow{\sim} \frac{C}{[C, C]}.$$

We can put a Λ-module structure on $\frac{C}{[C,C]}$ so that we can calculate $H_1(\tilde{X})$ from the knot group [7, pages 174–178].

4.1.1. *Presentation matrices.*

Let R be a unital commutative ring and let M be a finitely-presented $R-$module. Then $M \cong (x_1, \ldots, x_n | r_1, \ldots, r_m)$, where each relation r_i is a linear combination of the generators x_j's: $r_i = a_{i1}x_1 + \cdots + a_{in}x_n$. In other words, M is generated as an R module by the elements $x_1, \ldots, x_n$, and the $r_1 = 0, \ldots, r_m = 0$ are relations among the x_i's. We can then define a *presentation matrix* to be an $m \times n$ matrix with entries a_{ij} for $1 \leq i \leq m, 1 \leq j \leq n$.

An *Alexander matrix* is a presentation matrix for $H_1(\tilde{X})$ as a Λ-module. If the Alexander matrix has size $m \times n$, the ideal generated by all $n \times n$ minors of the matrix is the Alexander ideal of the knot. The Alexander matrix of a tame knot is a square matrix [7, page 207], so the Alexander ideal is principal. Any generator of this principal ideal is the Alexander polynomial Δ.

In other words, Δ is an element of Λ such that $H_1(\tilde{X}) = \Lambda/(\Delta)$. The only interesting Alexander invariant for classical knots is in dimension 1. That is, if X is a knot complement in S^3, and $\tilde{X}$ is its infinite cyclic cover, then $H_i(\tilde{X}) = 0$ for all $i \geq 2$, and $H_0(\tilde{X}) = \Lambda/(t-1)$ [, page 171]. Hence an Alexander polynomial encodes all the information about $H_*(\tilde{X})$ in a polynomial.

5. Some Properties of Alexander Polynomials

Proposition 5.1. *For any knot K, its Alexander polynomial Δ satisfies the following properties after being normailzed so that the lowest degree term is a constant:*

(1) $\Delta(1) = \pm 1$
(2) $\Delta(t) = \Delta(t^{-1})$.

For a proof using Seifert matrices, see [7, pages 207–208]
The converse is also true:

Proposition 5.2. *For every Laurent polynomial $p(t)$ such that $p(1) = \pm 1$ and $p(t) = p(t^{-1})$, there is a tame knot K in S^3 whose Alexander Polynomial is p.*

An explicit construction of such knots can be found in [7, pages 171–172]

Theorem 5.3. *Let $K = K_1 \# K_2$ be a composite knot in S^3, and let X, X_1, X_2 denote their respective knot complements. Then their Alexander invariants are connected by the Λ isomorphisms:*

$$H_i(\tilde{X}) \cong H_i(\tilde{X_1}) \oplus H_i(\tilde{X_2}) \textit{for all } i > 0.$$

assuming appropriate choices of generators $\tau : \tilde{X}_i \longrightarrow \tilde{X}_i$ of the covering translation group, determining the Λ-action.

One can prove this using Mayer-Vietoris sequence [7, page 179].

Alexander polynomials are very successful in distinguishing knot types. They completely classify all knots with 8 or fewer crossings. However, the Alexander polynomials cannot distinguish a knot from its mirror image. Moreover, there are some, in fact infinitely many, non-trivial knots whose Alexander polynomial is 1. In particular, any "double knot with twisting number 0" has a trivial Alexander polynomial. A double knot lies along a boundary of an annulus embedded in S^3, except that it hooks itself and double back. The twisting number is the linking number of the two boundary curves of the annulus. This result can be proven using Seifert surfaces [7, page 167]. Figure 6 shows such a knot.

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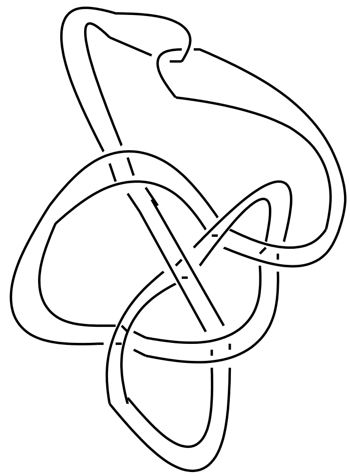

Figure 6. Knot with trivial Alexander polynomial

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