An introduction to cobordism

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1 Introduction

Cobordism theory is the study of manifolds modulo the cobordism relation: two manifolds are considered the same if their disjoint union is the boundary of another manifold. This may seem like a strange thing to study, but there appears to be (at least) two good reasons why one may want to take a look at such a thing. The first reason has to do with the homology of manifolds. One can imagine trying to setup a homology theory of manifolds by looking at chains built from embedded submanifolds. Because the boundary of a boundary of a manifold is empty, this gives a differential chain complex. To compute the homology of a manifold, one would then form the usual quotient $\ker \partial / \operatorname{im} \partial$. In order to see if two cycles are equal, one would need to know if their difference (which can be thought of as being some form of disjoint union) is the image of ∂, i.e., a boundary of another embedded manifold. This is precisely the question that cobordism theory answers.

The second reason is a little more vague, but perhaps just as important. There is no hope of classifying manifolds up to homeomorphism — a well known result of A. Markov tells us that producing an algorithm that classifies compact connected 4-manifolds up to homeomorphism is equivalent to producing an algorithm that solves the classification problem for finitely-presented groups, a problem for which no such algorithm can exist. So we have to set our sights a little lower. Classifying manifolds up to cobordism might then seem like a reasonable thing to try to do.

The notion of cobordism goes back to Poincaré, but the method that allows us to successfully study it is more recent: Pontrjagin noticed that the study of cobordism for framed manifolds is related to the study of certain homotopy groups; Thom generalized this approach and successfully completed the computation of the unoriented cobordism groups. Thom's method was then used by other people to compute other cobordism groups.

The method used by Thom to study unoriented cobordism consists of two steps. The problem is first reduced to a homotopy problem — the cobordism groups in question are shown to be related to the homotopy groups of a certain space, called the Thom space. These homotopy groups are then computed by whatever means it seems appropriate to use. This basic pattern is followed in the study of other cobordism groups; this pattern is how we will approach oriented cobordism in this paper.

2 Definitions

Let us define precisely the notions we introduced informally above. Unless otherwise stated, we will assume that all maps are smooth and that all manifolds are smooth and oriented — the focus will be on oriented cobordism theory. There are other cobordism theories, but we will defer talking about them until later.

Given an oriented manifold M, we will denote by $-M$ the manifold that has the same underlying topological and smooth structure as M, but with the opposite orientation. By ∂M, we mean the boundary of M with the induced orientation. By $M + N$, we mean the disjoint union of M and N; by $M - N$, we mean $M + (-N)$. By $M = N$, we mean that M is isomorphic to N as oriented manifolds (i.e., M is diffeomorphic to N via an orientation preserving diffeomorphism). What we are studying is the equivalence relation of cobordism:

Definition 1. Two compact boundaryless manifolds M and N are *cobordant* if there exists a compact manifold with boundary W such that $\partial W = M - N$. We might sometimes write this as $M \sim N$.

Let us check that this is an equivalence relation. First, it is easy to see that it is reflexive since $M - M$ is the boundary of $M \times [0, 1]$. It is also symmetric: if $\partial W = M - N$, then $\partial(-W) = N - M$. Finally, we check transitivity. Assume $\partial V = L - M$ and $\partial W = M - N$. Using the collar neighborhood theorem[1], we can define a new manifold X by gluing together the $-M$ component of ∂V and the M component of ∂W. We would then have $\partial X = L - N$. Note that our definition subsumes the equivalence relation of being isomorphic as oriented manifolds, since if M is isomorphic to N, then $M - N$ is isomorphic to $\partial(M \times [0, 1])$.

We can now define the oriented cobordism groups:

Definition 2. The n-th oriented cobordism group Ω_n is the set of compact boundaryless n-dimensional manifolds together with the group operation $+$ (i.e., disjoint union), modulo the equivalence relation of cobordism.

The associativity of $+$ is obvious. To simplify things, we will think of the empty set $\emptyset$ as being an n-manifold for every n. This allows us to set the identity element in Ω_n to be the equivalence class of $\emptyset$. What about inverses? Our notation suggests a natural choice: the inverse of M should be $-M$. And this is indeed the case: $M - M$, as we already mentioned, is the boundary of $M \times [0, 1]$, hence is cobordant to $\emptyset$.

Note that any manifold M is the boundary of $M \times [0, \infty)$. This is why we should only look at compact manifolds — we would otherwise be studying a completely trivial theory.

[1] [2] Ch. 4, Sec. 6 has a proof of this theorem.

Because disjoint union is a commutative operation, Ω_n is an abelian group. If we have an m-manifold M and n-manifold N, we can take their cartesian product $M \times N$. This gives an associative bilinear product $\Omega_m \times \Omega_n \to \Omega_{m+n}$. We can thus consider the graded ring $\Omega_* = \bigoplus_{n=0}^{\infty} \Omega_n$. This is in fact commutative in the graded sense: $M \times N$ is isomorphic to $(-1)^{mn} N \times M$ as an oriented manifold. Further, the ring possesses an identity element $1 \in \Omega_0$ — the cobordism class of a point.

The classification of compact 0, 1, and 2-manifolds allows us to easily compute Ω_n for $n = 0$, 1, or 2. For Ω_0, observe that a 0-manifold is just a collection of signed points, and that the difference in number between the positive and negative points determines the cobordism class of the manifold (since a positive point and a negative point is the boundary of $[0, 1]$). Thus, $\Omega_0 = \mathbb{Z}$. Ω_1 and Ω_2 are even easier to compute: there is only one compact boundaryless 1-manifold, S^1, and S^1 with any orientation bounds the 2-disk D^2 (with an appropriate choice of orientation); similarly, the classification theorem for surface allows us to list all possible compact orientable boundaryless surfaces, and each of these is clearly the boundary of a suitable 3-manifold (just think of the solid sphere, the solid torus, the solid doughnut with 2 holes, etc.). This means that Ω_1 and Ω_2 are both 0. Determining Ω_n for $n > 2$ turns out to be a little more difficult. For $n = 3$, it is a theorem of Rohlin [5] that any compact orientable 3-manifold is the boundary of a 4-manifold, allowing us to conclude that Ω_3 is also 0. For even larger n, we have to resort to the general theory described below.

3 Some differential topology

Let us begin our study of the oriented cobordism groups by asking a preliminary question: how does the cobordism relation behave under (smooth) homotopies? More precisely, if $f : M \to N$ and $g : M \to N$ are homotopic maps, how are $f^{-1}(A)$ and $g^{-1}(A)$ related[2]?

Before we discuss this question, let us recall some definitions from differential topology. Let $f : M \to N$ be a map between manifolds, and let $A \subset N$ be a submanifold. We say that f is *transverse* to A if for every $y \in A$ and every $x \in f^{-1}(y)$, the tangent space T_yN is spanned by T_yA and the image of $f_* : T_xM \to T_yN$. This is usually written $f \pitchfork A$. Note: if f is transverse to A, then $f^{-1}(A)$ is a submanifold of M. The key fact[3] about transversality is:

Theorem 3 (Transversality theorem). *Let $F : M \times S \to N$ be a map of manifolds, where S may have boundary. Assume A is a submanifold of N. If both F and ∂F are transverse to A, then for almost every $s \in S$, both $F(-, s)$ and $\partial F(-, s)$ are transverse to A. Here, by ∂F, we mean the restriction of F to the boundary of its domain, and by $F(-, s)$, we mean the map from M to N obtained from F by holding s constant.*

[2] The discussion in the next three sections loosely follows that in [2], Ch. 7.

[3] See [1] or [2] for a proof.

This theorem says that any map can be deformed by an arbitrarily small amount to make it transverse. This will be used more or less freely in what follows, usually by simply noting that certain maps may be chosen to be transverse.

Before we try to answer the question asked above, we need to mention how orientations work in our setup. Remember that we are considering the case of a map $f : M \to N$, and a submanifold $A \subset N$. Let ν_A be the normal bundle of A, and assume that this bundle is oriented. Assume that M is oriented and $f \pitchfork A$. It can be verified that ν_A induces a bundle over $f^{-1}(A)$ in M that can be identified with the normal bundle of $f^{-1}(A)$. The orientation of this bundle together with the orientation on M determine an orientation on $f^{-1}(A)$.

We can now answer the question:

Lemma 4. *Let $f : M \to N$ and $g : M \to N$ be homotopic maps between boundaryless manifolds, and let $A \subset N$ be a closed boundaryless submanifold with an oriented normal bundle ν_A. If both f and g are transverse to A, then the manifolds $-f^{-1}(A)$ and $g^{-1}(A)$ are cobordant.*

Proof. We can choose the homotopy $H : M \times I \to N$ between f and g to be transverse to A. Then $H^{-1}(A)$ is a submanifold of $M \times I$ whose boundary is $g^{-1}(A) - f^{-1}(A)$. □

4 Reduction to homotopy theory

Let us try to massage the problem of corbodism into a form where we can more readily apply Lemma 4. Assume the dimension n is fixed. We need to choose the manifold M in the lemma to be "big" enough so that any compact n-manifold can be thought of as an embedded manifold in M. This is easy to arrange: the Whitney embedding theorem guarantees that this can be done with $M = \mathbb{R}^{n+k}$ for sufficiently large k (in fact, for $k \geq n$). But we are going to go with the slightly different choice of $M = S^{n+k}$ for sufficiently large k because our goal is reduce things to homotopy theory, and the homotopy groups are built from maps of spheres into spaces. What should we choose for N and A? Whatever we choose has to satisfy a few conditions. First, $\dim f^{-1}(A) = \dim M - \dim N + \dim A$, so our choice will determine k. More importantly, under the assumptions of the lemma, and as was already mentioned, the normal bundle of $f^{-1}(A)$ in M is determined by the normal bundle of A in N[4]. Thus we need to make sure that the oriented normal bundle of any embedded manifold $X \subset M$ can be pulled back via a map $h : X \to A$ from the oriented normal bundle of A. What we need is precisely the property that the oriented universal bundle has: we should take our A to be $\tilde{G}_k(\mathbb{R}^s)$, the Grassmann manifold of oriented k-planes in $\mathbb{R}^s$, for $s \geq n + k$; and we should take our N to be the total space $E(\tilde{\gamma}_k(\mathbb{R}^s))$ of the oriented universal bundle $\tilde{\gamma}_k(\mathbb{R}^s)$ over $\tilde{G}_k(\mathbb{R}^s)$.

With our choices made as above, the situation is as follows. For any given compact n-manifold X, we embed X into S^{n+k}. A tubular neighborhood U of

[4] It is the pullback bundle.

X can be identified with the oriented normal bundle of X. By the properties of the oriented universal bundle, there is a bundle map α between the bundle $U \to X$ and the oriented universal bundle $E(\tilde{\gamma}_k(\mathbb{R}^s)) \to \tilde{G}_k(\mathbb{R}^s)$, provided that $s \geq n+k$. This map $\alpha : U \to E(\tilde{\gamma}_k(\mathbb{R}^s))$ is what we will try to apply Lemma 4 to. But one problem with it as it stands is that α is only defined on a subset of S^{n+k}; it needs to be extended to an $\tilde{\alpha}$ defined on all of S^{n+k}, and $\tilde{\alpha}$ must not send points not in U to $\tilde{G}_k(\mathbb{R}^s)$. The way to fix this problem is by adjoining a point at infinity, which leads us to the notion of a Thom space.

5 Thom spaces

Given a Euclidean vector bundle $\xi = (\pi, E, B)$, one can form a space $T(\xi)$, called the *Thom space* of ξ, as follows. Let $V \subset E$ be the the set of all vectors in each fiber whose norm is ≥ 1. $T(\xi)$ is defined to be the quotient space E/V. If the base B is compact, as it is in the case we will consider, the Thom space can just be thought of as the Alexandrov one-point compactification of E (identify V with the point ∞). This space is what will allow us to fix the problem mentioned in the previous section. The idea will be to send $S^{n+k} \setminus U$ to ∞.

We can define a homomorphism, called the *Thom homomorphism*,

$$\tau : \pi_{n+k}(T(\tilde{\gamma}_k(\mathbb{R}^s))) \to \Omega_n$$

as follows. Let $\alpha \in \pi_{n+k}(E(\tilde{\gamma}_k(\mathbb{R}^s)))$, i.e., α is a homotopy class of maps from S^{n+k} to $T(\tilde{\gamma}_k(\mathbb{R}^s))$. Define

$$\tau(\alpha) = [f^{-1}(\tilde{G}_k(\mathbb{R}^s))],$$

where f is any map in the homotopy class α such that $f \pitchfork \tilde{G}_k(\mathbb{R}^s)$, and where the orientation of $f^{-1}(\tilde{G}_k(\mathbb{R}^s))$ is determined, as we discussed earlier, by the pullback of the normal bundle of $\tilde{G}_k(\mathbb{R}^s)$ in $E(\tilde{\gamma}_k(\mathbb{R}^s))$, which is just the bundle $\tilde{\gamma}_k(\mathbb{R}^s)$ itself.

The key result in cobordism theory, due to Thom, is the following theorem[5].

Theorem 5. *τ is surjective if $k > n$ and $s \geq n+k$ and injective if $k > n+1$ and $s \geq n+k+1$. Thus, τ is an isomorphism if $k > n+1$ and $s \geq n+k+1$.*

Proof. To simplify the notation, we will set $E = E(\tilde{\gamma}_k(\mathbb{R}^s))$, $G = \tilde{G}_k(\mathbb{R}^s)$, and $\gamma = \tilde{\gamma}_k(\mathbb{R}^s)$. First, we will prove surjectivity. The Whitney embedding theorem lets us embed any compact boundaryless n-manifold M in $\mathbb{R}^{n+k}$ since $k > n$. Choose a tubular neighborhood U of M; this is diffeomorphic to the total space of the normal bundle, ν_M, of M. By using something similar to the generalized Gauss map, we can get a map β from this space to $E(\tilde{\gamma}_k(\mathbb{R}^{n+k}))$: for a point $(x, v) \in E(\nu_M)$, with $x \in M$ and v in the fiber over x in ν_M (which we will denote N_xM), we set

$$\beta(x, v) = (N_xM, v) \in E(\tilde{\gamma}_k(\mathbb{R}^{n+k})),$$

[5] The proof is pieced together from those given in [4] and [2].

where we can consider the oriented vector space N_xM as a point of $\tilde{G}_k(\mathbb{R}^{n+k})$ thanks to the embedding in $\mathbb{R}^{n+k}$. Using the obvious inclusion $E(\tilde{\gamma}_k(\mathbb{R}^{n+k})) \subset E$, this can be thought of as a map from $E(\nu_M)$ to E. The diffeomorphism between U and $E(\nu_M)$ lets us turn this into a map from U to E. Finally, there is a canonical map $E \to T(\gamma)$. Composing all these maps gives us a map $g : U \to T(\gamma)$ which is transverse to the zero-section G. We can extend g to $\hat{g}$ defined on all of $\mathbb{R}^{n+k}$ by sending $R^{n+k} \setminus U$ to ∞. By thinking of S^{n+k} as the one-point compactification of $\mathbb{R}^{n+k}$, we can think of $\hat{g}$ as a map from S^{n+k} to $T(\gamma)$ (we of course send $\infty \in S^{n+k}$ to $\infty \in T(\gamma)$). Clearly τ sends the homotopy class of $\hat{g}$ to the cobordism class of M.

Let us now prove injectivity. In order to do this, we need to recall some more definitions and results from differential topology. A submanifold $A \subset M$ is called a *neat submanifold* of M if $\partial A = A \cap \partial M$ and A is covered by charts (ϕ, U) of M such that $A \cap U = \phi^{-1}(R^k)$, where $k = \dim A$. The idea is that the boundary of a neat submanifold A is nicely placed in the boundary of M: $T_xA \not\subset T_x(\partial M)$ for $x \in \partial A$. An embedding $A \to M$ is called a *neat embedding* if its image is a neat submanifold. The result that we will need about neat submanifolds is the following[6]: if $A \subset M$ is a neat submanifold, then every tubular neighborhood of ∂A in ∂M is the intersection with ∂M of a tubular neighborhood of A in M.

Assume the homotopy class of some map $g : S^{n+k} \to T(\gamma)$ gets sent by τ to the 0 cobordism class. We need to fiddle with g to get it into a form where we can use our previous results. We start by using the transversality theorem to replace g with a map homotopic to g and transverse to G. Using this new map (which we will continue to call g), set $M = g^{-1}(G)$.

Next, we will adjust g so that there is tubular neighborhood U of M such that $U = g^{-1}(E)$ and such that $g|_U : U \to E$ is a map of vector bundles. This would mean that $g(S^{n+k} \setminus U) = \infty$. To accomplish this, set $U \subset g^{-1}(E)$. This is a tubular neighborhood of M in S^{n+k}. Let $X \subset U$ be a disk subbundle. By yet another result in differential topology[7], we can assume that g agrees in X with a vector bundle map $\psi : U \to E$. We can define a new map $h : S^{n+k} \to T(\gamma)$, by $h \cong \psi$ on U and $h \cong \infty$ on $S^{n+k} \setminus U$. By construction, g and h agree on X, in particular on ∂X. They also both map $S^{n+k} \setminus \operatorname{int} X$ into $T(\gamma) \setminus G$, which is contractible (we can contract to ∞). By a smooth version of the homotopy extension property, g and h are homotopic. For the rest of the proof, we will replace g with h.

Take M, U, and g as in the paragraph above. Since $\tau([g]) = 0$, M must be the boundary of some $(n+1)$-manifold W. The assumption $k > n + 1$ means that $n + k + 1 > 2(n + 1)$; by the Whitney embedding theorem, this allows us to embed W in D^{n+k+1}. It is easy to see that this embedding can be chosen so that it extends the inclusion of $M = \partial W$ into $S^{n+k} = \partial D^{n+k+1}$ and so that it is neat — the obvious way to do this is to construct a small collar neighborhood of M in D^{n+k+1} so that the "collar" is normal to the boundary sphere, and then to

[6] [2], Ch. 4, Sec. 6; this is sometimes taken to be part of the definition of a neat submanifold.
[7] Ibid

attach W along this collar. U extends to a tubular neighborhood $V \subset D^{n+k+1}$ of W by the result mentioned earlier about neat submanifolds. Remember that we can think of U as a k-plane bundle over the n-manifold M, and of g as a bundle map $g : U \to E$. We can similarly think V as a k-plane bundle over the $(n+1)$-manifold W. The assumption $s \geq k + (n+1)$ means that there is a bundle map from V to E. This map $\hat{g} : V \to E$ can be chosen so that it extends g; this is due to the fact that any classifying map $\partial W = M \to \tilde{G}_k(\mathbb{R}^s)$ for the bundle U extends to a classifying map $W \to \tilde{G}_k(\mathbb{R}^s)$ for the bundle V[8]. Extend $\hat{g}$ to all of D^{n+k+1} by sending $D^{n+k+1} \setminus V$ to ∞. $\hat{g}$ is defined on a contractible space and $\hat{g}|_{S^{n+k}} = g$. Thus, the homotopy class of g must be zero. □

6 Solving the homotopy problem

The only thing left to do is to compute the homotopy groups of the Thom space $T(\tilde{\gamma}_k(\mathbb{R}^s))$. We will not attempt to compute these groups completely; the full computation is perhaps best done with the machinery of the Adams spectral sequence[9], and so we will content ourselves with a computation modulo $\mathcal{C}$-isomorphism. By $\mathcal{C}$-isomorphism, we mean a homomorphism $h : G \to H$ between abelian groups whose kernel and cokernel are both are both finite abelian groups[10].

Let $\xi = (\pi, E, B)$ be a Euclidean k-plane bundle over a base space B. We can give the space $T(\xi)$ a CW-structure:

Lemma 6. *Given a CW-complex structure on B, $T(\xi)$ has a $(k-1)$-connected CW-complex structure having one $(n+k)$-cell for each n-cell of B, and one additional 0-cell (the point ∞).*

Proof. If e_a is an open n-cell of B, then $\pi^{-1}(e_a) \cap (E \setminus V)$ is an open $(n+k)$-cell of. These cells are pairwise disjoint, and they cover $E \setminus V \cong T(\xi) \setminus \{\infty\}$.

Let $c : D^n \to B$ be the characteristic map for e_a. The pullback bundle $c^*(\xi)$ is trivial since D^n is contractible. The vectors of length ≤ 1 in $c^*(\xi)$ thus forms a product $D^n \times D^k$. We can form the characteristic map of $\pi^{-1}(e_a)$ in $T(\xi)$ by taking the composition

$$D^n \times D^k \hookrightarrow c^*(\xi) \to E \to T(\xi).$$

It is easy to see that this gives the required CW-complex structure on $T(\xi)$. The fact that it is $(k-1)$-connected simply follows from the fact that this structure has no n-cells for $n = 1, \ldots, k-1$. □

We can also compute the homology of $T(\xi)$:

Lemma 7. *$H_{k+i}(T(\xi), \infty)$ is canonically isomorphic to $H_i(B)$.*

[8] [2] Thm. 4.3.4

[9] [3], Ch. 9 has a discussion on the use of the Adams spectral sequence to compute the homotopy groups of Thom spaces.

[10] This section is based on Ch. 18 of [4].

Proof. B is embedded in $E \backslash V \cong T(\xi) \backslash \{\infty\}$ as the zero-section. Let $T_0(\xi)$ be the complement of the zero-section in $T(\xi)$. $T_0(\xi)$ is contractible — we can contract to ∞. The exact sequence of the triple $(T(\xi), T_0(\xi), \infty)$ yields an isomorphism $H_n(T(\xi), \infty) \simeq H_n(T(\xi), T_0(\xi))$. By excision, $H_n(T(\xi), \infty) \simeq H_n(E, E_0)$, where E_0 is the complement of the zero-section in ξ. But then the Thom isomorphism tells us that $H_n(E, E_0) \simeq H_{n-k}(B)$. □

These homology groups will allow us to extract some information about the homotopy groups of the Thom space. The key to this is the following theorem, proven in [4] by putting together some results of Serre.

Theorem 8. *Let X is a finite, $(k-1)$-connected, CW-complex, with $k \geq 2$. Then the Hurewicz homomorphism $\pi_i(X) \to H_i(X; \mathbb{Z})$ is a $\mathcal{C}$-isomorphism for $i < 2k - 1$.*

We know the homology of $\tilde{G}_k(\mathbb{R}^{n+k})$: $H_i(\tilde{G}_k(\mathbb{R}^{n+k}))$ is finite if $i \not\equiv 0 (\mathrm{mod}\, 4)$, and is finitely generated of rank $p(r)$ if $i = 4r$, where $p(r)$ is the number of partitions of r. Together with the above theorem, this impllies that

$$\operatorname{rank} \pi_i(\tilde{G}_k(\mathbb{R}^{n+k}) = p(r)$$

if $i = 4r$; otherwise $\pi_i(\tilde{G}_k(\mathbb{R}^{n+k}))$ is finite.

We can combine all these results to get the main theorem, which is due to Thom:

Theorem 9. *Ω_n is finite for $n \not\equiv 0 (\mathrm{mod}\, 4)$, and is finitely generated of rank $p(r)$ for $n = 4r$.*

It is possible to prove part of this theorem by other means, namely, by looking at Pontrjagin numbers[11]. A compact oriented $4n$-manifold M cannot be the boundary of a $4n+1$ manifold if some Pontrjagin number $p_{i_1} \dots p_{i_r}[M]$ of M is non-zero. The Pontrjagin number of $\mathbb{C}P^{2k}$ is given by

$$p_{i_1} \dots p_{i_r}[\mathbb{C}P^{2k}] = \binom{2k+1}{i_1} \cdots \binom{2k+1}{i_r},$$

where $i_1, \dots, i_r$ is a partition of k. In particular, it is not zero; hence, $\mathbb{C}P^{2k}$ is not cobordant to zero. What is also true is that Pontrjagin numbers are additive:

$$p_{i_1} \dots p_{i_r}[M_1 + M_2] = p_{i_1} \dots p_{i_r}[M_1] + p_{i_1} \dots p_{i_r}[M_2].$$

This implies that the map $\Omega_{4n} \to \mathbb{Z}$ defined by $M \mapsto p_{i_1} \dots p_{i_r}[M]$ is a homomorphism for any choice of the partition $i_1, \dots, i_r$ of $n = \frac{1}{4} \dim M$. Finally, there is a theorem — appropriately enough, also due to Thom — about the linear independence of Pontrjagin numbers: if $M_{j_1}, \dots, M_{j_s}$ are oriented manifolds of dimensions $4k_{j_1}, \dots, 4k_{j_s}$ and such that the characteristic numbers

[11] See [4] Ch. 16, 17 for details.

$s_{k_{j_i}}(p)[M_{j_i}] \neq 0$, then the square matrix (with the number of rows equal to the number of partitions of $k_{j_1} + \cdots + k_{j_s}$)

$$\left[p_{i_1} \ldots p_{i_r}[M^{j_1} \times \cdots \times M^{j_s}]\right]$$

is non-singular. This theorem applies to the products $\mathbb{C}P^{2m_1} \times \cdots \times \mathbb{C}P^{2m_t}$, where $m_1, \ldots, m_t$ is a partition of n, to show that the set of all products $\mathbb{C}P^{2m_1} \times \cdots \times \mathbb{C}P^{2m_t}$, where $m_1, \ldots, m_t$ ranges over all partitions of n, represent linearly independent elements of the cobordism group Ω_{4n}. This is enough to show that

$$\operatorname{rank} \Omega_{4n} \geq p(n).$$

Combining this with the main theorem, we can also say something about $\Omega_* \otimes \mathbb{Q}$: it is a polynomial algebra over $\mathbb{Q}$ with independent generators $\mathbb{C}P^2, \mathbb{C}P^4, \ldots$

7 Other cobordism theories

There are many other types of cobordism theories. Instead of looking at oriented manifolds, we could look at unoriented manifolds. This would give us the unoriented cobordism groups $\mathfrak{N}_n$. We could also look at manifolds with additional structures: manifolds with an equivalence class of complex vector bundle structures on the normal bundle, manifolds with an equivalence class of quaternionic vector bundle structures on the normal bundle, manifolds with an equivalence class of Spin structures on the normal bundle, which would lead to theories of complex corbordism, symplectic cobordism, spin cobordism. (And these by no means exhaust all the possibilities.) The study of these groups follows the same overall pattern used to study oriented cobordism: the problem is reduced to a homotopy problem, which is solved by various methods. Thom spaces play the same role in these other theories as they did in the oriented case — Thom spaces are defined in the category of objects under consideration and the cobordism problem is reduced to a computation of the homotopy groups of these Thom spaces. [6] is a good reference for cobordism theory in general, and contains a discussion of the notion of a cobordism category, which serves as a unifying language for talking about the different cobordism theories.

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