A CURSORY INTRODUCTION TO SPIN STRUCTURE

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Abstract. In our brief summary paper, we give the basic algebraic definitions of $Spin(V)$ and $Spin^c(V)$, compute $Spin(\mathbb{R}^n)$ for small n, and describe some of $Spin$'s elementary properties, most importantly that it is the non-trivial double cover of $SO(V)$. From there, we propose two definitions for a Spin structure and show their equivalence. Lastly, we generalize the notion of spin structure to that of a $spin^c$ structure and roughly sketch why every four manifold has a $spin^c$ structure and how $spin^c$ structures are used to define the Seiberg-Witten invariant of a four manifold. As an afterthought, we give a short list of results that involve spin structures. Assuredly, much of the exposition below is muddled and confused. For clarification on the first two sections chapter 2 and the beginning of chapter 3 of [7] are recommended and for clarification on the the other sections, consult chapter 5 and the appendix for chapter 2 in [1].

1. The Players

For $n \in \mathbb{N}, n > 2$, $SO(n)$ has fundamental group $\mathbb{Z}_2$ and so possesses a unique non-trivial double cover := $Spin(n)$. It is of general interest to construct $Spin(n)$ explicitly.

Let V be a real finite dimensional vector space with an inner product (always assumed to be positive, non-degenerate).

Let $T(V)$ be defined as $\oplus_{0\leq n} V^{\otimes n}$ which is a real algebra with unit. Note that $T(V)$ has a natural $\mathbb{Z}$ grading.

The Clifford algebra, $Cl(V)$ is defined to be $T(V)$ modulo the relation $v\otimes v + \|v\|^2 \cdot 1 = 0$. The grading does not desend, as the relation is not homogeneous, however, it is homogenous modulo two, so the $\mathbb{Z}$ grading of $T(V)$ descends onto a $\mathbb{Z}_2$ grading of $Cl(V)$ so we have $Cl(V) = Cl_0(V) \oplus Cl_1$ where Cl_0 is a subalgebra and Cl_1 is a Cl_0 module.

Let $e_1, ..., e_k$ be an orthonormal basis for V. Then we may write $Cl(V)$ in terms of generators and relations. As $T(V)$ is generated by all products of e_i's. Obviously ${e_i}^2 = -1$ and as $\|e_i + e_j\|^2 = 2$ we see that $e_ie_j = -e_je_i$. So $Cl(V) = \{e_{i_1}e_{i_2}\cdots e_{i_j} | e_i^2 = -1 \text{ and } e_ie_j = -e_je_i\}$. Thus every element of $Cl(V)$ may be uniquely written as a sum of products of the form $e_{i_1}\cdots e_{i_j}$ with $i_l < \cdots < i_{l+1}$. Uniqueness can first be established for monomials i.e there does not exist $e_{i_1}\cdots e_{i_s} = e_{j_1}\cdots e_{j_t}$. To see this, multiply through by $e_{i_1}\cdots e_{i_s}$, and then square both sides. There is a contradiction (the right side will equal -1) if the length of the monomial being squared is of length one or two modulo four. But this is good enough, since if the length is zero or three, we can multiply the original monomials by, say, e_1

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to get our contradiction. For more general expressions, squaring both sides and invoking the uniqueness of monomial expressions gets us what we want. From this it follows that if V is of dimension k then $Cl(V)$ has dimension 2^k.
The multiplicative units of $Cl(V)$ form a group. $Pin(V)$ is the subgroup generated by the units of norm one in V. $Spin(V) := Pin(V) \cap Cl_0(V)$. The generators of $Pin(V)$ are in $Cl_1(V)$ so $Spin(V)$ is the subgroup of index two consisting of all elements in $Pin(V)$ expressible as a product of an even number of generators in $Pin(V)$.

There is a natural action of $SO(n)$ on V, $SO(V)$, that extends to an action on $Cl(V)$ and respects the grading. V can be naturally identified with a subspace of $Cl(V)$ that is invariant and orientation preserved under $SO(V)$. So $SO(V)$ is represented by the group of automorphisms of $Cl(V)$ that restrict to an orientation preserving linear automorphism of V. Every element of $Spin(V)$ acts on $Cl(V)$ by conjugation and each action is the same as an action induced by an element of $SO(V)$. In this way we get a map $\phi : Spin(V) \to SO(V)$ when $\dim V > 2$. We claim that ϕ is surjective and that the kernel of ϕ is $\{-1, 1\}$. Recall our definition of $Spin(V)$ as the subgroup of $Pin(V)$ consisting of all elements which can be written as a product on an even number of the given generators of $Pin(V)$ (elements in V with norm one). We can extend our representation of $Spin(V)$ to $Pin(V)$ by again taking an element of $Pin(V)$ to the automorphism of $Cl(V)$ that is conjugation by the element. To see how conjugation by an element of $Pin(V)$ restricts to V, we note that the generators of $Pin(V)$ are the elements of unit length in V. For any $d, v \in V$, $\|d\| = 1$, dvd^{-1} is going to be the reflection of v across the subspace which is normal to w. By our previous remarks then it follows that the actions induced on V by $Spin(V)$ are the compositions of an even number of reflections of V. These are exactly the automorphisms of V which comprise $SO(V)$. Surjectivity has been established. Note that the kernel of our representation is equal to the intersection of $Spin(V)$ with the center of $Cl(V)$. The next paragraph will give some idea of why the intersection of $Spin(V)$ with the center of $Cl(V)$ is $\{+1, -1\}$. We show this intersection must lie in $\mathbb{R}$, to see the whole argument consult Morgan [7].

The only elements in this intersection are 1 and -1. This is not obvious. First we must make a number of observations. Let $e_1, ..., e_n$ be an orthonormal basis for V. $e_j \cdot (e_{i_1} \cdots e_{i_t}) = (-1)^{d(t)} (e_{i_1} \cdots e_{i_t}) e_j$ where $d(t) = t$ provided $j \neq i_r$ for all $1 \leq r \leq t$ and $d(t) = t - 1$ otherwise. From this and the uniqueness of the representatives that if the dimension of V is even, then the center of $Cl(V)$ is the real multiples of the unit 1. If the dimension of V is odd, then the center is the real vector space spanned by 1 and $e_1 \cdots e_n$. In both cases, the intersection of the center with $Cl_0(V)$ is isomorphic to $\mathbb{R}$. Thus, the center of $Spin(V)$ is a subgroup of the multiplicative group $\mathbb{R}^*$. As the square of any generator of $Pin(V)$ is -1, we see that $\{+1, -1\} \subset Spin(V)$.

Next we want to verify that ϕ is a non-trivial double cover. To do so, it suffices to show that the kernel of ϕ is in the same component. Following Stong [9], consider the path

$$\lambda : [0, \pi] \to Spin(n)$$

$$t \mapsto \cos(t) + \sin(t)e_1e_2$$

From this we conclude that $Spin(V)$ is a non-trivial double cover of $SO(V)$. As $SO(V)$ is a compact Lie group of dimension $\frac{n(n-1)}{2}$, and $\pi_1(SO(V)) = \mathbb{Z}_2$ when $dim(V) > 2$, we conclude that $Spin(V)$ is a Lie group of dimension $\frac{n(n-1)}{2}$ and is simply connected when $dim(V) > 2$.

It will be useful for us to compute $Spin(\mathbb{R}^n)$ for $n = 1, 2, 3, 4$.

i) Clearly $Cl(\mathbb{R}) = \frac{\mathbb{R}[x]}{(x^2+1)} = \mathbb{C}$. Then $Pin(1)$ is the subgroup of $\mathbb{C}$ generated by i. $Spin(1)$ must be $\{\pm 1\}$.

ii) $Cl(\mathbb{R}^2)$ is the algebra generated by i and j such that $i^2 = j^2 = -1$ and $ij = -ji$. So $Cl(\mathbb{R}^2)$ is the quaternion algebra $\mathbb{H}$. $Pin(2)$ is generated by elements of the form $cos(t) + sin(t)i$ and $(cos(t) + sin(t)i) * j$ which we see is the union of two circles. So $Spin(2) = S^1$.

iii)A straightforward computation shows that $Cl(\mathbb{R}^3) \simeq \mathbb{H} \oplus \mathbb{H}$. It is generated by x, y, z with $x^2 = y^2 = z^2 = -1$ and anti-commutation between each other. The isomorphism from $\mathbb{H} \oplus \mathbb{H}$ is given by

$$(1_1, i_1, j_1, k_1, 1_2, i_2, j_2, k_2) \mapsto$$

$$\left(\frac{1+xyz}{2}, \frac{xy-z}{2}, \frac{yz-x}{2}, \frac{zx-y}{2}, \frac{1-xyz}{2}, \frac{xy+z}{2}, \frac{yz+x}{2}, \frac{zx+y}{2}\right)$$

Under this isomorphism, $\mathbb{R}^3$ is identified with pairs (a, a) with a having no real part. For any pair of unit imaginary quaternions, a, b, their product ab is an element of $Spin(V) \subset Cl_0(\mathbb{R}^3) = \mathbb{H}$ as the diagonal copy in $\mathbb{H} \oplus \mathbb{H}$. The set of these products generate S^3.

iv)It is not hard to show that $Cl(\mathbb{R}^n) \simeq Cl_0(\mathbb{R}^{n+1})$. Under this identification when $n = 3$, $Spin(3)$ becomes a subgroup of $Spin(4)$, where the embedding is determined by the embedding of $\mathbb{R}^3$ in $\mathbb{R}^4$. The union of the images of all these embeddings generates $S^3 \times S^3 \subset \mathbb{H} \oplus \mathbb{H}$. We conclude then that $Spin(4) \simeq SU(2) \times SU(2)$. To see the group isomorphism, given a quaternion $a + bi + cj + dk$, associate to it the matrix

$$\begin{pmatrix} a - di & -b + ci \\ b + ci & a + di \end{pmatrix}$$

When restricted to the unit quaternions, S^3 this gives the desired isomorphism with $SU(2) \times SU(2)$.

For four manifoldular reasons that will be explained later, we are actually more inclined to be interested in the complex version of $Spin(V)$, $Spin^c(V)$. This is defined as the subgroup of the multiplicative group of units of $Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$ generated by $Spin(V)$and the complex unit circle of complex scalars i.e. $Spin^c(V) \simeq Spin(V) \times_{\{\pm 1\}} S^1$. Most of the arguments made for $Spin(n)$ can be modified to work for $Spin^c(n)$, but for simplicity's sake we stick with $Spin(n)$ for as long as possible. For much more algebraic information on $spin(V)$ and $spin^c(V)$ refer to Morgan [7].

2. Principal Bundles and the Spin Structures they love

We give two definitions of spin structures. While they do not all have the same generality, they are equivalent in overlapping cases. One has been explained in class, and will be re-explained here. We also present the other and sketch why they are equivalent. (In fact there is another definition for Riemannian four manifolds).

Because we know the viewing audience at home can't wait, we waste no time in giving the first definition here: given a real finite dimensional vector space V with inner product and $dim(V) > 1$. Let X be an oriented manifold, and $P \to X$ a principal $SO(V)$ bundle. The bundle $P \to X$ has a spin structure if there is a principal $Spin(V)$ bundle $P_{Spin(V)} \to X$ such that $P_{Spin(V)}$ is a double cover of P. The important case to consider is when X is an oriented Riemannian manifold and we consider its frame bundle F which is a principal $SO(n)$ bundle. In this case, if the bundle lifts to a principal $Spin(n)$ bundle, then we say X along with a particular choice of lifting $P_{Spin(V)}$ of P is a spin manifold.

When does a $SO(n)$ bundle P lift to a $Spin(n)$ bundle? Is the lifting unique? We'll find out that:

Proposition 1. A principal $SO(n)$ bundle P lifts to a principal $Spin(n)$ bundle $P_{Spin(n)}$ iff its second Steifel-Whitney class $w_2(P) \in H^2(X; \mathbb{Z}_2) = 0$.

Proposition 2. If X is simply connected, then the lifting is unique. In general, the set of distinct spin structures is in a non-canonical bijection with $H^1(X; \mathbb{Z}_2)$.

3. Two definitions of spin structure that coincide

In the previous section we have given a definition for a spin structure. In this section we give an other definition, the definition briefly given in class which we shall review.

First we refresh our memory of what a frame bundle is. Let E be a vector bundle over B with fiber $\mathbb{R}_n$. We construct a new bundle over B where the fiber over a point p is the set of all frames in the fiber over P in E. (We say that a frame in $\mathbb{R}_n$ is any orthonormal basis.) We denote the frame bundle by $Fr(E)$. For most purposes, the tangent space TM over a manifold M plays the part of E.

Let X be a CW complex, and P an oriented real vector bundle over X with fiber a vector space of dimension greater than two. Let $Fr(P)$ denote the bundle of oriented frames over X. We define a spin structure on a bundle $P \to X$ to be an element of $H^1(Fr(P); \mathbb{Z}_2)$ which restricts to the generator on each fiber. Another way to say this is that there is a trivialization of P restricted to the two skeleton of X. We try to construct a trivializaton. As P is oriented, we may continuously pick an orientation for the fibers, and so we can extend a trivialization of P restricted to the zero skeleton (which obviously exists) to a trivialization of the one skeleton. Call the trivializaton σ.

The harder part is to extend this trivialization σ to the two skeleton. The attaching region of each 2-cell lifts to an element in $\pi_1(SO(n))\simeq\mathbb{Z}_2$. And as on any 2-cell, P must be trivial since a 2-cell is conctractible, we conclude σ extends over a 2-cell iff the 2-cell's corresponding element in $\pi_1 = 0$. By extending this map on the boundary of the 2-cells, we get a cochain in $C^2(X;\mathbb{Z}_2)$. We had some choice in σ however. Supposing we instead choose σ' as a trivialization over the one skeleton, σ and σ' lift to elements of $\pi_1(SO(n))$ along the one handles that differ by either 0 or 1. This difference, $\sigma - \sigma'$ gives a one cochain. Then that σ and σ' both extend over the same two handle is equivalent to saying their difference evaluated over the boundary of the two handle is zero. Switching the boundary operator, we get the condition that the coboundary of $\sigma - \sigma'$ evaluate to zero on the two handle. So changing our choice of σ affects the cochain in $C^2(X;\mathbb{Z}_2)$ by a coboundary. As P restricted to a three handle must be trivial, the cochain in $C^2(X;\mathbb{Z}_2)$ must be a cocycle. We conclude that in fact it is a representative of $H^2(X;\mathbb{Z}_2)$ and is the obstruction to to an oriented vector bundle admitting a spin structure.

To see how many spin structures we get, we work backwards. Given two spin structures, σ_1 and σ_2 we know from above they determine trivializations τ_1 and τ_2 up to homotopy on the 1 skeleton. We homotope the two trivializations to agree on the zero skeleton. Their difference once again gives a 1- cochain via lifting to the fundamental group of $SO(n)$. As both τ_1 and τ_2 extend over the two skeleton we see the difference is a cocycle that depended on the homotopy to ensure agreement on the zero skeleton. Changing the homotopy changes WOLOG τ_1 by the generator of $\pi_1(SO(n))$, and so the difference changes by a coboundary. The difference then defines a class in $H^1(X;\mathbb{Z}_2)$.

We have yet to show the equivalency of this definition of spin structure with our original definition. We know from the first section that when $n > 2$ and $i = 1$ that $\pi_1(Spin(n)) = 0$. In fact, as $\pi_2(SO(n)) = 0$, $\pi_2(Spin(n)) = 0$ also. Given a real vector bundle $E\to X$, let $P_{Spin(n)}\to X$ be the spin structure associated to it in the sense defined earlier. Because the zero, first, second fundamental groups are trivial, we conclude that $P_{Spin(n)}$ restricted to the two skeleton has only one section up to homotopy, as if it had more than one, its difference would induce a non-zero π_1 or π_2. Recalling that $P_{Spin(n)}$ projects as a double cover onto $P_{SO(n)}$ we obtain a section of $P_{SO(n)}$ restricted to the two skeleton. But this determines a trivialization of E on the two skeleton.

At this point in time, the viewing audience might want to keep in mind that: The non-trivial double covers of a space X correspond to the subgroups of $\pi_1(X)$ of index 2. These subgroups in turn are in correspondence with homomorphisms $\phi : \pi_1(X)\to\mathbb{Z}_2$. Each non-trivial ϕ is determined by $\phi' : H_1(X;\mathbb{Z})\to\mathbb{Z}_2$ which is in 1-1 correspondence with non-zero elements of $H^1(X;\mathbb{Z}_2)$.

From an exact sequence for bundles in [1]:

$$0\to H^1(X;\mathbb{Z}_2)\to H^1(P_{SO(n)};\mathbb{Z}_2)\to H^1(SO(n);\mathbb{Z}_2)\to H^2(X;\mathbb{Z}_2) \tag{3.1}$$

As $H^1(P_{SO(n)};\mathbb{Z}_2)\simeq\mathbb{Z}_2$, if there is a spin structure i.e. the last arrow in the sequence must map everything to zero, then the second to last map must be a surjection. By the above argument, the double covers of $P_{SO(n)}$ can be associated to $H^1(P_{SO(n)};\mathbb{Z}_2)$ and the spin structures are the preimages of 1 under the second to last arrow. This shows that the spin structures in the sense defined in earlier in this paper, like those defined in class, are in 1-1 correspondence with elements of $H^1(X;\mathbb{Z}_2)$.

With this is mind, that our map from "double cover" spin structures to the spin structures defined in class is clearly one-to-one suffices to show equivalence.

4. Every 4-manifold has a spin^c structure

The existence/definition of the Seiberg-Witten invariants depend on the presence of some sort of "spin structure" on a manifold. Not all four manifolds admit a true spin structure though. As it turns out though every four manifold admits a *spin*^{c} structure.

Recalling (out of nowhere) that $Spin^c(4)\simeq\{(A,B)\in U(2)\times U(2)|\det(A)=\det(B)\}$, we see there is a natural homomorphism $\lambda : Spin^c(4)\to S^1$ by representing an element of $Spin^c(4)$ as (A,B) and defining $\lambda((A,B)) := \det(A)$. We associate to a $Spin^c$ structure the line bundle L given by $L = P_{Spin^c(4)}\times_\lambda\mathbb{C}$

Proposition 3. Let X be an oriented four-manifold and let $P\to X$ be the frame bundle of the tangent bundle. Then there is a $Spin^c(4)$ bundle that is a cover of the frame bundle and on each fiber restricts to quotienting by the center of $Spin^c(4)$.

Proof. As $Spin^c(n)\simeq Spin(n)\times_{\{\pm1\}}S^1$ we obtain the following exact seqence:

$$\{e\} \to \mathbb{Z}_2\to Spin^c(4)\to S^1\times SO(4)\to\{e\} \tag{4.1}$$

From this we conclude that there exists a $Spin^c$ structure with determinant line bundle L iff the principal bundle $P_{S^1\times SO(4)}$ has a double cover which on each fiber has $Spin^c\to S^1\times SO(4)$. Remember that the double covers of a manifold X correspond to elements of $H^1(X;\mathbb{Z}_2)$. Applying this to our case, where $H_1(S^1\times SO(4);\mathbb{Z}_2) = \mathbb{Z}_2\times\mathbb{Z}_2$ we see there are three non-trivial double covers. Through trial and error we see they are $Spin^c(4)$, $S^1\times SO(4)$ where S^1 wraps around itself twice, and $S^1\times Spin(4)$. It turns out that the double cover extends to one of $SO(6)$ iff the double cover is $Spin^c(4)$. For a more detailed analysis consult the section (appendix to chapter 2) in Gompf and Stipsicz [1] on *spin*^{c} structures.
So there exists a spin^c structure with determinant line bundle L iff $P_{S^1\times SO(4)}$ itself admits a spin structure. But this condition we know is equivalent to $w_2(P_{S^1\times SO(4)}) = 0$. But $w_2(P_{S^1\times SO(4)}) = w_2(L) + w_2(X)$. As L is a complex bundle, $w_2(L)$ is the reduction of $c_1(L)$ mod two, and we see that existence of a spin^c structure with line bundle L is equivalent to $c_1(L)\equiv w_2(X)$ mod two. Every characteristic element is the first Chern class of the determinant line bundle for some spin^c structure. Thus, the set of *spin*^{c} structures on X surjects via $c_1(L)$ onto the set $\{K\in H^2(X;\mathbb{Z})|K\equiv w_2(X)\text{mod}2\}$. That this set is nonempty for compact oriented four-manifolds is a result of Hirzebruch and Hopf. Later, Teichner and Vogt extended the result to non-compact oriented four-manifolds. See [1] for more. ∎

5. Sketch on how to get to the Seiberg-Witten equations

It turns out (and not much is explained here...mostly definitions, and unjustified assertions) that when n is even, say $n = 2k$, $Cl(2k)\otimes_{\mathbb{R}}\mathbb{C}\simeq M_{2k}(\mathbb{C})$ and the corresponding complex 2^n-dimensional representation S_n of $Spin(n)$ splits into two non-isomorphic irreducible representations, $S_n := {S_n}^+\oplus S_n^-$.
From this complex representation S_n, if we have fixed a spin structure $P_{Spin(n)}\to X$ on an oriented Riemannian manifold, we can associate the bundle $S\to X$. Sections of S are called spinors. When X is even, the bundle splits into $S^+\oplus S^-$ from the splitting of the representation S_n. There is another bundle associated to $P_{(Spin(n)}\to X$ by the $Spin(n)$ representation $Cl(n)\otimes_{\mathbb{R}}\mathbb{C}$ This bundle is referred to as the Clifford bundle of the spin structure on X. The action of $Cl(n)\otimes_{\mathbb{R}}\mathbb{C}$ on S_n induces an action on the bundle S called Clifford multiplication. Now suppose in particular we are working on a spin manifold, $(P_{SO(n)} = TX)$. To the metric d on our manifold is associated the unique Levi-Civita connection $\Delta_d : \Gamma(X;TX)\to\Gamma(X;TX\otimes T^*X)$ where $\Gamma(X;Y)$ denotes the vector space of smooth sections of the vector bundle $Y\to X$. We pull this connection back to the bundle $P_{Spin(n)}\to X$ which in turn induces a differentiation $\Delta : \Gamma(X;S)\to\Gamma(X;S\otimes T^*X)$.
As T^*X is a subbundle of the Clifford bundle, T^*X acts on the spinor bundle S. Hence we get the map induced by the Clifford multiplication: $C : \Gamma(X;S\otimes T^*X)\to\Gamma(X;S)$.

Definition 1. For a given Riemannian manifold X with spin structure $P_{Spin(n)}$, the composition:

$$\delta = C\circ\Delta : \Gamma(X;S)\to\Gamma(X;S) \tag{5.1}$$

is called the Dirac operator of X.

The construction above can be carried out in a similar fashion for spin^c structures so that we get a Clifford multiplication and a Dirac operator. If we assume that X is a simply connected four-manifold, then the spin^c structure uniquely determines the determinant line bundle L. Any $U(1)$ connection on L combined with the Levi-Civita connection gives a covariant differentiation from smooth sections of the spinor bundle to smooth sections of the spinor bundle tensor the cotangent bundle. Composing with the Clifford multiplication, we get the coupled Dirac operator of the spin^c structure coupled to a $U(1)$ connection on the determinant line bundle which we will denote δ_A.
It ends up being important that in dimension four, $Lie(SO(4))$ splits as $Lie(SO(3)\oplus Lie(SO(3))$ coinciding with the splitting of two forms into self-dual and anti-dual two forms under the Hodge-* operator. Somehow from this we are able to induce a map, $i\sigma$ from positive spinors to imaginary-valued-self-dual-two-forms by noting that conjugation of i by a quaternion element is spin$^c(4)$ invariant.
Given a $U(1)$ connection, A on the determinant line bundle, the curvature $F_A$is a Lie algebra-valued two form. As $U(1)$ isomorphic to the imaginary line, it follows that F_A is a imaginary two form and hence splits by extending the splitting on two forms to imaginary valued two forms. Let ${F^+}_A$ denote the self-dual part of the curvature.

Lastly, the Gauge group is the group of bundle automorphisms of $P_{Spin^c(4)}\to X$ inducing the trivial action on the frame bundle. Alternatively it is defined as the group of maps from X to S^1. Any element in the Gauge group acts on a section in the positive spinor bundle by multiplication.

Definition 2. Given a $U(1)$ connection A on the determinant line bundle, and a section in the positive spin bundle, ψ the Seiberg-Witten equations are:

$$\delta_A(\psi) = 0 \tag{5.2}$$

and

$$F^+{}_A = i\sigma\psi \tag{5.3}$$

The set of all pairs (A, ψ) that satisfy the Seiberg-Witten equations is called the moduli space. This space is invariant under the Gauge group. And here we really don't know what we're talking about....so we should shut up before we say something bad.

6. Why We Should Care

Spin has some useful applications. It is used in the following results:

i) Every closed, oriented three-manifold embeds smoothly in m connected sums of $S^2 \times S^2$ for sufficiently large m.

ii) The Poincare homology sphere is not the boundary of any contractible smooth four-manifold [1].

iii) To define the Seiberg-Witten invariants it is necessary to have a spin^c structure on a four-manifold.

iv) If you are involved in K-theory, then spin has been partially responsible for your employment. It is a theorem attributed to Anderson-Brown-Peterson that two spin manifolds are cobordant iff their Stiefel-Whitney classes agree and all their K-theoretical classes agree too [9].

The Seiberg-Witten invariants which we outlined using spin-c structures have been the source of much active research. Previously the only finely tuned four manifold invariant was the Donaldson invariant which was difficult to work with for reasons of compactness (lack thereof). In the case of the Seiberg-Witten, the moduli space consisting of all of the solutions of the Seiberg-Witten equations up to Gauge transformation is compact. The invariants have been used, perhaps most notably by Kronheimer and Mrowka to give proof of the Thom conjecture which states that a holomorphic curve in $\mathbb{C}P^2$ minimizes the genus of any surface which represents the same homology class and Morgan, Szabo, and Taubes to prove that a smooth symplectic curve of nonnegative self-intersection in a symplectic four-manifold is genus minimizing, and have given a product formula for computing Seiberg-Witten invariants which allows them to give a non-vanishing result for the Seiberg-Witten invariants of certain generalized connected sum manifolds. Lastly, the invariants are computable for Kahler manifolds, and are used to gain insight into the enumeration of algebraic curves. For the real deal see for instance [3],[4],[8],[10],[11] and a survey, [1].

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