A brief glance at K-theory

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June 1, 2004

1 Introduction

1.1 History

K-theory has its origins in the late 1950s generalization by Grothendieck of the Riemann-Roch theorem [2]. Algebraic geometry utterly baffles me, making me unfit to summarize Grothendieck's achievement, but he appears to have associated to each X in some family of algebraic spaces a group $K(X)$ that turns out to be a natural repository for information about X and various objects defined "over" X. Within this framework, he recovered classical Riemann-Roch as a special case of a general theorem involving K-groups.

Hirzebruch and Atiyah soon realized that these ideas could be exported to the world of algebraic topology. The resulting K-theory of topological spaces— the subject of this paper— turns out to be quite powerful: among its early triumphs were Adams' determination of the maximum number of linearly independent vector fields on S^n, and the formulation and proof of the Atiyah-Singer index theorem. (Further work by many others has extended the ideas and constructions of K-theory beyond topological spaces to C^*-algebras and more general rings, but we will not discuss this here.)

1.2 Outline

Topological K is a functor from compact Hausdorff spaces to commutative rings. Roughly speaking, $K(X)$ is what you get when you turn the operations of direct sum and tensor product of vector bundles over X into the addition and multiplication of a ring. Along with K we have his "reduced" companion $\tilde{K}$ which is roughly K modulo trivial bundles.

By setting $\tilde{K}^{-i}(X) = \tilde{K}(S^iX)$, S^iX denoting the ith suspension of X, $\tilde{K} = \tilde{K}^0$ becomes the first of a sequence of functors. After also defining $\tilde{K}^{-i}(X,A) = \tilde{K}^i(X/A)$ for compact pairs (X,A), we get a long exact sequence

$$\cdots \longrightarrow \tilde{K}^{-1}(X,A) \longrightarrow \tilde{K}^{-1}(X) \longrightarrow \tilde{K}^{-1}(A) \longrightarrow \tilde{K}^0(X,A) \longrightarrow \tilde{K}^0(X) \longrightarrow \tilde{K}^0(A) \tag{1}$$

Remarkably enough, it turns out that $\tilde{K}^{-i}(X) \cong \tilde{K}^{-i-2}(X)$ for all X in a natural way, allowing one to extend the sequence $\tilde{K}^i$ to all integers i and turn the above sequence into one with only six terms:

$$\begin{array}{ccccc} \tilde{K}^0(X,A) & \longrightarrow & \tilde{K}^0(X) & \longrightarrow & \tilde{K}^0(A) \\ \uparrow & & & & \downarrow \\ \tilde{K}^1(A) & \longleftarrow & \tilde{K}^1(X) & \longleftarrow & \tilde{K}^1(X,A) \end{array} \tag{2}$$

These properties of K-theory make it computable; its relation to vector bundles makes it worth computing.

The first section of this paper defines K and $\tilde{K}$. The subsequent sections explain (1) and the isomorphism $\tilde{K}^{-i}(X) \cong \tilde{K}^{-i-2}(X)$ that gives us (2). With this basic knowledge in hand we turn to an application: Adams' result on the existence of maps with Hopf invariant 1 and its corollaries.

My general approach is to relate the ideas and skip the details. I give references wherever possible so that the interested reader may see every unsubstantiated claim in full detail. I will only deal with complex vector bundles over compact spaces although K-theory can certainly be developed in greater generality.

1.3 Aside: Why is it called K-theory?

For this bit of mathematical culture we turn to Grothendieck himself, as quoted in [5]:

> The way I first visualized a K-group was as a group of "classes of objects" of an abelian (or more generally, additive) category, such as coherent sheaves on an algebraic variety, or vector bundles, etc. I would presumably have called this group $C(X)$ (X being a variety or any other kind of "space"), C the initial letter of 'class,' but my past in functional analysis may have prevented this, as $C(X)$ designates also the space of continous functions on X (when X is a topological space). Thus, I reverted to K instead of C, since my mother tongue is German, Class = Klasse (in German), and the sounds corresponding to C and K are the same.

Fascinating.

2 $K(X)$ and $\tilde{K}(X)$

Throughout this section, X and Y will denote compact Hausdorff spaces.

2.1 Vector bundles

A *trivial bundle* over X is a product $X \times \mathbb{C}^n$ for some nonnegative[1] integer n, together with the projection $X \times \mathbb{C}^n \to X$. A *vector bundle* over X is a space E and a surjection $p : E \to X$ together with a complex vector space structure on each fiber $E_x = p^{-1}(x)$, such that for each $x \in X$ there is an open U containing x such that $p^{-1}(U)$ is isomorphic to a trivial bundle $U \times \mathbb{C}^n$. Here the isomorphism is understood in the sense appropriate to vector bundles over X: a morphism $E \to F$ is a map $E \to F$ taking E_x linearly to F_x for all $x \in X$. If the dimension of $p^{-1}(x)$ is constant over X it is called the *rank* of the bundle. A rank 1 bundle is a *line bundle*.

An isomorphism $p^{-1}(U) \to U \times \mathbb{C}^n$ is called a *trivialization over* U (or more loosely a *local trivialization*).

We will denote the set of isomorphism classes of vector bundles over X by $V(X)$, and $\mathbf{n} \in V(X)$ will denote the trivial bundle of rank n.

2.2 Operations on vector bundles

2.2.1 The pullback

Given a map $g : X \to Y$ the *pullback* construction associates to each vector bundle $q : E \to Y$ over Y a vector bundle $g^*(E)$ over X.

We define $g^*(E) = \{(x, v) \in X \times E : g(x) = q(v)\}$ together with $p : g^*(E) \to X$ given by $(x, v) \mapsto x$. The map g induces a morphism $g^*(E) \to E$ by way of $(x, v) \mapsto v$ and one has $g^*(E)_x \cong E_{g(x)}$ for all $x \in X$. It is convenient to know that the pullback is characterized by this property: if one has a bundle F over X and morphism $\phi : F \to E$ inducing isomorphisms $F_x \cong E_{g(x)}$ on each fiber, then $F \cong g^*(E)$. ([11], 19)

The pullback is the basis of the functoriality of the constructions to come. It is clear that g^* induces a map $V(Y) \to V(X)$ and that $g \mapsto g^*$ respects composition of maps. When we define K we will see that g^* also induces a map $K(Y) \to K(X)$ that plays the role of $K(g)$.

2.2.2 The direct sum

If E and F are vector bundles over X there is a natural notion of their *direct sum* $E \oplus F$. Letting $p : E \to X$ and $q : F \to X$ denote the bundle maps, we set $E \oplus F = \{(e, f) \in E \times F : p(e) = q(f)\}$ with the obvious map $E \oplus F \to X$ (induced by either p or q). It is readily checked that this is a vector bundle over X and that $(E \oplus F)_x = E_x \oplus F_x$. It is characterized by a universal property analogous to that of the vector space direct sum. We will need the following fact ([3], 27):

Fact 1. *Every vector bundle over a compact Hausdorff space is a direct summand of a trivial bundle.*

[1] $\mathbb{C}^0 = 0$.

2.2.3 The tensor product

If E and F are vector bundles over X there is also a *tensor product* bundle $E \otimes F$. At the fiber level it is clear that one should have $(E \otimes F)_x = E_x \otimes F_x$; the only question is how to topologize this collection. This is not as straightforward as the direct sum case, but it is elementary: the idea is to topologize $\bigcup_{x \in X}(E_x \otimes F_x)$ by patching together the toplogies induced by local trivializations of E and F. This is done in ([11], 15-15) or more generally in ([3], 6-9).

2.2.4 The kth exterior power, $k \geq 0$

Given a bundle E over X, and $k \geq 0$ one can form the kth exterior power $\Lambda^k(E)$. This is a bundle over X with $(\Lambda^k(E))_x = \Lambda^k(E_x)$; see ([3], 6-9) for the topology. Our main use of exterior powers will be in constructing the Adams operations needed in §5. It is readily checked that

- $\Lambda^0(E) = \mathbf{1}$ and $\Lambda^1(E) = E$ for any E,
- $\Lambda^k(E) = 0$ whenever k exceeds the dimension of any fiber of E,
- $\Lambda^k(E \oplus F) = \sum_{j=0}^{\infty} \Lambda^i(E) \otimes \Lambda^{k-i}(F)$.

It follows that if $E = L_1 \oplus \cdots \oplus L_j$ is a sum of line bundles then

$$\Lambda^k(E) = s_k(L_1, \ldots, L_j)$$

where $s_k \in \mathbb{Z}[x_1, \ldots, x_j]$ is the jth elementary symmetric polynomial in $x_1, \ldots, x_j$ (ie, $(-1)^j$ times the coefficient of X^{n-j} in $\prod_{k=1}^{j}(X - x_k)$), and the evaluation of s_k at $L_1, \ldots, L_j$ has the obvious meaning[2].

2.3 The semiring structure of $V(X)$

It is clear that $\oplus$ and $\otimes$ give rise to commutative and associative operations on $V(X)$ satisfying a distributive law $(E \oplus F) \otimes G = (E \otimes G) \oplus (F \otimes G)$. The bundles $\mathbf{0}$ and $\mathbf{1}$ serve as identities for $\oplus$ and $\otimes$ respectively. Thus $V(X)$ is a commutative *semiring* with identity. If $f : X \to Y$ is given then $f^* : V(Y) \to V(X)$ is readily checked to be a homomorphism of semirings[3] and this assignment is functorial (as in §2.2.1).

One might wonder why V is not the focus of attention— or, as V is the starting point of K-theory, why K-theory appeared only so recently. (Vector bundles are classical objects, and viewing $\oplus$ and $\otimes$ as kinds of "addition" and "multiplication" is not particularly deep.) The fact is that V is very difficult to compute. Thus, although *particular* vector bundles appear quite naturally in a variety of classical contexts, the aggregate of *all* vector bundles over a space appears rather intractable at first glance. It is somewhat miraculous that a quotient of V turns out to be both easy to compute and useful.

An elementary fact about V is that it is invariant under homotopy equivalence.

Fact 2. *If $f : X \to Y$ is a homotopy equivalence then $f^* : V(Y) \to V(X)$ is an isomorphism. ([3], 18)*

In the case of a point p we have an isomorphism $V(\{p\}) \to \mathbb{Z}_{\geq 0}$ given by dimension; we conclude

Corollary 1. *If X is contractible then $V(X) \cong \mathbb{Z}_{\geq 0}$.*

When $A \subseteq X$ is contractible the projection $X \to X/A$ is often a homotopy equivalence. Even when it isn't, we have the following fact, which is needed to construct (1):

Fact 3. *If A is a closed contractible subset of X then the quotient $X \to X/A$ induces an isomorphism $V(X/A) \to V(X)$. ([3], 19)*

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[2]For example if $j = 3$ we have $s_2(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3$ and $s_2(L_1, L_2, L_3) = (L_1 \otimes L_2) \oplus (L_1 \otimes L_3) \oplus (L_2 \otimes L_3)$.

[3]This is most easily done by checking that $f^*(E) \oplus f^*(F)$ satisfies the universal property required of $f^*(E \oplus F)$, and so on.

2.3.1 Vector bundles on spheres and clutching functions

In the following it will help to have some knowledge of vector bundles on spheres. First note that if $E \to S^n$ is a vector bundle, as S^n is connected the locally constant function $x \mapsto \dim E_x$ on S^n is in fact constant, so every element of $V(S^n)$ has a well-defined rank.

There is a bijection between the set of isomorphism classes of rank k vector bundles on S^n and the set of homotopy classes of maps $S^{n-1} \to \mathrm{GL}_k(\mathbb{C})$. Given a map $f : S^{n-1} \to \mathrm{GL}_k(\mathbb{C})$ (a "clutching function") we obtain a rank k vector bundle E_f on S^n via the following construction (the "clutching construction"). Let D_1 and D_2 denote the upper and lower hemispheres of S^n; note $D_1 \cap D_2 = S^{n-1}$. Let E_f denote the space obtained from the disjoint union of the trivial bundles $T_1 = D_1 \times \mathbb{C}^k$ and $T_2 = D_2 \times \mathbb{C}^k$ by identifying $(x, v) \in \partial D_1 \times \mathbb{C}^k$ with $(x, f(x)v) \in \partial D_2 \times \mathbb{C}^k$. The evident projection $E_f \to S^n$ makes E_f into a vector bundle, depending (up to isomorphism) only on the homotopy class of f ([11], 24).

Going the other way, if $E \to S^n$ is any rank k vector bundle, the restrictions E_1 and E_2 of E to D_1 and D_2 are each trivial by Corollary 1. Choosing trivializations $h_1 : E_1 \to D_1 \times \mathbb{C}^k$ and $h_2 : E_2 \to D_2 \times \mathbb{C}^k$, then $h_2 h_1^{-1}$ gives a map $S^{n-1} \to \mathrm{GL}_k(\mathbb{C})$ whose homotopy class is independent of the choice of h_1 and h_2. It is readily checked that this map is inverse to the clutching construction.

Since $\mathrm{GL}_k(\mathbb{C})$ is path connected we conclude that every complex vector bundle over S^1 is trivial so that $V(S^1) \cong \mathbb{Z}_{\geq 0}$ by dimension. This will be important in §4.

A bundle of particular importance to us will be the bundle $H \to S^2$ corresponding to the clutching function $f : S^1 \to \mathrm{GL}_1(\mathbb{C})$ given by $f(z) = z$. (Regarding S^2 as $\mathbb{C}P^1$, this is the "tautological" bundle whose fiber over a line in $\mathbb{C}P^1$ is just that line.) It will be significant that $(H \otimes H) \oplus 1 = H \oplus H$. This can be seen by noticing that the corresponding clutching functions $z \mapsto \left(\begin{smallmatrix} z^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $z \mapsto \left(\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right)$ are homotopic.

2.4 $K(X)$ and $\tilde{K}(X)$

We turn $V(X)$ into a ring in a purely formal way. Let $\Delta : V(X) \to V(X) \times V(X)$ be the diagonal homomorphism of semirings, and let $K(X)$ denote the set of cosets of $\Delta(V(X))$ in $V(X) \times V(X)$. This is clearly a quotient semiring, and it is readily checked that the map $(a, b) \mapsto (b, a)$ induces an inverse operation for the addition in $K(X)$ so that $K(X)$ is a commutative ring. If $f : X \to Y$ is given, the pullback $f^* : V(Y) \to V(X)$ induces a ring homomorphism $K(Y) \to K(X)$. Functoriality is evident.

In a suitable sense, $K(X)$ is the unique way of "making a ring out of" $V(X)$. Let $\iota : V(X) \to K(X)$ denote the composition of $E \mapsto (E, 0)$ with the projection $V(X) \times V(X) \to K(X)$. The pair $(K(X), \iota)$ has the property that if R is any ring and $\phi : V(X) \to R$ any semiring homomorphism there is a unique ring homomorphism $\Phi : K(X) \to G$ such that $\phi = \Phi\iota$. This property determines $K(X)$ up to isomorphism.

It is clear from Fact 2 that K is also invariant under homotopy equivalence. From Corollary 1 and the universal property of $K(X)$ we conclude that $K(X) \cong \mathbb{Z}$ for any contractible X.

To get a better picture of $K(X)$ let $[E]$ denote $\iota(E)$, for $E \in K(X)$. We have

$$[F] = [G] \text{ iff there is } B \in V(X) \text{ such that } F \oplus B = G \oplus B. \tag{3}$$

and

$$K(X) = \{[F] - [G] : F, G \in V(X)\} \tag{4}$$

Note that $[F] - [G] = [H] - [K]$ iff there is $B \in V(X)$ such that $F \oplus K \oplus B = G \oplus H \oplus B$. By Fact 1 both this B and the B in (3) may be taken to be trivial. Similarly, the RHS of (4) is equivalently $\{[F] - [T] : F \in V(X), T \text{ trivial}\}$. When there is a trivial B for which $F \oplus B = G \oplus B$, F and G are said to be *stably equivalent*. The above shows that one may equally well have defined $K(X)$ to be the collection of formal differences of stable equivalence classes of vector bundles over X with the obvious operations.

It will be convenient to have a "reduced" form of $K(X)$ based on a looser form of equivalence. Declare $E \sim F$ in $V(X)$ if there are trivial bundles T and T' (of potentially different ranks) such that $E \oplus T = F \oplus T'$. This is an equivalence relation and $\oplus$ induce a group structure on the set $\tilde{K}(X)$ of $\sim$-equivalence classes. One obtains a surjection $K(X) \to \tilde{K}(X)$ by sending $[E] - [T]$ (here $E \in V(X)$ and T is trivial) to the class of E. The kernel of this map is $\{[\mathbf{n}] - [\mathbf{m}] : m, n \in \mathbb{Z}\} \cong \mathbb{Z}$. We thus have an exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z} \longrightarrow K(X) \longrightarrow \tilde{K}(X) \longrightarrow 0$$

For any $x_0 \in X$, the inclusion $\{x_0\} \to X$ induces a map $K(X) \to K(\{x_0\}) \cong \mathbb{Z}$ which restricts to an isomorphism on $\ker(K(X) \to \tilde{K}(X))$; we thus have a splitting $K(X) \cong \tilde{K}(X) \oplus \mathbb{Z}$ determined by x_0. Regarding $\tilde{K}(X)$ as $\ker(K(X) \to K(x_0))$ allows us to define a ring structure on $\tilde{K}(X)$ by restricting the operations of $K(X)$. Hence we may regard $\tilde{K}$ as a functor from pointed compact spaces to commutative rings. This will be done in the following, although we will rarely make explicit mention of the choice of basepoint.

3 The long exact sequence

3.1 A very short exact sequence

The starting point for (1) is the definition of $\tilde{K}$ for pairs (X, A) (here X is compact Hausdorff and A is a closed subset of X: we set $\tilde{K}(X, A) = \tilde{K}(X/A)$ (taking $A \in X/A$ as a basepoint). The inclusion and quotient give us an exact sequence

$$A \longrightarrow X \longrightarrow X/A \tag{5}$$

Applying $\tilde{K}$ we get a sequence

$$\tilde{K}(X/A) \longrightarrow \tilde{K}(X) \longrightarrow \tilde{K}(A) \tag{6}$$

Fact 4. *The sequence* (6) *is exact.*

The proof is elementary ([11], 52). The non-obvious part is to show that anything in the kernel of $\tilde{K}(X) \to \tilde{K}(A)$ comes from something in $\tilde{K}(X/A)$. (Adopting the view of $\tilde{K}$ as a quotient of $V(X)$ by an equivalence relation, given a bundle E in the kernel, one constructs a bundle over X/A using the trivialization of any bundle $\sim E$ that is trivial over A.)

3.2 Cones and suspensions

For any space X we recall the notions the *cone* CX and the *suspension* SX of X. One obtains CX from $X \times [0, 1]$ by collapsing $X \times \{1\}$ to a point, and one obtains SX from $X \times [-1, 1]$ by collapsing $X \times \{1\}$ and $X \times \{-1\}$ to points. (In each case one regards $X = X \times \{0\}$ as contained in the resulting space.)

If one is given a pair (X, A), one may attach the cone on A to X in an obvious way, by identifying the image of A in CA with the image of A in X. We denote this space by $X \cup CA$. An iterated version of this construction gives rise to (1).

3.3 The long exact sequence

More precisely, (5) may be viewed as the beginning of the sequence

$$\begin{array}{ccccccccc} A & \longrightarrow & X & \longrightarrow & X \cup CA & \longrightarrow & (X \cup CA) \cup CX & \longrightarrow & ((X \cup CA) \cup CX) \cup C(X \cup CA) \\ & & & & \Big\downarrow{\scriptstyle \text{collapse } CA} & & \Big\downarrow{\scriptstyle \text{collapse } CX} & & \Big\downarrow{\scriptstyle \text{collapse } C(X \cup CA)} \\ & & & & X/A & & SA & & SX \end{array}$$

It is clear how to continue the first row: at each step, attach the cone on the space from two steps ago. Each vertical map collapses the newly attached cone to a point. Applying $\tilde{K}$ and making repeated use of (6) and Fact 3 we obtain the exact sequence

$$\cdots \longrightarrow \tilde{K}(SX) \longrightarrow \tilde{K}(SA) \longrightarrow \tilde{K}(X/A) \longrightarrow \tilde{K}(X) \longrightarrow \tilde{K}(A)$$

Making the definitions $\tilde{K}^{-i}(X) = \tilde{K}(S^i X)$ and $\tilde{K}^{-i}(X, A) = \tilde{K}(S^i(X/A))$ we obtain (1).

It is worth noting that the construction of (1) via cones and suspensions requires very little about $\tilde{K}$ beyond its formal properties, namely homotopy invariance and "half exactness"[4]. These properties and the constructions they permit are important in the axiomatization and characterization of K-theories (eg [8] in the case of C^*-algebras).

4 Periodicity and $\tilde{K}(S^n)$

In this section we outline one way of establishing the isomorphism $\tilde{K}^0(X) \cong \tilde{K}^{-2}(X)$ giving rise to (2). The details of this argument are somewhat involved (see ([11], 42-55) or ([3], 57-78) or the original paper [4]). Of primary interest to us will be the parts of the argument allowing us to compute $\tilde{K}(S^n)$.

4.1 The product theorem

Recall the bundle $H \to S^2$ from § 2.3.1. It satisfies $H \otimes H \oplus 1 = H \oplus H$ so that $H^2 + 1 = 2H$ in $K(S^2)$, or equivalently $(H-1)^2 = 0$ in $K(S^2)$. The ring homomorphism $\mathbb{Z}[x] \to K(S^2)$ given by $x \mapsto H$ thus induces a homomorphism $\phi : \mathbb{Z}[x]/(x-1)^2 \to K(S^2)$.

4.2 The external product

There is a natural multiplication $K(X) \otimes K(Y) \to K(X \times Y)$ induced by the projections $p_x : X \times Y \to X$ and $p_y : X \times Y \to Y$. Given $a \in K(X)$ and $b \in K(Y)$ we define

$$a * b = {p_x}^*(a){p_y}^*(b) \in K(X \times Y)$$

the *external product* of a and b. An involved argument shows that the composition

$$K(X) \otimes \mathbb{Z}[x]/(x-1)^2 \xrightarrow{\mathrm{id}\otimes\phi} K(X) \otimes K(S^2) \xrightarrow{*} K(X \times S^2)$$

is an isomorphism. When X is a point this shows that $K(S^2) = \mathbb{Z}[x]/(x-1)^2$. Thus the abelian group $K(S^2)$ is generated by H and 1; the kernel of $K(S^2) \to K(s_0)$ is thus generated as an abelian group by $H - 1$. Since $(H-1)^2 = 0$ in $K(S^2)$ we conclude that the multiplication in $\tilde{K}(S^2)$ is trivial. Thus the ring structure $\tilde{K}(S^2)$ is completely determined.

4.3 The reduced external product and periodicity

The external product $K(X) \otimes K(Y) \to K(X \times Y)$ induces a map $\tilde{K}(X) \otimes \tilde{K}(Y) \to \tilde{K}(X \wedge Y)$ called the *reduced external product.* (Recall that the wedge product $X \wedge Y$ of pointed spaces (X, x_0) and (Y, y_0) is the space obtained from $X \times Y$ by collapsing $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$ to a point.)

A corollary of the product theorem in the previous section is that multiplication by $H - 1$ induces an isomorphism $\tilde{K}(X) \to \tilde{K}(S^2 \wedge X)$ for any X ([11], 55). Now $S^2 \wedge X$ is a quotient of $S^2 X$ by a contractible subset so that $\tilde{K}(S^2 \wedge X) \cong \tilde{K}(S^2 X)$ in a natural way. Thus

Theorem 1. *The reduced product with $H - 1$ induces an isomorphism $\tilde{K}(X) \to \tilde{K}(S^2 X)$.*

As $S^n = S(S^{n-1})$ for any $n \geq 1$ we conclude that the group $\tilde{K}(S^{2n}) \cong \tilde{K}(S^2) \cong \mathbb{Z}$ with generator $(H-1) * (H-1) * \cdots * (H-1)$ (n times), and the multiplication in $\tilde{K}(S^{2n})$ is trivial. The n-fold reduced power of $H - 1$ will be referred to as "the" generator of $\tilde{K}(S^{2n})$. By §2.3.1 we conclude $\tilde{K}(S^{2n+1}) = 0$ for any n.

[4]The property that $\tilde{K}(A) \to \tilde{K}(B) \to \tilde{K}(C)$ is exact whenever $0 \to A \to B \to C \to 0$ is

5 Applications

One may view the preceding sections merely as the construction of a functor from compact Hausdorff spaces to abelian groups that vanishes on closed discs D^n but not on their boundaries— thus obtaining the nonexistence of retractions $D^n \to \partial D^n$ and the Brouwer fixed point theorem. Alternatively, one could develop the degree theory of maps on spheres: to each $f : S^n \to S^n$, regard $f^* : \tilde{K}(S^n) \to \tilde{K}(S^n)$ as a homomorphism $\mathbb{Z} \to \mathbb{Z}$, hence multiplication by some integer $d(f)$. This agrees with the usual definition in terms of $H^n(S^n)$ or $H_n(S^n)$, and the same elementary results (eg the "hairy ball theorem") follow.

5.1 The Bott-Milnor theorem

The above applications are a little underwhelming in that they exploit only the functorial and cohomological character of K-theory. We will now turn to a more impressive application where the role of vector bundles is more explicit, namely the result of Bott and Milnor [6] (and Kervaire [13]) of 1958:

Theorem 2. *The tangent bundle*[5] TS^{n-1} *to* S^{n-1} *is trivial only when* $n = 2$*,* 4*, or* 8*.*

To aid in the appreciation of this result we mention a related problem: for which n does there exist a division algebra of dimension n? The answer,

Theorem 3. *The dimension of a division algebra is* 1*,* 2*,* 4*, or* 8*.*

will be deduced along with Theorem 2 from calculations in the K-theory of spheres.

5.2 Division algebras

A *division algebra* is a finite dimensional real vector space with a bilinear multiplication such that for any $b \neq 0$ the maps $x \mapsto bx$ and $x \mapsto xb$ are bijective; the usual examples are $\mathbb{R}$, $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$ of dimensions 1, 2, 4, and 8. Classical results show that in various restricted senses these are the only possibilities: $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ are the only *associative* examples (Frobenius 1877; [9], 229), and $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the only *alternative*[6] examples (Hurwitz 1898; [9], §10.1). Other division algebras (eg $\mathbb{C}$ with the non-alternative $(x, y) \mapsto x\overline{y}$, and variations on this theme) were known in the 19th century, but all with the dimensions of known examples. That these are the only possible was not established until 1958, as a corollary of Theorem 2.

It is somewhat remarkable that all known proofs use algebraic topology ([10], p. 173). To see how it enters the picture, assume $n > 1$ and suppose $\mathbb{R}^n$ is a division algebra. One may assume without loss of generality ([11], p.60) that the multiplication has a two-sided identity e. Extending e to a basis $e, v_1, \ldots, v_{n-1}$ of $\mathbb{R}^n$ ones sees that the vector fields $x \mapsto \frac{d}{dt}|_{t=0}(x \cdot (e + tv_j))$ on S^{n-1}, $1 \leq j \leq n-1$, are linearly independent, so that the tangent bundle TS^{n-1} is trivial[7].

From the existence of $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ one thus deduces the triviality of the bundles mentioned in Theorem 2. More significantly, one can gets partial information on the general question from various algebraic invariants. Viewed this way, basic developments in homology and cohomology in the early 20th century represent further progress on the problem: if TS^{n-1} is trivial,

- The degree theory of maps on spheres (the "hairy ball theorem") tells us that n must be even (Brouwer, 1910s; [10], p.135)
- Calculations in the $\mathbb{Z}/2$ cohomology ring of $\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}$ (Hopf 1940 [12]; or [10], 222; [9], 283) or the Stiefel-Whitney classes of $T\mathbb{R}P^n$ (Stiefel 1940 [14]; or [9], 289) tell us that n must be a power of 2.

[5] If you are unfamiliar with the tangent bundle, it is exactly what you think it ought to be. For a precise definition see e.g. ([7], 55).

[6] An algebra is alternative if $x(yy) = (xy)y$ and $(xx)y = x(xy)$ for all x and y.

[7] For more information see ([9], 289)

The 1958 proof of Theorem 2 used rather involved calculations in cohomology. The proof we give here follows a 1966 paper of Adams and Atiyah [1], as simplified in [3] and [11].

With the long history in mind, we outline the proof of:

Theorem 4. *Suppose $n > 1$. If $\mathbb{R}^n$ is a division algebra, or if TS^{n-1} is trivial, then $n = 2$, 4, or 8.*

From the above more elementary results we may assume n is even[8]. So let us replace n with $2n$.

Either hypothesis implies existence of a $g : S^{2n-1} \times S^{2n-1} \to S^{2n-1}$ with two-sided identity element[9] (one says that S^{2n-1} is an *H-space*). An elementary construction ([11], p.62) now constructs from g a map $\widetilde{g} : S^{4n-1} \to S^{2n}$. The proof continues by examining what $\widetilde{g}$ does on K-theory.

5.3 The Hopf invariant

Given a general $f : S^{4n-1} \to S^{2n}$ let X denote the space obtained by attaching a $4n$-cell to S^{2n} via f. As $\tilde{K}^1(S^{2n}) = \tilde{K}^1(S^{4n}) = 0$ the sequence (2) for the pair (X, S^{2n}) has three terms:

$$0 \to \tilde{K}(S^{2n}) \longrightarrow \tilde{K}(X) \longrightarrow \tilde{K}(S^{4n}) \to 0$$

Let A and B denote the generators of $\tilde{K}(S^{2n})$ and $\tilde{K}(S^{4n})$ respectively. Let a denote the image of A in $\tilde{K}(X)$. If x is any element mapping to B then x^2 maps to zero (as any square in $\tilde{K}(S^{4n})$ is zero); thus $x^2 = ma$ for some $m \in \mathbb{Z}$. The coset $m + 2\mathbb{Z}$ does not depend on the choice of x [10]; we call this the *Hopf invariant* of f. If $f = \widetilde{g}$ from the previous section, a diagram chase ([11], p.62) shows that the Hopf invariant of f is 1.

5.4 Adams' theorem

It thus suffices to prove the following theorem of Adams:

Theorem 5. *If there is $f : S^{4n-1} \to S^{2n}$ with Hopf invariant 1 then n is 1, 2, or 4.*

The proof constructs for any X and $k > 0$ natural maps $\psi^k : K(X) \to K(X)$. (Here "natural" means that if $f : X \to Y$ one has $\psi^k f^* = f^* \psi^k$ for any $f : X \to Y$.)

Natural maps $K(X) \to K(X)$ defined for all X are called *operations* in K-theory. The Adams operations ψ^k are an extension of the notion of taking the kth tensor power of a line bundle.

5.4.1 Adams operations

More precisely, we have $\psi^k(L) = L^k$ for any line bundle L, and if $E = L_1 \oplus \cdots \oplus L_j$ is a sum of line bundles, we have

$$\psi^k(E) = L_1^k + L_2^k + \cdots + L_j^k \tag{7}$$

This property essentially serves as the definition of ψ^k. As a degree k symmetric polynomial in $L_1, \ldots, L_j$, $L_1^k + L_2^k + \cdots + L_j^k$ is a polynomial p_k in the elementary symmetric polynomials $s_1, \ldots, s_k$ in $L_1, \ldots, L_j$. From §2.2.4 we know that $s_q = \Lambda^q(E)$ so we have $L_1^k + L_2^k + \cdots + L_j^k = q_k(\Lambda^1(E), \ldots, \Lambda^k(E))$. As the right hand side of this makes sense for any bundle we define

$$\psi^k(E) = q_k(\Lambda^1(E), \ldots, \Lambda^k(E))$$

At the moment all we have is a map $V(X) \to K(X)$ defined for any X. This clearly satisfies $f^*\psi^k = \psi^k f$ for any $f : X \to Y$ as f^* commutes with Λ^k. To extend the definition of ψ^k to $K(X)$ we will need the following "splitting principle" for K-theory. See ([11], 66) or ([3], 110).

[8] One can also give a one-paragraph K-theoretic reduction to the even case ([11] p. 61).

[9] In the first case this is established by reducing to the case when the multiplication on $\mathbb{R}^n$ has a two sided identity element and setting $g(s,t) = (s \cdot t)/|s \cdot t|$). In the second one chooses vector fields $v_1, \ldots, v_{2n-1}$ on S^{2n-1} so that $x, v_1(x), \ldots, v_{2n-1}(x)$ is orthonormal for all x and such that $v_i(e_1) = e_{i+1}$ for $1 \leq i \leq 2n-1$, where e_j denotes the ith standard basis vector; then $g(s,t) = \alpha_s(t)$ where α_s is the matrix sending the standard basis to $x, v_1(x), \ldots, v_{2n-1}(x)$. ([11], p.61)

[10] If x' has the same image as x then $x' = ka + x$ for some $k \in \mathbb{Z}$; squaring and using the fact that $a^2 = 0$ we conclude that $(x')^2$ and x^2 differ by $2xka$.

Fact 5. *Given any bundle $E \to X$ there is a compact Hausdorff space Y and $p : Y \to X$ such that $p^* : K(X) \to K(Y)$ is injective and $p^*(E)$ is a sum of line bundles.*

We now verify that ψ^k, as a map $V(X) \to K(X)$, is additive. Fix bundles E and F and suppose that F is a sum of line bundles. Choose $p : Y \to X$ as in the theorem so that $p^*(E)$ is a sum of line bundles. Then $p^*(E)$ and $p^*(F)$ are each sums of line bundles, and clearly $\psi^k(p^*(E) \oplus p^*(F)) = \psi^k(p^*(E)) + \psi^k(p^*(F))$. Thus $p^*(\psi^k(E) \oplus \psi^k(F) - \psi^k(E) - \psi^k(F)) = 0$ in $K(Y)$. As p^* is injective we conclude that $\psi^k(E) \oplus \psi^k(F) = \psi^k(E) + \psi^k(F)$ in $K(X)$, verifying additivity in this case. For general E and F, choose $p : Y \to X$ as in the theorem so that $p^*(F)$ is a sum of line bundles. As we have just shown that $\psi^k(p^*(E) + p^*(F)) = \psi^k(p^*(E)) + \psi^k(p^*(F))$ the general claim follows from injectivity of p^*.

The additive map ψ^k therefore gives rise to an additive $K(X) \to K(X)$. Naturality is immediate; furthermore,

(A) Each ψ^k is a ring homomorphism,

(B) $\psi^k \psi^l = \psi^l \psi^k = \psi^{kl}$ for any k and l,

(C) If p is prime then $\psi^p(x) = x^p \bmod p$ (in the sense that for all x there is y with $\psi^p(x) - x^p = py$).

By the splitting principle it suffices to check these properties on sums of line bundles, where each is obvious.

It is clear that each ψ^k induces a homomorphism $\tilde{K}(X) \to \tilde{K}(X)$. To prove Adams' theorem we wil need to compute ψ^k on $\tilde{K}(S^{2n})$. In the case $n = 1$, consider the generator $H - 1$ of $\tilde{K}(S^2)$:

$$\begin{aligned} \psi^k(H-1) &= H^k - 1 && \text{definition of } \psi^k \text{ on line bundles} \\ &= ((H-1)+1)^k - 1 \\ &= (k(H-1)+1) - 1 && \text{binomial theorem and } (H-1)^j = 0,\ j \geq 2 \\ &= k(H-1) \end{aligned}$$

More generally we have that $\psi^k : \tilde{K}(S^{2n}) \to \tilde{K}(S^{2n})$ is multiplication by k^n. As it is clear that ψ^k respects the external product $\tilde{K}(S^2) \otimes \tilde{K}(S^{2n-2}) \to \tilde{K}(S^{2n})$, this follows by induction[11].

5.4.2 The proof of Adams' theorem

With these properties in mind, consider the setup of Section 5.3, where $f : S^{4n-1} \to S^{2n}$ is a map of Hopf invariant 1, a is the image of the generator of $\tilde{K}(S^{2n})$ in $\tilde{K}(X)$, and x is a chosen element in the preimage of the generator of $\tilde{K}(S^{4n})$.

By naturality and the properties of ψ^k on even-dimensional spheres we have $\psi^2(a) = 2^n a$ and $\psi^3(a) = 3^n a$. Similarly we have $r \in \mathbb{Z}$ and $s \in \mathbb{Z}$ such that $\psi^2(x) = 2^{2n}x + ra$ and $\psi^3(x) = 3^{2n}x + sa$. With these we can compute formulas for $\psi^2\psi^3(x)$ and $\psi^3\psi^2(x)$; by (B) we conclude

$$3^n(3^n - 1)r = 2^n(2^n - 1)s$$

Note that r is odd: $x^2 = ma$ for some odd m by definition of the Hopf invariant and $r - m$ is is even by (C). As 3^n is also odd we conclude that 2^n divides $3^n - 1$. It is then mere arithmetic ([11], p.66) to show that 2^n divides $3^n - 1$ only if n is 1, 2, or 4.

[11] For a the generator of $\tilde{K}(S^2)$ and b the generator of $\tilde{K}(S^{2n-2})$, $a * b$ is the generator for S^{2n} and $\psi^k(a*b) = \psi^k(a) * \psi^k(b) = k^n a * k^{n-1} b = k^n(a*b)$.

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