**Math 215a Homework #6, Due Wednesday 11/28 at 1:10 PM**

Recall that if $X$ and $Y$ are compact oriented connected $n$-manifolds with fundamental classes $[X] \in H_n(X)$ and $[Y] \in H_n(Y)$, then the *degree* of a map $f : X \to Y$ is defined to be the integer $\deg(f)$ such that $f_*[X] = \deg(f)[Y]$.

1. Hatcher section 3.2 exercises 4, 11.

2. Let $\Sigma_g$ denote the compact orientable surface of genus $g$. Show that one can identify the cohomology $H^*(\Sigma_g; \mathbb{Z})$ as

$$H^1(\Sigma_g; \mathbb{Z}) = \mathbb{Z}\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\},$$
$$H^2(\Sigma_g; \mathbb{Z}) = \mathbb{Z}\{\gamma\}$$

such that the cup product is given by

$$\alpha_i \smile \alpha_j = \beta_i \smile \beta_j = 0,$$
$$\alpha_i \smile \beta_j = -\beta_j \smile \alpha_i = \begin{cases} \gamma, & i = j, \\ 0, & i \neq j. \end{cases}$$

*Hint:* One can do this directly from the definition using a suitable delta-complex structure, or one can follow the suggestion in Hatcher, section 3.2, exercise 1. One can also use the fact that cup product is Poincare dual to intersection of submanifolds.

3. Show that if $g < h$, then any map $f : \Sigma_g \to \Sigma_h$ has degree zero. *Hint:* Show that there exists a nonzero $\alpha \in H^1(\Sigma_h; \mathbb{Z})$ with $f^*\alpha = 0$. Use the previous exercise (or general facts related to Poincare duality) to show that there exists $\beta \in H^1(\Sigma_h; \mathbb{Z})$ with $\alpha \smile \beta \neq 0$. Use naturality of cup product to conclude that $\deg(f) = 0$.

4. Let $A$ be an $n \times n$ matrix with integer entries. Then $A$ induces a map $\phi : \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$.

   (a) Show that under the obvious identification $H^1(T^n; \mathbb{Z}) \simeq \mathbb{Z}^n$, the pullback $\phi^* : H^1(T^n; \mathbb{Z}) \to H^1(T^n; \mathbb{Z})$ equals the transpose of $A$.

   (b) Show that the degree of $\phi$ is the determinant of $A$. *Hint:* Use naturality of cup product and part (a). Alternatively, you can do Hatcher, section 3.3, exercise 8 and then use that.