Math 215a Homework #3, Due Wednesday 10/17 at 1:10 PM

1. (a) Show that chain homotopy of chain maps is an equivalence relation.

(b) Show that composition of chain maps induces a well-defined map on equivalence classes.


3. Let
\[ 0 \to V_1 \to V_2 \to \cdots \to V_n \to 0 \]
be an exact sequence of finite dimensional vector spaces over a field \( F \). Show that \( \sum_{i=1}^{n} (-1)^i \dim(V_i) = 0 \).

4. Show that if
\[ 0 \to A \to B \to \mathbb{Z}^k \to 0 \]
is exact, then \( B \cong A \oplus \mathbb{Z}^k \).

For the next three problems, use the Mayer-Vietoris sequence.

5. Compute the homology of the space obtained by taking three copies of \( D^n \) and identifying their boundaries with each other.

6. Compute the homology of the nonorientable surface obtained by taking the connect sum\(^1\) of a genus \( g \) surface with a projective plane.

7. Let \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2 \mathbb{Z} \); this induces a self-homeomorphism \( \phi_A \) of \( \mathbb{R}^2/\mathbb{Z}^2 = S^1 \times S^1 \). Let \( Y \) be the 3-manifold\(^2\) obtained by taking two copies of \( S^1 \times D^2 \) and identifying the boundary tori via \( \phi_A \). Compute the homology of \( Y \), in terms of \( A \).

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\(^1\) The connect sum of two surfaces is defined by cutting an open disc out of each surface, and then gluing the boundary circles together.

\(^2\) For example, if \( A \) is the identity matrix, then \( Y \cong S^1 \times S^2 \); if \( A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) then \( Y \cong S^3 \). Can you see why?