Math 215a Homework #2 Solutions

Sorry, I didn't have time to draw pictures for this.

1. The first covering space is a "linear chain" of spheres connected by arcs. More precisely one can take $\widetilde{X}$ to be the subset of $\mathbb{R}^3$ consisting of spheres of radius 1 centered at the points $(3k, 0, 0)$ for $k \in \mathbb{Z}$, together with a line segment from $(3k+1, 0, 0)$ to $(3k+2, 0, 0)$ for each $k \in \mathbb{Z}$. The covering map sends each sphere to the standard sphere by translation, and each interval to a diameter of the standard sphere by translation along the x-axis and reflection.

   To make the second covering space, start with the universal cover of $S^1 \bigvee S^1$ as we described it in class. Now pull apart each vertex into a copy of S^2, with the two outgoing edges attached to the north pole and the two incoming edges attached to the south pole. The covering map sends each 'a' edge to one arc of the circle and each 'b' edge to the other arc of the circle.

2. The answer $\widetilde{X}$ looks like covering space number 8 on page 58 of Hatcher, except using an 8-gon instead of a 4-gon. To see that this works, let $\widetilde{x}_0$ be any vertex of $\widetilde{X}$, let $G := \pi_1(\widetilde{X}, \widetilde{x}_0)$, and let N denote the normal subgroup of F_2 generated by $(ab)^4$, a^2, and b^2; we need to show that $G = N$. The algorithm for computing π_1 of a graph shows that G is generated by $(ab)^4$ together with a^2, b^2, and some conjugates thereof. Thus $G \subset N$. Since $(ab)^4, a^2, b^2 \in G$, to show that $N \subset G$ it is enough to show that G is a normal subgroup of F_2. Equivalently, we need to show that the automorphism group of the covering $\widetilde{X}$ acts transitively on the vertices. One can rotate the picture by $\pi/2$; this acts on the vertices via two disjoint four-cycles. The picture also has a "reflectional symmetry" which exchanges the two four-cycles. (The automorphism group of the covering is the dihedral group of order 8.)

3. Let $X = \mathbb{RP}^2 \bigvee \mathbb{RP}^2$ with the natural base point x_0. The universal cover $\widetilde{X}$ can be identified with the subset of $\mathbb{R}^3$ consisting of radius 1 spheres centered at the points $(2k+1, 0, 0)$ for $k \in \mathbb{Z}$. Regarding $\mathbb{RP}^2$ as S^2 modulo the antipodal map, the covering map sends the sphere centered at $(4k+1, 0, 0)$ to the first $\mathbb{RP}^2$, and the sphere centered at $(4k+3, 0, 0)$ to the second $\mathbb{RP}^2$. We can take the base point $\widetilde{x}_0$ to be the origin.

To find the other coverings, note that in general, $\pi_1(X, x_0)$ acts on the left on the universal cover $\widetilde{X}$, where $[f] \in \pi_1(X, x_0)$ sends $[\gamma] \in \widetilde{X}$ to $[f \cdot \gamma]$. The connected covering $\widehat{X}$ corresponding to a subgroup $G \subset \pi_1(X, x_0)$ is obtained by modding out $\widetilde{X}$ by the restriction of this action to G.

In our particular case, the left $\pi_1(X, x_0)$ action on the universal cover $\widetilde{X}$ is given as follows. Let a and b denote the generators of π_1 of the first and second $\mathbb{RP}^2$'s, respectively. Then a reflects the whole picture through the point $(1, 0, 0)$, while b reflects through the point $(-1, 0, 0)$. In particular, ab translates everything 4 units to the right.

We now determine the nontrivial subgroups G of $\pi_1(X, x_0)$ and the corresponding covering spaces $\widehat{X}$. If G does not contain a reflection, then G is generated by $(ab)^k$ for some positive integer k, so $\widehat{X}$ is a cyclic chain of $2k$ spheres. Now suppose that G contains a reflection. If G has order 2, then $\widehat{X}$ is a half-infinite chain of spheres, with an $\mathbb{RP}^2$ attached to the end. If G also contains a translation $(ab)^k$ where the positive integer k is as small as possible, then $\widehat{X}$ is a chain of $k - 1$ spheres with an $\mathbb{RP}^2$ attached to each end.

4. This is one of those problems where it is probably easier to do it yourself than to read a solution.

5. By a previous homework problem, if X is a connected graph with v vertices and e edges, then $\pi_1(X, x_0) \simeq F(e - v + 1)$. Now let $(\widetilde{X}, \widetilde{x}_0)$ be the path connected based covering space of $\bigvee_n S^1$ corresponding to the subgroup G. Then $\widetilde{X}$ is a graph with k vertices and kn edges. So $G = \pi_1(\widetilde{X}, \widetilde{x}_0) \simeq F(kn - k + 1)$.

6. Since the universal covering space of T^n is $\mathbb{R}^n$, any path connected covering space $\widetilde{X}$ of T^n is isomorphic to $\mathbb{R}^n/G$ where G is a subgroup of $\mathbb{Z}^n$. By basic algebra, there are vectors $v_1, \ldots, v_m \in \mathbb{Z}^n$, which are linearly independent over $\mathbb{R}$, such that $G = \operatorname{span}(v_1, \ldots, v_m)$. We can perform a real change of basis sending v_i to the i^{th} standard basis vector e_i. Then $\widetilde{X}$ is homeomorphic to

$$\frac{\mathbb{R}^n}{\operatorname{span}(e_1, \ldots, e_m)} \simeq \frac{\mathbb{R}^m}{\operatorname{span}(e_1, \ldots, e_m)} \times \mathbb{R}^{n-m} = T^m \times \mathbb{R}^{n-m}.$$

7. We will first find the covering space $\widetilde{X}$ of $S^1 \bigvee S^1$ corresponding to the commutator subgroup $G \subset F(2)$, and then find a basis for $\pi_1(\widetilde{X})$, regarded as a subgroup of $F(2)$.

   Let $\widetilde{X}$ be the "grid"

$$\widetilde{X} := (\mathbb{Z} \times \mathbb{R}) \bigcup (\mathbb{R} \times \mathbb{Z}) \subset \mathbb{R}^2$$

   with $\widetilde{x}_0 := (0,0) \in \widetilde{X}$. Define a covering $p : \widetilde{X} \to S^1 \bigvee S^1$ by sending the horizontal edges of the grid to the 'a' circle and the vertical edges of the grid to the 'b' circle. I claim that $(\widetilde{X}, \widetilde{x}_0)$ corresponds to the commutator subgroup G of $F(2)$. To see this, given an element $f \in F(2) = \pi_1(S^1 \bigvee S^1, x_0)$, let m denote the total exponent of a and let n denote the total exponent of b in f. Then f lifts to a path in $\widetilde{X}$ from $(0,0)$ to (m,n). Now f is in $p_*\pi_1(\widetilde{X}, \widetilde{x}_0)$ iff f lifts to a loop in $\widetilde{X}$, i.e. $m = n = 0$. It is not hard to check that the latter condition holds iff f is in the commutator subgroup. (That is, the map sending $f \mapsto (m,n)$ defines an isomorphism from the abelianization of $F(2)$ to $\mathbb{Z}^2$.)

   Now we need to find a free basis for $\pi_1(\widetilde{X}, \widetilde{x}_0)$. Following the algorithm, we choose the maximal spanning tree of $\widetilde{X}$ consisting of the x-axis together with all of the vertical lines. Each remaining edge connects a pair of lattice points (m,n) and $(m-1,n)$ where $m \in \mathbb{Z}$ and $n \in \mathbb{Z} \setminus \{0\}$. We then get the basis

$$\{a^m b^n a^{-1} b^{-n} a^{-1-m} \mid m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\}\}. \tag{1}$$

   One can then do a change of basis to obtain the nicer basis

$$\{[a^m, b^n] \mid m, n \in \mathbb{Z} \setminus \{0\}\}. \tag{2}$$

   To see this, note that the $m = 1$ elements in (1) agree with the $m = 1$ elements in (2). The $m > 1$ elements in (1) are given in terms of the nice basis (2) by

$$a^m b^n a^{-1} b^{-n} a^{-1-m} = [a^m, b^n][a^{m-1}, b^n]^{-1}.$$

   The $m \leq 0$ elements in (1) are handled similarly.

8. No extra credit for me this week.