1. (a) Let \( g \) and \( h \) be two paths from \( x_0 \) to \( x_1 \). Then the composition

\[
\pi_1(X, x_0) \xrightarrow{\beta_g} \pi_1(X, x_1) \xrightarrow{\beta_h} \pi_1(X, x_0)
\]

sends

\[
[f] \mapsto [h \cdot \overline{g} \cdot f \cdot \overline{g}] = [h \cdot \overline{g}] [f] [h \cdot \overline{g}]^{-1}.
\]

So \( \beta_g = \beta_h \) for all \( g, h \) iff conjugation by all elements of the form \( [h \cdot \overline{g}] \) acts trivially on \( \pi_1(X, x_0) \). But any path \( f \) from \( x_0 \) to itself is homotopic to a loop of the form \( h \cdot \overline{g} \); to see this, let \( h \) be any path from \( x_0 \) to \( x_1 \) and let \( g = \overline{h} \cdot f \). So \( \beta_g = \beta_h \) for all \( g, h \) iff conjugation by any element of \( \pi_1(X, x_0) \) acts trivially, i.e. \( \pi_1(X, x_0) \) is abelian.

(b) Let \([S^1, X]\) denote the set of homotopy classes of free loops, i.e. continuous maps \( S^1 \to X \) without any basepoint conditions. There is an obvious map \( \pi_1(X, x_0) \to [S^1, X] \), and this descends to a map

\[
\Phi : \{ \text{conjugacy classes in } \pi_1(X, x_0) \} \to [S^1, X].
\]

The reason is that if \( f, g : [0, 1] \to X \) send \( 0, 1 \mapsto x_0 \), then the path \( g \cdot f \cdot \overline{g} \) is homotopic through free loops to the path \( \overline{g} \cdot g \cdot f \) (by rotating the domain 1/3 turn), which is homotopic to \( f \).

To see that \( \Phi \) is surjective, consider a map \( S^1 \to X \), regarded as a map \( f : [0, 1] \to X \) with \( f(0) = f(1) \). Since \( X \) is path connected we can choose a path \( g \) from \( x_0 \) to \( f(0) \). Then as above \( g \cdot f \cdot \overline{g} \) is homotopic through free loops to \( f \), so \( \Phi[g \cdot f \cdot \overline{g}] = [f] \).

To see that \( \Phi \) is injective, let \( f_0, f_1 : [0, 1] \to X \) send \( 0, 1 \mapsto x_0 \). Suppose these are homotopic through free loops, so that there is a map \( F : [0, 1] \times [0, 1] \to X \) with \( F(0, t) = f_0(t), F(1, t) = f_1(t), \) and \( F(s, 0) = F(s, 1) \). Define \( g(s) := F(s, 0) \). Then \( F \) induces a homotopy from \( f_0 \) to \( g \cdot f_1 \cdot \overline{g} \). [Draw picture.] Hence \([f_0]\) and \([f_1]\) are conjugate in \( \pi_1(X, x_0) \).

2. If \( \iota : A \to X \) denotes the inclusion, and if \( r : X \to A \) is a retraction, i.e. \( r \circ \iota = \text{id}_A \), then \( r_\ast \circ \iota_\ast \) is the identity on \( \pi_1(A, \ast) \).

In particular, \( \iota_\ast \) is injective. We can rule this out in cases (a), (b), (d), and (e) just by computing the groups. Namely, in case (a), \( \pi_1(X) = \{1\} \)
and \( \pi_1(A) = \mathbb{Z} \). In case (b), \( \pi_1(X) = \mathbb{Z} \) and \( \pi_1(A) = \mathbb{Z}^2 \). In case (d), \( \pi_1(X) = \{1\} \) and \( \pi_1(A) = \mathbb{Z} \times \mathbb{Z} \). In case (e), \( \pi_1(X) = \mathbb{Z} \) (because \( X \) deformation retracts onto the \( S^1 \) obtained by taking half of the boundary of \( D^2 \) and identifying the endpoints), while \( \pi_1(A) = \mathbb{Z} \times \mathbb{Z} \).

To rule out case (c), note that the solid torus \( X = S^1 \times D^2 \) has \( \pi_1 \simeq \mathbb{Z} \). The projection of the circle \( A \) from \( S^1 \times D^2 \) to \( S^1 \) is constant, so \( \iota_\ast: \pi_1(A) \to \pi_1(X) \) is zero. Since \( \pi_1(A) \simeq \mathbb{Z} \) is nontrivial, \( \iota_\ast \) is not injective.

To rule out case (f), observe that the Möbius band deformation retracts onto its core circle, and therefore has \( \pi_1(X) \simeq \mathbb{Z} \). Moreover, the boundary circle winds around twice, i.e. the map \( \iota_\ast: \mathbb{Z} \simeq \pi_1(A) \to \pi_1(X) \simeq \mathbb{Z} \) is multiplication by \( \pm 2 \). Then \( r_\ast \circ \iota_\ast \) is multiplication by some even integer, so it cannot equal the identity.

3. (This will be a bit sketchy because I am too lazy to draw the pictures.) Let \( X \) denote the complement of \( \alpha \) and \( \beta \) in \( D^2 \times I \), and let \( x_0 \) be the center point of \( D^2 \times I \). One can see in various ways that \( \pi_1(X, x_0) \simeq F_2 \), the free group on two generators \( a, b \). For example, one can see by isotoping \( \alpha \) and \( \beta \) to straight lines that \( X \) is homeomorphic to a twice-punctured disc cross an interval. In any event, the two generators \( a \) and \( b \) are represented by small loops around \( \alpha \) and \( \beta \). We then see that with suitable orientation conventions, \( [\gamma] = aba^{-1}b^{-1} \). This is not equal to the identity in the free group, and so \( \gamma \) is not nullhomotopic in \( X \).

4. Take the base point to be \((x_0, y_0)\) where \( x_0 \) and \( y_0 \) are both irrational. Given another pair of irrational numbers \((x_1, y_1)\), define a path \( R(x_0, y_0; x_1, y_1) \) by starting at \((x_0, y_0)\), and then traversing a rectangle by moving horizontally to \((x_1, y_0)\), vertically to \((x_1, y_1)\), horizontally to \((x_0, y_1)\), and vertically back to \((x_0, y_0)\). As \((x_1, y_1)\) ranges over all pairs of irrational numbers, we obtain uncountably many paths \( R(x_0, y_0, x_1, y_1) \). I claim that the homotopy classes of these paths are all distinct in \( \pi_1(\mathbb{R}^2 \setminus \mathbb{Q}^2, (x_0, y_0)) \). To see this, let \((x_1, y_1)\) and \((x'_1, y'_1)\) be distinct pairs of irrational numbers with corresponding rectangles \( R \) and \( R' \). Then there is a rational point \((p, q)\) \( \in \mathbb{Q}^2 \) which is enclosed by one rectangle but not the other. Hence the path \( \overline{R} \cdot R' \) has winding number \( \pm 1 \) around the point \((p, q)\). That is, the inclusion
$\mathbb{R}^2 \setminus \mathbb{Q}^2 \to \mathbb{R}^2 \setminus \{(p, q)\}$ induces a map on fundamental groups

$$\pi_1(\mathbb{R}^2 \setminus \mathbb{Q}^2, (x_0, y_0)) \to \pi_1(\mathbb{R}^2 \setminus \{(p, q)\}, (x_0, y_0)) \simeq \mathbb{Z}$$

sending $[R]^{-1}[R'] \mapsto \pm 1 \neq 0$. Since a homomorphism sends the identity to the identity, it follows that $[R] \neq [R']$ in $\pi_1(\mathbb{R}^2 \setminus \mathbb{Q}^2, (x_0, y_0))$.

5. One can show by induction, either on the number of vertices or on the number of edges, that $\chi(T) = 1$ if $T$ is a (finite) tree. This is pretty straightforward so I will omit the details. Now if $X$ is a connected (finite) graph, let $T$ be a maximal spanning tree, and let $E$ denote the set of edges in $X \setminus T$. We know from class or Hatcher that the Seifert-van Kampen theorem implies that $\pi_1(X)$ is free with one generator for each edge in $E$, so $\text{rk} \pi_1(X) = |E|$. On the other hand, by definition $\chi(X) = \chi(T) - |E|$. We have seen that $\chi(T) = 1$, and combining these three equations gives $\text{rk} \pi_1(X) = 1 - \chi(X)$.

6. Here are two solutions. The first seems easier to think of, while the second is cleverer and shorter.

First solution: If $f$ and $g$ are paths starting and ending at 1, let $f \cdot g$ denote their concatenation as usual, and define a new path $f \star g$ using the group multiplication by

$$(f \star g)(t) := f(t)g(t).$$

This gives a well-defined operation on $\pi_1(\mathbb{G}, 1)$, because if $F$ is a homotopy from $f_0$ to $f_1$ and if $G$ is a homotopy from $g_0$ to $g_1$, then a homotopy from $f_0 \star g_0$ to $f_1 \star g_1$ is given by

$$(F \star G)(s, t) := f(s, t)g(s, t).$$

Now we compute that

$$f \cdot g = (f \cdot 1) \star (1 \cdot g) \sim f \star g \sim (1 \cdot f) \star (g \cdot 1) = g \cdot f.$$  

Here the first and last equalities are immediate from the definitions, while the middle homotopies exist because $\star$ is well-defined on homotopy classes.

Second solution: If $f$ and $g$ are paths starting and ending at 1, define a map $H : [0, 1]^2 \to \mathbb{G}$ by

$$H(s, t) := f(s)g(t).$$
Since \( f(0) = f(1) = g(0) = g(1) = 1 \), it follows that \( H \) sends the boundary of the square (starting at \((0,0)\) and going counterclockwise) to the concatenation \( f \cdot g \cdot \overline{f} \cdot \overline{g} \). Since \( H \) is defined continuously on the whole square, it follows that \([f][g][f]^{-1}[g]^{-1}\) is the identity in \( \pi_1(G,1) \).

7. My solution to this problem is going to be a bit more demanding of the reader than those to the previous problems. In particular I will quote some theorems and ideas which you might not know. I did say “extra for experts” after all! Maybe one of you found a more elementary solution.

The problem is a little easier when \( \gamma \) is a smooth embedding. Then you can cut \( T^2 \) along \( \gamma \) to obtain a surface with boundary \( X \). One can then apply the classification of surfaces with boundary to deduce that if \( X \) is connected then \( X \) is homeomorphic to a cylinder, and otherwise \( X \) is homeomorphic to a disc union a punctured torus. The second case is impossible by our assumption that \( \gamma \) is not nullhomotopic. So we can identify \( X \) with the cylinder \( S^1 \times I \); but then when you glue the boundary circles back together to get a torus \( S^1 \times S^1 \), you see immediately that the boundary curve gets glued to a circle representing the class \((0,1) \in \pi_1(S^1 \times S^1)\), which is not divisible in \( \mathbb{Z}^2 \).

If all we know is that \( \gamma \) is continuous and injective, it is a bit trickier. Still, you can lift \( \gamma \) to the universal cover to get a map \( \widetilde{\gamma} : \mathbb{R} \to \mathbb{R}^2 \) such that \( \widetilde{\gamma}(t + 1) = \widetilde{\gamma} + (a,b) \). Now identify \( \mathbb{R}^2 \simeq \text{int}(D^2) \) by, in polar coordinates, sending \((r, \theta) \in \text{int}(D^2)\) to \((r/(1-r), \theta) \in \mathbb{R}^2\). Under this identification, since \((a,b) \neq (0,0)\), it follows that \( \widetilde{\gamma} \) compactifies to an arc \( \gamma' : [0,1] \to D^2 \). One then use the Jordan curve theorem to conclude that \( D^2 \setminus \gamma' \), and hence \( \mathbb{R}^2 \setminus \widetilde{\gamma} \), has two components. Now suppose that \((a,b) = k(a',b')\) where \(k, a', b'\) are integers and \(k > 1\). Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) denote the translation by \((a', b')\). Observe that \( \widetilde{\gamma} \) cannot intersect \( T \circ \widetilde{\gamma} \). (Proof: if \( \widetilde{\gamma}(t) = \widetilde{\gamma}(t') + (a', b') \), then \( t' - t \in \mathbb{Z} \) since \( \gamma \) is injective, but then the periodicity property of \( \widetilde{\gamma} \) implies that \((a', b')\) is an integer multiple of \((a,b)\), which is a contradiction.) Thus \( T \circ \widetilde{\gamma} \) is in one component of \( \mathbb{R}^2 \setminus \widetilde{\gamma} \). Moreover, \( T \) sends the component of \( \mathbb{R}^2 \setminus \widetilde{\gamma} \) that contains \( T \circ \widetilde{\gamma} \) to the component of \( \mathbb{R}^2 \setminus T \circ \widetilde{\gamma} \) that does not contain \( \widetilde{\gamma} \). (One can see this by noting that under our identification of \( \mathbb{R}^2 \) with \( \text{int}(D^2) \), \( T \) extends to a self-homeomorphism of \( D^2 \) which fixes the boundary.) In particular, if \( A \subset \mathbb{R}^2 \) denotes the union of \( \widetilde{\gamma} \).
with the component of $\mathbb{R}^2 \setminus \tilde{\gamma}$ that contains $T \circ \tilde{\gamma}$, then $T$ sends $A$ to $A \setminus \tilde{\gamma}$. Then $T^k \circ \tilde{\gamma}$ does not intersect $\tilde{\gamma}$, but this contradicts the fact that $T^k \circ \tilde{\gamma}$ is a reparametrization of $\tilde{\gamma}$.

By the way, if we replace $T^2$ by a genus $g$ oriented surface $\Sigma$, then one can still show that a non-nullhomotopic injective loop $\gamma$ must represent an indivisible element of $\pi_1(\Sigma)$. (This does not work for an unorientable surface. Do you see why?)