Math 215a Homework #1 Solutions

1. (a) Let g and h be two paths from x_0 to x_1. Then the composition

$$\pi_1(X,x_0) \xrightarrow{\beta_{\overline{g}}} \pi_1(X,x_1) \xrightarrow{\beta_h} \pi_1(X,x_0)$$

sends

$$[f] \longmapsto [h \cdot \overline{g} \cdot f \cdot g \cdot \overline{h}] = [h \cdot \overline{g}][f][h \cdot \overline{g}]^{-1}.$$

So $\beta_g = \beta_h$ for all g, h iff conjugation by all elements of the form $[h \cdot \overline{g}]$ acts trivially on $\pi_1(X,x_0)$. But any path f from x_0 to itself is homotopic to a loop of the form $h \cdot \overline{g}$; to see this, let h be any path from x_0 to x_1 and let $g = \overline{h} \cdot f$. So $\beta_g = \beta_h$ for all g, h iff conjugation by any element of $\pi_1(X,x_0)$ acts trivially, i.e. $\pi_1(X,x_0)$ is abelian.

(b) Let $[S^1, X]$ denote the set of homotopy classes of *free loops*, i.e. continuous maps $S^1 \to X$ without any basepoint conditions. There is an obvious map $\pi_1(X,x_0) \to [S^1,X]$, and this descends to a map

$$\Phi : \{\text{conjugacy classes in } \pi_1(X,x_0)\} \longrightarrow [S^1, X].$$

The reason is that if $f, g : [0,1] \to X$ send $0,1 \mapsto x_0$, then the path $g \cdot f \cdot \overline{g}$ is homotopic through free loops to the path $\overline{g} \cdot g \cdot f$ (by rotating the domain 1/3 turn), which is homotopic to f.

To see that Φ is surjective, consider a map $S^1 \to X$, regarded as a map $f : [0,1] \to X$ with $f(0) = f(1)$. Since X is path connected we can choose a path g from x_0 to $f(0)$. Then as above $g \cdot f \cdot \overline{g}$ is homotopic through free loops to f, so $\Phi[g \cdot f \cdot \overline{g}] = [f]$.

To see that Φ is injective, let $f_0, f_1 : [0,1] \to X$ send $0,1 \mapsto x_0$. Suppose these are homotopic through free loops, so that there is a map $F : [0,1] \times [0,1] \to X$ with $F(0,t) = f_0(t)$, $F(1,t) = f_1(t)$, and $F(s,0) = F(s,1)$. Define $g(s) := F(s,0)$. Then F induces a homotopy from f_0 to $g \cdot f_1 \cdot \overline{g}$. [Draw picture.] Hence $[f_0]$ and $[f_1]$ are conjugate in $\pi_1(X,x_0)$.

2. If $\imath : A \to X$ denotes the inclusion, and if $r : X \to A$ is a retraction, i.e. $r \circ \imath = \mathrm{id}_A$, then $r_* \circ \imath_*$ is the identity on $\pi_1(A,*)$.

In particular, $\imath_*$ is injective. We can rule this out in cases (a), (b), (d), and (e) just by computing the groups. Namely, in case (a), $\pi_1(X) = \{1\}$

and $\pi_1(A) = \mathbb{Z}$. In case (b), $\pi_1(X) = \mathbb{Z}$ and $\pi_1(A) = \mathbb{Z}^2$. In case (d), $\pi_1(X) = \{1\}$ and $\pi_1(A) = \mathbb{Z} * \mathbb{Z}$. In case (e), $\pi_1(X) = \mathbb{Z}$ (because X deformation retracts onto the S^1 obtained by taking half of the boundary of D^2 and identifying the endpoints), while $\pi_1(A) = \mathbb{Z} * \mathbb{Z}$.

To rule out case (c), note that the solid torus $X = S^1 \times D^2$ has $\pi_1 \simeq Z$. The projection of the circle A from $S^1 \times D^2$ to S^1 is constant, so $\imath_* : \pi_1(A, *) \to \pi_1(X, *)$ is zero. Since $\pi_1(A) \simeq \mathbb{Z}$ is nontrivial, $\imath_*$ is not injective.

To rule out case (f), observe that the Möbius band deformation retracts onto its core circle, and therefore has $\pi_1(X) \simeq \mathbb{Z}$. Moreover, the boundary circle winds around twice, i.e. the map $\imath_* : \mathbb{Z} \simeq \pi_1(A) \to \pi_1(X) \simeq \mathbb{Z}$ is multiplication by ± 2. Then $r_* \circ \imath_*$ is multiplication by some even integer, so it cannot equal the identity.

3. (This will be a bit sketchy because I am too lazy to draw the pictures.) Let X denote the complement of α and β in $D^2 \times I$, and let x_0 be the center point of $D^2 \times I$. One can see in various ways that $\pi_1(X, x_0) \simeq F_2$, the free group on two generators a, b. For example, one can see by isotoping α and β to straight lines that X is homeomorphic to a twice-punctured disc cross an interval. In any event, the two generators a and b are represented by small loops around α and β. We then see that with suitable orientation conventions, $[\gamma] = aba^{-1}b^{-1}$. This is not equal to the identity in the free group, and so γ is not nullhomotopic in X.

4. Take the base point to be (x_0, y_0) where x_0 and y_0 are both irrational. Given another pair of irrational numbers (x_1, y_1), define a path $R(x_0, y_0; x_1, y_1)$ by starting at (x_0, y_0), and then traversing a rectangle by moving horizontally to (x_1, y_0), vertically to (x_1, y_1), horizontally to (x_0, y_1), and vertically back to (x_0, y_0). As (x_1, y_1) ranges over all pairs of irrational numbers, we obtain uncountably many paths $R(x_0, y_0, x_1, y_1)$. I claim that the homotopy classes of these paths are all distinct in $\pi_1(\mathbb{R}^2 \setminus \mathbb{Q}^2, (x_0, y_0))$. To see this, let (x_1, y_1) and (x_1', y_1') be distinct pairs of irrational numbers with corresponding rectangles R and R'. Then there is a rational point $(p, q) \in \mathbb{Q}^2$ which is enclosed by one rectangle but not the other. Hence the path $\overline{R} \cdot R'$ has winding number ± 1 around the point (p, q). That is, the inclusion

$\mathbb{R}^2 \setminus \mathbb{Q}^2 \to \mathbb{R}^2 \setminus \{(p,q)\}$ induces a map on fundamental groups

$$\pi_1(\mathbb{R}^2 \setminus \mathbb{Q}^2, (x_0, y_0)) \longrightarrow \pi_1(\mathbb{R}^2 \setminus \{(p,q)\}, (x_0, y_0)) \simeq \mathbb{Z}$$

sending $[R]^{-1}[R'] \mapsto \pm 1 \neq 0$. Since a homomorphism sends the identity to the identity, it follows that $[R] \neq [R']$ in $\pi_1(\mathbb{R}^2 \setminus \mathbb{Q}^2, (x_0, y_0))$.

5. One can show by induction, either on the number of vertices or on the number of edges, that $\chi(T) = 1$ if T is a (finite) tree. This is pretty straightforward so I will omit the details. Now if X is a connected (finite) graph, let T be a maximal spanning tree, and let E denote the set of edges in $X \setminus T$. We know from class or Hatcher that the Seifert-van Kampen theorem implies that $\pi_1(X)$ is free with one generator for each edge in E, so $\operatorname{rk} \pi_1(X) = |E|$. On the other hand, by definition $\chi(X) = \chi(T) - |E|$. We have seen that $\chi(T) = 1$, and combining these three equations gives $\operatorname{rk} \pi_1(X) = 1 - \chi(X)$.

6. Here are two solutions. The first seems easier to think of, while the second is cleverer and shorter.

   First solution: If f and g are paths starting and ending at 1, let $f \cdot g$ denote their concatenation as usual, and define a new path $f \star g$ using the group multiplication by

$$(f \star g)(t) := f(t)g(t).$$

   This gives a well-defined operation on $\pi_1(G, 1)$, because if F is a homotopy from f_0 to f_1 and if G is a homotopy from g_0 to g_1, then a homotopy from $f_0 \star g_0$ to $f_1 \star g_1$ is given by

$$(F \star G)(s,t) := f(s,t)g(s,t).$$

   Now we compute that

$$f \cdot g = (f \cdot 1) \star (1 \cdot g) \sim f \star g \sim (1 \cdot f) \star (g \cdot 1) = g \cdot f.$$

   Here the first and last equalities are immediate from the definitions, while the middle homotopies exist because $\star$ is well-defined on homotopy classes.

   Second solution: If f and g are paths starting and ending at 1, define a map $H : [0,1]^2 \to G$ by

$$H(s,t) := f(s)g(t).$$

Since $f(0) = f(1) = g(0) = g(1) = 1$, it follows that H sends the boundary of the square (starting at $(0,0)$ and going counterclockwise) to the concatenation $f \cdot g \cdot \overline{f} \cdot \overline{g}$. Since H is defined continuously on the whole square, it follows that $[f][g][f]^{-1}[g]^{-1}$ is the identify in $\pi_1(G,1)$.

7. My solution to this problem is going to be a bit more demanding of the reader than those to the previous problems. In particular I will quote some theorems and ideas which you might not know. I did say "extra for experts" after all! Maybe one of you found a more elementary solution.

   The problem is a little easier when γ is a smooth embedding. Then you can cut T^2 along γ to obtain a surface with boundary X. One can then apply the classification of surfaces with boundary to deduce that if X is connected then X is homeomorphic to a cylinder, and otherwise X is homeomorphic to a disc union a punctured torus. The second case is impossible by our assumption that γ is not nullhomotopic. So we can identify X with the cylinder $S^1 \times I$; but then when you glue the boundary circles back together to get a torus $S^1 \times S^1$, you see immediately that the boundary curve gets glued to a circle representing the class $(0,1) \in \pi_1(S^1 \times S^1)$, which is not divisible in $\mathbb{Z}^2$.

   If all we know is that γ is continuous and injective, it is a bit trickier. Still, you can lift γ to the universal cover to get a map $\widetilde{\gamma} : \mathbb{R} \to \mathbb{R}^2$ such that $\widetilde{\gamma}(t+1) = \widetilde{\gamma} + (a,b)$. Now identify $\mathbb{R}^2 \simeq \operatorname{int}(D^2)$ by, in polar coordinates, sending $(r,\theta) \in \operatorname{int}(D^2)$ to $(r/(1-r),\theta) \in \mathbb{R}^2$. Under this identification, since $(a,b) \neq (0,0)$, it follows that $\widetilde{\gamma}$ compactifies to an arc $\gamma' : [0,1] \to D^2$. One then use the Jordan curve theorem to conclude that $D^2 \setminus \gamma'$, and hence $\mathbb{R}^2 \setminus \widetilde{\gamma}$, has two components. Now suppose that $(a,b) = k(a',b')$ where k, a', b' are integers and $k > 1$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ denote the translation by (a',b'). Observe that $\widetilde{\gamma}$ cannot intersect $T \circ \widetilde{\gamma}$. (Proof: if $\widetilde{\gamma}(t) = \widetilde{\gamma}(t') + (a',b')$, then $t' - t \in \mathbb{Z}$ since γ is injective, but then the periodicity property of $\widetilde{\gamma}$ implies that (a',b') is an integer multiple of (a,b), which is a contradiction.) Thus $T \circ \widetilde{\gamma}$ is in one component of $\mathbb{R}^2 \setminus \widetilde{\gamma}$. Moreover, T sends the component of $\mathbb{R}^2 \setminus \widetilde{\gamma}$ that contains $T \circ \widetilde{\gamma}$ to the component of $\mathbb{R}^2 \setminus T \circ \widetilde{\gamma}$ that does not contain $\widetilde{\gamma}$. (One can see this by noting that under our identification of $\mathbb{R}^2$ with $\operatorname{int}(D^2)$, T extends to a self-homeomorphism of D^2 which fixes the boundary.) In particular, if $A \subset \mathbb{R}^2$ denotes the union of $\widetilde{\gamma}$

with the component of $\mathbb{R}^2 \setminus \widetilde{\gamma}$ that contains $T \circ \widetilde{\gamma}$, then T sends A to $A \setminus \widetilde{\gamma}$. Then $T^k \circ \widetilde{\gamma}$ does not intersect $\widetilde{\gamma}$, but this contradicts the fact that $T^k \circ \widetilde{\gamma}$ is a reparametrization of $\widetilde{\gamma}$.

By the way, if we replace T^2 by a genus g oriented surface Σ, then one can still show that a non-nullhomotopic injective loop γ must represent an indivisible element of $\pi_1(\Sigma)$. (This does not work for an unorientable surface. Do you see why?)