8/27 Math 215A

Office hour Wed 9-12, 923 Evans

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Classification Questions

Given two topological spaces X and Y, how can you decide if they are homeomorphic?

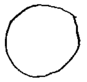
Def : An n-dimensional manifold is a (second countable, Hausdorff) top space X which is locally homeomorphic to $\mathbb{R}^n$.

i.e. $\forall p \in X \ \exists$ nbhd $p \in U \subset X$ s.t. $U \underset{\text{homeo}}{\simeq} \mathbb{R}^n$

Problem : Classify compact, connected, n-manifolds up to homeomorphism.

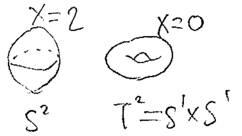
$n=0$ • point

$n=1$

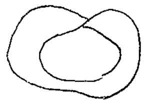


circle

$n=2$ orientable v.s. non orientable.



Möbius band



$\chi = 2-2g$

Euler characteristic $\chi = V - E + F$ ← # of faces.

take a triangulation ↑ # of edges

\# of vertices

Non-orientable surfaces : (genus g surface)$/\mathbb{Z}_2$

$\chi = 1-g$

$g=0$: $\mathbb{R}P^2$

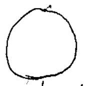
$g=1$: Klein bottle (cannot be embedded in $\mathbb{R}^3$)

$n=3$ Poincaré Conjecture

Any simply connected (compact, connected) 3-manifold is homeomorphic to $S^3=\{x\in\mathbb{R}^4 \mid \|x\|=1\}$

Def X is simply connected if $\forall f : S^1 \xrightarrow[\text{cont.}]{} X$, then $\exists\, u : D^2 \longrightarrow X$ s.t. $u|_{\partial D^2 = S^1} = f$

Def A (smooth) knot is a subset $K \subset \mathbb{R}^3$ such that there is a homeomorphism $f : S^1 \longrightarrow K$ (which is a smooth embedding) f is smooth injective and its derivative is never zero.


unknot


right trefoil


left trefoil


figure 8

Two knots $K_0, K_1 \subset \mathbb{R}^3$ are isotopic if there is a smooth map $F : [0,1] \times S^1 \longrightarrow \mathbb{R}^3$ s.t.

- $F(t, \bullet) : S^1 \longrightarrow \mathbb{R}^3$ is a smooth embedding $\forall t$
- $F(0, \bullet) : S^1 \xrightarrow{\cong} K_0$
- $F(1, \cdot) : S^1 \xrightarrow{\cong} K_1$

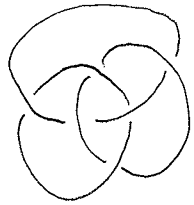
Problem : Classify knots up to isotopy.

Links : Like knots, but use

$\bigsqcup_n S^1$ instead of S^1


unlink


Hopf link


Borromean ring.

Challenge : Find n-component analogue.

Existence question.

Brower Fixed Point theorem

For any (continuous) map $f : \underset{\substack{\| \\ \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}}}{D^n} \longrightarrow D^n$

there is a fixed point, i.e. $\exists\, x \in D^n$ s.t. $f(x) = x$

(False if you replace D^n by $\text{int}(D^n)$). (exercise).

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Ham Sandwich Thm

If $A_1, \dots, A_n$ are bounded measurable sets in $\mathbb{R}^n$ then $\exists$ hyperplane $H \subset \mathbb{R}^n$ which cuts each A_i into two halves of equal volume.

Borsuk-Ulam Thm :

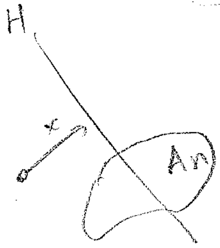
$$\begin{array}{cccc} \nexists f : & S^n & \longrightarrow & S^{n-1} \\ & \cap & & \cap \\ & \mathbb{R}^{n+1} & & \mathbb{R}^n \end{array} \quad \text{s.t.} \quad f(-x) = -f(x) \ \ \forall x$$

Claim : This implies Ham Sandwich thm in $\mathbb{R}^{n+1}$!

Proof of HST (assuming Borsuk)

Suppose no such H exists.

Define $f : S^{n-1} \longrightarrow S^{n-2}$ as follows.



Given $x \in S^{n-1}$, take a hyperplane $H \subset \mathbb{R}^n$ orthogonal to x, s.t. H cuts A_n into 2 halves of equal volume.

For $i=1, \dots, n-1$, let $z_i = \dfrac{\operatorname{vol}(A_i \cap H^+)}{\operatorname{vol}(A_i)} - \dfrac{1}{2} \neq 0$

define $f(x) = \dfrac{(z_1, \dots, z_{n-1})}{\sqrt{\sum_{i=1}^{n-1} z_i^2}} \in S^{n-2}$

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1) Fundamental Group

2) Homology Group

3) Cohomology ring

Top space $\rightarrow$ $\boxed{\text{Machine}}$ $\longrightarrow$ Alg. object

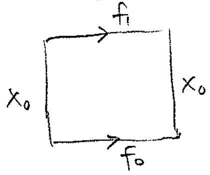
Def let X be a top. space and $x_0 \in X$ a "base point"

Define fundamental group $\pi_1(X, x_0)$ as follows,

As a set, $\pi_1(X, x_0) = \{ f : [0,1] \longrightarrow X \mid f(0) = f(1) = x_0 \} / \sim$

where $f_0 \sim f_1$ if f_0 and f_1 are "homotopic rel endpoints" i.e.

$\exists$ "homotopy" $F : [0,1] \times [0,1] \longrightarrow X$ such that

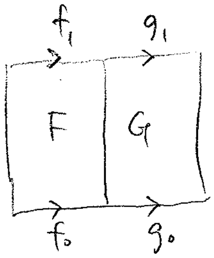


$$F(0,t) = f_0(t)$$
$$F(1,t) = f_1(t)$$
$$F(s,0) = F(s,1) = x_0$$

Group operation is <u>concatenation</u>

$$(f \cdot g)(t) = \begin{cases} f(2t) & 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

This is well-defined, i.e. $f_0 \underset{F}{\sim} f_1$ and $g_0 \underset{G}{\sim} g_1$, then $f_0 \cdot g_0 \underset{H}{\sim} f_1 \cdot g_1$

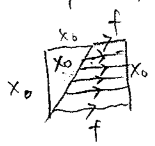


$$H(s,t) = \begin{cases} F(s, 2t) & \text{if} \quad 0 \leq t \leq \frac{1}{2} \\ G(s, 2t-1) & \text{if} \quad 1/2 \leq t \leq 1 \end{cases}$$

Prop This is a group.

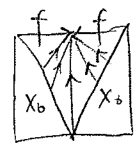
Pf 1) identity $= x_0$ (const map)

$x_0 \cdot f \sim f \sim f \cdot x_0$

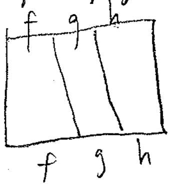


2) inverse of $f : [0,1] \rightarrow X$ is reverse path $\bar{f}(t) := f(1-t)$

$f \cdot \bar{f} \sim x_0$



(3) Associativity $(f \cdot g) \cdot h = f \cdot (g \cdot h)$



Example 1 $\pi_1(\mathbb{R}^n, 0) = \{1\}$

Proof Any path $f:[0,1] \to \mathbb{R}^n$ with $f(0)=f(1)=0$ is homotopic rel endpts to 0 via homotopy $F(s,t) = s \cdot f(t)$

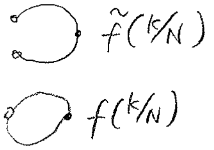
Example 2 $S^1 = \mathbb{R}/\mathbb{Z}$ $\pi_1(S^1, 0) = \mathbb{Z}$

Lemma Let $f:[0,1] \to S^1$ with $f(0)=0$. Then $\exists !$ lift $\tilde{f}:[0,1] \to \mathbb{R}$ s.t. $\tilde{f}(0)=0$ and $\pi \circ \tilde{f} = f$

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\pi"] \\ {[0,1]} \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & S^1 \end{tikzcd}$$

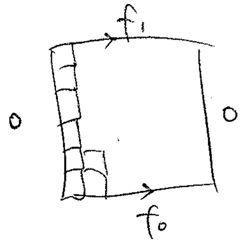
Pf Existence: Since $[0,1]$ is compact, f is uniformly continuous.
so $\exists$ pos. int. N s.t. $\forall x,y \in [\frac{k}{N}, \frac{k+1}{N}]$ we have $d(f(x), f(y)) < \frac{1}{2}$

Now define $\tilde{f}$ on $[0, \frac{k}{N}]$ by induction on K.



Uniqueness: Let $\tilde{f}, \tilde{g} : [0,1] \to \mathbb{R}$ be two such lifts
Define $h:[0,1] \to \mathbb{R}$ by $h(t) = \tilde{f}(t) - \tilde{g}(t)$
Since $\pi \circ \tilde{f} = \pi \circ \tilde{g}$, $h:[0,1] \to \mathbb{Z}$. $(\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z} \approx S^1)$
Since h is continuous, h is constant.
$\because h(0)=0$, $h(t)=0$ $\forall t$.

Define the degree $\deg : \pi_1(S^1, 0) \to \mathbb{Z}$ by $\deg(f) = \tilde{f}(1)$.
This is well-defined because if $F:[0,1]\times[0,1] \to S^1$ is a homotopy rel endpts from f_0 to f_1, then
similarly to above lemma,
$\exists \tilde{F} : [0,1]\times[0,1] \to \mathbb{R}$ s.t. $\pi \circ \tilde{F} = F$ and $\tilde{F}(s,0)=0$
Since $F(s,1)=0$, $\tilde{F}(s,1)$ is a const.
so $\tilde{f}_0(1) = \tilde{F}(0,1) = \tilde{F}(1,1) = \tilde{f}_1(1)$.



- deg is a homomorphism because
$$\widetilde{f \cdot g} = \tilde{f} \cdot (\tilde{g}(\cdot) + \tilde{f}(1))$$

- deg is surj. if $k \in \mathbb{Z}$, then $f(t) = kt$ has $\deg(f) = k$

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- deg is inj. because if $\deg(f)=0$, then $\tilde{f}(1)=\tilde{f}(0)=0$,

$\Rightarrow$ $\tilde{f}$ is a loop in $\mathbb{R}$.

$\Rightarrow$ Since $\pi_1(\mathbb{R},0)=\{1\}$, $\exists$ homotopy $F:[0,1]\times[0,1]\to\mathbb{R}$ from $\tilde{f}$ to 0,

Then $\pi\circ F$ is a homotopy from f to 0. ▨

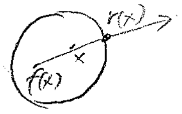
Remark: deg is defined similarly for any map $S^1 \xrightarrow{f} S^1$.

If $\exists\, u: D^2\to S^1$ s.t. $f=u|_{\partial D^2=S^1}$ then $\deg(f)=0$

Pf of 2-d Brouwer fixed pt thm

suppose $f: D^2\to D^2$ has no fixed pt.

Define $r: D^2\to S^1=\partial D^2$ as follows, $\longrightarrow$



If $x\in S^1=\partial D^2$, then $r(x)=x$

So $r|_{S^1}=\mathrm{id}_{S^1}$ so $\deg(r|_{S^1})=1$

since r extends over D^2, $\deg(r|_{S^1})=0$ $\rightarrow\leftarrow$

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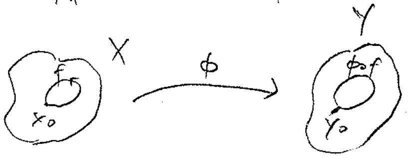
General properties of the fundamental group.

Induced map Suppose $\phi : (X, x_0) \longrightarrow (Y, y_0)$

i.e. $\phi : X \to Y$, $\phi(x_0) = y_0$. Define a homomorphism $\phi_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ as follows:

suppose $f : [0,1] \to X$ w/ $f(0) = f(1) = x_0$

define $\phi_*[f] := [\phi \circ f]$



This is well-defined because if $f_0 \underset{F}{\sim} f_1$, then $\phi \circ f_0 \underset{\phi\circ F}{\sim} \phi \circ f_1$

ϕ_* is a homomorphism because $\phi \circ (f \cdot g) = (\phi \circ f) \cdot (\phi \circ g)$ by def'n concatenation.

Example: If $\phi : (S^1, \circ) \longrightarrow (S^1, \circ)$, then $\phi_* : \mathbb{Z} \longrightarrow \mathbb{Z}$ is multiplication by $\deg(\phi)$.

Properties
- $(id_X)_* = id_{\pi_1(X, x_0)}$
- If $\psi : (Y, y_0) \longrightarrow (Z, z_0)$, then $(\psi \circ \phi)_* = \psi_* \circ \phi_*$

i.e. π_1 is a functor from pointed topological space to groups.

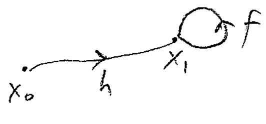
Cor If $\phi : (X, x_0) \xrightarrow{\simeq} (Y, y_0)$, then $\pi_1(X, x_0) \simeq \pi_1(Y, y_0)$

Prop If X is path connected and $x_0, x_1 \in X$, then $\exists$ (noncannonical) isomorphism $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$

Cor If X, Y are path connected and $\pi_1(X, x_0) \not\simeq \pi_1(Y, y_0)$, then $X \not\simeq Y$.

Proof of prop: let $h : [0,1] \longrightarrow X$ with $h(0) = x_0$, $h(1) = x_1$

Define $\beta_h : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$ by $\beta_h[f] := [h \cdot f \cdot \bar{h}]$



This is clearly well-defined.

β_h is a homomorphism because

$$\beta_h[f \cdot g] = [h \cdot f \cdot g \cdot \bar{h}] = [h \cdot f \cdot \bar{h} \cdot h \cdot g \cdot \bar{h}] = \beta_h[f]\beta_h[g]$$

properties

1) If h and h' are homotopic rel. endpts, then $\beta_h = \beta_{h'}$

2) $\beta_{const} = id_{\pi_1(X, x_0)}$

3) If
$$\begin{tikzcd} x_0 \arrow[r, dash, bend left, "h"] & x_1 \arrow[r, dash, bend left, "h'"] & x_2 \end{tikzcd}$$
, then

$$\beta_{h \cdot h'} = \beta_h \circ \beta_{h'} : \pi_1(X, x_2) \longrightarrow \pi_1(X, x_0)$$

(1), (2), (3) $\Rightarrow$ β_h is an iso.

$$\beta_h \beta_{\bar{h}} = id \qquad \beta_{\bar{h}} \beta_h = id$$

Exercise: The isomorphism $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$ is canonical (i.e. independent of h) iff $\pi_1(X, x_0)$ is abelian.

HW1

Exercise: If X is path connected, then

$$\{\text{map } S^1 \to X\} \big/ \text{free homotopy} = \pi_1(X, x_0) \big/ \text{conjugation}$$

(no base pt condition)

Def: Maps $f_0, f_1 : X \to Y$ are homotopic (not just paths)
if $\exists F : [0,1] \times X \to Y$ s.t.
$F(0, x) = f_0(x) \quad F(1, x) = f_1(x)$
i.e. $f_0 \sim f_1$

Def: $f : X \to Y$ is a homotopy equivalence if $\exists g : Y \to X$ s.t. $f \circ g \sim id_Y$ and $g \circ f \sim id_X$

Write $X \simeq Y$.

e.g. $\mathbb{R}^n \simeq pt$ (if $X \simeq pt$, say X is contractible)

e.g. $X \simeq$ Möbius band $\simeq Y = S^1$

f: contract onto core circle

g: include core circle

$f \circ g = id_{S^1}$, $g \circ f \sim id_X$

Prop: If $\phi : X \to Y$ is a homotopy equivalence then $\phi_* : \pi_1(X, x_0) \longrightarrow (Y, y_0)$ is an isom.

Lemma: let $\Phi : [0,1] \times X \to Y$ be a homotopy from ϕ_0 to ϕ_1

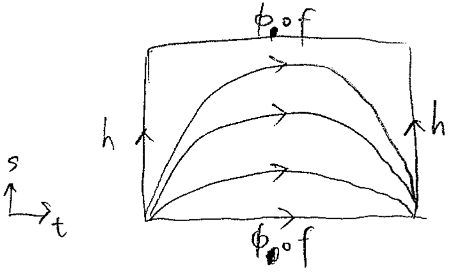
Define $h : [0,1] \to Y$ by $h(t) = \Phi(t, x_0)$

Then the following diagram commutes

$$\begin{tikzcd} & \pi_1(Y, \phi_0(x_0)) \\ \pi_1(X, x_0) \arrow[ur, "(\phi_0)_*"] \arrow[dr, "(\phi_1)_*"'] & \\ & \pi_1(Y, \phi_1(x_0)) \arrow[uu, "\cong", "\beta_h"'] \end{tikzcd}$$

Proof let $f : [0,1] \longrightarrow X$ $f(0) = f(1) = x_0$

Consider the map $(s, t) \longmapsto \Phi(s, f(t))$



h : path from $\phi_0(x_0)$ to $\phi_1(x_0)$

□

In particular, if $(\phi_0)_*$ is an iso. then so is $(\phi_1)_*$. The prop follows:

$\psi : Y \to X \quad \psi \circ \phi \sim id_X \quad \phi \circ \psi \sim id_Y$

$\Rightarrow$ $\psi_* \circ \phi_*$ is an isom, and $\phi_* \circ \psi_*$ is an isom.

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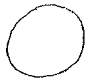
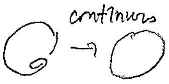
Defn : A knot will be a smoothly embeded closed curve in $\mathbb{R}^3$ (S^3)
($f : S^1 \to \mathbb{R}^3$, $f \in C^\infty$) ($Df \neq 0$)





2 knots equivalent if ∃ a smooth isotopy of $\mathbb{R}^3$ taking one to the other

A link is a disjoint union of knots (orientation preserving diffeomorphism taking knot 1 → knot 2)

Invariants of knots

1) The complement of the knot (up to homeomorphism)
The knot group of K is called $\pi_1(\mathbb{R}^3 \setminus K)$ $\quad \pi_1(\mathbb{R}^3 - \overset{*}{\bigcirc}) = \mathbb{Z}$

eg $x(t) = \cos(n_x t + \varphi_x)$
$y(t) =$ —"— $x \to y$
$z(t) =$ —"— $y \to z$

Theorem : Any knot (link) admits a projection onto $\mathbb{R}^2$ with at most double pts.

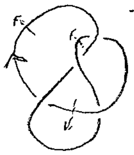
Wirtinger presentation
Dehn presentation

Base point (0, 0, large #)

overpass on a knot diagram

call an overpass α and use same letter for the loop in π_1 going round it from base point in a certain way...

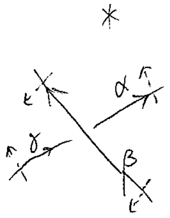
To choose the "way" systematically, first orient the knot.



Draw arrows under each overpass in the right hand rule. and π_1 generators go along arrow, tail to head.

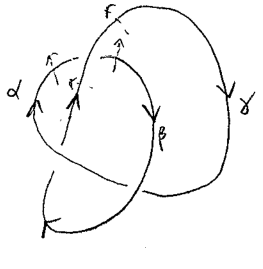
Claim these elements of π_1 generate it.

$\pi_1 = \langle \alpha, \beta, \gamma, \ldots \mid$ relations $\gamma\beta = \beta\alpha$ at each crossing$\rangle$



$\gamma \cdot \beta \simeq \beta\alpha$

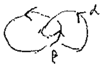
π_1 (trefoil)



$\langle \alpha, \beta, \gamma \mid \gamma\alpha = \beta\gamma , \ \gamma\alpha = \alpha\beta ,$ consequence $\rangle$

$\alpha = \gamma^{-1}\beta\gamma$ ↗ ↘ $\beta\gamma = \gamma^{-1}\beta\gamma\beta$

$\gamma\beta\gamma = \beta\gamma\beta$



$\langle \alpha, \beta \mid \alpha\beta = \beta\alpha \rangle$

$\pi_1(\text{trefold}) = \langle \gamma, \beta \mid \gamma\beta\gamma = \beta\gamma\beta \rangle$

S_3 $\quad$ (12) $\quad$ (23)
$\quad$ a $\quad$ b

$aba = bab$

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Seifert-van Kampan Thm

Suppose $X = A \cup B$ where

- A, B are open sets
- $A \cap B$ is path connected.

Let $x_0 \in A \cap B$. Then

$$\pi_1(X, x_0) = \pi_1(A, x_0) \underset{\pi_1(A \cap B, x_0)}{*} \pi_1(B, x_0)$$

If G and H are groups, define the free product $G * H$ as follows.

- In terms of generators and relations if $G = \langle g_i \mid r_j \rangle$, $H = \langle h_k \mid s_\ell \rangle$ then $G * H = \langle g_i, h_k \mid r_j, s_\ell \rangle$

$$\begin{tikzcd} G \arrow[r] \arrow[rr, bend left] & G*H \arrow[r, dashed, "\exists !"] & K \\ H \arrow[u, phantom, ""] \arrow[ur] \arrow[urr, bend right] & & \end{tikzcd}$$

Example

- $\mathbb{Z} * \mathbb{Z} = F_2$
- $F_m * F_n = F_{m+n}$
- $\mathbb{Z}/2 * \mathbb{Z}/2 = \langle a, b \mid a^2, b^2 \rangle$
  infinite dihedral group.

a = reflection around 0
$n \longmapsto -n$

b = reflection around $1/2$
$n \longmapsto 1-n$

$ab: n \longmapsto n-1$
$ba: n \longmapsto n+1$

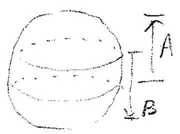
suppose
$$\begin{tikzcd} K \arrow[r, "i_K^G"] \arrow[dr, "i_K^H"'] & G \\ & H \end{tikzcd}$$

Define $G \underset{K}{*} H = \dfrac{G * H}{\text{normal subgroup gen by } \{ i_K^G(x)\, i_K^H(x)^{-1} \mid x \in K \}}$

$G *_K H$ is $G * H$ with one extra relation for each $x \in K$ saying that $i_K^G(x) = i_K^H(x)$.

Example: $X = S^n$ $n > 1$

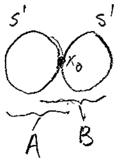
$A = \{$ above tropic of capricorn $\}$
$B = \{$ below tropic of cancer $\}$



$\pi_1(A, x_0) = \{1\}$
$\pi_1(B, x_0) = \{1\}$
$\Rightarrow \pi_1(S^n, x_0) = \{1\}$

$A \cap B \simeq S^{n-1}$ path connected $n > 1$

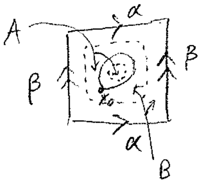
Example $\pi_1(S^1 \vee S^1, x_0) = \mathbb{Z} * \mathbb{Z} = F_2$



$\pi_1(A, x_0) = \mathbb{Z}$
$\pi_1(B, x_0) = \mathbb{Z}$
$\pi_1(A \cap B, x_0) = \{1\}$

More generally, if (X, x_0), (Y, y_0) are "reasonable" pointed spaces, then
$$\pi_1(X \vee Y, *) = \pi_1(X, x_0) * \pi_1(Y, y_0)$$

Example $\pi_1(T^2)$ fact: $\pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0)$



A = large ball $\simeq$ pt
B = complement of smaller ball $\simeq S^1 \vee S^1$

$A \cap B \simeq S^1$

$$\pi_1(T^2, x_0) = \{1\} \underset{\mathbb{Z}}{*} F_2 = \langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta^{-1} \rangle = \mathbb{Z} \times \mathbb{Z}$$

$A \subset X$ is a deformation retract if $\exists H: [0,1] \times X \to X$
$H(0, x) = x$ $H(1, x) \in A$
$H(t, a) \in A$ $\forall a \in A$
$A \simeq X$ (exercise)

Proof Three steps

There is a map $\Phi: \pi_1(A, x_0) * \pi_1(B, x_0) \longrightarrow \pi_1(X, x_0)$

1) Φ is surjective
2) $\operatorname{Ker}\Phi \supset$ Normal subgp gen by $\{ i_{A \cap B}^A(\gamma)\, i_{A \cap B}^B(\gamma)^{-1} \mid \gamma \in \pi_1(A \cap B, x_0) \}$
3) $\subset$

2) clear
1) see Hatcher
3) think about it

9/12

How to compute π_1 of a CW-complex

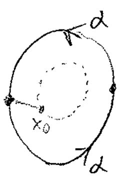
Seifert-van Kampen thm

$X = A \cup B$ A, B open

$x_0 \in A \cap B$ path connected

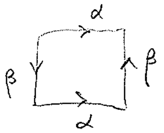
$$\pi_1(X, x_0) = \pi_1(A, x_0) \underset{\pi_1(A\cap B, x_0)}{*} \pi_1(B, x_0)$$

$\mathbb{R}P^2$



$$\pi_1(\mathbb{R}P^2, x_0) = \langle \alpha \mid \alpha^2 \rangle = \mathbb{Z}/2$$

Klein Bottle



$$\pi_1(KB) = \langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta \rangle = G$$

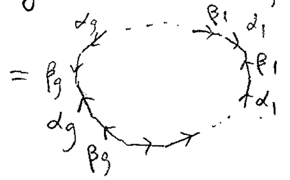
$$G \xrightarrow{\Phi} \mathbb{Z}/2$$

$$w \longmapsto \text{total exponent of } \alpha, \text{ mod } 2$$

$\operatorname{Ker}\Phi \cong \mathbb{Z}^2$ generators α^2, β

Genus g orientable surface Σg

$\Sigma_g =$



$$\pi_1(\Sigma_g) = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \mid \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1} \rangle$$

If G is a group, its abelienization $Ab(G)$ is obtained by adding relation saying that everything commutes.

$$Ab(G) = \frac{G}{\substack{\text{normal subgp} \\ \text{generated by} \\ \text{the commutators} \\ xyx^{-1}y^{-1}}} = \frac{G}{[G,G]}$$

$$Ab(\pi_1(KB)) = \frac{\mathbb{Z}\{\alpha, \beta\}}{2\beta = 0} = \mathbb{Z} \oplus \mathbb{Z}/2$$

$$Ab(\pi_1(\Sigma_g)) = \mathbb{Z}^{2g}$$

<u>CW-complex</u> is a topological space X built as follows: $X = \bigcup_{n=0}^{\infty} X^{(n)}$, where $X^{(n)}$ is constructed by induction.

$X^{(0)}$ is a set with the discrete topology. (Points in $X^{(0)}$ are called 0-cells)

$X^{(n)}$ is obtained from $X^{(n-1)}$ as follows.

Let $Y = \coprod_{\alpha \in I} D^n_\alpha$ D^n_α is a copy of D^n.

For each $\alpha \in I$, let $\varphi_\alpha : \partial D^n_\alpha = S^{n-1} \longrightarrow X^{(n-1)}$ (attaching map)

$$X^{(n)} := \frac{X^{(n-1)} \amalg Y}{\sim}$$

$$\forall \alpha \in I, \ \forall y \in \partial D^n_\alpha, \ y \sim \varphi_\alpha(y).$$

$X^{(n)}$ is called the n-skeleton.

D^n_α is called an n-cell.

Topology on X: A is open (closed) in X iff $A \cap X^{(n)}$ is open (closed) in $X^{(n)}$, $\forall n$.

X is called n-dimensional if $X^{(n)} \neq X^{(n-1)}$ but $X^{(m)} = X^{(n)}$ $\forall m \geq n$, so $X = X^{(n)}$.

<u>Example</u>

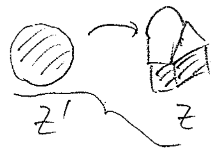
- $\mathbb{R}P^n$ has a CW structure. with one i-cell for $i = 0, \ldots, n$
- $\mathbb{C}P^n$ has a CW structure. with one $2i$-cell for $i = 0, \ldots, n$

<u>Punchline</u>: If X is a CW complex with one 0-cell x_0, then $\pi_1(X, x_0)$ has a presentation with one generator for each 1-cell, and one relation for each 2-cell.

<u>Proof</u>: For simplicity, assume only finitely many cells compute $\pi_1(X^{(n)}, x_0)$ by induction on n.

$$X^{(1)} = \bigvee_{\text{1-cells}} S^1 \qquad \pi_1(X^{(1)}, x_0) = \mathop{*}_{\text{1-cell}} \pi_1(S^1).$$

If Z is a subcomplex of X and Z' is a subcomplex obtained from Z by attaching an n-cell for $n \geq 2$, how does $\pi_1(Z', x_0)$ relate to $\pi_1(Z, x_0)$?



Want to show

- if $n=2$, then $\pi_1(Z',x_0) = \frac{\pi_1(Z,x_0)}{\text{new relation}}$
- if $n>2$, then $\pi_1(Z',x_0) = \pi_1(Z,x_0)$

Apply Seifert van Kampen thm with

$A = Z' \setminus \{\text{center of the } n\text{-cells}\}$

$B = \{\text{interior of the } n\text{-cells}\}$

$A \simeq Z \qquad B \simeq pt$

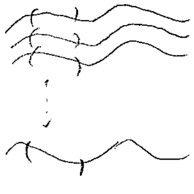
$A \cap B \simeq S^{n-1}$ $\qquad n=2 \quad \pi_1(A\cap B) = \mathbb{Z}$

$n>2 \quad \pi_1(A \cap B) = \{1\}$

9/14 **Def** A covering space a map $\pi : \tilde{X} \to X$ such that $\forall p \in X$, $\exists$ nbhd $p \in U \subset X$ s.t. $\pi^{-1}(U) = \bigsqcup_{\alpha \in I} V_\alpha$ and

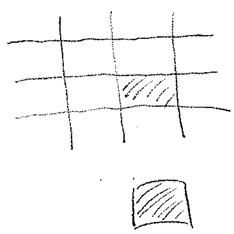
$\pi|_{V_\alpha} : V_\alpha \to U_\alpha$ is a homeomorphism

$$\begin{tikzcd}\tilde{X} \arrow[d, "\pi"'] \\ X\end{tikzcd}$$



Examples: $\pi : \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$

- $\pi : \mathbb{R}^2 \to T^2 = \mathbb{R}^2/\mathbb{Z}^2$



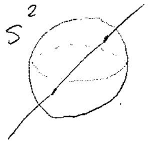
$$\begin{tikzcd}\tilde{X} = \mathbb{R}/3\mathbb{Z} \arrow[d, "\pi"] \\ X = \mathbb{R}/\mathbb{Z}\end{tikzcd}$$

- $S^2 \xrightarrow{\pi} \mathbb{R}P^2 = (\mathbb{R}^2 \setminus \{0\})/\sim$ $\qquad V \sim \lambda V$

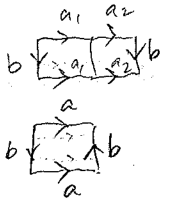
$V \in \mathbb{R}^3 \setminus \{0\}$

$\lambda \in \mathbb{R} \setminus \{0\}$

$S^2 = \{v \in \mathbb{R}^3 \mid \|v\| = 1\}$



- T^2 $\downarrow$ Klein bottle



$\Sigma_g =$

$\downarrow$

$\Sigma_g/\mathbb{Z}_2$

Any manifold X has an orientable double cover

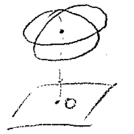
$\tilde{X} = \{(x, o) \mid x \in X,\ o \text{ is a local orientation of } X \text{ at } x\}$

$\downarrow$

X

Not quite a covering space "branched cover"

$$\begin{tikzcd}\mathbb{C} \arrow[d, "\pi"] & z \arrow[d, mapsto] \\ \mathbb{C}^2 & z^2\end{tikzcd}$$



o has only 1 preimage.

covering space condition holds for $p \neq 0$.

A branched cover of surface is a map $\pi : \tilde{X} \to X$ where $\tilde{X}, X$ are surfaces and $\forall p \in X$ $\exists$ nbhd $p \in U \subset X$ s.t. $\pi^{-1}(U) = \bigsqcup_{\alpha \in I} V_\alpha$ where

$$\begin{tikzcd}V_\alpha \simeq \text{a ball in } \mathbb{C} \arrow[d, "\pi"'] & z \arrow[d, mapsto] \\ U \simeq \text{a ball in } \mathbb{C} & z^k\end{tikzcd}$$

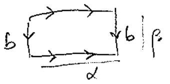
$k \in \{1, 2, \ldots\}$

Prop: let $p:(\tilde{X},\tilde{x}_0)\longrightarrow(X,x_0)$ be covering space. Then, the map $p_*:\pi_1(\tilde{X},\tilde{x}_0)\longrightarrow\pi_1(X,x_0)$ is injective.

eg

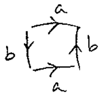
$$\begin{tikzcd} \mathbb{R}/3\mathbb{Z} \arrow[d] \\ \mathbb{R}/\mathbb{Z} \end{tikzcd}$$

$$\begin{tikzcd} \pi_1(\tilde{X}) \arrow[r, phantom, "="] \arrow[d, "p_*"'] & \mathbb{Z} \arrow[d, "3\cdot"] \\ \pi_1(X) \arrow[r, phantom, "="] & \mathbb{Z} \end{tikzcd}$$



$\pi_1(T^2)=\langle\alpha,\beta\mid\alpha\beta=\beta\alpha\rangle$

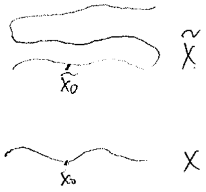
$$\begin{tikzcd} \pi_1(T^2) \arrow[d] & \alpha \arrow[d, mapsto] & \beta \arrow[d, mapsto] \\ \pi_1(KB) & a^2 & b \end{tikzcd}$$



$\pi_1(KB)=\langle a,b\mid aba^{-1}=b^{-1}\rangle$

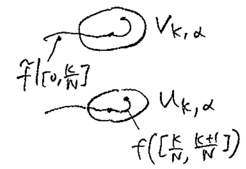
Lemma: path Lifting let $p:\tilde{X}\longrightarrow X$ be a covering space. let $f:[0,1]\longrightarrow X$, $f(0)=x_0$ $p(\tilde{x}_0)=x_0$. Then $\exists!\ \tilde{f}:[0,1]\longrightarrow\tilde{X}$ s.t. $p\circ\tilde{f}=f$, $\tilde{f}(0)=\tilde{x}_0$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ {[0,1]} \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & X \end{tikzcd}$$



Proof: By compactness, $\exists$ pos int. N s.t. f sends $[\frac{k}{N},\frac{k+1}{N}]$ to a set U_k s.t. $\underset{p}{\pi}^{-1}(U_k)=\bigcup_{\alpha\in I_k}V_{k,\alpha}$, $\pi|_{V_{k,\alpha}}:V_{k,\alpha}\longrightarrow U_k$ is a homeo.

construct $\tilde{f}$ by induction on k, If $\tilde{f}$ has been defined on $[0,\frac{k}{N}]$, then $\tilde{f}(\frac{k}{N})\in V_{k,\alpha}$



Define $\tilde{f}$ on $[\frac{k}{N},\frac{k+1}{N}]$ by $\tilde{f}:=(\pi|_{V_{k,\alpha}})^{-1}\circ f$

Homotopy Lifting Lemma

Given $f:[0,1]\times Y\longrightarrow X$ and $g:\{0\}\times Y\longrightarrow\tilde{X}$ s.t. $p\circ g=f(0,\cdot)$, $\exists!\ \tilde{f}:[0,1]\times Y\longrightarrow\tilde{X}$ s.t. $p\circ\tilde{f}=f$ and $\tilde{f}|_{\{0\}\times Y}=g$.

$$\begin{tikzcd} \{0\}\times Y \arrow[r, "g"] \arrow[d, hook] & \tilde{X} \arrow[d] \\ {[0,1]}\times Y \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & X \end{tikzcd}$$

9/17 Path Lifting Lemma

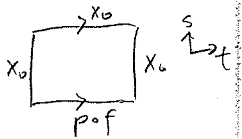
Homotopy Lifting Lemma

Pf: For each $y\in Y$, define $\tilde{f}$ on $[0,1]\times\{y\}$ by the path lifting Lemma. Can check that $\tilde{f}$ is continuous. □

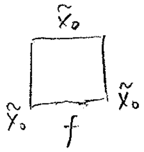
Proposition: let $p:\tilde{X}\to X$ be a covering space, $x_0\in X$, $\tilde{x}_0\in\tilde{X}$, $p(\tilde{x}_0)=x_0$. Then $p_*:\pi_1(\tilde{X},\tilde{x}_0)\longrightarrow\pi_1(X,x_0)$ is inj.

Pf: let $f:[0,1]\longrightarrow\tilde{X}$, $f(0)=f(1)=\tilde{x}_0$ suppose $p_*[f]=0\in\pi_1(X,x_0)$.

Then there is a homotopy $H:[0,1]\times[0,1]\longrightarrow X$ with $H(0,\cdot)=p\circ f$
$H(1,\cdot)=H(\cdot,0)=H(\cdot,1)=x_0$



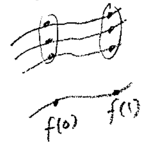
By homotopy lifting lemma, $\exists\ \tilde{H}:[0,1]\times[0,1]\longrightarrow X$ w/ $p\circ\tilde{H}=H$ and $\tilde{H}(0,\cdot)=f$



By uniqueness part of the path lifting lemma, $\tilde{H}(1,\cdot)=\tilde{H}(\cdot,0)=\tilde{H}(\cdot,1)=\tilde{x}_0$

So, $\tilde{H}$ is a homotopy from f to x_0 rel. endpts. so, $[f]=0\in\pi_1(X,x_0)$

Observatn: If $p:\tilde{X}\longrightarrow X$ is a covering space, then a path $f:[0,1]\longrightarrow X$ induces a bijection $\phi(f):p^{-1}(f(0))\longrightarrow p^{-1}(f(1))$



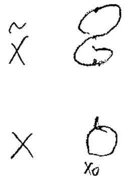
This is defined by lifting f.

$\phi(f)(\tilde{x}_0):=\tilde{f}(1)$

where $\tilde{f}$ is the lift of f with $f(0)=\tilde{x}_0$

$\phi(\tilde{f})=\phi(f)^{-1}$

In particular, there is a right action of $\pi_1(X,x_0)$ on $p^{-1}(x_0)$



Thm let X be path connected and "reasonable". let $x_0\in X$. Then {path connected base coverings up to iso. $p:(\tilde{X},\tilde{x}_0)\longrightarrow(X,x_0)$} $\overset{\text{bij.}}{\longleftrightarrow}$ {subgroups of $\pi_1(X,x_0)$}

$p\longmapsto p_*\pi_1(\tilde{X},\tilde{x}_0)$

$$\begin{tikzcd} (\tilde{X}, \tilde{x}_0) \arrow[r, "\simeq"] \arrow[d, "p"] & (\tilde{X}', \tilde{x}_0') \arrow[dl, "p'"] \\ X & \end{tikzcd}$$

Examples: $(X, x_0) = (S^1 = \mathbb{R}/\mathbb{Z}, 0)$

$$\underbrace{\left\{ \begin{matrix} \text{path conn} \\ \text{based coverings} \\ \text{of } (S^1, 0) \end{matrix} \right\}}_{\text{iso}} = \{ \text{subgroups of } \mathbb{Z} \}$$

$Id: S^1 \to S^1 \longleftrightarrow \mathbb{Z}$

$\mathbb{R} \to S^1 \longleftrightarrow \{0\}$

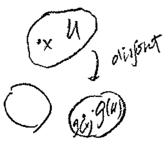
$\mathbb{R}/k\mathbb{Z} \xrightarrow{\cdot k} \mathbb{R}/\mathbb{Z} \longleftrightarrow k\mathbb{Z} \quad k \in \{1, 2, \dots\}$

$(X, x_0) = (\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2, (0,0))$

$$\underbrace{\left\{ \begin{matrix} \text{path conn.} \\ \text{based coverings} \\ \text{of } \mathbb{T}^2, (0,0) \end{matrix} \right\}}_{\text{iso}} = \{ \text{subgroups of } \mathbb{Z}^2 \}$$

$$\left(\mathbb{R}^2/H \to \mathbb{R}^2/\mathbb{Z}^2 \right) \longleftrightarrow H \subset \mathbb{Z}^2$$

More generally, A covering space action (properly disconit) is an action of a group G on X s.t. $\forall x \in X$ $\exists$ nbhd $x \in U \subset X$ s.t. $\{ g(U) \mid g \in G \}$ are disjoint

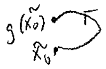


In this case, the quotient map

$X \xrightarrow{p} X/G$ is a covering space.

Prop: Suppose X is simply connected and path conn. Then $\pi_1(X/G, x_0) = G$

$$\begin{tikzcd} X \arrow[d] \\ X/G \end{tikzcd}$$



$$\left\{ \begin{matrix} \text{path conn.} \\ \text{based covering} \\ \text{of } X/G \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{subgoup} \\ \text{of } G \end{matrix} \right\}$$

$X/H \longleftrightarrow (H \subset G)$

Not all covering spaces can be described as $X = \tilde{X}/G$.

Def: A deck transformation of a covering space $p: \tilde{X} \to X$ is a map

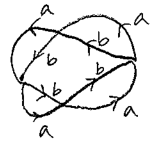
$$\begin{tikzcd} \tilde{X} \arrow[r, "f"] \arrow[dr, "p"'] & \tilde{X} \arrow[d, "p"] \\ & X \end{tikzcd}$$

Def: A covering space is normal if deck transformation act transitively on each fiber.

If $X = \tilde{X}/G$, then this is a normal covering.

Examples: (coverings of $S^1 \vee S^1 = X$)

(oriented graph in which each edge is labeled "a" or "b" and each vertex has one coming 'a' edge, one outgoing 'a' edge, —"— "b" —— "b" edge.)



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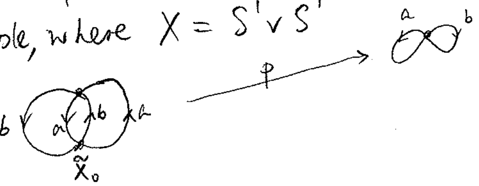
Thm If (X, x_0) is path connected and "reasonable" then

$$\underbrace{\left\{ \begin{matrix} \text{based covering} \\ \text{space} \\ p: (\tilde{X}, \tilde{x}_0) \to (X, x_0) \end{matrix} \right\}}_{\text{isom}} \longleftrightarrow \left\{ \begin{matrix} \text{subgroups of} \\ \pi_1(X, x_0) \end{matrix} \right\}$$

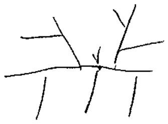
$p \longleftrightarrow p_* \pi_1(\tilde{X}, \tilde{x}_0)$

Example, where $X = S^1 \vee S^1$

$\tilde{X} =$



If G is a graph, v a vertex, $\pi_1(G, v)$.



$T =$ maximal spanning tree.

$\pi_1(G, v) =$ Free group generated by edges in $G \setminus T$.

$p_* \pi_1(\tilde{X}, \tilde{x}_0) = \langle ab^{-1}, a^2, ab \rangle \subset \underset{\langle a, b \rangle}{\overset{\shortparallel}{\pi_1(X, x_0)}}$

$= \{\text{even length words}\}$

This is a normal subgroup. of index 2.

In general

- $[\pi_1(X,x_0) : p_*\pi_1(\tilde{X},\tilde{x}_0)] = |p^{-1}(x_0)|$

i.e. there is a bijection

$$\phi : \left\{ \begin{array}{c} \text{"right" cosets of } p_*\pi_1(\tilde{X},\tilde{x}_0) \\ \text{in } \pi_1(X,x_0) \end{array} \right\} \xrightarrow{\cong} p^{-1}(x_0)$$

Given $[f] \in \pi_1(X,x_0)$. Lift f to a path.

$\tilde{f} : [0,1] \longrightarrow \tilde{X} \quad p \circ \tilde{f} = f \quad \tilde{f}(0) = \tilde{x}_0$

Define $\phi([f]) := \tilde{f}(1)$.

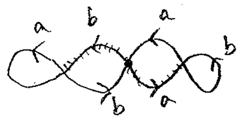
$\phi([f]) = \phi([g])$

$\Leftrightarrow f\bar{g} \in p_*\pi_1(\tilde{X}, x_0)$

$\Leftrightarrow f \in p_*\pi_1(\tilde{X},\tilde{x}_0)g$ so ϕ is well-defined

ϕ is surjective because $\tilde{X}$ is path connected.

- $p_*\pi_1(\tilde{X},\tilde{x}_0)$ is a normal subgroup of $\pi_1(X,x_0) \Leftrightarrow$ the group of deck transformation acts transitively on $p^{-1}(x_0)$

 $= \tilde{X}$

$p_*\pi_1(\tilde{X},\tilde{x}_0) = \langle a^2, b^2, aba^{-1}, bab^{-1} \rangle$

$$\begin{gathered} a^2 b^3 a^{-1} b \in p_*\pi_1(\hat{X}, \hat{x}_0) \\ \shortparallel \\ a^2 b^2 (bab^{-1})^{-1} \end{gathered}$$

- Group of deck transformations

$$= \frac{N(p_*\pi_1(\tilde{X},\tilde{x}_0))}{p_*\pi_1(\tilde{X},\tilde{x}_0)}$$

Thm: Any subgroup of a free group is free

pf: Any covering space of a wedge of circles is a graph.

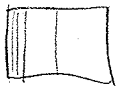
Lifting Criterion

$$\begin{tikzcd} & (\tilde{X},\tilde{x}_0) \arrow[d, "p = \text{covering space}"] \\ (Y,y_0) \arrow[r, "f"] \arrow[ur, dashed, "\exists!\, \tilde{f}"] & (X,x_0) \end{tikzcd}$$

path conn. (under (Y,y_0))

path connected and locally path connected (under (X,x_0))

Def X is locally path connected if $\forall x \in X$, $\forall$ nbhd $x \in U \subset X$, $\exists$ nbhd $x \in V \subset U$ s.t. V is path connected.



Thm: $\exists$ lift $\tilde{f}$ of f as above

$\Leftrightarrow f_*\pi_1(Y,y_0) \subseteq p_*\pi_1(\tilde{X},\tilde{x}_0)$

Proof: "$\Rightarrow$" if $\tilde{f}$ exists, then $f_* = p_* \circ \tilde{f}_*$

"$\Leftarrow$" $\operatorname{Im}(f_*) \subset \operatorname{Im}(p_*)$

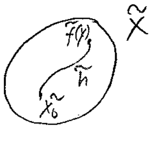
construct $\tilde{f}$

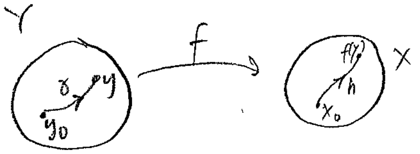
Given $y \in Y$, let $\gamma : [0,1] \longrightarrow Y$ be a path with $\gamma(0) = y_0$ and $\gamma(1) = y$

Then $h = f \circ \gamma : [0,1] \rightarrow X$ has $(f\circ\gamma)(0) = x_0$, $(f\circ\gamma)(1) = f(y)$

$\exists!\ \tilde{h} : [0,1] \longrightarrow \tilde{X} \quad p \circ \tilde{h} = h \quad \tilde{h}(0) = \tilde{x}_0$

Define $\tilde{f}(y) := \tilde{h}(1)$





Need to check that $\tilde{f}$ is well-defined and continuous.

Well-defined; let γ' be a different path from y_0 to y with h', $\tilde{h}'$.

Why is $\tilde{h}'(1) = \tilde{h}(1)$?

Because $\operatorname{Im}(f_*) \subseteq \operatorname{Im}(p_*)$

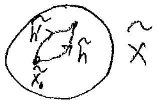
$\tilde{f}$ is continuous because X is locally path conn.

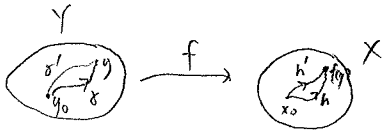
9/21 Lifting Criterion

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, dashed, "\exists!\tilde{f}"] \arrow[r, "f"] & (X, x_0) \end{tikzcd}$$

path conn. — path connected and locally path conn.

$\tilde{f}$ exists $\Leftrightarrow$ $\text{Im} f_* \subset \text{Im}\, p_*$





The path $h\bar{h}'$ is a loop in X based at x_0 which is in $\text{Im}(f_*) \subset \text{Im}(p_*)$

So $h\bar{h}'$ lifts to a loop in $\tilde{X}$, ...

Why is $\tilde{f}$ cont? let $y \in Y$ and let U be a nbhd of $\tilde{f}(y)$ in $\tilde{X}$.

NTS: $\tilde{f}^{-1}(U)$ contains a nbhd of y.

By shrinking U, we can assume,

- p maps U homeomorphically to $p(U)$
- $U \simeq p(U)$ is path connected.

(......

Thm Suppose (X, x_0) is path conn. locally path conn. and semi-locally simply conn. Then

$$\left\{ \begin{matrix} \text{path conn.} \\ \text{base coverings} \\ p: (\tilde{X}, \tilde{x}_0) \to (X, x_0) \end{matrix} \right\} / \text{iso} \xrightarrow{\simeq} \left\{ \begin{matrix} \text{subgroups of} \\ \pi_1(X, x_0) \end{matrix} \right\}$$

Def X is semi-locally simply conn. if $\forall x \in X$ $\forall$ nbhd $x \in U \subset X$ $\exists$ nbhd $x \in V \subset U$ s.t. if $i: V \to X$ then $i_* \pi(V, x) = \underset{\pi_1(X,x)}{\underset{\cap}{\{1\}}}$

Pf: injectivity: suppose $\text{Im}(p_*) = \text{Im}(p'_*)$

$$\begin{tikzcd} (\tilde{X}, \tilde{x}_0) \arrow[rr, dashed, bend left, "f"] \arrow[dr, "p"'] & & (\tilde{X}', \tilde{x}_0') \arrow[ll, dashed, bend left, "f'"] \arrow[dl, "p'"] \\ & (X, x_0) & \end{tikzcd}$$

$p = p' \circ f$

$p' = p \circ f'$

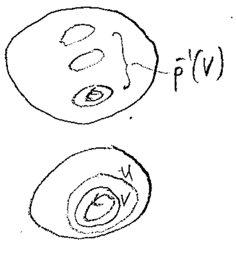
Since $\text{Im}(p_*) \subset \text{Im}(p'_*)$, we can lift p to f, since $\text{Im}(p'_*) \subset \text{Im}(p_*)$, we can lift p' to f'

$f' \circ f = id_{\tilde{X}}$ because $f' \circ f$ is a lift of id and $(f' \circ f)(\tilde{x}_0) = \tilde{x}_0$, so by uniqueness of lifting $f' \circ f = id_{\tilde{X}}$; Likewise $f \circ f' = id_{\tilde{X}'}$

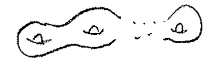
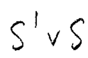
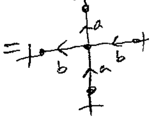
surjectivity:

special case: Prove that (X, x_0) has a simply conn. covering space. (This is called universal cover of (X, x_0))

Remark: if X has a universal cover then X is semi-locally simply conn.



Example of univ. covers

$X = S^1$	$\tilde{X} = \mathbb{R}$
$X = S^2$	$\tilde{X} = S^2$
$X = \mathbb{R}P^2$	$\tilde{X} = S^2$
$X = T^2$	$\tilde{X} = \mathbb{R}^2$
$X =$  Σ_g	$\tilde{X} = \mathbb{R}^2$
$X = S^1 \vee S^1$ 	$\tilde{X} =$ 

Construction of universal cover

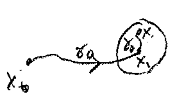
$p: (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ as a set

$$\tilde{X} = \left\{ (x, [\gamma]) \;\middle|\; \begin{matrix} x \in X \\ [\gamma] \text{ is a homotopy class} \\ \text{of paths from } x_0 \text{ to } x \end{matrix} \right\}$$

$\tilde{x}_0 = (x_0, [\text{const}])$

If U is path conn. open set in X and $x_1 \in U$ and γ_0 is a path from x_0 to x_1

define $A_{U,[\gamma]} = \left\{ (x, \gamma) \;\middle|\; \begin{matrix} x \in U \\ \gamma = \gamma_0 \cdot \gamma_1 \end{matrix} , \begin{matrix} \gamma_1: [0,1] \to U \\ \gamma_1(0) = x_1 \end{matrix} \right\}$



one can check these sets $A_{U,[\gamma]}$ are basis for a topology on $\tilde{X}$

$p: \tilde{X} \longrightarrow X$ is a covering space.

$\tilde{X}$ is path conn.

Why is $\pi_1(\tilde{X}, \tilde{x}_0) = \{1\}$?

let $f: [0,1] \to \tilde{X}$, $f(0) = f(1) = \tilde{x}_0$

9/24 (X, x_0) path conn., LPC, SLSC

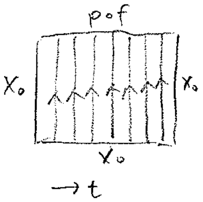
Construct universal cover $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ as follows. As a set,

$$\tilde{X} = \{ (x, [\gamma]) \mid x \in X,\ \gamma: [0,1] \to X \quad \gamma(0) = x_0,\ \gamma(1) = x \}$$

$p(x, [\gamma]) = x \qquad \tilde{x}_0 = (x_0, [\text{const}])$

Why $\tilde{X}$ is simply conn?

let $f: [0,1] \to \tilde{X}$, $f(0) = f(1) = \tilde{x}_0$



$F: [0,1] \times [0,1] \longrightarrow X$

$f(t) = (F(1,t), [F(\cdot, t)])$

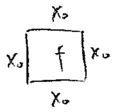
Homology

$\pi_1(\mathbb{R}^3 \setminus \{0\}) = \{1\}$

Higher homotopy group $\pi_n(X, x_0)$

Generator of $\pi_n(X, x_0)$ is a map $f: (I^n, \partial I^n) \longrightarrow (X, x_0)$ up to homotopy.

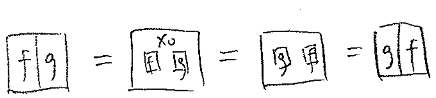
eg



$$\boxed{f}\ \boxed{g} = \boxed{f|g}$$

Good news

- π_n is abelian for $n > 1$,

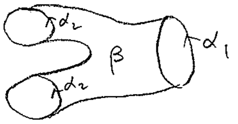


- $\pi_n(S^n) = \mathbb{Z}$

Bad news : hard to compute in general

Idea of homology $H_n(X)$

- Generator is an "n-dimensional object in X" (chain) without boundary
- Two generators α_1, α_2 are equivalent if there is an $(n+1)$-dimensional chain β with $\partial\beta = \alpha_1 - \alpha_2$

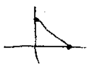
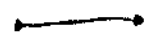


Simplicial homology of Delta-complex

n-simplex

$$\Delta_n = \{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid 0 \le x_i \le 1 \quad \sum_{i=0}^{n} x_i = 1 \}$$

$\Delta_0 = \bullet$

$\Delta_1 =$  $=$  interval

$\Delta_2 =$  $=$ 

$\Delta_3 =$ 

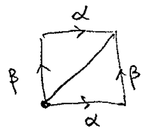
For each $i = 0, \dots, n$ $\{(x_0, \dots, x_n) \in \Delta_n \mid x_i = 0\} \cong \Delta_{n-1}$

So Δ_n has $(n+1)$ faces $\cong \Delta_{n-1}$

A Delta-complex is a special kind of CW complex in which

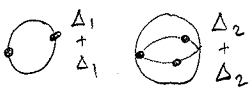
- Every n-cell is identified with Δ_n
- For each n-cell, the attaching map $\Delta_n \longrightarrow X^{(n-1)}$ identifies each face of Δ_n with an $(n-1)$ cell via an isomorphism

T^2



1 0-simplex
3 1-simplices
2 2-simplices.

S^n take 2 copies of Δ_n and identify their boundary



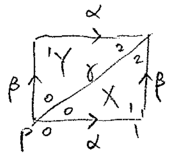
Definition of simplical homology

$C_n^\Delta(X) :=$ free abelian group generated by the n-cells in X

$$\partial : C_n^\Delta \longrightarrow C_{n-1}^\Delta$$

if σ is an n-cell, then for $i = 0, \dots, n$ let σ_i denote the i^{th} face of σ

Define $\partial\sigma := \sum_{i=0}^{n} (-1)^i \sigma_i$



$\partial p = 0$

$\partial\alpha = p - p = 0$

$\partial\beta = p - p = 0$

$\partial\gamma = p - p = 0$

$\partial X = \beta - \gamma + \alpha$

$\partial Y = \alpha - \gamma + \beta$

Lemma $\partial \circ \partial = 0$

Pf: In the expansion of $\partial(\partial\sigma)$, each $(n-2)$-cells appear twice with opposite signs.

Def $H_n^\Delta(X) = \dfrac{Ker(\partial : C_n^\Delta \longrightarrow C_{n-1}^\Delta)}{Im(\partial : C_{n+1}^\Delta \longrightarrow C_n^\Delta)}$ ← cycles ← boundaries

(an abelian group)

$H_0^\Delta = \mathbb{Z}\{p\}$

$H_1^\Delta = \dfrac{\mathbb{Z}\{\alpha,\beta,\gamma\}}{\langle\alpha+\beta-\gamma\rangle} = \mathbb{Z}\{\alpha,\beta\} \cong \mathbb{Z}^2$

$H_2^\Delta = \mathbb{Z}\{x-y\} \simeq \mathbb{Z}$

Good news — $H_n^\Delta(X)$ is easy to compute

Bad news — Want $H_n^\Delta(X)$ to depend only on the top space X and not on the choice of Delta-complex structure.

9/26 Homology of a Δ-complex

$C_n^\Delta :=$ free $\mathbb{Z}$-module generated by the n-cells

$$\partial : C_n^\Delta \longrightarrow C_{n-1}^\Delta$$

$(*)$ — $\partial\sigma := \sum_{i=0}^{n} (-1)^i \sigma|_{[0,\dots,\hat{i},\dots,n]}$

$\sigma : \Delta_n \longrightarrow X$

$\Delta_n = [0,\dots,n] = \left\{(x_0,\dots,x_n) \in \mathbb{R}^{n+1} \;\middle|\; \begin{matrix} x_i \in [0,1] \\ \sum x_i = 1 \end{matrix}\right\}$

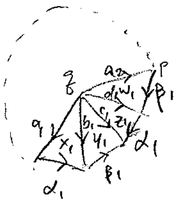
$[0,\dots,\hat{i},\dots,n] := \{(x_0,\dots,x_n) \in \Delta_n \mid x_i = 0\}$

Lemma: $\partial^2 = 0$

Pf:
$$\partial(\partial\sigma) = \sum_{i=0}^{n} (-1)^i \partial\left(\sigma|_{[0,\dots,\hat{i},\dots n]}\right)$$
$$= \sum_{i=0}^{n} (-1)^i \left(\sum_{j<i} (-1)^j \sigma|_{[0,\dots,\hat{j},\dots,\hat{i},\dots,n]} + \sum_{j>i} (-1)^{j-1} \sigma|_{[0,\dots,\hat{i},\dots\hat{j},\dots,n]} \right)$$
$$= \sum_{0\le j<i\le n} (-1)^{i+j} \sigma|_{[0,\dots,\hat{i},\dots,\hat{j},\dots,n]} + \sum_{0\le i<j\le n} (-1)^{i+j-1} \sigma|_{[0,\dots,\hat{i},\dots\hat{j},\dots,n]}$$
$$= 0$$

$$H_n^\Delta = \frac{Ker(\partial : C_n^\Delta \longrightarrow C_{n-1}^\Delta)}{Im(\partial : C_{n+1}^\Delta \longrightarrow C_n^\Delta)}$$

eg $X = \Sigma_g$



$\partial q = 0$

$\partial p = 0$

$\partial\alpha_i = p - p = 0$

$\partial\beta_i = p - p = 0$

$\partial x_i = \alpha_i - b_i + a_i$

$\partial y_i = \beta_i - c_i + b_i$

$\partial z_i = \alpha_i - c_i + d_i$

$\partial w_i = \beta_i - d_i + a_{i+1}$ $\quad i \in \mathbb{Z}/g$

$\partial a_i = p - q$

$\partial b_i = p - q$

$\partial c_i = p - q$

$\partial d_i = p - q$

$H_0 = \mathbb{Z}\{p,q\}/(p-q) \cong \mathbb{Z}$

$H_2 = \mathbb{Z}\left\{\sum_{i=1}^{g} (x_i + y_i - z_i - w_i)\right\} \cong \mathbb{Z}$

Claim $H_1 = \mathbb{Z}\{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\} \cong \mathbb{Z}^{2g}$

Pf we know that $\partial\alpha_i = \partial\beta_i = 0$

Need to show: - any 1-cycle is homologous to linear combination of α_i's and β_i's

- no non-trivial linear combination of α_i's and β_i's is nullhomotopic

Problem Want to show that $H_n^{\Delta}(X)$ depends only on the topological space X, and not on the Δ-complex structure.

idea: Give a different defn of $H_n(X)$ that takes into account all possible maps $\Delta_n \to X$. (Singular homology)

Def let X be any topological space,

$C_n(X) :=$ free $\mathbb{Z}$-module generated by all continuous maps $\sigma : \Delta_n \to X$.

$\partial : C_n(X) \to C_{n-1}(X)$

defined by $(*)$

$$H_n(X) = \frac{\operatorname{Ker}(\partial : C_n(X) \to C_{n-1}(X))}{\operatorname{Im}(\partial : C_{n+1}(X) \to C_n(X))}$$

Thm: If X is a Δ-complex then there is a canonical isomorphism

$$\underset{\text{manifestly a topological invar.}}{H_n(X)} = \underset{\text{computable}}{H_n^{\Delta}(X)}$$

Proof Later

Prop : If X is path conn. then $H_0(X) = \mathbb{Z}$

Pf: Define a map $\varepsilon : H_0(X) \to \mathbb{Z}$

$$\sum a_p \cdot p \longmapsto \sum a_p \in \mathbb{Z}$$

Need to check
- ε is well-defined
- ε is surjective
- ε is injective.

Consider $\varepsilon : C_0(X) \to \mathbb{Z}$

need to check $\varepsilon(\operatorname{Im}(\partial : C_1(X) \to C_0(X))) = 0$.

If $\sigma : \Delta_1 \to X$ then $\partial\sigma = \sigma_0 - \sigma_1$ so

$\varepsilon(\partial\sigma) = 0$

$(\Rightarrow \operatorname{Ker}\varepsilon \supset \operatorname{Im}\partial)$

surj: pick $x_0 \in X$ $\varepsilon(kx_0) = k$

inj: Suppose $\varepsilon(\sum a_p p) = 0$

For each p, choose $\sigma^p : \Delta_1 \to X$ with $\sigma_1^p = x_0$ and $\sigma_0^p = p$. Then

$$\partial\left(\sum_p a_p \sigma^p\right) = \sum_p a_p(\sigma_0^p - \sigma_1^p)$$
$$= \sum_p a_p \cdot p \quad \text{but } \partial\left(\sum_p a_p \sigma^p\right) = 0 \text{ in } H_0(X)$$

$(\Rightarrow \operatorname{Im}\partial \supset \operatorname{Ker}\varepsilon)$

9/28 <u>Singular homology</u> of a top. X

$C_n(X) :=$ free $\mathbb{Z}$-module generated by maps $\sigma : \Delta_n \to X$

$\partial : C_n(X) \to C_{n-1}(X)$

$$\partial\sigma := \sum_{i=0}^{n} (-1)^i \sigma|_{[0,\dots,\hat{i},\dots,n]}$$

$\partial^2 = 0$ $\quad H_n(X) = \operatorname{Ker}\partial|_{C_n} / \operatorname{Im}\partial|_{C_{n+1}}$

<u>Last time</u>: if X is path conn, then $H_0(X) = \mathbb{Z}$

<u>Note</u>: in general $H_n(X) = \bigoplus_{X' \text{ a path component of } X} H_n(X')$.

<u>Pf</u>: if $\sigma : \Delta_n \to X$, then since Δ_n is path conn, σ maps Δ_n to a single path component of X.

Thus, $C_n(X) = \bigoplus_{X' \text{ path comp.}} C_n(X')$

∂ respects this splitting. ▯

<u>Thm</u>: If X is nonempty path conn., then $H_1(X) = \mathrm{Abs}(\pi_1(X, x_0))$

<u>Pf</u>: Step 1: Define a map $\Phi : \pi_1(X, x_0) \to H_1(X)$

Identify $\Delta_1 \simeq [0,1]$ $\quad \left(\begin{array}{l}[\hat{0},1] \leftrightarrow 0 \\ [0,\hat{1}] \leftrightarrow 1\end{array}\right)$

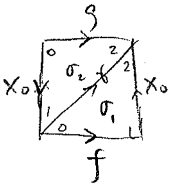
If $f : \overset{\Delta_1}{[0,1]} \to X$ $\quad f(0) = f(1) = x_0$

define $\Phi([f]) := [f]$

Note that $\partial f = f(0) - f(1) = x_0 - x_0 = 0 \in C_0(X)$

$f \in C_1(X)$

Need to check that if $[f] = [g]$ in $\pi_1(X, x_0)$ then $[f] = [g]$ in $H_1(X)$

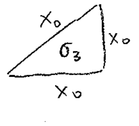


$\partial(\sigma_1) = x_0 - \gamma + f$

$\partial(\sigma_2) = \gamma - g + x_0$

$\partial(\sigma_3) = x_0$

$\partial(\sigma_1 + \sigma_2 - 2\sigma_3) = f - g$



$\Rightarrow$ homotopic paths leads to homologous cycles.

Since $H_1(X)$ is abelian, Φ descends to $\mathrm{Ab}(\pi_1(X, x_0))$.

Example $X = \{x_0\}$

$C_n(X) = \mathbb{Z}\{\sigma_n\}$ let $\sigma_n : \Delta_n \longrightarrow x_0$

$$\partial \sigma_n = \sum_{i=0}^{n} (-1)^i \sigma_{n-1} = \begin{cases} \sigma_{n-1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

$$\operatorname{Ker}(\partial|_{C_n}) = \begin{cases} C_n & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

$$\operatorname{Im}(\partial|_{C_{n+1}}) = \begin{cases} C_n & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$H_n = \begin{cases} \mathbb{Z} & \text{if } n=0 \\ 0 & \text{if } n>0 \end{cases}$$

---

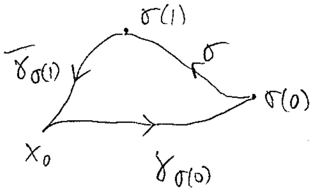
continued the proof

Step 2: Define $\Psi : C_1(X) \longrightarrow Ab(\pi_1(X, x_0))$

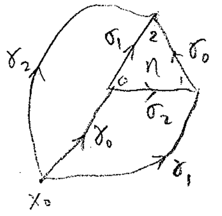
For every point $p \in X$, choose a path $\gamma_p : [0,1] \longrightarrow X$ with $\gamma(0) = x_0$, $\gamma(1) = p$

Take $\gamma_{x_0} = $ const.

Given $\sigma : \underset{\substack{\cong \\ {[0,1]}}}{\Delta_1} \longrightarrow X$, define $\Psi(\sigma) := [\gamma_{\sigma(0)} \sigma \bar{\gamma}_{\sigma(1)}]$



Claim: Ψ annihilates the image of $\partial : C_2(X) \longrightarrow C_1(X)$



$$\Psi(\partial\eta) = \Psi(\sigma_0) - \Psi(\sigma_1) + \Psi(\sigma_2)$$

$$= [\gamma_1 \sigma_0 \bar{\gamma}_2] - [\gamma_0 \sigma_1 \bar{\gamma}_2] + [\gamma_0 \sigma_2 \bar{\gamma}_1]$$

$$= [\gamma_1 \sigma_0 \cancel{\bar{\gamma}_2 \gamma_2} \bar{\sigma}_1 \cancel{\bar{\gamma}_0 \gamma_0} \sigma_2 \bar{\gamma}_1]$$

$$= [\gamma_1 \sigma_0 \bar{\sigma}_1 \sigma_2 \bar{\gamma}_1] = [x_0]$$

η gives a homotopy $\gamma_1 \sigma_0 \bar{\sigma}_1 \sigma_2 \bar{\gamma}_1 \sim x_0$

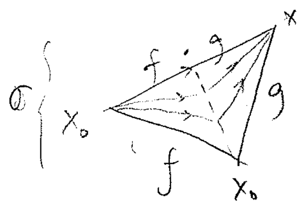
$\Rightarrow \Psi \circ \partial = 0$

$$\begin{tikzcd}
C_1(X) \arrow[r] \arrow[d] & Abs(\pi_1(X, x_0)) \\
C_1(X)/\operatorname{Im}\partial \arrow[ur] & \\
H(X) \arrow[u, phantom, "\cong" description] \arrow[uur] &
\end{tikzcd}$$

So, we now have map

$$\begin{tikzcd}
Ab(\pi_1(X, x_0)) \arrow[r, bend left, "\Phi"] & H_1(X) \arrow[l, bend left, "\Psi"]
\end{tikzcd}$$

Proof that Φ is a homomorphism



$\partial\sigma = g - f \cdot g + f$

so, $[f + g] = [f \cdot g] \in H_1(X)$

Step 3 Show $\Psi \circ \Phi = id_{Ab(\pi_1(X, x_0))}$

let $f : [0,1] \rightarrow X$, $f(0) = f(1) = x_0$

Then $\Psi(\Phi[f]) = \Psi[f] = [\gamma_{x_0} f \bar{\gamma}_{x_0}] = [x_0 f x_0] = [f]$

Step 4 Show $\Phi \circ \Psi = id_{H_1(X)}$

let $\alpha = \sum_\sigma a_\sigma \cdot \sigma \in C_1(X)$ be a cycle.

($\sigma : \Delta_1 \longrightarrow X$, $a_\sigma \in \mathbb{Z}$, only finitely many $a_\sigma \neq 0$)

$$\Psi(\alpha) = \sum_\sigma a_\sigma \cdot [\gamma_{\sigma(0)} \sigma \bar{\gamma}_{\sigma(1)}]$$

$$\Phi \circ \Psi(\alpha) = \left[\sum_\sigma a_\sigma [\gamma_{\sigma(0)} \sigma \bar{\gamma}_{\sigma(1)}]\right]$$

$$= \sum_\sigma a_\sigma (\gamma_{\sigma(0)} + \sigma - \gamma_{\sigma(1)})$$

$$0 = \partial\alpha = \sum_\sigma a_\sigma (\sigma(0) - \sigma(1))$$

$\Rightarrow$ Each γ_p appears with total coeff zero in

10/1 Last time If X is path connected, then $H_1(X) = Ab(\pi_1(X, x_0))$

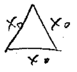
Hurewicz Isomorphism Thm

If X is path connected and $\pi_i(X, x_0) = (1)$ for $1 \leq i < k$ then $H_k(X) = \pi_k(X, x_0)$

eg If $X = \Sigma_g$ $g > 0$ then $H_2(X) \cong \mathbb{Z}$ but $\pi_2(X, x_0) = 0$.

(If universal cover of X is contractible, then $\pi_k(X, x_0) = 0$ $\forall k > 1$)

$$\pi_2(X, x_0) \longrightarrow H_2(X)$$

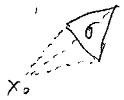


Prop If X is contractible, then

$$H_i(X) = \begin{cases} \mathbb{Z} & \text{if } i=0 \\ 0 & \text{if } i>0 \end{cases}$$

Pf: Fix $x_0 \in X$. Since X is contractible there is $H: [0,1]\times X \to X$ s.t. $H(0,x)=x_0$, $H(1,x)=x$

Define $K: C_n(X) \to C_{n+1}(X)$ as follows



If $\sigma: \Delta_n \to X$, $K(\sigma): \Delta_{n+1} \to X$ is defined by $K(\sigma)(t_0,\dots,t_{n+1}) :=$

$$H\left(1-t_0;\ \sigma\left(\frac{t_1}{1-t_0},\dots,\frac{t_{n+1}}{1-t_0}\right)\right)$$

Claim: If $n>0$, then $\partial K + K\partial = id_{C_n(X)}$

Pf of claim

$$\partial K(\sigma) = \sum_{i=0}^{n+1}(-1)^i K(\sigma)|_{[0,\dots,\hat{i},\dots,n+1]}$$

$$= \sigma + \sum_{i=1}^{n+1}(-1)^i \underbrace{K(\sigma)|_{[0,\dots,\hat{i},\dots,n+1]}}_{K(\sigma|_{[0,\dots,\widehat{i-1},\dots,n]})}$$

let $j=i-1$

$$= \sigma + \sum_{i=0}^{n}(-1)^{i+1} K(\sigma|_{[0,\dots,\widehat{i-1},\dots,n]})$$

$$= \sigma - K(\partial\sigma) \quad \boxed{\text{claim}}$$

If $n=0$, $\partial K + K\partial = id_{C_0(X)} - \phi$

where ϕ (every 0 simplex) $= x_0$

Why is $H_n(X)=0$ for $n>0$?

Let $\alpha \in C_n(X)$ with $\partial\alpha = 0$

$$\partial K\alpha + K\underset{0}{\cancel{\partial\alpha}} = \alpha$$

Since $\alpha = \partial$(something), $[\alpha]=0 \in H_n(X)$

Def A chain-complex is a sequence of abelian groups $\{C_n : n\in\mathbb{Z}\}$ and homomorphisms $\partial = \partial_n : C_n \to C_{n-1}$ s.t. $\partial^2 = 0$ i.e. $\partial_{n-1}\circ\partial_n : C_n \to C_{n-2}$ is 0.

Given a chain complex, define the homology $H_n := Ker(\partial_n)/Im(\partial_{n+1})$

If (C_n', ∂') is another chain complex, a chain map ϕ from (C_n,∂) to (C_n',∂') consists of homomorphisms $\phi_n : C_n \to C_n'$ s.t. $\partial_n' \circ \phi_n = \phi_{n-1}\circ\partial_n$

$$\begin{tikzcd}
\cdots \arrow[r] & C_n \arrow[r, "\partial_n"] \arrow[d, "\phi_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "\phi_{n-1}"] & C_{n-2} \arrow[r] \arrow[d, "\phi_{n-2}"] & \cdots \\
\cdots \arrow[r] & C_n' \arrow[r, "\partial_n'"] & C_{n-1}' \arrow[r, "\partial_{n-1}'"] & C_{n-2}' \arrow[r] & \cdots
\end{tikzcd}$$

A chain map $\phi : C_* \to C_*'$ induces a map on homology $\phi_* : H_n \to H_n'$

If $\alpha \in C_n$ and $\partial\alpha = 0$, then $\phi_*[\alpha] := [\phi(\alpha)]$

Check − $\phi(\alpha)$ is a cycle

$\because \partial'\phi\alpha = \phi(\partial\alpha) = 0$

− If $\alpha' \in C_n$ is another cycle s.t. $[\alpha]=[\alpha']$, i.e. $\alpha - \alpha' = \partial\beta$, then $\phi(\alpha)-\phi(\alpha') = \phi(\partial\beta) = \partial'\phi\beta$

$\Rightarrow [\phi(\alpha)] = [\phi(\alpha')]$

If $\phi, \phi' : C_* \to C_*'$ are two chain maps. a chain homotopy from ϕ to ϕ' consists of homomorphisms $K_n : C_n \to C_{n+1}'$ s.t. $\partial' K + K\partial' = \phi - \phi'$

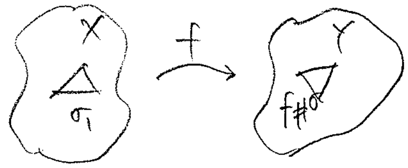
Observe: If ϕ is chain homotopic to ϕ', then $\phi_* = \phi_*' : H_n \to H_n'$

Pf: if $\alpha \in C_n$ and $\partial\alpha = 0$, then

$$\partial' K\alpha + K\underset{0}{\cancel{\partial\alpha}} = \phi\alpha - \phi'\alpha$$

Obser 1) A continuous map $f: X \to Y$ induces a chain map $f_\#: C_*(X) \to C_*(Y)$

If $\sigma: \Delta_* \to X$, then $f_\#(\sigma) = f \circ \sigma : \Delta_n \to Y$



Easy to see that $f_\#$ is a chain map

So we get $f_*: H_n(X) \to H_n(Y)$

- If $g: Y \to Z$ then $(g\circ f)_* = g_* \circ f_*$
- $(id_X)_* = id_{H_n(X)}$

$\Rightarrow$ Singular homology is a functor from (Top spaces, continuous maps) $\to$ (Graded abelian groups).

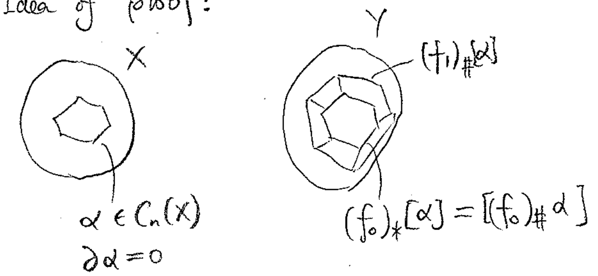
Prop: If $f, f': X \to Y$ are homotopic, then $f_\#, f'_\# = C_*(X) \to C_*(Y)$ are chain homotopic, so $f_* = f'_*: H_n(X) \to H_n(Y)$.

$\Rightarrow H_n$ is invariant under homotopy equiv.

10/3 Homotopy Invariance of homology

Prop: If $F: [0,1]\times X \to Y$ is a homotopy from f_0 to f_1 then $(f_0)_* = (f_1)_*: H_*(X) \to H_*(Y)$

Idea of proof:



Difficulty $I \times \Delta_n$ is not a simplex

For $i = 0, \dots, n$. let $\phi_i: \Delta_{n-1} \to \Delta_n$

$(t_0, \dots, t_{n-1}) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$

$\sigma: \Delta_n \to X$

$$\partial\sigma = \sum_{i=0}^n (-1)^i \sigma\circ\phi_i$$

Lemma: For $n = 0, 1, \dots$

$\exists\ P_n \in C_{n+1}(I \times \Delta_n)$ s.t.

- $\partial P_0 = \{1\} - \{0\}$ $\quad (P_0 = I \cong \Delta_1)$
- If $n \geq 0$, then $(P_{-1} = 0)$

$$\partial P_n = \{1\}\times id_{\Delta_n} - \{0\}\times id_{\Delta_n} - \sum_{i=0}^n (-1)^i (id_I \times \phi_i)_\# P_{n-1} \quad ①$$

n=2


Proof: Later

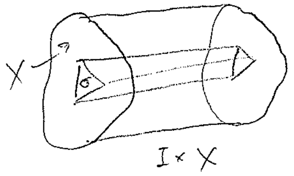
Notation: $i_0: X \to I \times X \quad x \mapsto (0, x)$
$i_1: X \to I \times X \quad x \mapsto (1, x)$

Lemma: $\exists\ K: C_*(X) \to C_{*+1}(I \times X)$ s.t.

$$\partial K + K\partial = (i_1)_\# - (i_0)_\# \quad ②$$

(so, $(i_0)_* = (i_1)_*: H_*(X) \to H_*(I \times X)$)

Proof:



Given $\sigma: \Delta_n \to X$. define

$K(\sigma) := (id\times\sigma)_\#(P_n) \in C_{n+1}(I\times X)$

Eqn (2) eval at σ follows by applying $(id\times\sigma)_\#$ to eqn (1)

$$\begin{aligned}\partial K\sigma &= \partial(id\times\sigma)_\# P_n \\ &= (id\times\sigma)_\# \partial P_n \\ &= (id\times\sigma)_\#(\{1\}\times id_{\Delta_n}) - (id\times\sigma)_\#(\{0\}\times id_{\Delta_n}) \\ &\quad - (id\times\sigma)_\# \sum_{i=0}^n (-1)^i (id\times\phi_i)_\# P_{n-1} \\ &= (i_1)_\#\sigma - (i_0)_\#\sigma - \underbrace{\sum_{i=0}^n (-1)^i \underbrace{(id\times\sigma\circ\phi_i)_\# P_{n-1}}_{K(\sigma\circ\phi_i)}}_{K\partial\sigma}\end{aligned}$$

Proof of prop: Define $J: C_*(X) \to C_{*+1}(Y)$ by $J := F_\# K$

claim $\partial J + J\partial = (f_1)_\# - (f_0)_\#$

$$\begin{aligned}\partial J + J\partial &= \partial F_\# K + F_\# K\partial \\ &= F_\#(\partial K + K\partial) \\ &= F_\#((i_1)_\# - (i_0)_\#) \\ &= (F\circ i_1)_\# - (F\circ i_0)_\# \\ &= (f_1)_\# - (f_0)_\#\end{aligned}$$

easier ↘

$(i_0)_* = (i_1)_*: H_*(X) \to H_*(I\times X)$

$F_*\circ(i_0)_* = F_*\circ(i_1)_*: H_*(X) \to H_*(Y)$

$\Rightarrow (f_0)_* = (f_1)_*$ ∎

Proof of Lemma1: We will prove that a required P_n exists by induction on n.

$n=0$. $P_0 : \Delta_1 \longrightarrow I$ is our identification.

let $n>0$, Suppose $P_0, \dots, P_{n-1}$ have been constructed satisfying eqn (1)

We need to show that $\exists\, P_n$ satisfies (1).

Since $I \times \Delta_n$ is contractible, $H_n(I\times\Delta_n)=0$

Then enough to show that RHS of (1) is a cycle.

$$\partial\,(\text{RHS of }(1)) = \sum_{i=0}^{n} (-1)^i \{1\}\times\phi_i - \sum_{i=0}^{n}(-1)^i\{0\}\times\phi_i - \sum_{i=0}^{n}(-1)^i (id_I\times\phi_i)_{\#}\,\partial P_{n-1}$$

$$= \sum_{i=0}^{n}(-1)^i\Big(\cancel{\{1\}\times\phi_i} - \cancel{\{0\}\times\phi_i} - (id_I\times\phi_i)_{\#}\Big[\cancel{\{1\}\times id_{\Delta_{n-1}}} - \cancel{\{0\}\times id_{\Delta_{n-1}}} - \cancel{\sum_{j=0}^{n-1}(-1)^j (id_I\times\phi_j)_{\#} P_{n-2}}\Big]\Big)$$

$\swarrow$ 0 as $\partial^2=0$

$= 0$

Acyclic models (this trick's name).

10/5 Mayer-Vietoris Sequence

$X = A\cup B$ $\quad$ A, B open

relate $H_*(X)$ to $H_*(A)$, $H_*(B)$ and $H_*(A\cap B)$

Def A sequence of abelian groups and homomorphisms

$$\cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \rightarrow \cdots$$

is exact if for each n, $\operatorname{Ker}(f_n) = \operatorname{Im}(f_{n+1})$

- If $0\rightarrow A \rightarrow 0$ is exact, then $A=0$
- If $0 \rightarrow A \xrightarrow{f} B \rightarrow 0$ is exact, then $A \cong B$ via f
- A short exact sequence is an exact sequence $0\rightarrow A\rightarrow B\rightarrow C\rightarrow 0$

If A, B, C are torsion free, then $B \simeq A\oplus C$ (noncanonically)

Note that for any A, C, the seq.

$$\begin{tikzcd} & a \arrow[r, mapsto] & (a,0) & & \\ 0 \arrow[r] & A \arrow[r] & A\oplus C \arrow[r] & C \arrow[r] & 0 \\ & & (a,c) \arrow[r, mapsto] & c & \end{tikzcd}$$

is exact

eg

$$\begin{tikzcd} & & {[x]} \arrow[r, mapsto] & {[x]} & \\ 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/4 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \\ & {[x]} \arrow[r, mapsto] & {[2x]} & & \end{tikzcd}$$

exact

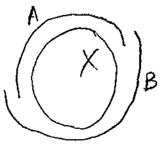
Thm There is a (long) exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A\cap B) \arrow[r, "\ominus"] & H_n(A)\oplus H_n(B) \arrow[r, "\oplus"] & H_n(X) \arrow[dll, "\delta"] & \\ & H_{n-1}(A\cap B) \arrow[r] & \cdots & \cdots & \\ & & \cdots \arrow[r] & H_0(X) \arrow[r] & 0 \end{tikzcd}$$

Rmk: If $0\rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is a short exact seq., then $C \cong B/f(A)$

$$\begin{tikzcd} & (\alpha,\beta) \arrow[r, mapsto] & (i_A^X)_*\alpha + (i_B^X)_*\beta \\ H_n(A\cap B) \arrow[r] & H_n(A)\oplus H_n(B) \arrow[r] & H_n(X) \\ \alpha \arrow[r, mapsto] & \big((i_{A\cap B}^A)_*\alpha,\ -(i_{A\cap B}^B)_*\alpha\big) & \end{tikzcd}$$

Example: $X = S^1$.



$A \simeq pt$

$B \simeq pt$

$A\cap B \simeq pt \sqcup pt$

$$\begin{tikzcd} & \cancel{H_2(A)\oplus H_2(B)} \arrow[r] & H_2(S^1) \simeq 0 \arrow[dll] \\ \cancel{H_1(A\cap B)} \arrow[r] & \cancel{H_1(A)\oplus H_1(B)} \arrow[r] & H_1(S^1) \simeq \mathbb{Z} \arrow[dll] \\ H_0(A\cap B) \arrow[r] & H_0(A)\oplus H_0(B) \arrow[r] & H_0(S^1) \arrow[r] & 0 \\ \mathbb{Z}\oplus\mathbb{Z} \arrow[u, phantom, "\cong"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[u, phantom, "\cong"] & \simeq \mathbb{Z} \end{tikzcd}$$

0 $\quad$ 0 $\quad$ 0

$(m,n) \longmapsto (m+n, -m-n)$

$(a,b) \longmapsto a+b$

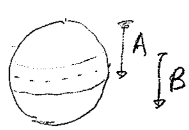
Conclusion $H_n(S^1) \simeq \begin{cases} \mathbb{Z} & \text{if } n=0,1 \\ 0 & \text{if } n>1 \end{cases}$

Example If $k\geq 1$, then $H_*(S^k) \cong \begin{cases} \mathbb{Z} & \text{if } *=0,k \\ 0 & \text{otherwise} \end{cases}$

Pf Induction on k, $k=1$ done.

let $k>1$, assume true for $k-1$.

$X = S^k$



$A \simeq pt$

$B \simeq pt$

$A\cap B \simeq S^{k-1}$

Assume $n\geq 2$.

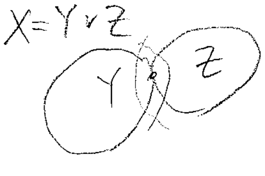
$$\begin{tikzcd} H_n(A)\oplus H_n(B) \simeq 0 \arrow[r] & H_n(S^k) \arrow[dl] & \\ H_{n-1}(A\cap B) \arrow[r] & H_{n-1}(A)\oplus H_{n-1}(B) \simeq 0 \arrow[r] & H_{n-1}(S^k) \arrow[dll] \\ \cdots & & \end{tikzcd}$$

So, $H_n(S^k) \simeq H_{n-1}(S^{k-1})$ ∎

Example: Suppose Y, Z path connected, "reasonable" spaces

Then $H_n(Y \vee Z) = \begin{cases} \mathbb{Z} & \text{if } n=0 \\ H_n(Y) \oplus H_n(Z) & \text{if } n>0 \end{cases}$

$X = Y \vee Z$



Y, Z are "reasonable", $\exists$ nbhds A of Y, B of Z s.t.

$A \simeq Y$, $B \simeq Z$ $A \cap B \simeq pt$

$$\begin{tikzcd} H_n(pt) \arrow[r] & H_n(Y) \oplus H_n(Z) \arrow[r, "f"] & H_n(Y \vee Z) \arrow[dll, "\delta"'] \\ H_{n-1}(pt) \arrow[r, "g"] & H_{n-1}(Y) \oplus H_{n-1}(Z) & \end{tikzcd}$$

If $n \geq 2$, then clear

If $n=1$, then f is inj. ($\because H_1(pt)=0$)

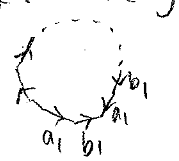
f is surj. since g is inj.

$$\begin{tikzcd} H_0(pt) \arrow[r, "g"] \arrow[d, phantom, "\cong"] & H_0(Y) \oplus H_0(Z) \arrow[d, phantom, "\cong"] \\ \mathbb{Z} & \mathbb{Z} \oplus \mathbb{Z} \\ m \arrow[r, mapsto] & (m, -m) \end{tikzcd}$$

$$\underset{H_1(Y \vee Z)}{\overset{\shortparallel}{\mathrm{Im} f}} = \ker \delta \Leftarrow \mathrm{Im}\,\delta = \ker g = 0$$

---

eg. $H_n(\Sigma_g) \cong \begin{cases} \mathbb{Z} & \text{if } n=0,2 \\ \mathbb{Z}^{2g} & \text{if } n=1 \\ 0 & \text{if } n>2 \end{cases}$



A = interior of $4g$-gon

B = complement of center of $4g$-gon

$A \simeq pt$ $B \simeq \bigvee_{2g} S^1$ $A \cap B \simeq S^1$

$$\begin{tikzcd} & \cancel{H_2(pt) \oplus H_2(\bigvee_{2g} S^1)} \arrow[r] \arrow[d, phantom, "0"] & H_2(\Sigma_g) \simeq \mathbb{Z} \arrow[dll] \\ H_1(S^1) \simeq \mathbb{Z} \arrow[r, "0"] & H_1(pt) \oplus H_1(\bigvee_{2g} S^1) \simeq \mathbb{Z}^{2g} \arrow[r, "\text{surj}"'] \arrow[d, phantom, "0"] & H_1(\Sigma_g) \simeq \mathbb{Z}^{2g} \arrow[dll] \\ H_0(S^1) \arrow[r, "\text{inj.}"'] \arrow[d, phantom, "\cong"] & H_0(pt) \oplus H_0(\bigvee_{2g} S^1) \arrow[r] \arrow[d, phantom, "\cong"] & H_0(\Sigma_g) \arrow[r] \arrow[d, phantom, "\shortparallel"] & 0 \\ \mathbb{Z} & \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} & \end{tikzcd}$$

Exercise: δ is well-defined and the seq is exact.

10/8 Mayer-Vietoris Seq.

$X = A \cup B$ A, B open

long exact seq

$$\begin{tikzcd} H_n(A \cap B) \arrow[r, "\ominus"] & H_n(A) \oplus H_n(B) \arrow[r, "\oplus"] & H_n(A \cup B) \arrow[dll, "\delta"'] \\ H_{n-1}(A \cap B) \arrow[r] & \cdots & \end{tikzcd}$$

let $C'_*(X)$ denote the subcomplex of $C_*(X)$ generated by maps $\sigma : \Delta_n \to A$ and $\sigma : \Delta_n \to B$

Subdivision Lemma The inclusion $C'_*(X) \to C_*(X)$ induces an isomorphism on homology. i.e. every element of $H_*(X)$ can be represented by a cycle in $C'_*(X)$ and if a cycle in $C'_*(X)$ is nullhomologous in $C_*(X)$ then it is nullhomologous in $C'_*(X)$

Proof (Later, some of it)

There is a short exact sequence of chain complexes.

$$0 \to C_*(A \cap B) \xrightarrow{\ominus} C_*(A) \oplus C_*(B) \xrightarrow{\oplus} C'_*(X) \to 0$$

$$\alpha \longmapsto (\alpha, -\alpha)$$

$$(\alpha, \beta) \longmapsto (\alpha + \beta)$$

Lemma A short exact sequence of chain complexes

$$0 \to C_* \xrightarrow{f} C'_* \xrightarrow{g} C''_* \to 0$$

induces a long exact sequence in homology.

$$\begin{tikzcd} H_n \arrow[r, "f_*"] & H'_n \arrow[r, "g_*"] & H''_n \arrow[dll, "\delta"'] \\ H_{n-1} \arrow[r] & \cdots & \end{tikzcd}$$

These two lemmas imply MVS.

Definition of δ. let $\alpha \in C''_n$ $\partial\alpha = 0$.

$$\begin{tikzcd} 0 \arrow[r] & C_n \arrow[r, "f"] \arrow[d, "\partial"] & C'_n \arrow[r, "g"] \arrow[d, "\partial'"] & \alpha \in C''_n \arrow[r] \arrow[d, "\partial''"] & 0 \\ 0 \arrow[r] & \gamma \in C_{n-1} \arrow[r] \arrow[d, "\partial"] & C'_{n-1} \arrow[r] \arrow[d, "\partial'"] & C''_{n-1} \arrow[r] \arrow[d, "\partial''"] & 0 \\ 0 \arrow[r] & C_{n-2} \arrow[r] & C'_{n-2} \arrow[r] & C''_{n-2} \arrow[r] & 0 \end{tikzcd}$$

Define $\delta[\alpha] = \gamma$

$\alpha = g(\beta)$ $g(\partial\beta) = \partial g\beta = \partial\alpha = 0$

$\partial\beta = f(\gamma)$ $f(\partial\gamma) = \partial f\gamma = \partial\partial\beta = 0 \Rightarrow \partial\gamma = 0$

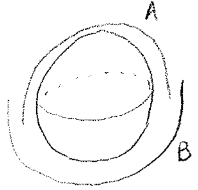
$\delta : H_n(X) \longrightarrow H_n(A\cap B)$ is described as follows. Given $\eta \in H_n(X)$. use subdivision lemma to represent

$\eta = [\alpha+\beta]$ where $\alpha \in C_n(A)$, $\beta \in C_n(B)$.

$\partial(\alpha+\beta) = 0 \Rightarrow \partial\alpha = -\partial\beta \in C_{n-1}(A\cap B)$

$\delta\eta = [\partial\alpha] \in H_{n-1}(A\cap B)$

Example $X = S^2$



$\delta : H_2(S^2) \longrightarrow H_1(A\cap B) \simeq H_1(S^1)$

Here is an element $\eta \in H_2(S^2)$

$\eta = \alpha + \beta$

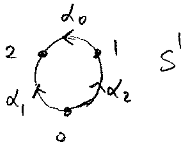
$\alpha : \Delta_2 \xrightarrow{\simeq}$ northern hemisphere

$\beta =$ minus $(\Delta_2 \xrightarrow{\simeq}$ southern hemisphere$)$

$\alpha \in C_2(A)$, $\beta \in C_2(B)$

$\delta[\eta] = [\partial\alpha]$

$= [\alpha_0 - \alpha_1 + \alpha_2]$

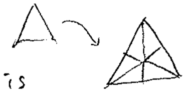


$=$ a generator of $H_1(S^1)$

$\Rightarrow [\eta]$ is a generator of $H_2(S^2)$

Lemma There is a chain map

$S : C_*(X) \longrightarrow C_*(X)$ with the following properties.



1) For any $\alpha \in C_*(X)$, if n is sufficiently large, then $S^n\alpha \in C'_*(X)$
2) There is a chain homotopy
   $T : C_*(X) \longrightarrow C_{*+1}(X)$ with
   $\partial T + T\partial = 1 - S$
3) S sends C'_* to C'_* and
   T sends C'_* to C'_{*+1}

Proof Tedious.

Claim: Tedious Lemma $\Rightarrow$ Subdivison Lemma.

10/10 Subdivision Lemma

$X = A \cup B$ A, B open

$C'_*(X) =$ image of $C_*(A) \oplus C_*(B)$ in $C_*(X)$

then the inclusion $C'_*(X) \longrightarrow C_*(X)$ induces an isomorphism on homology

SubLemma There exist a chain map $s : C_*(X) \circlearrowleft$ and a chain homotopy $T : C_*(X) \longrightarrow C_{*+1}(X)$ s.t.

1) $\forall \alpha \in C_*(X)$ if $n >> 0$ then $S^n\alpha \in C'_*(X)$
2) $\partial T + T\partial = 1 - s$
3) S sends C'_* to C'_* and
   T sends C'_* to C'_{*+1}

Pf of subdivision lemma

Note $\exists T_n : C_*(X) \longrightarrow C_{*+1}(X)$ s.t.

$\partial T_n + T_n\partial = 1 - S^n$

and $T_n : C'_*(X) \longrightarrow C'_{*+1}(X)$

$(T_n = TS^{n-1} + TS^{n-2} + \dots + T)$

A) Show $H'_* \longrightarrow H_*(X)$ is surjective

let $[\alpha] \in H_k(X)$, so $\alpha \in C_k(X)$ $\partial\alpha = 0$

By (1), $S^n\alpha \in C'_*(X)$ for some n.

$$(\partial T_n + T_n\partial)\alpha = (1 - S^n)\alpha$$
$$\partial T_n\alpha = \alpha - S^n\alpha$$

so $[\alpha] = [S^n\alpha] \in H_k(X)$

B) Show $H'_* \longrightarrow H_*(X)$ is inj.

suppose $\alpha = \partial\beta$ where $\alpha \in C'_k$, $\beta \in C_{k+1}$

By (1), $S^n\beta \in C'_{k+1}$ for some n.

$$\partial(\partial T_n + T_n\partial)\beta = \partial(1 - S^n)\beta$$
$$0 + \partial T_n\alpha = \alpha - \partial(S^n\beta)$$
$$\alpha = \partial\underbrace{(\underset{\substack{\uparrow \\ C'_*}}{T_n\alpha} + \underset{\substack{\uparrow \\ C'_*}}{S^n\beta})}_{\in C'_{k+1}}$$

▨

Generalized Jordan Curve Thm

If $A \subseteq S^n$ and $A \simeq S^k$, then

(1) $\tilde{H}_*(S^n \setminus A) \cong \begin{cases} \mathbb{Z} & \text{if } * = n-k-1 \\ 0 & \text{otherwise.} \end{cases}$ $\boxed{\tilde{H}_0(S^2 \setminus S^1) \cong \mathbb{Z}}$

$\tilde{H}_*(X)$ "reduced homology" is the homology of "augmented chain complex"

$$C_n(X) \xrightarrow{\partial} C_{n-1}(X) \longrightarrow \dots \longrightarrow C_0(X) \longrightarrow \mathbb{Z} \rightarrow$$
$$p \longmapsto 1$$

$\tilde{H}_n(X) = H_n(X)$ when $n > 0$

If X has k path connected component, then $\tilde{H}_0(X) \simeq \mathbb{Z}^{k-1}$

MVS also works with $\tilde{H}_*$

Proof: Will also show that if $A \subseteq S^n$ and

(2) $A \simeq I^k$, then $\tilde{H}_*(S^n \setminus A) = 0$

Will prove (1) & (2) together by induction on k.

If $k=0$, then both statements are known. because $S^n \setminus \{pt\} \approx \mathbb{R}^n$ and $S^n \setminus \{2\,pt\} \simeq S^{n-1}$

Let $k > 0$ and suppose we know (1) and (2) for $k-1$. Prove (2): suppose

$W \subseteq S^n$, $W \simeq I^k$ $\tilde{H}_j(S^n \setminus W) \neq 0$ for some j

Write $I^k = [0, \frac{1}{2}] \times I^{k-1} \cup [\frac{1}{2}, 1] \times I^{k-1}$

$A = S^n \setminus ([0, \frac{1}{2}] \times I^{k-1})$

$B = S^n \setminus ([\frac{1}{2}, 1] \times I^{k-1})$

$A \cup B = S^n \setminus (\{\frac{1}{2}\} \times I^{k-1})$

$A \cap B = S^n \setminus W$

$$\dots \longrightarrow \overset{\overset{0}{\|} \leftarrow \text{by ind}^{\text{n}}}{H_{j+1}(S^n \setminus \{\tfrac{1}{2}\} \times I^{k-1})}$$
$$\hookrightarrow \underset{\text{by assumption } \neq 0}{H_j(S^n \setminus W)} \longrightarrow H_j(A) \oplus H_j(B) \longrightarrow \dots$$

Conclusion from MVS, α maps to something nonzero in at least one of

$H_j(S^n \setminus [0, \frac{1}{2}] \times I^{k-1})$, $H_j(S^n \setminus [\frac{1}{2}, 1] \times I^{k-1})$

is nonzero. Repeat this argument.

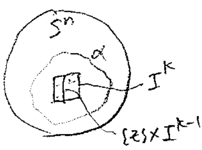
obtain a nested sequence of intervals.

$I = I_0 \supset I_1 \supset I_2 \supset \dots$ s.t. I_m has length 2^{-m}

and $H_j(S^n \setminus I_m \times I^{k-1}) \neq 0$.

$\exists z \in \bigcap_{i=0}^{\infty} I_i$ By induction, $H_j(S^n \setminus \{z\} \times I^{k-1}) = 0$

(Take $\underset{0 \neq}{[\alpha]} \in H_j(S^n \setminus I^k)$.)



$\exists \beta \in C_{k+1}(S^n \setminus \{z\} \times I^{k-1})$

with $\partial \beta = \alpha$ β is a finite linear combination of simplices.

By compactness argument, if $m >> 0$, then

$\beta \in C_{j+1}(S^n \setminus I_m \times I^{k-1})$

$\Rightarrow [\alpha] = 0$ in $H_j(S^n \setminus I_m \times I^{k-1})$ $\rightarrow\leftarrow$

so statement (2) holds for k.

To prove (1) for k $S^k \approx W \subset S^n$

$A = S^n \setminus$ (closed northern hemisphere in W)

$B = S^n \setminus$ (closed southern hemisphere in W)

$A \cup B = S^n \setminus$ (equatorial S^{k-1} in W)

$A \cap B = S^n \setminus W$

By MVS, it works.

10/12 Thm If $A \subseteq S^n$ and $A \simeq S^k$, then

$\tilde{H}_*(S^n \setminus A) = \begin{cases} \mathbb{Z} & \text{if } * = n-k-1 \\ 0 & \text{otherwise} \end{cases}$

We proved if $A \subseteq S^n$ and $A \approx I^k$ then $\tilde{H}_*(S^n \setminus A) = 0$.

Proof of Thm $U = S^n \setminus \{\text{closed northern hemis in } A \simeq S^k\}$

$V = S^n \setminus \{\text{closed southern hemis in } A \approx S^k\}$

$U \cup V = S^n \setminus \{\text{equatorial } S^{k-1} \text{ in } A\}$

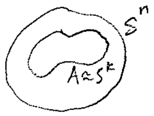
$U \cap V = S^n \setminus A$

$$\tilde{H}_{*+1}(U \cap V) \longrightarrow \overbrace{\tilde{H}_{*+1}(U) \oplus \tilde{H}_{*+1}(V)}^{\overset{0}{\|}} \longrightarrow \tilde{H}_{*+1}(U \cup V)$$
$$\hookrightarrow H_*(U \cap V) \longrightarrow \underbrace{\tilde{H}_*(U) \oplus \tilde{H}_*(V)}_{\underset{0}{\|}}$$

$$\tilde{H}_*(S^n \setminus \underset{S^k}{\underset{\approx}{A}}) \simeq \tilde{H}_{*+1}(S^n \setminus \underset{S^{k-1}}{\underset{\approx}{B}})$$

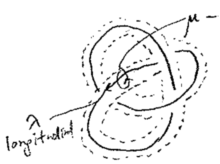
Thm follows by induction on k. ∎

Special case: A is a smooth submanifold of S^n, A diffeomorphic to S^k, i.e.
A is the image of a smooth embedding $S^k \to S^n$



Tubular neighborhood Thm
⇒ ∃ nbhd $\mathcal{U}$ of A such that

$$\begin{tikzcd} \mathcal{U} \arrow[r, phantom, "\simeq"] & S^1\times\mathbb{R}^2 \\ A \arrow[u, phantom, sloped, "\subset"] \arrow[r] & S^1\times\{0\} \arrow[u, phantom, sloped, "\subset"] \end{tikzcd}$$



let $V = S^3\setminus A$ $\quad \mathcal{U}\cup V = S^3$

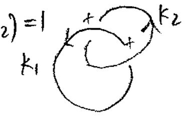
$\mathcal{U}\cap V \simeq S^1\times(\mathbb{R}^2\setminus\{0\})$
$\approx S^1\times S^1$

$$\begin{tikzcd} & & H_2(S^3)=0 \arrow[dll, bend right] \\ H_1(\mathcal{U}\cap V) \arrow[r, "\cong"] \arrow[d, equal] & H_1(\mathcal{U})\oplus H_1(V) \arrow[r] \arrow[d, equal] & H_1(S^3) \arrow[d, equal] \\ \mathbb{Z}\times\mathbb{Z} & \mathbb{Z}\qquad\mathbb{Z} & 0 \end{tikzcd}$$

$\mu \longmapsto (0, \pm 1) \Rightarrow \mu$ generates $H_1(S^3\setminus A)$
$\lambda \longmapsto (1, ?)$

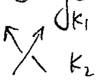
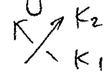
<u>Linking number</u>

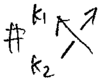
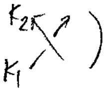
let $L = K_1 \sqcup K_2$ be a 2-component-(smooth) oriented link in S^3.

$lk(K_1,K_2)=1$ 

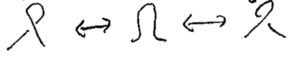
 $lk(K_1,K_2)=0$

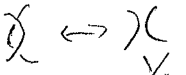
Define the linking number $lk(K_1,K_2)\in\mathbb{Z}$ as follows: Choose a generic projection of L.

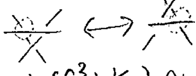
$lk(K_1,K_2) = ($#  + # 

− #  − #  $)/2.$

<u>Claim</u> This does not depend on the projection.

Deidemeister move : I  ✓

II  ✓

III  ✓

<u>Claim</u> The isomorphism $H_1(S^3\setminus K_1)\simeq\mathbb{Z}$ can be chosen so that $[K_2] = lk(K_1,K_2)$

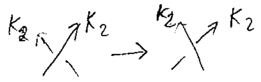
$$\begin{tikzcd} {[K_2]} \\ H_1(S^3\setminus K_1)\simeq\mathbb{Z} \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

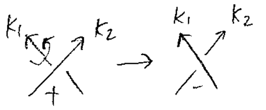
<u>Proof</u> Choose isom. $H_1(S^3\setminus K_1)\simeq\mathbb{Z}$ so that

 ↔ 1

Enough to show: observe

(1)  ⇒ $lk(K_1,K_2)=0$ and $[K_2]=0$

(2)  doesn't change $lk(K_1,K_2)$ or $[K_2]\in H_1(S^3\setminus K_1)$ ✓

(3)  decreases both $lk(K_1,K_2)$ and $[K_2]\in H_1(S^3\setminus K_1)$ by 1 ✓

The 3 observations imply the claim.

10/15 <u>Relative homology</u>

<u>Idea</u> Given $A \subset X$, relate $H_*(A)$ to $H_*(X)$ and $H_*(X/A)$ where X/A means $a\sim a'$ $\forall a,a'\in A$

<u>Def</u> $C_*(X,A) = C_*(X)/C_*(A)$
Since $\partial: C_*(A)\circlearrowleft$, it induces $\partial: C_*(X,A)\circlearrowleft$

<u>Def</u> <u>relative homology</u>

$$H_n(X,A) := \frac{\operatorname{Ker}(\partial: C_n(X,A)\to C_{n-1}(X,A))}{\operatorname{Im}(\partial: C_{n+1}(X,A)\to C_n(X,A))}$$

<u>Prop</u> There is a long exact sequence

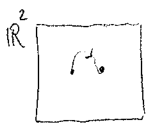
$$\begin{tikzcd} H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[dll, "\partial"] \\ H_{n-1}(A) \arrow[r] & \cdots & \\ & \cdots \arrow[r] & H_0(X,A) \arrow[r] & 0 \end{tikzcd}$$

<u>Proof</u>: There is a short exact sequence of chain complexes $0\to C_*(A)\to C_*(X)\to C_*(X,A)\to 0$
Apply homological algebra.
let $[\alpha]\in H_n(X,A)$, $\alpha\in C_n(X)$, $\partial\alpha\in C_{n-1}(A)$
$\partial[\alpha] = [\partial\alpha]\in H_{n-1}(A)$ ▨

<u>Examples</u> Suppose X is path connected, $A\neq\phi$.
Then $H_0(X,A)=0$

$\partial\sigma = p-a$
$[p]=0\in H_0(X,A)$

<u>Example</u> $X=\mathbb{R}^2$, $A=\{(0,0),(1,0)\}$



$H_1(X,A)=\mathbb{Z}$

$$\begin{tikzcd} H_1(\mathbb{R}^2) \arrow[r] \arrow[d, equal] & H_1(\mathbb{R}^2,A) \arrow[dl, bend left] \arrow[r, phantom, "\simeq\mathbb{Z}\ni a"] & {} \\ 0 & & \\ H_0(A) \arrow[r] \arrow[d, equal] & H_0(\mathbb{R}^2) \arrow[r] \arrow[d, equal] & H_0(\mathbb{R}^2,A) \arrow[d, equal] \\ \mathbb{Z}^2 & \mathbb{Z} & 0 \\ (a,b) \arrow[r, mapsto] & a+b & \end{tikzcd}$$

$a \longmapsto (a,-a)$

Def The pair (X,A) is "good" if there is an open set V s.t. $\bar{A} \subseteq V \subseteq X$ and V deformation retracts onto A.

Thm If (X,A) is good and $A \neq \emptyset$, then

$$H_n(X,A) = \tilde{H}_n(X/A)$$

Example Recompute $H_*(S^n)$

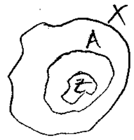
Use $S^n = D^n/\underbrace{\partial D^n}_{S^{n-1}}$

$$H_i(D^n) \longrightarrow \overset{\tilde{H}_i(S^n)}{\overset{\parallel}{H_i(D^n, S^{n-1})}} \hookrightarrow H_{i-1}(S^{n-1}) \longrightarrow H_{i-1}(D^n)$$

$\Rightarrow H_i(S^n) \cong H_{i-1}(S^{n-1})$ if $i > 1$

Excision $Z \subset A \subset X$ Suppose $\bar{Z} \subseteq \text{int}(A)$

Then $H_n(X,A) = H_n(X \setminus Z, A \setminus Z)$



Proof Use the subdivision lemma for $X = (X \setminus \bar{Z}) \cup \text{int}(A)$ (Details = exercise)

Long exact sequence of a triple

$A \subseteq B \subseteq X$

short exact sequence of chain complexes.

$$0 \to C_*(B,A) \to C_*(X,A) \to C_*(X,B) \to 0$$

$$\Rightarrow H_n(B,A) \to H_n(X,A) \to H_n(X,B) \hookrightarrow H_{n-1}(B,A) \to \cdots$$

Pf of Thm

$$\begin{tikzcd} H_n(X,A) \arrow[r, "(1)"] \arrow[d, "(*)"] & H_n(X,V) \arrow[d] & H_n(X-A, V-A) \arrow[l, "(2)"'] \arrow[d, "(3)"] \\ H_n(X/A, A/A) \arrow[r, "(5)"'] \arrow[d, phantom, "\cong" rotate=90] & H_n(X/A, V/A) & H_n(X/A - A/A, V/A - A/A) \arrow[l, "(4)"] \\ \tilde{H}_n(X/A) & & \end{tikzcd}$$

(1) is an isom: look at

$$H_n(V,A) \longrightarrow H_n(X,A) \to H_n(X,V) \hookrightarrow H_{n-1}(V,A)$$

$H_*(V,A) = 0$ because V deformation retracts onto A

(2) is an isom by excision.

(3) is induced by a homeomorphism of pairs.

(4) is an isom by excision.

(5) LES of triple.

Since they are all isom, $(*)$ is an isom.

10/17

Thm If X is a Δ-complex, then $H_*(X) = H_*^{\Delta}(X)$

Lemma $H_*(\Delta_n, \partial\Delta_n) \simeq \begin{cases} \mathbb{Z} & \text{if } * = n \\ 0 & \text{otherwise} \end{cases}$

Moreover, $[\text{id}_{\Delta_n} : \Delta_n \to \Delta_n]$ is a generator of $H_n(\Delta_n, \partial\Delta_n)$

Proof: $(\Delta_n, \partial\Delta_n)$ is a "good pair"

so $H_*(\Delta_n, \partial\Delta_n) \simeq \tilde{H}_*(\Delta_n/\partial\Delta_n) \cong \tilde{H}_*(S^n)$

$$\simeq \begin{cases} \mathbb{Z} & \text{if } * = n \\ 0 & \text{otherwise} \end{cases}$$

To prove the claim about generator,

let $\Lambda := \partial\Delta_n \setminus \text{int}(\underbrace{0\text{-face}}_{\approx \Delta_{n-1}})$

Δ_n deformation retracts onto Λ

$\partial\Delta_n = \{(t_0, \dots, t_n) \in \Delta_n \mid \exists i \text{ s.t. } t_i = 0\}$

$\Lambda = \{(t_0, \dots, t_n) \in \partial\Delta_n \mid t_0 \neq 0 \text{ unless } t_i = 0 \text{ for some } i > 0\}$

$F : I \times \Delta_n \longrightarrow \Delta_n$

$$F(s, t_0, \dots, t_n) = \begin{cases} (t_0 + s, t_1 - \frac{s}{n}, \dots, t_n - \frac{s}{n}) & \text{if } s \le \min(nt_1, \dots, nt_n) = s_0 \\ (t_0 + s_0, t_1 - \frac{s_0}{n}, \dots, t_n - \frac{s_0}{n}) & \end{cases}$$

$F(0, t_0, \dots, t_n) = (t_0, \dots, t_n)$

$F(1, t_0, \dots, t_n) \in \Lambda$

if $(t_0, \dots, t_n) \in \Lambda$ then $F(s, t_0, \dots, t_n) = (t_0, \dots, t_n)$

Consequently $H_*(\Delta_n, \Lambda) = 0$

(General fact if $(X,A) \underset{\uparrow}{\simeq} (Y,B)$
homotopy equivalence pair

then $H_*(X,A) \simeq H_*(Y,B)$

If X deformation retracts onto A, then $(X,A) \simeq (X,X)$ $H_*(X,X) = 0$)

Long exact sequence of triple $(\Delta_n, \partial\Delta_n, \Lambda)$

$$H_*(\partial\Delta_n, \Lambda) \longrightarrow \underset{\overset{\parallel}{0}}{H_*(\Delta_n, \Lambda)} \longrightarrow H_*(\Delta_n, \partial\Delta_n) \overset{\partial}{\hookrightarrow} H_{*-1}(\partial\Delta_n, \Lambda) \longrightarrow \underset{\overset{\parallel}{0}}{H_{*-1}(\Delta_n, \Lambda)}$$

$$\partial : H_*(\Delta_n, \partial\Delta_n) \xrightarrow{\simeq} H_{*-1}(\partial\Delta_n, \Lambda) \simeq \tilde{H}_{*-1}(\partial\Delta_n/\Lambda)$$

$$\cong \tilde{H}_{*-1}(\Delta_{n-1}/\partial\Delta_{n-1})$$

$$\cong H_{*-1}(\Delta_{n-1}, \partial\Delta_{n-1})$$

Prove claim by induction on n.

n=1 ✓

let $n>1$, assume true for $n-1$

ETS the above isomorphism

$$H_n(\Delta_n, \partial\Delta_{n-1}) \xrightarrow{\simeq} H_{n-1}(\Delta_{n-1}, \partial\Delta_{n-1})$$

sends $[id_{\Delta_n}] \longmapsto \pm[id_{\Delta_{n-1}}]$

$$\begin{tikzcd}
 & & \Delta_{n-1}/\partial\Delta_{n-1} \arrow[d, phantom, "\cong"] \\
H_*(\Delta_n, \partial\Delta_n) \arrow[r, "\partial"] \arrow[d, phantom, "\ni"] & H_{*-1}(\partial\Delta_n, \Lambda) \arrow[r] & \tilde{H}_{*-1}(\partial\Delta_n/\Lambda) \\
{[id_{\Delta_n}]} \arrow[r, mapsto] & {[\varphi_0 : \Delta_{n-1} \to \partial\Delta_n]} \arrow[r, mapsto] & {[\Delta_{n-1} \xrightarrow{q} \frac{\Delta_{n-1}}{\partial\Delta_{n-1}}]}
\end{tikzcd}$$

<u>Proof of thm</u> There is a chain map

$$\Phi : C_*^{\Delta}(X) \longrightarrow C_*(X) \quad \text{by def}^{\underline{n}}$$

Induces $\Phi_* : H_*^{\Delta}(X) \longrightarrow H_*(X)$

will show that Φ_* is an isomorphism

Use induction on $\dim(X)$

$X^{(n)} :=$ n-skeleton of X

$= \{$union of i-simplices in X for $i \leq n\}$

Show $\Phi_* : H_*^{\Delta}(X^{(n)}) \longrightarrow H_*(X^{(n)})$

is an isom. by induction on n.

If A is a sub Δ-complex of X,

define $C_*^{\Delta}(X, A) = C_*^{\Delta}(X)/C_*^{\Delta}(A)$

$H_*^{\Delta}(X, A) :=$ homology of $C_*^{\Delta}(X, A)$

$$\begin{tikzcd}
0 \arrow[r] & C_*^{\Delta}(X^{(n-1)}) \arrow[r] \arrow[d, "\Phi"] & C_*^{\Delta}(X^{(n)}) \arrow[r] \arrow[d, "\Phi"] & C_*^{\Delta}(X^{(n)}, X^{(n-1)}) \arrow[r] \arrow[d, "\Phi"] & 0 \\
0 \arrow[r] & C_*(X^{(n-1)}) \arrow[r] & C_*(X^{(n)}) \arrow[r] & C_*(X^{(n)}, X^{(n-1)}) \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & H_{*+1}^{\Delta}(X^{(n)}, X^{(n-1)}) \arrow[r] \arrow[d, "\Phi_*", "\cong"', "\text{by Lemma}"'] & H_*^{\Delta}(X^{(n-1)}) \arrow[r] \arrow[d, "\Phi_*", "\cong"', "\text{by inductn}"'] & H_*^{\Delta}(X^{(n)}) \arrow[r] \arrow[d, "\Phi_*"] & H_*^{\Delta}(X^{(n)}, X^{(n-1)}) \arrow[r] \arrow[d, "\Phi_*", "\cong"', "\text{by Lemma}"'] & H_{*-1}^{\Delta}(X^{(n-1)}) \arrow[d, "\Phi_*"', "\text{by inductn}"] \\
\cdots \arrow[r] & H_{*+1}(X^{(n)}, X^{(n-1)}) \arrow[r] & H_*(X^{(n-1)}) \arrow[r] & H_*(X^{(n)}) \arrow[r] & H_*(X^{(n)}, X^{(n-1)}) \arrow[r] & H_{*-1}(X^{(n-1)})
\end{tikzcd}$$

This diagram commutes because LES on homology induced by SES is natural

<u>5-Lemma</u> If the diagram

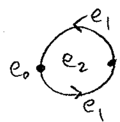
$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"'] & E \arrow[d, "\varepsilon"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

commutes and the row are exact, and if $\alpha, \beta, \gamma, \delta, \varepsilon$ are isom., then γ is an isom. (exercise)

10/29

<u>Goal</u> How to compute homology of a CW-complex

<u>Example</u> $X = \mathbb{RP}^2$



chain complex should have one generator for each cell

$\partial e_0 = 0$

$\partial e_1 = e_0 - e_0 = 0$

$\partial e_2 = 2e_1$

$$H_*(\mathbb{RP}^2) = \left\{ \begin{array}{ll} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2\mathbb{Z} & * = 1 \\ 0 & * \geq 2 \end{array} \right\} (*)$$

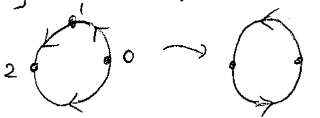
<u>Proof of (*)</u> $S^1 \subset \mathbb{RP}^2$ coming from boundary of disc

$$\begin{tikzcd}
0 \simeq H_2(S^1) \arrow[r] & \overset{=0}{H_2(\mathbb{RP}^2)} \arrow[r] & H_2(\mathbb{RP}^2, S^1) \simeq \mathbb{Z} \arrow[dll, bend left] \\
\mathbb{Z} \simeq H_1(S^1) \arrow[r] & \underset{\simeq \mathbb{Z}/2\mathbb{Z}}{H_1(\mathbb{RP}^2)} \arrow[r] & H_1(\mathbb{RP}^2, S^1) \simeq 0
\end{tikzcd}$$

$$\begin{tikzcd}
H_*(\mathbb{RP}^2, S^1) \arrow[r, phantom, "\simeq"] & \tilde{H}_*(\mathbb{RP}^2/S^1) \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\parallel"] & \tilde{H}_*(S^2) \\
H_*(D^2, \partial D^2) \arrow[r, phantom, "\simeq"] \arrow[u, "q", "\cong"'] & \tilde{H}_*(D^2/\partial D^2) &
\end{tikzcd}$$

Compute $\partial : \underbrace{H_2(\mathbb{RP}^2, S^1)}_{\text{generated by an identification } \Delta_2 \to D^2} \longrightarrow H_1(S^1)$

$1 \longmapsto \pm 2$



<u>Def</u> If $f : S^n \longrightarrow S^n$, define the <u>degree</u> $\deg(f) \in \mathbb{Z}$ to be the integer s.t. $f_* : \underset{\mathbb{Z}}{\underset{\cong}{\tilde{H}_n(S^n)}} \longrightarrow \underset{\mathbb{Z}}{\underset{\cong}{\tilde{H}_n(S^n)}}$ is multiplication by $\deg(f)$.

Note: A matrix $A \in GL(n+1, \mathbb{R})$ induces a map

$$f_A : S^n \longrightarrow S^n \quad f_A(v) = \frac{Av}{\|Av\|}$$

<u>Claim</u> - If $\det(A) > 0$, then $\deg(f_A) = 1$

- If $\det(A) < 0$, then $\deg(f_A) = (-1)$

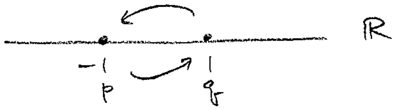
Note: $GL(n+1, \mathbb{R})$ has just 2 components $\{\det A > 0\}$, $\{\det A < 0\}$.

Enough to compute $\deg(f_A)$ for one A in each component: $\deg(f_1) = \deg(id_{S^1}) = 1$ ✓

let $A = \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ claim: $\deg(f_A) = -1$

Induction on n

$n=0$



$\tilde{H}_0(S^0)$ is generated by $[p-q]$

$(f_A)_*[p-q]=[q-p]=-[p-q]$

$n>0$, assume true for $n-1$

$S^n = D^n_+ \cup D^n_-$

$D^n_+ = \{(x_1,\dots,x_{n+1})\in S^n \mid x_{n+1} \geq 0\}$

$D^n_- = \{(x_1,\dots,x_{n+1})\in S^n \mid x_{n+1} \leq 0\}$

f_A sends $D^n_\pm$ to itself.

$$\begin{tikzcd}
\tilde{H}_n(S^n) \arrow[r, phantom, "\simeq"] \arrow[d, "(f_A)_*"] & \tilde{H}_n(S^n, D^n_-) \arrow[r, phantom, "\simeq"] & \tilde{H}_n(D^n_+, \partial D^n_+) \arrow[r, phantom, "\simeq"] & \tilde{H}_{n-1}(S^{n-1}) \arrow[d, bend left, "(f_A)_*"] \\
\tilde{H}_n(S^n) \arrow[r, phantom, "\simeq"] & \tilde{H}_n(S^n, D^n_-) \arrow[r, phantom, "\simeq"] & \tilde{H}_n(D^n_+, \partial D^n_+) \arrow[r, phantom, "\simeq"] & \tilde{H}_{n-1}(S^{n-1})
\end{tikzcd}$$

$(f_A)_* = -1$ by induction

Example let $a: S^n \to S^n$ be the antipodal map $a(x_1,\dots,x_{n+1}) := (-x_1,\dots,-x_{n+1})$

Then $\deg(a) = (-1)^{n+1}$.

Hairy ball Thm If V is a continuous vector field on S^n and n is even, then V has a zero.

Recall a vector field on S^n assign to each $x \in S^n \subset \mathbb{R}^{n+1}$, a vector $V(x) \in \mathbb{R}^{n+1}$ with $\langle x, V(x)\rangle = 0$

Pf Suppose V is a nowhere vanishing vector field on S^n. Define a homotopy from $\mathrm{id}: S^n \to S^n$ to $a: S^n \to S^n$. Existence of such a homotopy implies $1 = \deg(\mathrm{id}) = \deg(a) = (-1)^{n+1}$, so $n = \text{odd}$ $\rightarrow\leftarrow$

Homotopy rotates from x to $-x$ in direction of $V(x)$. $H: I \times S^n \to S^n$

$$H(t,x) = \cos(\pi t)x + \sin(\pi t)\frac{V(x)}{\|V(x)\|}$$

$H(0,x) = x$

$H(1,x) = -x$

$$\|H(t,x)\|^2 = \cos^2(\pi t)\|x\|^2 + \sin^2(\pi t)\left\| V(x)/\|V(x)\| \right\|^2 + 2\cos\pi t \sin\pi t \langle x, V(x)/\|V(x)\|\rangle = 1$$

How to compute $\deg(f: S^n \to S^n)$ when f is smooth. Choose a regular value of f. (i.e. $q \in S^n$, $f^{-1}(q)$ consists of finitely many pts $p_1,\dots,p_k \in S^n$ s.t. $\det(df_{p_i}) \neq 0$.)

Thm $\deg(f) = \sum_{p\in f^{-1}(q)} \underbrace{\mathrm{sign}(\det(df_p))}_{\in\{\pm 1\}}$

10/22 Homology of a CW-complex

$X = \bigcup_{n=0}^{\infty} X^{(n)}$ $\quad X^{(0)} = \bigsqcup_\alpha e_{0,\alpha}$ — 0 cells (pts)

n-cell $e_{n,\alpha}$ (a copy of D^n)

attaching map $\phi_{n,\alpha}: S^{n-1} \to X^{(n-1)}$

$X^{(n)} = X^{(n-1)} \bigsqcup_\alpha e_{n,\alpha} /$ if $p \in \partial e_{n,\alpha} = S^{n-1}$ then $p \sim \phi_{n,\alpha}(p)$

Cellular homology

$C^{cell}_n(X) := \bigoplus_{n\text{-cells}} \mathbb{Z}$

Observe $C^{cell}_n(X) = H_n(X^{(n)}, X^{(n-1)})$ because $(X^{(n)}, X^{(n-1)})$ is a good pair and $X^{(n)}/X^{(n-1)} = \bigvee_{n\text{-cell}} S^n$

$\partial: C^{cell}_n \to C^{cell}_{n-1}$ is the connecting homomorphism in the long exact sequence of the triple $(X^{(n)}, X^{(n-1)}, X^{(n-2)})$. That is, if $[\alpha] \in H_n(X^{(n)}, X^{(n-1)})$, then $\partial[\alpha] := [\partial\alpha] \in H_{n-1}(X^{(n-1)}, X^{(n-2)})$

Note $\partial^2 = 0 : C^{cell}_n \to C^{cell}_{n-2}$ because $\partial^2 = 0$ in singular homology

Define $H^{cell}_n(X) := \dfrac{Ker(\partial: C^{cell}_n \to C^{cell}_{n-1})}{Im(\partial: C^{cell}_{n+1} \to C^{cell}_n)}$

Thm $H_n(X) = H^{cell}_n(X)$

Pf

$$\begin{tikzcd}
 & H_n(X^{(n-1)})^{=0} \arrow[ddr] & & & & H_n(X^{(n+1)}, X^{(n)}) = 0 \\
 & & & H_n(X^{(n+1)}) = H_n(X) \arrow[urr] & & \\
 & & H_n(X^{(n)}) \arrow[ur] \arrow[dr, "g"] \arrow[d, phantom, "\wr"] & & & \\
 & H_{n-1}(X^{(n+1)}, X^{(n)}) \arrow[ur, "f"] \arrow[rr, "\partial"] & {} & H_n(X^{(n)}, X^{(n-1)}) \arrow[rr, "\partial"] \arrow[dr] & & H_{n-1}(X^{(n-1)}, X^{(n-2)}) \\
 & & & & H_{n-1}(X^{(n-1)}) \arrow[ur] \arrow[dr] & \\
 & & & 0 = H_{n-1}(X^{(n-2)}) \arrow[ur] & & H_{n-1}(X^{(n)})
\end{tikzcd}$$

Note: $H_n(X^{(n+1)}) = H_n(X)$

Reason: let $k \geq 1$.

$$\begin{tikzcd}
H_{n+1}(X^{(n+k+1)}, X^{(n+k)}) \arrow[r] \arrow[d, equal] & H_n(X^{(n+k)}) \arrow[r, "\simeq"] & H_n(X^{(n+k+1)}) \arrow[d] \\
0 & & H_n(X^{(n+k+1)}, X^{(n+k)}) \arrow[d, equal] \\
 & & 0
\end{tikzcd}$$

So, $H_n(X^{(n+1)}) = H_n(X^{(n+k)}) \quad \forall k \geq 1$

Note: $H_n(X^{(n-1)}) = 0$ because $H_n(X^{(n-k)}) \simeq H_n(X^{(n-k+1)}) \quad \forall k \geq 1$

Hence, $H_n(X) = H_n(X^{(n)}) / f(H_{n+1}(X^{(n+1)}, X^{(n)}))$

$$g : \frac{H_n(X^{(n)})}{f(H_{n+1}(X^{(n+1)}, X^{(n)}))} \to \frac{\operatorname{Ker}(\partial : H_n(X^{(n)}, X^{(n-1)}) \dots)}{\operatorname{Im}(\partial : H_{n+1}(X^{(n+1)}, X^{(n)}) \dots)}$$

Check g 1-1, onto

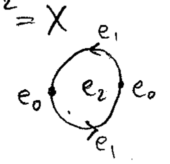
Nicer formula for $\partial : C_n^{cell}(X) \to C_{n-1}^{cell}(X)$

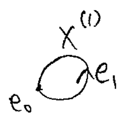
Given an n-cell $e_{n,\alpha}$ denote the corresponding generator of $C_n^{cell}(X)$ by $e_{n,\alpha}$

Prop

$$\partial e_{n,\alpha} = \sum_\beta \deg\Big(S^{n-1} \xrightarrow{\phi_{n,\alpha}} X^{(n-1)} \longrightarrow \underbrace{\frac{X^{(n-1)}}{X^{(n-1)} \setminus \operatorname{int}(e_{n-1,\beta})} = \frac{e_{n-1,\beta}}{\partial e_{n-1,\beta}} = S^{n-1}\Big)}_{\in \mathbb{Z}} \cdot e_{n-1,\beta}$$

eg $\mathbb{RP}^2 = X$

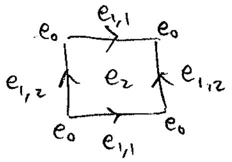


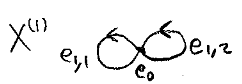


$\phi_2 : S^1 \xrightarrow{2:1} X^{(1)} = S^1$

$\partial e_2 = 2e_1$

eg $X = T^2$ $\qquad \partial e_2 = a_1 e_{1,1} + a_2 e_{1,2}$ $\quad a_1, a_2 \in \mathbb{Z}$





$\Phi_2 : S^1 \longrightarrow X^{(1)}$

$a_1 = \deg(S^1 \xrightarrow{\Phi_2} X^{(1)} \longrightarrow X^{(1)}/e_{1,2} = S^1) = 0$

similarly, $a_2 = 0$

eg $X = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n = I^n/\sim$

The faces of I^n (of all dims) make $\mathbb{T}^n$ into a CW-complex with $\binom{n}{k}$ k-cells

$-(\partial : C_k^{cell} \longrightarrow C_{k-1}^{cell}) = 0 \qquad H_k(T^n) \simeq \mathbb{Z}^{\binom{n}{k}}$

10/24

Examples of cellular homology

$- \ X = \mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus 0)/\sim \qquad x \sim \lambda x \quad \lambda \in \mathbb{R}^*$

$= S^n/_{x \sim -x}$

This has a CW-structure with one i-cell for $i = 0, \dots, n$.

Proof by induction.

$$\begin{tikzcd}
\mathbb{RP}^{n-1} \arrow[r, hook] & \mathbb{RP}^n \\
{[x_0 : \dots : x_{n-1}]} \arrow[r, mapsto] & {[x_0 : \dots : x_{n-1} : 0]}
\end{tikzcd}$$

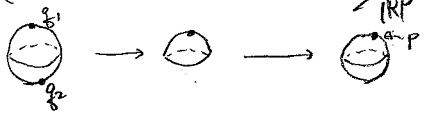
Attaching map $\varphi_n : S^{n-1} \longrightarrow \mathbb{RP}^{n-1} = S^{n-1}/_{x \sim -x}$ is the quotient map.

$\mathbb{RP}^n = \mathbb{RP}^{n-1} \sqcup$ (closed northern hemisphere of S^n) $/ \ \varphi_n(x) \sim x \in S^{n-1} = \text{equator}$

$$C_i^{cell}(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & i = 0, \dots, n \\ 0 & \text{otherwise.} \end{cases}$$

compute $\partial_k :$
$$\begin{tikzcd}
C_k^{cell} \arrow[r] \arrow[d, equal] & C_k^{cell} \arrow[d, equal] \\
\mathbb{Z} & \mathbb{Z}
\end{tikzcd}$$

$$\partial_k = \deg\big(\overbrace{S^{k-1} \longrightarrow \mathbb{RP}^{k-1} \longrightarrow \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2}}^{\varphi_k} = S^{k-2}\big)$$



$\varphi_k^{-1}(p) = \{q_1, q_2\}$

$\deg(\varphi_k) = \operatorname{sign}(d\varphi_k|_{q_1}) + \operatorname{sign}(d\varphi_k|_{q_2})$

$$\begin{tikzcd}
q_1 \in S^{k-1} \arrow[r, "\varphi_k"] \arrow[d, "a"] & S^{k-1} \\
q_2 \in S^{k-1} \arrow[ur, "\varphi_k"'] &
\end{tikzcd}$$

$\deg \varphi_k = 1 + \deg(a : S^{k-1} \circlearrowleft) = 1 + (-1)^k$

$$\partial_k = \begin{cases} 2 & \text{if } k = \text{even} \\ 0 & \text{otherwise} \end{cases}$$

$$0 \to \overset{C_n^{cell}}{\overset{\|}{\mathbb{Z}}} \to \cdots \xrightarrow{\partial=0} \mathbb{Z} \xrightarrow{\partial=2} \mathbb{Z} \xrightarrow{\partial=0} \overset{C_0^{cell}}{\overset{\|}{\mathbb{Z}}} \to 0$$

$$\tilde{H}_k(\mathbb{R}P^n) = \begin{cases} 0 & k \text{ even} \\ \mathbb{Z}/2\mathbb{Z} & k \text{ odd}, \ k < n \\ \mathbb{Z} & k \text{ odd}, \ k = n \\ 0 & \text{otherwise.} \end{cases}$$ ∎

let X be a top space such that

$$\sum_n \operatorname{rk} H_n(X) < \infty \quad (*)$$

(If G is an abelian group, $G/_{\text{Torsion}} \simeq \mathbb{Z}^k$ for some k (or not, then $\operatorname{rk}(G) = \infty$)

define $\operatorname{rk}(G) := k$

(OR : $\operatorname{rk}(G) = \dim_{\mathbb{Q}}(G \otimes \mathbb{Q})$)

Define the <u>Euler characteristic</u>

$$\chi(X) := \sum_n (-1)^n \operatorname{rk} H_n(X)$$

<u>Thm</u> If X is a CW complex with finitely many cells, then $\chi(X) = \sum_n (-1)^n \cdot (\# \text{ of } n\text{-cells})$

<u>Lemma</u> If $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated abelian groups then $\operatorname{rk} A - \operatorname{rk} B + \operatorname{rk} C = 0$

<u>Proof</u> Tensoring with $\mathbb{Q}$ preserves exactness. (more about this later)

$$0 \to A \otimes \mathbb{Q} \to B \otimes \mathbb{Q} \to C \otimes \mathbb{Q} \to 0$$

HW $\Rightarrow$ ∎

<u>Proof of Thm</u> let $C_n := C_n^{cell}$

$$Z_n = \operatorname{Ker}(\partial : C_n \to C_{n-1})$$

$$B_n = \operatorname{Im}(\partial : C_{n+1} \to C_n)$$

$$H_n = \text{homology} = Z_n / B_n$$

By definition, short exact sequence

$$0 \to B_n \to Z_n \to H_n \to 0$$

$$0 \to Z_n \to C_n \xrightarrow{\partial} B_{n-1} \to 0$$

$$\operatorname{rk}(Z_n) = \operatorname{rk}(B_n) + \operatorname{rk}(H_n)$$

$$\operatorname{rk}(C_n) = \operatorname{rk}(Z_n) + \operatorname{rk}(B_{n-1})$$

$$\Rightarrow \operatorname{rk}(C_n) = \operatorname{rk}(B_n) + \operatorname{rk}(B_{n-1}) + \operatorname{rk}(H_n)$$

$$\underbrace{\sum_n (-1)^n \operatorname{rk}(C_n)}_{\sum_n (-1)^n (\# \, n\text{-cells})} = \underbrace{\sum_n (-1)^n \operatorname{rk}(H_n)}_{\chi(X)}$$

Suppose X is as in $(*)$ and $f : X \to X$

Define the <u>Lefschetz number</u>

$$L(f) = \sum_n (-1)^n \operatorname{Tr}(f_* : H_n(X) \circlearrowleft)$$

$$\operatorname{Tr}(f_*) := \operatorname{Tr}(f_* \otimes \mathbb{Q} : H_n(X) \otimes \mathbb{Q} \circlearrowleft)$$

$$= \operatorname{Tr}\left(f_* : \frac{H_n(X)}{\text{Torsion}} \longrightarrow \frac{H_n(X)}{\text{Torsion}}\right) \in \mathbb{Z}$$

<u>Example</u> $L(id_X) = \chi(X)$ because $\operatorname{Tr}(id_G) = \operatorname{rk}(G)$

<u>Lefschetz fixed pt thm (for Δ-complexes)</u>

let X be a finite Δ-complex suppose $f : X \to X$ has $L(f) \neq 0$. Then f has a fixed pt.

eg suppose $f : D^n \to D^n$. Then $L(f) = 1$

So, Lefschetz FPT $\Rightarrow$ Brouwer FPT.

<u>Pf Sketch</u> Suppose f has no fixed pt. we can subdivide X to get a different Δ-complex structure on X, and homotope f to a map $f' : X \to X$ s.t. f' sends each new simplex Δ to a subcomplex not containing Δ. Then $\underset{\substack{\| \\ L(f)}}{L(f')} = 0$.

10/26 **Lefschetz fixed pt thm**

If X is a finite simplicial complex, $f: X \to X$, and $L(f) \neq 0$ then f has a fixed pt

Here, $L(f) = \sum_n (-1)^n \operatorname{Tr}(f_*: H_n(X) \circlearrowleft)$

Pf Step 1 Assume f has no fixed pt
By subdivision and homotopy, we can assume that

(a) f sends each n-simplex to a subcomplex ($\leq n$-dim)

Simplical Approx. Thm.

Need: if σ is an n-simplex and σ' is an n'-simplex with $n < n'$, then $f(\sigma) \cap \operatorname{int}(\sigma') = \phi$, i.e. $f(\sigma) \subseteq X^{(n)}$
i.e. $f(X^{(n)}) \subseteq X^{(n)}$

(b) if σ is a simplex, then $f(\sigma) \cap \sigma = \phi$

Step 2 Deduce from (a) and (b) that $L(f) = 0$.

Def Let X, Y be CW-complexes
$f: X \to Y$ is cellular if $f(X^{(n)}) \subseteq Y^{(n)}$ $\forall n$

Def If f is cellular, define

$$\begin{tikzcd} f_\#^{cell}: C_n^{cell}(X) \arrow[r] \arrow[d, phantom, "\shortparallel"] & C_n^{cell}(Y) \arrow[d, phantom, "\shortparallel"] \\ H_n(X^{(n)}, X^{(n-1)}) & H_n(Y^{(n)}, Y^{(n-1)}) \end{tikzcd}$$

to be the map f_* induced by the map of pair

$$f|_{X^{(n)}}: (X^{(n)}, X^{(n-1)}) \to (Y^{(n)}, Y^{(n-1)})$$

Def $f_*^{cell}: H_*^{cell}(X) \to H_*^{all}(Y)$ to be the map on homology induced by $f_\#^{cell}$.

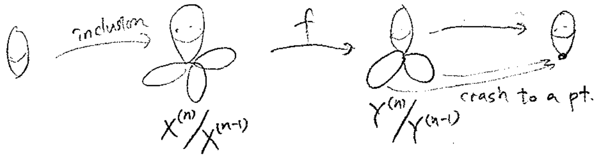
Lemma The following diagram commutes.

$$\begin{tikzcd} H_*^{all}(X) \arrow[r, "f_*^{cell}"] \arrow[d, phantom, "\cong"] & H_*^{all}(Y) \arrow[d, phantom, "\cong"] \\ H_*(X) \arrow[r, "f_*"] & H_*(Y) \end{tikzcd}$$

Proof HW #5

Nicer description of $f_\#^{cell}$

$$\begin{tikzcd} C_n^{all}(X) \arrow[r, "f_\#^{cell}"'] \arrow[d, phantom, "\shortparallel"] & C_n^{all}(Y) \arrow[ddd, no head, "="] \\ H_n(X^{(n)}, X^{(n-1)}) \arrow[d, phantom, "\shortparallel"] & \\ \tilde{H}_n(X^{(n)}/X^{(n-1)}) \arrow[d, phantom, "\shortparallel"] & \\ \tilde{H}_n(\bigvee_{\substack{n\text{-cell}\\ \text{in } X}} S^n) \arrow[r, "\text{''degree matrix''}"'] & \tilde{H}(\bigvee_{\substack{n\text{-cell}\\ \text{in } Y}} S^n) \end{tikzcd}$$



Returning to proof of LFPT,
Diagonal matrix of $f_\#^{cell}$ are 0. ($\because$ no fixed pt)
In particular, $\operatorname{Tr}(f_\#^{cell}: C_n^{all}(X) \circlearrowleft) = 0$

Lemma $\sum_n (-1)^n \operatorname{Tr}(f_*^{cell}: H_n^{all} \circlearrowleft) = L(f)$

$$\shortparallel$$

$$\sum_n (-1)^n \operatorname{Tr}(f_\#^{all}: C_n^{cell} \circlearrowleft)$$

Sublemma If we have a commutative diagram with exact row

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\ 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

where A, B, C are finitely generated, then
$\operatorname{Tr}(\beta) = \operatorname{Tr}(\alpha) + \operatorname{Tr}(\gamma)$

Pf $\otimes \mathbb{Q}$, $B \otimes \mathbb{Q} \cong (A \otimes \mathbb{Q}) \oplus (C \otimes \mathbb{Q})$

$$\beta \leftrightarrow \left(\begin{array}{c|c} \alpha & ? \\ \hline 0 & \gamma \end{array} \right)$$ ∎

To prove lemma, apply sublemma to

$$\begin{tikzcd} 0 \arrow[r] & B_i^{cell} \arrow[r] \arrow[d, "f_\#^{cell}"'] & Z_i^{all} \arrow[r] \arrow[d, "f_\#^{cell}"'] & H_i^{cell} \arrow[r] \arrow[d, "f_\#^{cell}"'] & 0 \\ 0 \arrow[r] & B_i^{all} \arrow[r] & Z_i^{cell} \arrow[r] & H_i^{cell} \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & Z_i^{all} \arrow[r] \arrow[d, "f_\#^{cell}"'] & C_i^{cell} \arrow[r, "\partial"] \arrow[d, "f_\#^{cell}"'] & B_{i-1}^{cell} \arrow[r] \arrow[d, "f_\#^{cell}"'] & 0 \\ 0 \arrow[r] & Z_i^{cell} \arrow[r] & C_i^{cell} \arrow[r] & B_{i-1}^{all} \arrow[r] & 0 \end{tikzcd}$$ ∎

Homology with coefficients

Let G be an abelian group, X a top. space.

$C_n(X; G) := C_n(X) \otimes G$
$= \{$finite linear comb. of n-simplices $\sigma: \Delta_n \to X$ with coefficient in $G\}$

eg $C_n(X; \mathbb{Z}) = C_n(X)$

$\partial: C_n(X; G) \to C_{n-1}(X; G)$ is defined as before.

$$H_n(X;G) = \frac{\text{Ker}(\partial | C_n(X;G))}{\text{Im}(\partial | C_{n+1}(X;G))}$$

Popular choice of G : $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

$H_n(X;G)$ is determined algebraically from $H_n(X)$

Universal Coefficient Theorem

There is a natural short exact sequence

$$0 \to H_n(X)\otimes G \to H_n(X;G) \to \text{Tor}(H_{n-1}(X);G) \to 0$$

This seq. splits, but not naturally.

Properties of Tor (of abelian groups)

- $\text{Tor}(A,B) = \text{Tor}(B,A)$
- $\text{Tor}(A,B) = 0$ if A or B is torsion free.
- $\text{Tor}(\mathbb{Z}/m, \mathbb{Z}/n) = \mathbb{Z}/\gcd(m,n)$
- $\text{Tor}(\bigoplus_{i=1}^{k} A_i, B) = \bigoplus_{i=1}^{k} \text{Tor}(A_i, B)$

eg $X = \mathbb{R}P^2$ $\quad H_0(X) = \mathbb{Z}$ $\quad H_1(X) = \mathbb{Z}/2$

$H_k(X) = 0 \quad \forall k \geq 2$

Compute $H_*(\mathbb{R}P^2; \mathbb{Z}/2)$

$H_0(\mathbb{R}P^2; \mathbb{Z}/2) = \mathbb{Z}/2$

$$0 \to \underbrace{H_1(\mathbb{R}P^2)\otimes\mathbb{Z}/2}_{\mathbb{Z}/2} \to \underbrace{H_1(\mathbb{R}P^2, \mathbb{Z}/2)}_{\mathbb{Z}/2} \to \underbrace{\text{Tor}(\underset{\mathbb{Z}}{\overset{\shortparallel}{H_0(\mathbb{R}P^2)}}, \mathbb{Z}/2)}_{0} \to 0 \qquad \mathbb{Z}/2 \Rightarrow \mathbb{Z}/2$$

$$0 \to \underbrace{\underbrace{H_2(\mathbb{R}P^2)}_{0}\otimes\mathbb{Z}/2}_{0} \to \underbrace{H_2(\mathbb{R}P^2, \mathbb{Z}/2)}_{\mathbb{Z}/2} \to \underbrace{\text{Tor}(\underbrace{H_1(\mathbb{R}P^2)}_{\mathbb{Z}/2}, \mathbb{Z}/2)}_{\mathbb{Z}/2} \to 0 \qquad \mathbb{Z}/2 \Leftarrow \mathbb{Z}/2$$

Thm If X is CW, then $H_n(X;G) = H_n^{cell}(X;G)$

10/29 Universal Coefficient theorem $\quad G =$ Abl. gp ($\mathbb{Z}$-module)

Natural short exact sequence

$$0 \to H_n(X)\otimes G \to H_n(X;G) \to \text{Tor}(H_{n-1}(X), G) \to 0$$

and splits but not naturally, so

$$H_n(X;G) \underset{\substack{\uparrow \\ \text{uncanonically}}}{\simeq} (H_n(X)\otimes G) \oplus \text{Tor}(H_{n-1}(X), G)$$

Tor

Lemma If $0 \to A \xrightarrow{f} B \xrightarrow{h} C \to 0$ is a short exact sequence of abelian groups, then $A\otimes G \xrightarrow{f\otimes 1} B\otimes G \xrightarrow{h\otimes 1} C\otimes G \to 0$ is exact

Pf (exercise)

Definition of $\text{Tor}(A,B)$ $\quad$ (A, B ab. gps)

Choose a free resolution

$$0 \to F_1 \to F_0 \to A \to 0$$

(This sequence is exact, and F_i's are free.)

(Need: F_i - projective

$$\begin{tikzcd} & F \arrow[dl, dashed] \arrow[d] & \\ G \arrow[r] & H \arrow[r] & 0 \end{tikzcd}$$

)

$$0 \to F_1\otimes B \to F_0\otimes B \to A\otimes B \to 0$$

is a chain complex.

Define $\text{Tor}(A,B) := \text{Ker}(F_1\otimes B \to F_0\otimes B)$.

$= H_1$ (this chain complex)

Lemma $\text{Tor}(A,B)$ does not depend on the projective resolution.

Sublemma Given two projective resolutions

$\cdots \to F_0 \to A \to 0$ and $\cdots \to F_0' \to A \to 0$,

1) there is a chain map extending 1_A,

$$\begin{tikzcd} \cdots \arrow[r] & F_1 \arrow[r] \arrow[d] & F_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "1"] & 0 \\ \cdots \arrow[r] & F_1' \arrow[r] & F_0' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

2) Any two such chain maps are chain homotopic.

Pf of sublemma : Repeatedly use the fact that the F_i's are projective.

Pf of lemma : Given two proj. res., take chain maps from sublemma and tensor with 1_B.

$\Rightarrow$ Two complexes defining Tor are chain homotopy equiv.

Example Compute $\text{Tor}(\mathbb{Z}/m, \mathbb{Z}/n)$

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m \to 0$$

$$\to \underbrace{\mathbb{Z}/n\otimes\mathbb{Z}}_{\mathbb{Z}/n} \xrightarrow{m\otimes 1} \underbrace{\mathbb{Z}\otimes\mathbb{Z}/n}_{\mathbb{Z}/n} \to \mathbb{Z}/m\otimes\mathbb{Z}/n \to 0$$

$$\text{Tor}(\mathbb{Z}/m, \mathbb{Z}/n) = \text{Ker}(m : \mathbb{Z}/n \circlearrowleft)$$

$$= \{k \in \mathbb{Z}/n \mid n \mid mk\}$$

$d = \gcd(m,n)$ $\quad mk = an$

$m = dm'$ $\quad m'k = an'$

$n = dn'$ $\quad n' \mid k$

$$\therefore \text{Tor}(\mathbb{Z}/m, \mathbb{Z}/n) = \{\text{multiple of } n'\} \subseteq \mathbb{Z}/n$$

$$\cong \mathbb{Z}/d$$

Facts
- $\mathrm{Tor}(A,B) = \mathrm{Tor}(B,A)$
- $\mathrm{Tor}(\bigoplus_{i=1}^{k} A_i, B) = \bigoplus_{i=1}^{k} \mathrm{Tor}(A_i, B)$
- If A or B is torsion free, then $\mathrm{Tor}(A,B) = 0$
- If $0 \to A \to B \to C \to 0$ is exact, then there is a long exact sequence

$$\begin{tikzcd} \mathrm{Tor}(A,G) \arrow[r] & \mathrm{Tor}(B,G) \arrow[r] & \mathrm{Tor}(C,G) \arrow[dll, rounded corners] \\ A\otimes G \arrow[r] & B\otimes G \arrow[r] & C\otimes G \arrow[r] & 0 \end{tikzcd}$$

Pf of UCT : we will prove the following algebraic statm

If $C_i \xrightarrow{\partial} C_{i-1} \to \cdots$ is a chain complex where the C_i's are free abelian groups, then there is a natural (split) short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H_n \otimes G \arrow[r] & H_n(C_* \otimes G, \partial \otimes 1_G) \arrow[d, rounded corners] \\ & & \mathrm{Tor}(H_{n-1}, G) \arrow[r] & 0 \end{tikzcd}$$

consider

$$\begin{tikzcd} 0 \arrow[r] & Z_i \arrow[r] \arrow[d, "0"] & C_i \arrow[r, "\partial"] \arrow[d, "\partial"] & B_{i-1} \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & Z_{i-1} \arrow[r] & C_{i-1} \arrow[r, "\partial"] & B_{i-2} \arrow[r] & 0 \end{tikzcd}$$

short exact sequence of free chain complexes.

$$\begin{tikzcd} 0 \arrow[r] & Z_i\otimes G \arrow[r] \arrow[d, "0"] & C_i\otimes G \arrow[r] \arrow[d, "\partial"] & B_{i-1}\otimes G \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & Z_{i-1}\otimes G \arrow[r] & C_{i-1}\otimes G \arrow[r] & B_{i-2}\otimes G \arrow[r] & 0 \end{tikzcd}$$

$\Rightarrow$ long exact sequence on homology

$$B_n \otimes G \xrightarrow{\delta = i_n \otimes 1_G} Z_n \otimes G \longrightarrow H_n(C_* \otimes G) \to B_{n-1} \otimes G \xrightarrow{\delta = i_{n-1} \otimes 1_G} Z_{n-1} \otimes G$$

$\delta : B_i \otimes G \longrightarrow Z_i \otimes G$ is the inclusion.

$$0 \longrightarrow \mathrm{Coker}(i_n) \longrightarrow H_n(C_* \otimes G) \to \mathrm{Ker}(i_{n-1}) \to 0$$

$$0 \to B_n \xrightarrow{i_n} Z_n \to H_n \to 0$$

This is a free resolution of H_n.

$$0 \to \mathrm{Tor}(H_n, G) \to B_n \otimes G \xrightarrow{i_n \otimes 1_G} Z_n \otimes G \to H_n \otimes G \to 0$$

plug in

$\mathrm{Ker}(i_n \otimes 1_G) = \mathrm{Tor}(H_n, G)$

$\mathrm{Coker}(i_n \otimes 1_G) = H_n \otimes G$ ▨

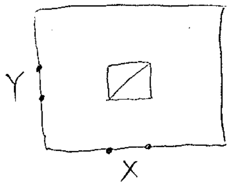
10/31 Künneth formula for $H_*(X \times Y)$

There is a natural (split) short exact sequence

"homology cross product"

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \arrow[r, "\times"] & H_n(X \times Y) \arrow[d, rounded corners] \\ & & \bigoplus_{i+j=n-1} \mathrm{Tor}(H_i(X), H_j(Y)) \arrow[r] & 0 \end{tikzcd}$$

Idea of homology cross product



Prop One can define a map

$$\times : C_i(X) \otimes C_j(Y) \longrightarrow C_{i+j}(X \times Y)$$

for all spaces X, Y such that

1) If $x \in X$, $y \in Y$, then $x \times y = (x,y) \in X \times Y$

2) If $\alpha \in C_i(X)$ and $\beta \in C_j(X)$, then
$$\partial(\alpha \times \beta) = (\partial\alpha) \times \beta + (-1)^i \alpha \times \partial\beta$$

3) Naturality : if $f : X \to X'$, $g : Y \to Y'$ then $(f_\# \alpha) \times (g_\# \beta) = (f \times g)_\# (\alpha \times \beta)$

Observe that $\times$ induces a map

$$\times : H_i(X) \otimes H_j(Y) \longrightarrow H_{i+j}(X \times Y)$$
$$[\alpha] \otimes [\beta] \longmapsto [\alpha \times \beta]$$

if $\alpha - \alpha' = \partial\gamma$, then $\partial(\gamma \times \beta) = \alpha \times \beta - \alpha' \times \beta$

Proof of Prop : Acyclic models

let $i_p : \Delta_p \to \Delta_p$ denote the identity map.

If we define $i_p \times i_q \in C_{p+q}(\Delta_p \times \Delta_q)$ satisfying (1) and (2), then for $\sigma : \Delta_p \xrightarrow{\sigma = \sigma_\# i_p} X$, $\sigma' : \Delta_q \xrightarrow{\sigma' = \sigma'_\# i_q} Y$, define $\sigma \times \sigma' := (\sigma \times \sigma')_\# (i_p, i_q) \in C_{p+q}(X \times Y)$

This will satisfy everything.

Define $i_p \times i_q$ by induction on $p+q$.

$p=0$, $q=0$, given by (1).

Suppose $p+q>0$, and $i_{p'} \times i_{q'}$ defined when $p'+q' < p+q$

we need

$$\partial(i_p \times i_q) = (\partial i_p) \times i_q + (-1)^p i_p \times \partial i_q$$

we know $\Delta_p \times \Delta_q$ is contractible,

If $p+q>1$, enough to check that RHS is a cycle.

($p+q=1$: not so hard)

$$\partial(RHS) = \cancel{\partial\partial i_p \times i_q} + \cancel{(-1)^{p-1}\partial i_p \times \partial i_q}$$
$$+ \cancel{(-1)^p \partial i_p \times \partial i_q} + (-1)^p \cancel{i_p \times \partial\partial i_q} = 0$$

Def let (C_*, ∂) and (C'_*, ∂') be chain complexes. Define the tensor product by

$$(C_* \otimes C'_*)_n := \bigoplus_{i+j=n} C_i \otimes C'_j$$

If $\alpha \in C_i$ and $\beta \in C_j$, then

$$\partial(\alpha \otimes \beta) := (\partial\alpha) \otimes \beta + (-1)^i \alpha \otimes \partial\beta$$

Check $\partial^2 = 0$

$$\partial\partial(\alpha\otimes\beta) = (\partial\partial\alpha)\otimes\beta + (-1)^{i-1}\partial\alpha\otimes\partial\beta$$
$$+ (-1)^i \partial\alpha \otimes \partial\beta + \alpha\otimes\partial\partial\beta = 0$$

Prop (restated)

One can define a natural chain map $\times : C_*(X) \otimes C_*(Y) \to C_*(X\times Y)$ for all X, Y which is the obvious map in degree 0.

This induces a map

$$\begin{tikzcd} H_*(X)\otimes H_*(Y) \arrow[r] & H_*(C_*(X)\otimes C_*(Y)) \arrow[r] & H_*(X\times Y) \\ & & \arrow[from=1-1, to=1-3, bend right, "\times"'] \end{tikzcd}$$

want to show that this does not depend on the choice in the proposition.

(due to Eilenberg-Zilber Thm :

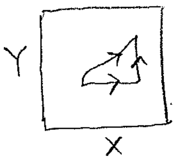
Any two natural chain maps ϕ, ϕ' : $C_*(X) \otimes C_*(Y) \to C_*(X\times Y)$, that are the cannonical maps in deg 0, are naturally chain homotopic)

Proof Exercise using acyclic models ∎

Prop $\exists$ natural chain map

$$\theta : C_*(X\times Y) \to C_*(X) \otimes C_*(Y)$$

s.t. $\theta(p, q) = p \otimes q$



Pf Still more acyclic models.

let $d_p : \Delta_p \to \Delta_p \times \Delta_p$ be the diagonal map $d_p(z) = (z, z)$

Key is to define $\theta(d_p) \in C_*(\Delta_p) \otimes C_*(\Delta_p)$ with the corresponding boundary

$$\begin{tikzcd} \text{Given } \sigma : \Delta_p \arrow[rr] \arrow[dr, "d_p"] & & X\times Y \\ & \Delta_p \times \Delta_p \arrow[ur, "\sigma_1\times\sigma_2"'] & \end{tikzcd}$$

$\sigma = \sigma_1 \times \sigma_2$

See Bredon for details.

Naturality forces $\theta(\sigma) = (\sigma_1 \times \sigma_2)_\# \theta(d_p)$ ∎

Prop $H_*(X\times Y) = \underbrace{H_*(C_*(X) \otimes C_*(Y))}_{}$
$\simeq (H_*(X) \oplus H_*(Y)) \oplus \mathrm{Tor}(\cdots)$

11/2 **Künneth formula**

Natural (split) short exact sequence

$$0 \to \bigoplus_{i+j=n} H_i(X)\otimes H_j(Y) \to H_n(X\times Y) \to \bigoplus_{i+j=n-1} \mathrm{Tor}(H_i(X), H_j(X)) \to 0$$

Proof ① natural isomorphism
$H_n(X\times Y) \simeq H_n(C_*(X) \otimes C_*(Y))$

② If C_*, C'_* are two free chain complexes then natural (split) short exact seq.

$$0 \to \bigoplus_{i+j=n} H_i \otimes H'_j \to H_n(C_* \otimes C'_*) \to \bigoplus_{i+j=n-1} \mathrm{Tor}(H_i, H'_j) \to 0$$

Pf② : Short exact seq. of chain complexes

$$0 \to Z_* \to C_* \to B_{*-1} \to 0$$

Tensor with C'_* to get SES

$$0 \to Z_* \otimes C'_* \to C_* \otimes C'_* \to B_{*-1} \otimes C'_* \to 0$$

$\Rightarrow$ Long exact sequence

$$\begin{tikzcd} \bigoplus_{i+j=n} B_i \otimes H'_j \arrow[r, hook, "i_n"'] & \bigoplus_{i+j=n} Z_i \otimes H'_j \arrow[r] & H_n(C_* \otimes C'_*) \arrow[r] & \bigoplus_{i+j=n-1} B_i \otimes H'_j \arrow[r, "i_{n-1}"] & \bigoplus_{i+j=n-1} Z_i \otimes H'_j \end{tikzcd}$$

(inclusion)

$$0 \to \mathrm{Coker}(i_n) \to H_n(C_* \otimes C'_*) \to \mathrm{Ker}(i_{n-1}) \to 0$$

consider $0 \to B_i \to Z_i \to H_i \to 0$,

$$\mathrm{Coker}(i_n) = \bigoplus_{i+j=n} H_i \otimes H'_j$$

$$\mathrm{Ker}(i_{n-1}) = \bigoplus_{i+j=n-1} \mathrm{Tor}(H_i, H'_j)$$

Cohomology

let X be a topological space, G an abelian gp.
Define cohomology $H^n(X;G)$ as follows.

$$C^n(X;G) := \mathrm{Hom}(C_n(X), G)$$
$$= \{ \underline{fn} : \{\sigma : \Delta_n \to X\} \to G \}$$

$$\delta : C^n(X;G) \to C^{n+1}(X;G)$$ called coboundary operator

if $\phi \in C^n(X;G)$, then if $\sigma : \Delta_{n+1} \to X$,

$$(\delta\phi)(\sigma) = \phi(\partial\sigma) \quad \text{i.e.} \quad \delta\phi = \phi\partial$$

$$\delta^2 \equiv 0 \quad \because \quad \delta(\delta\phi) = \delta(\phi\circ\partial) = (\phi\circ\partial)\circ\partial = 0$$

$$H^n(X;G) = \frac{\mathrm{Ker}(\delta : C^n(X;G) \to C^{n+1}(X;G))}{\mathrm{Im}(\delta : C^{n-1}(X;G) \to C^n(X;G))}$$

A map $f : X \to Y$ induces a <u>pullback</u>

$$f^* : H^n(Y;G) \to H^n(X;G)$$
$$f^*[\phi] = [\phi \circ f_\#]$$

Example $H^0(X;G) = \{ \underline{fcn}\ \underset{\{\text{path components of } X\}}{\pi_0(X)} \to G \}$

$$= \prod_{\pi_0(X)} G$$

$$\left(\text{Recall } H_0(X;G) = \bigoplus_{\pi_0(X)} G \right)$$

If X is a smooth manifold, then

$H^n(X;\mathbb{R}) \cong H^n_{dR}(X)$ deRham cohomology

If R is a commutative ring, we can define the <u>cup product</u>

$$\smile : H^i(X;R) \otimes_R H^j(X;R) \to H^{i+j}(X;R)$$

[if X is a smooth mfd and $R = \mathbb{R}$, then this is the wedge product of differential forms]

<u>Universal coefficient ~~thm for cohomology~~</u>

$$0 \to \mathrm{Ext}(H_{n-1}(X), G) \to H^n(X;G) \to \mathrm{Hom}(H_n(X), G) \to 0$$

Note If $0 \to A \to B \to C \to 0$ is exact, then the sequence

$$0 \to \mathrm{Hom}(C,G) \to \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G)$$

is exact.

eg $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$

$$0 \to \underset{\overset{\parallel}{0}}{\mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z})} \to \underset{\overset{\parallel}{\mathbb{Z}}}{\mathrm{Hom}(\mathbb{Z},\mathbb{Z})} \xrightarrow{\cdot 2} \underset{\overset{\parallel}{\mathbb{Z}}}{\mathrm{Hom}(\mathbb{Z},\mathbb{Z})}$$

<u>Define Ext(A,B)</u>

Choose a projective resolution $0 \to F_1 \to F_0 \to A \to 0$

$$0 \to \mathrm{Hom}(A,B) \to \mathrm{Hom}(F_0,B) \to \mathrm{Hom}(F_1,B) \to \mathrm{Ext}(A,B) \to 0$$

<u>Properties of Ext</u> $\left(\begin{tikzcd} 0 \arrow[r] & G \arrow[r] \arrow[d] & H \arrow[dl, dashed] \\ & B & \end{tikzcd} \right)$ inj. module.

If B is <u>divisible</u> $(\forall b \in B, \forall n \in \mathbb{Z}\setminus 0, \exists b' \in B, nb' = b)$ then $\mathrm{Ext}(A,B) = 0$

If A is free (projective), then $\mathrm{Ext}(A,B) = 0$

- $\mathrm{Ext}(\mathbb{Z}/n, G) = G/nG$

<u>Pf</u> $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0$

$$0 \to \mathrm{Hom}(\mathbb{Z}/n, G) \to \underset{\overset{\parallel}{G}}{\mathrm{Hom}(\mathbb{Z},G)} \xrightarrow{n} \overset{G}{\overset{\parallel}{\mathrm{Hom}(\mathbb{Z},G)}} \to \mathrm{Ext}(\mathbb{Z},G) \to 0 \quad \square$$

<u>eg</u> $$H^*(\mathbb{RP}^2, \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & * = 0 \\ \mathbb{Z}/2 & * = 1,2 \\ 0 & * > 2 \end{cases}$$

<u>Remark</u> $\mathrm{Ext}(A,B)$ is the set of short exact sequence

$$0 \to B \to C \to A \to 0$$

modulo the obvious equivalence relat<u>n</u>

eg $\mathrm{Ext}(\mathbb{Z}/3, \mathbb{Z}/3) = \mathbb{Z}/3$.

11/5 <u>Cup product</u>

let R be a commutative ring

Define the cup product

$$\smile : C^i(X;R) \otimes C^j(X;R) \to C^{i+j}(X;R)$$

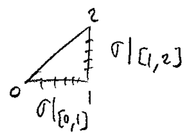
If $\alpha \in C^i(X;R)$, $\beta \in C^j(X;R)$, $\sigma : \Delta_{i+j} \to X$,

$$(\alpha \smile \beta)(\sigma) = \alpha(\sigma|_{[0,\dots,i]})\,\beta(\sigma|_{[i,\dots,i+j]})$$

$$[0,\dots,i] = \{ (t_0,\dots,t_{i+j}) \in \Delta_{i+j} \mid t_{i+1} = \dots = t_{i+j} = 0 \}$$

$$[i,\dots,i+j] = \{ (t_0,\dots,t_{i+j}) \in \Delta_{i+j} \mid t_0 = \dots = t_{i-1} = 0 \}$$

eg $i = j = 1$



<u>Lemma</u> If $\alpha \in C^i(X;R)$, $\beta \in C^j(X;R)$, then

$$\delta(\alpha \smile \beta) = (\delta\alpha) \smile \beta + (-1)^i \alpha \smile \delta\beta$$

<u>Proof</u> let $\sigma : \Delta_{i+j} \to X$

$$\delta(\alpha \smile \beta)(\sigma) = (\alpha \smile \beta)(\partial\sigma)$$
$$= (\alpha \smile \beta) \sum_{k=0}^{i+j+1} (-1)^k \sigma|_{[0,\dots,\hat{k},\dots,i+j+1]}$$

$$((\delta\alpha)\cup\beta)(\sigma) = (\delta\alpha)(\sigma|_{[0,\dots,i+1]})\,\beta(\sigma|_{[i+1,\dots,i+j+1]})$$
$$= \sum_{k=0}^{i+1}(-1)^k\alpha(\sigma|_{[0,\dots,\hat{k},\dots,i+1]})\,\beta(\sigma|_{[i+1,\dots,i+j+1]})$$

$$(((-1)^i\alpha\cup(\delta\beta))(\sigma) = (-1)^i\alpha(\sigma|_{[0,\dots,i]})\sum_{k=0}^{j}(-1)^k\beta(\sigma|_{[i,\dots,i+\hat{k},\dots,i+j+1]})$$
$$= \alpha(\sigma|_{[0,\dots,i]})\sum_{k=i}^{i+j+1}(-1)^k\beta(\sigma|_{[i,\dots,\hat{k},\dots,i+j+1]})$$ ∎

Corollary $\cup$ induces a map
$$\cup: H^i(X;R)\otimes H^j(X;R)\longrightarrow H^{i+j}(X;R)$$

Prop This is associative.

Proof Immediate from def (at the chain level)

Prop. If $\alpha\in H^i(X,R)$, $\beta\in H^j(X,R)$, then
$$\alpha\cup\beta = (-1)^{ij}\beta\cup\alpha$$
(not at chain level.)

Proof Define $\tau: C^i(X,R)\longrightarrow C^i(X,R)$

by $(\tau(\alpha))(\sigma) = (-1)^{i(i+1)/2}\alpha(\bar{\sigma})$

$\bar{\sigma}(t_0,\dots,t_i) = \sigma(t_i,\dots,t_0)$

By HW, τ induces the identity map on $H^i(X,R)$

By defn of cup product, if $\alpha\in C^i(X,R)$ and $\beta\in C^j(X,R)$, then $(\alpha\cup\beta)(\sigma) =$
$$(-1)^{\frac{i(i+1)}{2}+\frac{j(j+1)}{2}}(\tau(\beta)\cup\tau(\alpha))(\bar{\sigma})$$
$$\Rightarrow \alpha\cup\beta = (-1)^{\frac{i(i+1)}{2}+\frac{j(j+1)}{2}+\frac{(i+j)(i+j+1)}{2}}\tau(\tau(\beta)\cup\tau(\alpha))(\sigma)$$

so on homology
$$\alpha\cup\beta = (-1)^{(\dots)}\beta\cup\alpha$$

Check $(-1)^{\frac{i(i+1)}{2}+\frac{j(j+1)}{2}+\frac{(i+j)(i+j+1)}{2}} = (-1)^{ij}$

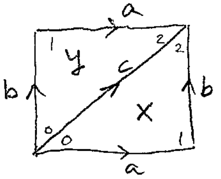
CUP PRODUCT IS POINCARE DUAL TO INTERSECTION OF SUBMANIFOLDS

Example There is a "unit" $1\in H^0(X;R)$ (if $1\in R$)
$$1\cup\alpha = \alpha\cup 1 = \alpha$$

Example Compute cup product on $H^*(T^2;R)$
$$H^*(T^2;R) = \begin{cases} R & *=0\\ R\oplus R & *=1\\ R & *=2\end{cases}$$

Only interesting cup product is
$$H^1(T^2;R)\otimes H^1(T^2;R)\longrightarrow H^2(T^2;R)$$

compute in terms of a Δ-complex structure,



$\partial x = a+b-c$

$\partial y = a+b-c$

H^1 is generated by $[\alpha]$, $[\beta]$ where

$\alpha(a)=1$, $\alpha(b)=0$, $\alpha(c)=1$

$\beta(a)=0$, $\beta(b)=1$, $\beta(c)=1$

H^2 is generated by $[\gamma]$ where

$\gamma(x)=1$ $\gamma(y)=0$

$(\alpha\cup\alpha)(x) = \alpha(a)\,\alpha(b) = 0$

$(\alpha\cup\alpha)(y) = \alpha(b)\,\alpha(a) = 0$

$(\alpha\cup\beta)(x) = \alpha(a)\,\beta(b) = 1$

$(\alpha\cup\beta)(y) = \alpha(b)\,\beta(a) = 0$

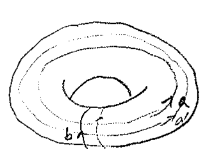
$\Rightarrow [\alpha]\cup[\beta] = [\gamma]$

$(\beta\cup\alpha)(x) = \beta(a)\,\alpha(b) = 0$

$(\beta\cup\alpha)(y) = \beta(b)\,\alpha(a) = 1$

$[\beta\cup\alpha] = -[\gamma]$

The cochain γ' sending $x\mapsto 0$, $y\mapsto 1$ satisfies $[\gamma]+[\gamma']=0$ because
$$\gamma+\gamma' = \delta\left(\substack{c\mapsto -1\\ a,b\mapsto 0}\right)$$



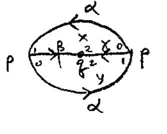
Note that a intersects b in one pt $\Leftrightarrow [\alpha]\cup[\beta]=1$

a does not intersect a' $\Leftrightarrow [\beta]\cup[\beta]=0$

b does not intersect b' $\Leftrightarrow [\alpha]\cup[\alpha]=0$

11/7 Compute cup product on $H^*(\mathbb{R}P^2;\mathbb{Z}/2)$

Δ-complex structure



$\partial\alpha = 0$ $\partial x = \beta-\gamma+\alpha$

$\partial\beta = q-p$ $\partial y = \gamma-\beta+\alpha$

$\partial\gamma = q-p$

$H^0(\mathbb{R}P^2;\mathbb{Z}/2) = \mathbb{Z}/2$

$H^1(\mathbb{R}P^2;\mathbb{Z}/2) = \mathbb{Z}/2$ gen. by $\phi\in C^1_\Delta(\mathbb{R}P^2;\mathbb{Z}/2)$ where $\phi(\alpha)=1$, $\phi(\beta)=1$, $\phi(\gamma)=0$

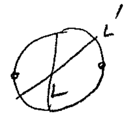
$H^2(\mathbb{R}P^2;\mathbb{Z}/2) = \mathbb{Z}/2$ gen. by $x\mapsto 1$, $y\mapsto 0$ or $x\mapsto 0$, $y\mapsto 1$

$H^1(\mathbb{RP}^2;\mathbb{Z}/2)\otimes H^1(\mathbb{RP}^2;\mathbb{Z}/2)\longrightarrow H^2(\mathbb{RP}^2;\mathbb{Z}/2)$

$[\phi]\smile[\phi]=?$

$(\phi\smile\phi)(x)=\phi(\alpha)\phi(\beta)=1$

$(\phi\smile\phi)(y)=\phi(\alpha)\phi(\gamma)=0$



L and L' intersect in one pt.

ϕ is "dual" to a line L in $\mathbb{RP}^2$

ψ is "dual" to a point

Thm let e_i denote the generator of $H^i(\mathbb{RP}^n;\mathbb{Z}/2)$ for $i=0,\dots,n$. Then

$$e_i\smile e_j=\begin{cases} e_{i+j} & \text{if } i+j\le n\\ 0 & \text{otherwise}\end{cases}$$

Prove later

Borsuk-Ulam Thm There does not exist

$f:S^n\longrightarrow S^{n-1}$ with $f(-x)=-f(x)$

(This implies Ham Sandwich Thm in $\mathbb{R}^{n+1}$)

Proof $n=1$ easy, assume $n>1$

Suppose such an f exists.

Since $f(-x)=-f(x)$, it follows that f descends to a map $g:\mathbb{RP}^n\longrightarrow\mathbb{RP}^{n-1}$

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d] & S^{n-1} \arrow[d] \\ \mathbb{RP}^n \arrow[r, "g"] & \mathbb{RP}^{n-1} \end{tikzcd}$$

Consider $g^*:H^*(\mathbb{RP}^{n-1};\mathbb{Z}/2)\to H^*(\mathbb{RP}^n;\mathbb{Z}/2)$

Claim $g^*(e_1)=e_1$. Given this,

By naturality of cup product,

$$0=g^*(0)=g^*(\underbrace{e_1\smile\cdots\smile e_1}_{n})=\underbrace{g^*e_1\smile\cdots\smile g^*e_1}_{n}$$

$$=e_1\smile\cdots\smile e_1=e_n$$

$\rightarrow\leftarrow$

Proof of claim:

Pairing between cohomology and homology

$H^i(X;G)\otimes H_i(X)\longrightarrow G$

$[\phi]\otimes[\alpha]\longmapsto\phi(\alpha)=\langle[\phi],[\alpha]\rangle$

This is the map $H^i(X;G)\longrightarrow\mathrm{Hom}(H_i(X),G)$ in the Universal coefficient thm.

This is natural if $f:X\to Y$, $\alpha\in H_i(X)$, $\phi\in H^i(Y;G)$

then $\langle f^*\phi,\alpha\rangle=\langle\phi,f_*\alpha\rangle$

let γ_n denote the generator of $H_1(\mathbb{RP}^n)$.

ETS $\langle g^*e_1,\gamma_n\rangle\overset{\langle e_1,g_*\gamma_n\rangle}{=}1$. Then by naturality,

ETS $g_*\gamma_n=\gamma_{n-1}$

γ_n is represented by a path in S^n from x to $-x$.

f sends this to a path in S^{n-1} from $f(x)$ to $-f(x)$

This represents γ_{n-1} ∎

Künneth formula for cohomology

If $H_*(X)$, $H_*(Y)$ are free & $H_*(Y)$ is finitely gen. then for a ring R,

$$H^n(X\times Y;R)=\bigoplus_{i+j=n}H^i(X;R)\otimes H^j(Y;R)$$

Pf UCT: $H^n(X\times Y;R)=\mathrm{Hom}(H_n(X\times Y),R)$

$$=\mathrm{Hom}\Big(\bigoplus_{i+j=n}H_i(X)\otimes H_j(Y),R\Big)$$

$$=\bigoplus_{i+j=n}\mathrm{Hom}(H_i(X)\otimes H_j(Y),R)$$

$$=\bigoplus_{i+j=n}\mathrm{Hom}(H_i(X),R)\otimes\mathrm{Hom}(H_j(Y),R)$$

WARNING In general,

$\mathrm{Hom}(A\otimes B,R)\ne\mathrm{Hom}(A,R)\otimes\mathrm{Hom}(B,R)$

There is always a map

$$(*)\quad \underset{\alpha}{\overset{\cup}{\mathrm{Hom}(A,R)}}\otimes\underset{\beta}{\overset{\cup}{\mathrm{Hom}(B,R)}}\longrightarrow\mathrm{Hom}(A\otimes B,R)$$

$$\longmapsto(a\otimes b\mapsto\alpha(a)\beta(b))$$

If B is free and finitely gen., then this is an isom.

let $b_1,\dots,b_n$ be a basis for B,

$A\otimes B=\{a_1\otimes b_1+\dots+a_n\otimes b_n\mid a_1,\dots,a_n\in A\}$

$=\underbrace{A\oplus\cdots\oplus A}_{n}$

Counterexample

Suppose A and B free each have one generator for each positive integer. Then the map $(*)$ sends an element $\alpha\otimes\beta$ to a rank 1 matrix.

Image of $(*)$ is the set of finite rank matrices $\Rightarrow$ $(*)$ not surj.

$$=\bigoplus_{i+j=n}H^i(X;R)\otimes H^j(Y;R)$$

11/9

Künneth formula for cohomology

If R is a ring and $H^*(Y, R)$ is free and finitely generated R-module, then

$$H^n(X\times Y;R) = \bigoplus_{i+j=n} H^i(X;R)\otimes H^j(Y;R)$$

If R is a ring, then w/o any assumptions on X and Y, define the cohomology cross product

$$\times:\ H^i(X,R)\otimes H^j(Y,R)\to H^{i+j}(X\times Y,R)$$

let $\theta: C_*(X\times Y,R)\to C_*(X)\otimes C_*(Y)$ be a natural chain map which is the obvious map in degree 0.

Given $\alpha\in C^i(X;R)$ and $\beta\in C^j(Y;R)$

define $[\alpha]\times[\beta]:=[(\alpha\otimes\beta)\circ\theta]$

let
$$\begin{tikzcd} & X\times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

Prop If $\alpha\in H^i(X;R)$ and $\beta\in H^j(Y;R)$, then

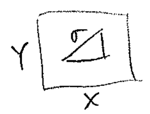
$$\alpha\times\beta = \pi_X^*\alpha\smile\pi_Y^*\beta$$

Pf Choose θ as follows,

Given $\sigma: \Delta_n\to X\times Y$, define.

$$\theta(\sigma)=\sum_{k=0}^{n}(\pi_X)_\#(\sigma|_{[0,\dots,k]})\otimes(\pi_Y)_\#(\sigma|_{[k,\dots,n]})$$

we can check that θ is a chain map, similar to the proof that



$$\delta(\alpha\smile\beta)=(\delta\alpha)\smile\beta+(-1)^{|\alpha|}\alpha\smile(\delta\beta)$$

Then if $\alpha\in C^i(X;R)$ and $\beta\in C^j(Y;R)$,

then $[\alpha]\times[\beta]=[(\alpha\otimes\beta)\circ\theta]$

$$=[\sigma\longmapsto\alpha((\pi_X)_\#\sigma|_{[0,\dots,i]})\,\beta((\pi_Y)_\#\sigma|_{[i,\dots,n]})]$$

$$=[\sigma\longmapsto\pi_X^\#\alpha(\sigma|_{[0,\dots,i]})\,\pi_Y^*\beta(\sigma|_{[i,\dots,n]})]$$

$$=\pi_X^*[\alpha]\smile\pi_Y^*[\beta]\qquad\blacksquare$$

Under assumptions of Künneth formula, cup product on $H^*(X\times Y;R)$ is given as follows. If $\alpha\in H^i(X;R)$, $\beta\in H^j(Y;R)$, $\alpha'\in H^{i'}(X;R)$, $\beta'\in H^{j'}(Y;R)$

then $(\alpha\times\beta)\smile(\alpha'\times\beta')=(-1)^{ji'}(\alpha\smile\alpha')\times(\beta\smile\beta')$

Pf $(\alpha\times\beta)\smile(\alpha'\times\beta')=(\pi_X^*\alpha\smile\pi_Y^*\beta)\smile(\pi_X^*\alpha'\smile\pi_Y^*\beta')$

$$=(-1)^{ji'}\pi_X^*(\alpha\smile\alpha')\smile\pi_Y^*(\beta\smile\beta')$$

$$=(-1)^{ji'}(\alpha\smile\alpha')\times(\beta\smile\beta')$$

Example $H^*(T^n;R)=?$

$$H^*(T^n;R)=\underbrace{H^*(S^1;R)\otimes\dots\otimes H^*(S^1;R)}_{n}$$

let $x_i = 1\otimes 1\otimes\dots\otimes(\text{generator of } H^1(S^1;R) \text{ in the } i\text{th factor})\otimes\dots\otimes 1$

Then, as a module, $H^k(T^n;R)$ is generated by monomial $x_{i_1}, x_{i_2},\dots, x_{i_k}$ where $i_1<i_2<\dots<i_k$

Claim As a ring, $H^*(T;R)$ is gen. by $x_1,\dots,x_n$ with the relatns

$x_i^2=0$ and $x_ix_j=-x_jx_i$

$H^*(T^n;R)=\Lambda^*R^n$

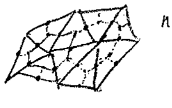
Poincaré Duality

Thm let X be a compact, R-oriented, n-dimensional topological manifold. Then $H_i(X;R)=H^{n-i}(X;R)$.

Rmk Every manifold is $\mathbb{Z}/2$-oriented. If X is smooth, then a $\mathbb{Z}$-orientatn is equivalent to an orientation in the diff top. sense (i.e. an orientation of T_pX for each $p\in X$ varying continuously with p.)

Idea Take $R=\mathbb{Z}/2$

Suppose X has a "triangulation".



Dual CW structure on X

one k-cell in the dual CW structure for each $(n-k)$-simplex in the triangulatn

$$H_i^\Delta(X;\mathbb{Z}/2)=H^{n-i}_{cell}(X;\mathbb{Z}/2)\Leftarrow(C_*^\Delta(X),\partial)=(C^{n-*}_{cell},\delta)\text{ over }\mathbb{Z}/2$$

eg If X is connected, then

$$H_n(X;\mathbb{Z}/2) = H^0(X;\mathbb{Z}/2) = \mathbb{Z}/2$$

Generator of $H_n(X;\mathbb{Z}/2)$ is the sum of all the n-simplices in the triangulation

$$H^n(X;\mathbb{Z}/2) = H_0(X;\mathbb{Z}/2) = \mathbb{Z}/2$$

Generator of $H^n(X;\mathbb{Z}/2)$ is given by cochain sending an n-chain α to the coefficient in α of some fixed n-simplex.

The pairing $H^n_\Delta(X;\mathbb{Z}/2)\otimes H^\Delta_n(X;\mathbb{Z}/2)\to\mathbb{Z}/2$ sends

$$\phi\otimes\alpha \mapsto \sum_{\substack{\sigma \text{ simplex}\\ \text{in triang}^{\underline{n}}}} \underset{\substack{\uparrow\\ \text{coeff of } \sigma \text{ in } \phi}}{\phi_\sigma}\ \underset{\substack{\nwarrow\\ \text{coeff of } \sigma \text{ in } \alpha}}{\alpha_\sigma}$$

11/14 <u>Thm</u> let M be a compact, connected, oriented n-dimensional manifold. Then $H_n(M) = \mathbb{Z}$.

<u>Def</u> The element of $H_n(M)$ corresponding to $1\in\mathbb{Z}$ is called the fundamental class of M and is denoted by $[M]\in H_n(M)$

<u>Idea</u> if M has a triangulation (eg if M is smooth) then $[M]$ is represented by the sum of all the n-simplices in the triangulation, with appropriate signs.

<u>Def</u> let $x\in M$. An orientation of M at x is a generator of $H_n(M, M\setminus\{x\})$

$$\begin{aligned}&\ \ \ \text{IS}\\ H_n(B^n&, B^n\setminus\{0\}) \simeq \mathbb{Z}\end{aligned}$$

An orientation of M is a choice of orientation $O_x\in H_n(M, M\setminus\{x\})$ for each $x\in M$, such that $\forall\, x\in M\ \exists$ nbhd $x\in U\simeq B^n$ and a generator $O\in H_n(M, M\setminus U)$ such that for all $y\in U$, O_y is the image of O under the map

$$H_n(M, M\setminus U)\longrightarrow H_n(M, M\setminus\{y\})$$

<u>Fact</u> If M is smooth, this agrees with diff. top. version.

<u>Def</u> let $\tilde{M} = \left\{(x,O)\ \middle|\ \begin{matrix}x\in M\\ O\in H_n(M, M\setminus\{x\}) \text{ gen.}\end{matrix}\right\}$

Obvious 2-to-1 map $\tilde{M}\to M$ sending $(x,O)\mapsto x$

There is a natural topology on $\tilde{M}$ such that π is a covering space.

A section of π is a map $\psi: M\to\tilde{M}$ s.t. $\pi\circ\psi = id_M$ $\{$orientation of $M\} = \{$section of $\pi\}$

<u>Remark</u> $\tilde{M}$ is a canonically oriented manifold

<u>Def</u> let $\tilde{M}_{\mathbb{Z}} := \left\{(x,\alpha)\ \middle|\ \begin{matrix}x\in M\\ \alpha\in H_n(M, M\setminus\{x\}) \text{ not necessarily a gen.}\end{matrix}\right\}$

$$\begin{matrix}\downarrow\\ M\end{matrix}$$

If $A\subseteq M$, let $\Gamma(A,\tilde{M}_{\mathbb{Z}})$ denote the set of section of $\tilde{M}_{\mathbb{Z}}$ over A.

<u>Prop</u> If M is an n-dimensional manifold and $A\subseteq M$ is compact, then (a) $H_{n+1}(M, M\setminus A) = 0$

(b) $H_n(M, M\setminus A) = \Gamma(A, \tilde{M}_{\mathbb{Z}})$

$$\overset{\Psi}{\beta}\longmapsto\left(\begin{matrix}\psi: A\to\tilde{M}_{\mathbb{Z}} \text{ sending}\\ x\in A \text{ to the image of } \beta \text{ under the}\\ \text{map } H_n(M, M\setminus A)\to H_n(M, M\setminus\{x\})\end{matrix}\right)$$

Thm follows if M is compact and oriented and $A=M$

<u>Pf of Prop</u> "Induction" on A, Enough to show

1) Prop is true if A is a convex set in a Euclidean nbhd in M. ✓ excision
2) If Prop is true for A, B and $A\cap B$ then true for $A\cup B$.
3) If Prop is true for $A_1\supseteq A_2\supseteq\dots\dots$ then true for $A=\bigcap_{i=1}^{\infty} A_i$

Why is this enough?

(1), (2), (3) $\Rightarrow$ Prop is true whenever $A\subseteq$ Euclidean nbhd.

(2) $\Rightarrow$ true for all A.

<u>Part (2)</u> Apply Five Lemma

$$\begin{tikzcd}
H_{n+1} \arrow[r] \arrow[d, "\simeq"] & H_n(M, M\setminus(A\cup B)) \arrow[r] \arrow[d, "\text{iso. by 5 lemma}"] & H_n(M, M\setminus A)\oplus H_n(M, M\setminus B) \arrow[r] \arrow[d, "\simeq"'] & H_n(M, M\setminus(A\cap B)) \arrow[d, "\simeq"] \\
0 \arrow[r] & \Gamma(A\cup B, \tilde{M}_{\mathbb{Z}}) \arrow[r, "+"] & \Gamma(A, \tilde{M}_{\mathbb{Z}})\oplus\Gamma(B, \tilde{M}_{\mathbb{Z}}) \arrow[r, "-"] & \Gamma(A\cap B, \tilde{M}_{\mathbb{Z}})
\end{tikzcd}$$

Def A directed set is a set I with a partial order $\leq$ s.t. $\forall\, i, j \in I$ $\exists\, k \in I$, $i \leq k$, $j \leq k$

$$\begin{tikzcd} & k & \\ i \arrow[ur] & & j \arrow[ul] \end{tikzcd}$$

A directed system (of groups) consists of

- for each $i \in I$, a group G_i
- for each $i \leq j$, a map $\phi_{ij} : G_i \to G_j$ s.t. if $i \leq j \leq k$ then $\phi_{jk}\phi_{ij} = \phi_{ik}$

Define $\varinjlim G_i = \bigsqcup_{i \in I} G_i/\sim$

if $i \leq j$ and $x \in G_i$ then $x \sim \phi_{ij}(x) \in G_j$

11/16 M (compact) n-dim mfld (connected)

$$\begin{array}{c} \tilde{M} \\ \downarrow \\ M \end{array} \quad \text{double cover } \tilde{M} = \left\{ (x, o) \;\middle|\; \begin{array}{l} x \in M \\ o \in H_n(M, M \setminus \{x\}) \\ \text{a generator} \end{array} \right\}$$

An orientation of M is a section of $\tilde{M}$.

$$\begin{array}{c} \tilde{M}_{\mathbb{Z}} = \{ (x, o) \mid x \in M,\ o \in H_n(M, M \setminus \{x\}) \} \\ \downarrow \\ M \end{array}$$

Prop If A is a compact subset of M, then

$$H_n(M, M \setminus A) = \Gamma(A; \tilde{M}_{\mathbb{Z}}), \quad \underset{\overset{\|}{0}}{H_{n+1}(M, M \setminus A)}$$

$\alpha \longmapsto \psi(x) =$ image of α under the map $H_n(M, M \setminus A) \to H_n(M, M \setminus x)$

If A is not necessarily compact, then $H_n(M, M \setminus A) = \{$compactly supported section of $\tilde{M}_{\mathbb{Z}}$ over $A\}$

Cor If M is compact, connected and oriented, then $H_n(M) = \mathbb{Z}$

If M is compact, connected, not orientable, then $H_n(M) = 0$

(If M is connected, not compact, then $H_n(M) = 0$)

Claim If $A_1 \supset A_2 \supset \dots$ and if Prop is true for each A_i, then it is true for $A = \bigcap_{i=1}^{\infty} A_i$.

Let I be a directed set.

Let $\{C_*(i) \mid i \in I\}$ be a directed system of chain complexes.

Useful fact $H_*(\varinjlim C_*(i)) = \varinjlim H_*(i)$ exercise

Warning: Homology does not commute with inverse limi

Cor Let $X_1 \subseteq X_2 \subseteq \dots$ be a sequence of topological spaces. Let $X = \bigcup_{i=1}^{\infty} X_i$. Suppose that every compact subset of X is contained in some X_i. Then

$$H_*(X) = \lim_{i \to \infty} H_*(X_i)$$

Proof Define $\Phi : \lim_{i \to \infty} H_*(X_i) \longrightarrow H_*(X)$ as follows. If $\alpha \in H_*(X_i)$ then $\Phi[\alpha] := (f_i)_*\alpha$ where $f_i : X_i \to X$

well-def b/c if $i \leq j$ and $f_{ij} : X_i \hookrightarrow X_j$ then $f_i = f_j \circ f_{ij}$ so $(f_i)_*\alpha = (f_j)_*(f_{ij})_*\alpha$

Φ is surj. b/c any chain in X is contained in some compact set, hence lives in some X_i

Φ is inj. b/c if $\alpha \in C_*(X)$, $\partial\alpha = 0$ and $\Phi[\alpha] = 0$, then $\alpha = \partial\beta$ for some $\beta \in C_{*+1}(X)$ but $\beta \in C_{*+1}(X_j)$ for some j.

$\exists\, k$ s.t. $i, j \leq k$ $[\alpha]$ maps to 0 in $H_*(X_k)$.

So, $[\alpha] = 0$ in $\varinjlim H_*(X_i)$. ∎

Example If X is an ∞-dim CW-complex then $H_*(X) = \lim_{i \to \infty} H_*(X^{(i)})$

so, $H_n(X) = H_n(X^{(n+1)}) \xrightarrow{id} H_n(X^{(n+2)}) \xrightarrow{id} \dots$

Proof of Claim

Since $H_{n+1}(M, M\setminus A_i) = 0$

$$\left.\begin{aligned} 0 &= \lim_{i\to\infty} H_{n+1}(M, M\setminus A_i) \\ &= H_{n+1}(M, M\setminus A) \end{aligned}\right\} \text{similar to corollary}$$

Now, we have a commutative diagram

$$\begin{tikzcd} \alpha \in H_n(M, M\setminus A_i) \arrow[r, "f"] \arrow[d, "\phi_i"', "\simeq"] & H_n(M, M\setminus A_{i+1}) \arrow[d, "\phi_{i+1}"', "\simeq"] \\ \Gamma(A_i, \tilde{M}_{\mathbb{Z}}) \arrow[r, "g"] & \Gamma(A_{i+1}, \tilde{M}_{\mathbb{Z}}) \end{tikzcd}$$

let $x \in A_{i+1}$, $\phi_{i+1}(f(\alpha))(x)$ is the image of α under the map $H_n(M, M\setminus A_i) \to H_n(M, M\setminus A_{i+1}) \to H_n(M, M\setminus\{x\})$

$g(\phi_i(\alpha))(x)$ is the image of α under the map $H_n(M, M\setminus A_i) \to H_n(M, M\setminus\{x\})$

so, same by naturality of relative homology.

Hence,

$$\begin{tikzcd} \lim\limits_{i\to\infty} H_n(M, M\setminus A_i) \arrow[r, "\simeq", "\text{inclusion}"'] \arrow[d, "\{\phi_i\}", "\simeq"'] & H_n(M, M\setminus A) \arrow[d, "\phi"] \\ \lim\limits_{i\to\infty} \Gamma(A_i, \tilde{M}_{\mathbb{Z}}) \arrow[r, "\text{restriction}"] & \Gamma(A, \tilde{M}_{\mathbb{Z}}) \end{tikzcd}$$

NTS it is an iso.

11/19 If M is a compact conn. oriented n-manifold, then $H_n(M) = \mathbb{Z}$

Same argument shows, $H_n(M; R) = R$ — comm. w/ 1

$H_n(M; \mathbb{Z}/2) = \mathbb{Z}/2$ even without the orientation assumptn

---

Cap product

$i \geq j$

Define $\frown : C_i(X;R) \otimes C^j(X;R) \to C_{i-j}(X;R)$

if $\sigma : \Delta_i \to X$, $\phi \in C^j(X;R)$, then

$$\sigma \frown \phi := \phi(\sigma|_{[0,\dots,j]})\, \sigma|_{[j,\dots,i]}$$

Can check $\partial(\alpha \frown \phi) = \pm(\partial\alpha) \frown \phi \pm \alpha \frown \delta\phi$

Therefore, $\frown$ induces a map

$$H_i(X;R) \otimes H^j(X;R) \to H_{i-j}(X;R)$$

eg if $i=j$ and X is path conn., then

$$\frown : H_i(X;R) \otimes H^j(X;R) \to H_0(X;R) = R$$

is the usual pairing of homology-cohomology

$\langle -, - \rangle$

Poincaré duality If M is compact, conn. oriented, then $H^i(M;R) \simeq H_{n-i}(M;R)$

$$\alpha \xmapsto{D} [M] \frown \alpha$$

Observe: Under the pairing $\langle -, - \rangle$, cup product is the "adjoint" of cup product.

$\alpha \in H_i(X;R)$ Assume $j+k=i$

$\phi \in H^j(X;R)$ $\langle \alpha \frown \phi, \psi \rangle = \langle \alpha, \phi \smile \psi \rangle$

$\psi \in H^k(X;R)$ This holds at the chain level by defn.

If $\sigma : \Delta_i \to X$, $\phi \in C^j(X;R)$, $\psi \in C^k(X;R)$

$$\langle \sigma \frown \phi, \psi \rangle = \psi(\phi(\sigma|_{[0,\dots,j]})\,\sigma|_{[j,\dots,i]})$$

$$\langle \sigma, \phi \smile \psi \rangle = \phi(\sigma|_{[0,\dots,j]})\,\psi(\sigma|_{[j,\dots,i]})$$

If M is compact, connected, oriented, define the cup product pairing (intersection pairing)

$(i+j=n)$ $H^i(M;R) \otimes H^j(M;R) \to R$

$$\alpha \otimes \beta \longmapsto \langle [M], \alpha \smile \beta \rangle$$

Corollary of P.D.

If $H^j(M;R) = \mathrm{Hom}(H_j(M;R), R)$

(eg if R is a field or if $R=\mathbb{Z}$ and $H_{j-1}(M)$ torsion free)

then the cup product pairing is nondegenerate.

i.e. it defines an isomorphism

$$H^j(M;R) \longrightarrow \mathrm{Hom}(H^i(M;R), R) = H_i(M;R)$$

Point: $\langle [M], \alpha \smile \beta \rangle = \langle [M] \frown \alpha, \beta \rangle = \langle D(\alpha), \beta \rangle$

Given $\underset{\mathrm{Hom}(H_j(M;R),R)}{\overset{\parallel}{\beta \in H^j(M;R)}}$, if $\beta \neq 0$, then $\exists\, \eta \in H_j(M;R)$ such that $\langle \eta, \beta \rangle \neq 0$. Since D is an iso. $\eta = D(\alpha)$ for some $\alpha \in H^i(M;R)$.

Then $\langle [M], \alpha \smile \beta \rangle = \langle D(\alpha), \beta \rangle = \langle \eta, \beta \rangle \neq 0$

So, the map $H^j(M;R) \to \mathrm{Hom}(H^i(M;R), R)$ is inj.

Fact If M is a compact mfd, then rank $H_*(M) < \infty$

So, above map is surj. also.

Example Suppose $n=2k$, and k is odd

$$H^k(M;\mathbb{R}) \otimes H^k(M;\mathbb{R}) \longrightarrow \mathbb{R}$$

This pairing is skew-symmetric

$\Rightarrow$ $\operatorname{rk} H^k(M;\mathbb{R})$ is even

Example This is almost enough to compute cup product on $\mathbb{C}P^n$

$$H^i(\mathbb{C}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0,2,\dots,2n \\ 0 & \text{otherwise} \end{cases}$$

let e_i be a gen. of $H^{2i}(\mathbb{C}P^n, \mathbb{Z})$

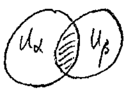
if $i+j=n$, then $e_i \cup e_j = \pm e_n$

In fact, if $i+j \leq n$, then $e_i \cup e_j = \pm e_{i+j}$

Proof inclusion $\mathbb{C}P^{i+j} \hookrightarrow \mathbb{C}P^n$. this induces isomorphism on cohomology in deg $\leq i+j$.

$$f^*(e_i \cup e_j) = \pm e_i \cup \pm e_j = \pm e_{i+j}$$

11/21 A smooth manifold is a topological manifold X with a collection of coordinate charts $\varphi_\alpha : \underset{\underset{X}{\cap}}{U_\alpha} \xrightarrow{\sim} \mathbb{R}^n$ such that

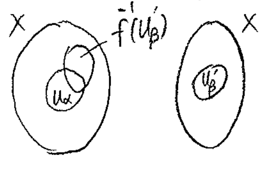


- $\bigcup_\alpha U_\alpha = X$
- The transition map $\psi_{\alpha,\beta}$ are smooth.

$$\begin{tikzcd} & U_\alpha \cap U_\beta \arrow[dr, "\varphi_\beta"] & \\ \varphi_\alpha(U_\alpha \cap U_\beta) \arrow[ur, "\varphi_\alpha^{-1}"] \arrow[rr, "\psi_{\alpha,\beta}"] & & \varphi_\beta(U_\alpha \cap U_\beta) \end{tikzcd}$$

- $\{\varphi_\alpha\}$ maximal w.r.t. above properties.

A map $f: X \to X'$ of smooth manifold is smooth if



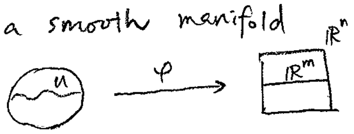
$$\begin{tikzcd} f^{-1}(U'_\beta) \cap U_\alpha \arrow[r, "f"] & U'_\beta \arrow[d, "\varphi'_\beta"] \\ \text{a subset of } \mathbb{R}^n \arrow[u, "\varphi_\alpha^{-1}"] \arrow[r, "\text{smooth}"'] & \mathbb{R}^n \end{tikzcd}$$

f is a diffeomorphism if $\exists f' : X' \to X$ smooth such that $f' \circ f = id_X$ and $f \circ f' = id_{X'}$.

eg $\mathbb{R}^n$ is canonically a smooth mfd.

A submanifold of a smooth manifold X is a subset $Y \subset X$ s.t. $\forall p \in Y$ $\exists$ coord chart $\varphi : \underset{\underset{p}{\in}}{U} \longrightarrow \mathbb{R}^n$ s.t. φ identifies $U \cap Y$ with $\mathbb{R}^m \subseteq \mathbb{R}^n$

$\Rightarrow$ Y is a smooth manifold



let X be a smooth manifold and $p \in X$.

A tangent vector to X at p is an $\mathbb{R}$-linear map $D: \{\text{smooth fns } X \to \mathbb{R}\} \longrightarrow \mathbb{R}$ such that $D(fg) = (Df)\,g(p) + f(p)\,(Dg)$

eg if $X = \mathbb{R}^n$ then a tangent vector to $\mathbb{R}^n$ at p is equivalent to a linear combination $a_1 \frac{\partial}{\partial x^1} + \dots + a_n \frac{\partial}{\partial x^n}$ for some $a_1, \dots, a_n \in \mathbb{R}$.

Tangent space $T_pX = \{\text{tangent vectors to } X \text{ at } p\}$

This is an n-dim real vector space.

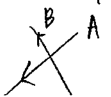
If $\gamma : \mathbb{R} \to X$ is a smooth curve then for $t \in \mathbb{R}$, velocity vector $\gamma'(t) \in T_{\gamma(t)}X$ defined by $\gamma'(t) f = \frac{d}{ds} f(\gamma(s)) \Big|_{s=t}$

in local coordinates $\{x^1, \dots, x^n\}$

$$\gamma(t) = (x^1(t), \dots, x^n(t)) \qquad \gamma'(t) = \sum_{i=1}^n \frac{d}{dt} x^i(t) \frac{\partial}{\partial x^i}$$

let X be a smooth manifold, let A, B be submanifolds. A and B intersect transversely if $\forall p \in A \cap B$, $T_pX = \operatorname{span}(T_pA, T_pB)$

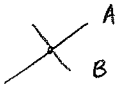
eg $X = \mathbb{R}^2$



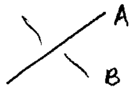
transverse

not transverse

eg $X = \mathbb{R}^3$



not transverse



transverse ($\because A \cap B = \phi$)

Thm If A and B intersect transversely, then $\forall p \in A \cap B$, there is a coord chart $\varphi : \underset{\underset{p}{\in}}{U} \to \mathbb{R}^n$ such that φ sends $A \cap U$ and $B \cap U$ to linear subspaces of $\mathbb{R}^n$ that span $\mathbb{R}^n$.

Consequently,

- $A \cap B$ is a submanifold of X.
- $\forall p \in A \cap B$, there is a short exact sequence

$$0 \longrightarrow T_p(A \cap B) \longrightarrow T_pA \oplus T_pB \longrightarrow T_pX \to 0$$

Thm If A and B are any submanifolds of X then A is isotopic to a submanifold A' which intersects B transversely.

(A is isotopic to A' if there is a smooth map $F : A \times [0,1] \to X$ s.t. $F|_{A \times \{0\}} = id_A$

$F|_{A \times \{t\}}$ is an embedding for each t

$F(A \times \{1\}) = A'$)

(If A is compact, we can find A' arbitrarily close to A.)

- If A is compact and A intersects B transversely ($A \pitchfork B$) then if A' is sufficiently (C^1)-close to A, then $A' \pitchfork B$.

An orientation of a finite dim. real vector space V is an isomorphism $V \simeq \mathbb{R}^n$ modulo the relation two orientation $\phi_1, \phi_2 : V \xrightarrow{\simeq} \mathbb{R}^n$ are the same if the map $\mathbb{R}^n \xrightarrow[\phi_1^{-1}]{} V \xrightarrow[\phi_2]{} \mathbb{R}^n$ has +ve det.

Any V has exactly two orientations.

An orientation of V can be described by an ordered basis $\{v_1, \dots, v_n\}$ for V.

If X is a smooth manifold and $p \in X$, an orientation of X at p is an orientation of T_pX in the above sense.

An orientation of X is an orientation of T_pX for each $p \in X$ which "varies continuously with p"

If $\varphi : U \xrightarrow{\simeq} \mathbb{R}^n$ is a coord chart $\{x^1, \dots, x^n\}$ then either $O_p = (\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n})$ for each $p \in U$ or $O_p = (-\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^n})$ for each $p \in U$.

(Bredon) Can check this agrees with $H_n(M, M \setminus \{p\})$ business

If A is a compact oriented $\overset{m\text{-dim}}{\wedge}$ submanifold of X, then A defines a homology class $[A] \in H_m(X)$

It turns out: "most" homology classes in X can be represented this way.

11/26

Duality between cup product and intersection

Let M^n be a compact oriented smooth manifold. Let A^k, B^l be compact oriented submanifold of $M=X$ intersecting transversely.

Short exact sequence for $p \in A \cap B$,

$$0 \to T_p(A \cap B) \xrightarrow{\ominus} T_pA \oplus T_pB \xrightarrow{\oplus} T_pX \to 0$$

Define an orientation of $A \cap B$ as follows.

Let $p \in A \cap B$. Choose vectors

$u_1, \dots, u_{k+l-n} \in T_p(A \cap B)$

$v_1, \dots, v_{n-l} \in T_pA$

$w_1, \dots, w_{n-k} \in T_pB$

s.t. $\{u_1, \dots, u_{k+l-n}, v_1, \dots, v_{n-l}\}$ is an oriented basis for T_pA

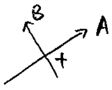
$\{u_1, \dots, u_{k+l-n}, w_1, \dots, w_{n-k}\}$ is an oriented basis for T_pB

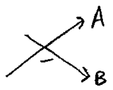
$\{u_1, \dots, u_{k+l-n}, v_1, \dots, v_{n-l}, w_1, \dots, w_{n-k}\}$ is an oriented basis for T_pX

Declare that $\{u_1, \dots, u_{k+l-n}\}$ is an oriented basis for $T_p(A \cap B)$

Example if $k+l=n$, so $\dim(A \cap B) = 0$, then a point $p \in A \cap B$ is a positive intersection iff the canonical isom. $T_pA \oplus T_pB = T_pX$ is orientation preserving.

e.g. A, B 1-manifold in $\mathbb{R}^2$ — (std orientation)





orientation on $B \cap A$ differs from that of $A \cap B$ by $(-1)^{(n-k)(n-l)}$

Fundamental classes $[A] \in H_k(M)$

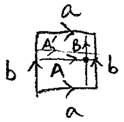
$[B] \in H_l(M)$

$[A \cap B] \in H_{k+l-n}(M)$

If $\alpha \in H_i(M)$, let $\alpha^* \in H^{n-i}(M; \mathbb{Z})$ denote its Poincaré dual, so $[M] \cap \alpha^* = \alpha$

Thm $[A \cap B]^* = [A]^* \cup [B]^*$

Example $M = T^2$ $\quad H_1(T^2) = \mathbb{Z}\{a, b\}$



$a^* \cup b^* = [A]^* \cup [B]^* = [A \cap B]^* = [pt]^*$

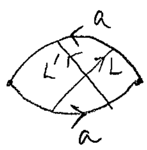
$b^* \cup a^* = [B \cap A]^* = -[pt]^*$

$a^* \cup a^* = [A \cap A'] = 0$

Remark without orientation assumptions, the thm holds with $\mathbb{Z}/2$ coeffs.

Example $M=\mathbb{RP}^2$ $H_*(\mathbb{RP}^2;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & *=0,1,2 \\ 0 & \text{otherwise} \end{cases}$

$$\overset{\parallel}{H^*(\mathbb{RP}^2;\mathbb{Z}/2)}$$



$a \in H_1(\mathbb{RP}^2;\mathbb{Z}/2)$ generator

$a^* \cup a^* = [L \cap L']^* = [pt]^*$

Rmk In the T^2 example, $H^1(T^2;\mathbb{Z}) = Hom(H_1(T^2),\mathbb{Z})$

Let $\alpha, \beta \in H^1(T^2;\mathbb{Z})$ be the hom-dual basis of a, b. $\langle a,\alpha\rangle = 1$, $\langle b,\alpha\rangle = 0$

$\langle a,\beta\rangle = 0$, $\langle b,\beta\rangle = 1$

Note $\alpha \neq a^*$, $\beta \neq b^*$

$\langle a, a^*\rangle = \langle [T^2]\cap a^*, a^*\rangle = \langle [T^2], a^*\cup a^*\rangle = 0$

$\langle a, b^*\rangle = \langle [T^2]\cap a^*, b^*\rangle = \langle [T^2], a^*\cup b^*\rangle = 1$

similarly, $\langle b, a^*\rangle = -1$

$\langle b, b^*\rangle = 0$

So, $a^* = -\beta$, $b^* = \alpha$

$k+l=n$

$$\langle [A], [B]^*\rangle = \langle [M]\cap[A]^*, [B]^*\rangle$$
$$= \langle [M], [A]^*\cup[B]^*\rangle$$
$$= \langle [M], [A\cap B]^*\rangle$$
$$= \#(A\cap B) = \text{sign count of intersection}$$

The poincaré dual of $[B]$ defines an element of $Hom(H_k(M),\mathbb{Z})$ which sends $[A]$ to the signed count of intersections of A and B.

Rmk A complex manifold is an even-dimensional smooth manifold with an atlas of coord charts $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ such that the transition maps

$$\begin{tikzcd} & U_\alpha \cap U_\beta \arrow[dr, "\varphi_\beta"] & \\ \varphi_\alpha(U_\alpha\cap U_\beta) \arrow[ur, "\varphi_\alpha^{-1}"] \arrow[rr] & & \varphi_\beta(U_\alpha\cap U_\beta) \end{tikzcd}$$

are holomorphic. (differential is $\mathbb{C}$-linear)

A complex manifold has a canonical orientation because $GL(n,\mathbb{C})$ is connected.

std orientation for $\mathbb{C}^n$:

complex basis $e_1, \dots, e_n$

oriented real basis $(e_1, ie_1, e_2, ie_2, \dots, e_n, ie_n)$

$\Rightarrow$ Cup product on $\mathbb{CP}^n$ is as claimed.

11/28 Let X^n be a compact oriented manifold

Define the intersection product

$$H_i(X)\otimes H_j(X) \longrightarrow H_{i+j-n}(X)$$
$$\alpha\otimes\beta \longmapsto [X]\cap(\alpha^*\cup\beta^*) =: \alpha\cdot\beta$$

where $[X]\cap\alpha^* = \alpha$ and $[X]\cap\beta^* = \beta$.

Thm If X is smooth, and if A and B are compact oriented submanifolds intersecting transversely then $[A\cap B] = [A]\cdot[B]$

Example $X = \mathbb{CP}^m$

Let $\alpha_k \in H_{2k}(\mathbb{CP}^m)$ denote the homology class of a $\dim_{\mathbb{C}} = k$ linear subspace.

$$\alpha_i \cdot \alpha_j = \alpha_{i+j-m}$$
$$\alpha_{m-i}\cdot\alpha_{m-j} = \alpha_{m-i-j}$$

$H^{2k}(\mathbb{CP}^m;\mathbb{Z})$ is generated by α^*_{m-k}

X^n = compact smooth manifold

$f: X \to X$ smooth

Def A fixed pt p of f is nondegenerate if $df_p : T_pX \circlearrowleft$ does not have 1 as an eigenvalue.

In this case, define the Lefschetz sign $\varepsilon(p) \in \{+1,-1\}$ to be the sign of $\det(1-df_p)$

Fact
- Nondegenerate fixed pts are isolated.
- If f is "generic" then all fixed pts are nondegenerate.

If all fixed pts are nondegenerate, define

$$\#Fix(f) := \sum_{p\in\text{fixed pts}} \varepsilon(p) \in \mathbb{Z}$$

letschefz fixed pt thm

If X is a compact smooth manifold and $f: X\to X$ is a smooth map with nondegenerate fixed pt, then

$$\#Fix(f) = L(f) = \sum_i (-1)^i \, Tr(f_* : H_i(X;\mathbb{Q})\circlearrowleft)$$

Example Let A be 2×2 integer matrix with $\det(1-A)\neq 0$, A induces a map $f: T^2 \circlearrowleft$ $''\mathbb{R}^2/\mathbb{Z}^2$

Fixed pts of f are nondegenerate b/c if p is a fixed pt, then $df_p = A$ so $\det(1-df_p)\neq 0$ by assumption

LFPT says $\#\mathrm{Fix}(f) = 1 - \mathrm{Tr}(A) + \det A$

Compute $\#\mathrm{Fix}(f)$ directly:

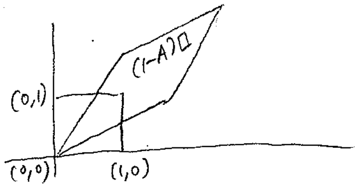
$p \in \mathbb{R}^2$ represents a fixed point of f

if $Ap - p \in \mathbb{Z}^2$, i.e. $(1-A)p \in \mathbb{Z}^2$

There are $|\det(1-A)|$ fixed points in T^2.

The sign of each fixed point is $\mathrm{sign}(\det(1-A))$

$\Rightarrow \#\mathrm{Fix}(f) = \det(1-A)$.



Poincaré-Hopf index theorem

A smooth vector field on X is an assignment, to each $p \in X$, of a tangent vector $V(p) \in T_pX$, depending smoothly on p.

(Smooth: in local coordinates

$$V(x^1, \dots, x^n) = \sum_{i=1}^{n} \underset{\text{smooth}}{v^i(x^1, \dots, x^n)} \frac{\partial}{\partial x^i} \quad)$$

let p be an isolated zero of V.

Define $\mathrm{ind}_p(V) \in \mathbb{Z}$ as follows.

Choose local coordinate $(x^1, \dots, x^n)$ with $p \leftrightarrow (0, \dots, 0)$

locally, V defines a fn $\mathbb{R}^n \to \mathbb{R}^n$

$V(0) = 0$, $V(q) \neq 0$ for $q \neq 0$ close to 0.

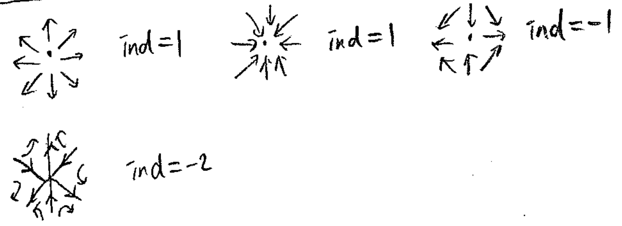
Pick $\varepsilon > 0$ so that V is nonzero on $B(0,\varepsilon) \setminus \{0\}$.

$$\begin{tikzcd} S^{n-1} \arrow[r, "V"] \arrow[d, phantom, "\shortparallel"] & \mathbb{R}^n \setminus \{0\} \arrow[r, "\text{normalize}"] & S^{n-1} \\ \partial B(0,\varepsilon) & & \end{tikzcd}$$

Define $\mathrm{ind}_p(V)$ to be the degree of this map.

If V is "generic", then $\mathrm{ind}_p(V) \in \{\pm 1\}$

Example on $\mathbb{R}^2$



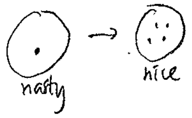
Poincaré-Hopf index thm

If X^n is a compact smooth manifold and V is a (continuous) vector field on X with isolated zeros, then

$$\sum_{p \text{ a zero of } V} \mathrm{ind}_p(V) = \cancel{(-1)^n}\, \chi(X)$$

Note: if n is odd, then $\chi(X) = 0$ by Poincaré duality.

Pf WLOG, assume V is smooth and near each zero, in local coords, the differential of $V: \mathbb{R}^n \to \mathbb{R}^n$ at the zero is invertible



By fundamental ODE, there exist smooth maps $\phi_t : X \to X$ for each $t \in \mathbb{R}$ s.t.

$\phi_0 = \mathrm{id}_X$ and $\frac{d}{dt}\phi_t(p) = V(\phi_t(p))$

Summary If $\varepsilon > 0$ is small, then

$\{\text{fixed pts of } \phi_\varepsilon\} = \{\text{zeros of } V\}$

Lefschetz sign $\longleftrightarrow$ $(-1)^n$ ind

$$\sum_{p \in V^{-1}(0)} \mathrm{ind}_p(V) = (-1)^n L(\phi_\varepsilon) = (-1)^n L(\mathrm{id}_X)$$
$$= (-1)^n \chi(X)$$

11/30 M closed oriented smooth manifold (compact & no boundary)

Use $\mathbb{Q}$ coeffs today.

$\alpha \in H_*(M)$ $\quad \alpha^* \in H^*(M)$ characterized by $[M] \frown \alpha^* = \alpha$

$\alpha, \beta \in H_*(M)$

$$\alpha \cdot \beta = [M] \frown (\alpha^* \smile \beta^*) = ([M] \frown \alpha^*) \frown \beta^*$$
$$= \alpha \frown \beta^*$$

Thm If A, B closed oriented submanifolds of M intersects transversely, then $[A \cap B] = [A] \cdot [B]$

Lefschetz FPT If $f: M \to M$ is a smooth map with nondegenerate find pts, then

$\#\mathrm{Fix}(f) = L(f)$

[can drop assum that M is orientable]

Def The graph of f is the set

$\Gamma(f) := \{(x, f(x)) \mid x \in M\} \subset M \times M$

$\Gamma(f)$ is a submanifold of $M \times M$

$\mathbb{1}_M \times f : M \to M \times M$

is an embedding with image $\Gamma(f)$

Use this diffeo $M \xrightarrow{\simeq} \Gamma(f)$ to orient $\Gamma(f)$

If $f = \mathbb{1}_M$, $\Gamma(\mathbb{1}_M) = \Delta \subset M \times M$ diagonal.

<u>Lemma</u> f has nondegenerate fixed pts
$\Leftrightarrow$ $\Gamma(f)$ intersects Δ transversely

In this case, $\# \mathrm{Fix}(f) = [\Gamma(f)] \cdot [\Delta]$

<u>Pf</u> There is an obvious identif<u>n</u> of sets

$$\mathrm{Fix}(f) = \Gamma(f) \cap \Delta$$
$$x \longleftrightarrow (x, x)$$

Let $p \in \mathrm{Fix}(f)$. Show p is a nondegenerate fixed pt $\Leftrightarrow$ $\Gamma(f)$ is transverse to Δ at (p,p)

Let $\{v_1, \ldots, v_n\}$ be an oriented basis for T_pM

Then $\{(v_1,v_1), \ldots, (v_n,v_n)\}$ is an oriented basis for $T_{(p,p)}\Delta$

And, $\{(v_1, df_p(v_1)), \ldots, (v_n, df_p(v_n))\}$ is —"— $T_{(p,p)}\Gamma(f)$

Transverse $\Leftrightarrow$ these $2n$ vectors are lin ind.

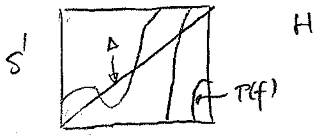
$$\det\left(\begin{array}{c|c} \mathbb{1}_{T_pM} & \mathbb{1}_{T_pM} \\ \hline df_p & \mathbb{1}_{T_pM} \end{array}\right) = \det\left(\begin{array}{c|c} \mathbb{1}_{T_pM} - df_p & 0 \\ \hline df_p & \mathbb{1}_{T_pM} \end{array}\right)$$
$$= \det(\mathbb{1}_{T_pM} - df_p)$$

$$\therefore \text{ Transverse } \Leftrightarrow \underbrace{\det(\mathbb{1}_{T_pM} - df_p)}_{\text{nondegenerate fixed pts}} \neq 0$$

This calculation also shows that the Lefschetz sign of the fixed pt equals the sign of the intersection

<u>To prove LFPT</u>, need to show $[\Gamma(f)] \cdot [\Delta] = L(f)$

<u>Example</u> $M = S^1$ $f: S^1 \to S^1$ let $d = \deg(f)$



$H_1(S^1 \times S^1) = \underset{\text{hori}}{\mathbb{Q}} \oplus \underset{\text{vert}}{\mathbb{Q}}$

$(1,0) \cdot (1,0) = 0$
$(1,0) \cdot (0,1) = 1$
$(0,1) \cdot (1,0) = -1$
$(0,1) \cdot (0,1) = 0$

$[\Delta] = (1,1)$
$[\Gamma(f)] = (1,d)$

$$[\Gamma(f)] \cdot [\Delta] = (1,d) \cdot (1,1) = 1 \cdot 1 - d \cdot 1 = 1 - d = L(f)$$

X, Y any top space

$$\times : H_i(X) \otimes H_j(Y) \longrightarrow H_{i+j}(X \times Y)$$
$$\times : H^i(X) \otimes H^j(Y) \longrightarrow H^{i+j}(X \times Y)$$
$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$
$$\phi \times \psi \longmapsto \pi_1^* \phi \smile \pi_2^* \psi$$

<u>Lemma 1</u> Let $\alpha \in H_*(X)$, $\beta \in H_*(Y)$, $\phi \in H^*(X)$, $\psi \in H^*(Y)$

Then
$$(\alpha \times \beta) \frown (\phi \times \psi)$$
$$= (-1)^{|\alpha \frown \phi||\psi|} (\alpha \frown \phi) \times (\beta \frown \psi)$$

<u>Pf</u> Immediate from the chain level def<u>n</u> of homology cross product. $\square$

<u>Lemma 2</u> If $\alpha, \beta \in H_*(M)$, then
$$(\alpha \times \beta)^* = (-1)^{|\alpha|(n-|\beta|)} \alpha^* \times \beta^*$$

<u>Pf</u> ETS both sides have the same cap product with $[M \times M]$.

$[M \times M] \frown (\alpha \times \beta)^* = \alpha \times \beta$ by def<u>n</u>

$$[M \times M] \frown (\alpha^* \times \beta^*) = ([M] \times [M]) \frown (\alpha^* \times \beta^*) \quad \text{by HW}$$
$$= (-1)^{|[M] \frown \alpha^*||\beta^*|} ([M] \frown \alpha^*) \times ([M] \times \beta^*)$$
$$= (-1)^{|\alpha|(n-|\beta|)} \alpha \times \beta \qquad \square$$

<u>Lemma 3</u> Let $\alpha, \beta, \gamma, \delta \in H_*(M)$. Then
$$(\alpha \times \beta) \cdot (\gamma \times \delta) = (-1)^{(n-|\alpha|)(n-|\delta|)} (\alpha \cdot \gamma) \times (\beta \cdot \delta)$$

<u>Note</u> If $\alpha, \beta, \gamma, \delta$ are represented by sumanifolds A, B, C, D intersecting transversely, this corr. to $(A \times B) \cap (C \times D) = (A \times C) \cap (B \times D)$

<u>Pf</u>
$$(\alpha \times \beta) \cdot (\gamma \times \delta) = (\alpha \times \beta) \frown (\gamma \times \delta)^*$$
$$= \pm (\alpha \times \beta) \frown (\gamma^* \times \delta^*) \quad \text{by Lemma 2}$$
$$= \pm (\alpha \frown \gamma^*) \times (\beta \frown \delta^*) \quad \text{by Lemma 1}$$
$$= \pm (\alpha \cdot \gamma) \times (\beta \cdot \delta) \qquad \square$$

<u>Lemma 4</u> If $f: M \to M$ and $\alpha, \beta \in H_*(M)$, then
$$[\Gamma(f)] \cdot (\alpha \times \beta) = (-1)^{|\alpha|(n-|\beta|)} f_*(\alpha \cdot \beta)$$

(As sets, $\Gamma(f) \cap (A \times B) = \{(x,y) \in A \times B \mid f(x) = y\}$

<u>Pf</u>
$$[\Gamma(f)] \cdot (\alpha \times \beta) = (\mathbb{1}_M \times f)_*[M] \frown (\alpha \times \beta)^*$$
$$= (-1)^{|\alpha|(n-|\beta|)} (\mathbb{1}_M \times f)_*[M] \frown (\alpha^* \times \beta^*)$$
$$= \pm (\mathbb{1}_M \times f)_*[M] \frown (\pi_1^* \alpha^* \smile \pi_2^* \beta^*)$$
$$= \pm [M] \frown (\mathbb{1}_M \times f)^* (\pi_1^* \alpha^* \smile \pi_2^* \beta^*)$$
$$= \pm [M] \frown \left((\mathbb{1}_M \times f)^* \pi_1^* \alpha^* \smile (\mathbb{1}_M \times f)^* \pi_2^* \beta^* \right)$$
$$= \pm [M] \frown (\alpha^* \smile f^* \beta^*)$$
$$= \pm ([M] \frown \alpha^*) \frown f^* \beta^*$$
$$= \pm \alpha \frown f^* \beta^*$$
$$= \pm f_* \alpha \frown \beta^*$$
$$= \pm f_* \alpha \cdot \beta \qquad \square$$

12/3 M^n closed (oriented, connected, smooth mfd

$f: M \to M$ smooth map with nondegenerate fixed points ($\mathbb{Q}$ coefs)

<u>Thm</u> $\#\mathrm{Fix}(f) = L(f)$

Last time – $\#\mathrm{Fix}(f) = [\Gamma(f)] \cdot [\Delta]$

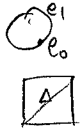
- If $\alpha, \beta, \gamma, \delta \in H_*(M)$, then in $M \times M$
$$(\alpha\times\beta)\cdot(\gamma\times\delta) = (-1)^{(n-|\alpha|)(n-|\delta|)} (\alpha\cdot\gamma)\times(\beta\cdot\delta)$$
- If $\alpha, \beta \in H_*(M)$, then
$$[\Gamma(f)]\cdot(\alpha\times\beta) = (-1)^{|\alpha|(n-|\beta|)} f_*(\alpha\cdot\beta)$$

Choose a basis $\{e_i\}$ for $H_*(M)$ of pure degree elements.

By Poincaré duality, the intersection pairing $H_*(M)\otimes H_{n-*}(M) \to \mathbb{Q}$ is nondegenerate.

let $\{e_j'\}$ be the dual basis with respect to this pairing, i.e. If $|e_i| + |e_j'| = n$, $e_i \cdot e_j' = \delta_{ij}$

<u>Lemma</u> $[\Delta] = \sum_k e_k \times e_k'$ eg $M = S^1$ $e_0' = e_1$ $e_1' = e_0$



<u>Pf</u> Know $H_n(M\times M) \overset{\times}{=} \bigoplus_{i+j=n} H_i(M)\otimes H_j(M)$ $\times$ ← homology cross prod.

so, $\{e_i' \times e_j \mid |e_i'| + |e_j| = n\}$ is a basis for $H_n(M\times M)$

To prove lemma, show both sides have the same intersection with $e_i' \times e_j$

$$[\Delta]\cdot(e_i'\times e_j) = [\Gamma(\mathbb{1}_M)]\cdot(e_i'\times e_j)$$
$$= (-1)^{|e_i'|(n-|e_j|)} e_i' \cdot e_j$$
$$= (-1)^{|e_i'|n} e_j \cdot e_i' = (-1)^{|e_i'|n} \delta_{ij}$$

$$\sum_k (e_k \times e_k')\cdot(e_i'\times e_j) = \sum_k (-1)^{|e_k'||e_i'|} (e_k\cdot e_i')\times(e_k'\cdot e_j)$$
$$= \sum_k (-1)^{|e_k'||e_i'| + |e_k'||e_j|} \delta_{ik}\delta_{jk}$$
$$= (-1)^{|e_i'||e_i'| + |e_i'||e_i'|} \delta_{ij} = (-1)^{|e_i'|n} \delta_{ij}.$$ ▨

<u>Pf of LFPT</u>

$$\#\mathrm{Fix}(f) = [\Gamma(f)]\cdot[\Delta]$$
$$= [\Gamma(f)]\cdot\sum_k e_k \times e_k'$$
$$= \sum_k (-1)^{|e_k|\cancel{(n-|e_k|)}} f_* e_k \cdot e_k'$$
$$= \sum_{i=0}^n (-1)^i \sum_{k:|e_k|=i} f_*(e_k\cdot e_k') = \sum_{i=0}^n (-1)^i \mathrm{Tr}(f_* : H_i(M) \circlearrowleft)$$
$$= L(f)$$ ▨

<u>Poincaré–Hopf index thm</u>

let M be a closed (connected, oriented) smooth manifold. let V be a continuous vector field on M with isolated zeros. Then

$$\sum_{p\in V^{-1}(0)} \mathrm{ind}_p(V) = (-1)^n \chi(M)$$

<u>Pf</u> (Sketch) ① WLOG, V is smooth, and at each zero p of V, the derivative of V at p,

$$\nabla V(p) : T_pM \to T_pM \quad \text{is invertible.}$$

② By fundamental thm of ODE's, there exists smooth maps $\phi_t : M \to M$ for each $t \in \mathbb{R}$ depending smoothly on ϕ_t, such that

$$\phi_0 = \mathbb{1}_M \qquad \frac{d}{dt}\phi_t(x) = V(\phi_t(x)) \quad \text{for each } x \in M.$$

③ Claim. If $\varepsilon > 0$ is sufficiently small then the obvious injection $\{\text{Zeros of } V\} \hookrightarrow \{\text{Fixed pt of } \phi_\varepsilon\}$ is a bijection, and for $p \in V^{-1}(0)$

$$\mathrm{ind}_p(V) = (-1)^n \operatorname{sign}\det(\mathbb{1} - d(\phi_\varepsilon)_p)$$

<u>Idea</u> in local coord. $\phi_\varepsilon(x) = x + \varepsilon V(x) + \text{error}$

so, if $V(x) \neq 0$, then $\phi_\varepsilon(x) \neq x$ for ε small.

If $V(p) = 0$, choose local coord. $(x^1, \dots, x^n)$ centered at p. For small fixed $\varepsilon > 0$, and $|x|$ small,

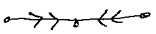
$$\phi_\varepsilon(x) = x + \varepsilon \nabla V(p) x + \text{error}$$
$$d\phi_\varepsilon(x) = \mathbb{1} + \varepsilon \nabla V(p)$$
$$\Rightarrow \varepsilon \nabla V(p) = \mathbb{1} - d\phi_\varepsilon(x)$$
$$\Rightarrow \underset{\substack{\| \\ \mathrm{ind}_p(V)}}{\operatorname{signdet} \nabla V(p)} = (-1)^n \operatorname{signdet}(\mathbb{1} - d\phi_\varepsilon(x))$$

④ The family of maps $\{\phi_t\}$ define a homotopy from ϕ_ε to $\mathbb{1}_M$, so $L(\phi_\varepsilon) = L(\mathbb{1}_M) = \chi(M)$ ▨

Another approach to prove PHIT:

Choose a triangulation of M.

Define a continuous vector field on M using a standard vector field on the n-simplex.



points towards the center of each simplex.

This vector field V on M has one zero p for each simplex σ in the triangulation

Can show $\mathrm{ind}_p(V) = (-1)^{\dim\sigma}$

$$\Rightarrow \sum_{p\in V^{-1}(0)} \mathrm{ind}_p(V) = \sum_\sigma (-1)^{\dim\sigma} = \chi(M)$$

12/5

let X^n be a smooth manifold

A k-form α on X is an assignment, to each $p \in X$, an element $\alpha(p) \in \Lambda^k T_p^* X$ depending smoothly on p. // dim = $\binom{n}{k}$

$T_p^* X = \mathrm{Hom}(T_p X, \mathbb{R})$ Alternating map $\otimes^k T_p X \to \mathbb{R}$

$\Lambda^k V$ is spanned by $v_1 \wedge \dots \wedge v_k$ w/ $v_i \wedge v_j = -v_j \wedge v_i$

In local coordin $(x^1, \dots, x^n)$

a basis for $T_p X$ is $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$

a dual basis (for $T_p^* X$) is $(dx^1, \dots, dx^n)$

a basis for $\Lambda^k T_p^* X$ is $\{dx^{i_1} \wedge \dots \wedge dx^{i_k} \mid i_1 < \dots < i_k\}$

A 0-form is a smooth fn $f : X \to \mathbb{R}$

If α is a k-form, $d\alpha$ is a (k+1)-form defined as follows in local coords,

$$d(f\, dx^{i_1} \wedge \dots \wedge dx^{i_k}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

Note if $k=0$, $df(V) = Vf$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$$

Lemma $d^2 = 0$

Pf $\frac{\partial^2}{\partial x^i \partial x^j} = \frac{\partial^2}{\partial x^j \partial x^i}$ but $dx^i dx^j = -dx^j dx^i$ □

let $\Omega^k(M) = \{\text{k-forms on } M\}$

(co)chain complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M)$$

Define de Rham cohomology

$$H^k_{dR}(M) = \frac{\ker(d : \Omega^k \to \Omega^{k+1})}{\mathrm{Im}(d : \Omega^{k-1} \to \Omega^k)}$$

Example $M = \mathbb{R}$

$$\Omega^0 \xrightarrow{d} \Omega^1 \qquad df = f'dx$$

$$H^0(\mathbb{R}) = \ker(d : \Omega^0 \to \Omega^1) = \mathbb{R}$$

$$H^1(\mathbb{R}) = \frac{\Omega^1}{d(\Omega^0)} = 0 \quad \text{b/c}$$

$$g\,dx = df \quad \text{where } f(x) = \int_0^x g(y)\,dy$$

$$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{if } k=0 \\ 0 & \text{otherwise.} \end{cases}$$

Mayer-Vietoris sequence

If $U, V \subset M$ are open sets, then ∃ LES

$$H^k(U \cup V) \xrightarrow{\oplus} H^k(U) \oplus H^k(V) \xrightarrow{\ominus} H^k(U \cap V) \to H^{k+1}(U \cup V)$$

Pf ETS ∃ short exact sequence of chain complexes.

$$0 \to \Omega^*(U \cup V) \xrightarrow{\oplus} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{\ominus} \Omega^*(U \cap V) \to 0$$

Clear that $\oplus$ is inj. & $\ker(\ominus) = \mathrm{Im}(\oplus)$

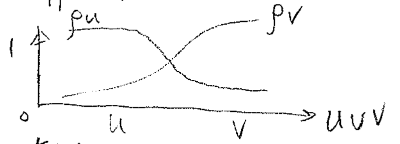
Pf that $\ominus$ is surj.

Smooth partition of unity

$\rho_U : U \cup V \to [0,1]$ supported in U

$\rho_V : U \cup V \to [0,1]$ supported in V

$\rho_U + \rho_V = 1$



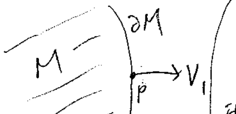
Given $\alpha \in \Omega^k(U \cap V)$

$\rho_V \alpha \in \Omega^k(V)$ $\rho_U \alpha \in \Omega^k(U)$

$\ominus(\rho_U \alpha - \rho_V \alpha) = (\rho_U + \rho_V)\alpha = \alpha$ ∎

Integration If M^n is a compact, oriented smooth mfld, possibly with boundary, and $\alpha \in \Omega^n(M)$ then $\int_M \alpha \in \mathbb{R}$ is defined.

Stroke's Thm If M^n is as above and ∂M is its boundary with induced orientation ("outward first")



(choose $v_1 \in T_p M$ points outwards, $v_2, \dots, v_n$ is an oriented basis for $T_p \partial M$ if $(v_1, \dots, v_n)$ is an —〃— for $T_p M$)

and $\alpha \in \Omega^{n-1}(\partial M)$ then $\int_{\partial M} \alpha = \int_M d\alpha$

Thm If M is a (compact) smooth manifold (with a "finite good cover", eg if M is compact) then $H^*_{dR}(M) = H^*_{sing}(M; \mathbb{R})$

Pf Step 1. let $C^{sm}_*(M)$ denote the subcomplex of $C^{sing}_*(M)$, generated by smooth maps $\Delta_k \longrightarrow M$.

Claim: The inclusion $C^{sm}_*(M) \hookrightarrow C^{sing}_*(M)$ induces an isom. on homology.

Also, the restriction $C^*_{sing}(M; \mathbb{R}) \to C^*_{sm}(M; \mathbb{R})$ induces isom. on cohomology.

Pf of claim: check that it is true for $\mathbb{R}^n$, that (co)homology with smooth chains satisfies Mayer-Vietoris, use induction.

(Finite good cover: $M=\bigcup_{i=1}^{m} U_i$ U_i open
$U_{i_1}\cap\cdots\cap U_{i_k}$ is ϕ or $\simeq\mathbb{R}^n$.)

Step 2. Define $\Phi:\Omega^k(M)\longrightarrow C^*_{sm}(M;\mathbb{R})$

If $\alpha\in\Omega^k(M)$, $\sigma:\Delta_k\longrightarrow\mathbb{R}$ smooth, then

$$\Phi(\alpha)(\sigma):=\int_{\Delta_k}\sigma^*\alpha.$$

By Stoke's Thm, Φ is a chain map.

$$\Phi(d\alpha)(\sigma)=\int_{\Delta_k}\sigma^*d\alpha=\int_{\Delta_k}d(\sigma^*\alpha)=\int_{\partial\Delta_k}\sigma^*\alpha$$
$$=\Phi(\alpha)(\partial\sigma)=\delta\Phi(\alpha)(\sigma)$$

Step 3. Use Mayer-Vietoris induction to show that Φ induces isom. on cohomology. ∎

Last time: if M is a smooth manifold (with a finite good cover) then there is a canonical isomorphism

$$H^*_{deRham}(M)=H^*_{sing}(M;\mathbb{R})$$

given by integration of differential forms over smooth singular chains.

Fact This isomorphism identifies the product

$$H^k_{dR}(M)\otimes H^l_{dR}(M)\longrightarrow H^{k+l}_{dR}(M)$$

given by wedge product of diff. forms with the cup product

$$H^k_{sing}(M;\mathbb{R})\otimes H^l_{sing}(M;\mathbb{R})\longrightarrow H^{k+l}_{sing}(M;\mathbb{R})$$

Not true at the chain level

Proof involves relating both to Čech cohomology. (Later)

Poincaré duality for de Rham cohomology

Thm If M^n is a closed oriented smooth mfld then $H^k_{(dR)}(M)=H^{n-k}(M)^*$.

Compactly supported de Rham cohomology $H^*_c(M)$

If ω is a differential form, define its support

$$\mathrm{supp}(\omega):=\{x\in M\mid \omega(x)\neq 0\}$$

ω is compactly supported if $\mathrm{supp}(\omega)$ is compact.

Then $d\omega$ is also compactly supported because $\mathrm{supp}(d\omega)\subset\mathrm{supp}(\omega)$

This compactly supported diff forms are a subcomplex of $\Omega^*(M)$. Homology of this is $H^*_c(M)$

Example $M=\mathbb{R}$ $H^0_c(M;\mathbb{R})=0$

Claim $H^1_c(M;\mathbb{R})=\mathbb{R}$

Define $\Phi:\Omega^1_c(M;\mathbb{R})\longrightarrow\mathbb{R}$ by

$$\Phi(f(t)dt)=\int_{-\infty}^{\infty}f(t)dt\in\mathbb{R}$$

This induces a map $H^1_c(M;\mathbb{R})\to\mathbb{R}$ b/c if $fdt=dg$ where g is compactly supported fn then $f=g'$ and

$$\int_{-\infty}^{\infty}f(t)\,dt=g(\infty)-g(-\infty)=0.$$

The map $H^1_c(M;\mathbb{R})\to\mathbb{R}$ is inj b/c if $\int_{-\infty}^{\infty}f(t)dt=0$, pick $a\in\mathbb{R}$ s.t. $f(t)=0$ $\forall t\leq a$ define $g(t)=\int_a^t f(s)ds$, then g is compactly supported and $dg=fdt$, Easy to see this is surj.

With a bit more work, one can show

$$H^*_c(\mathbb{R}^n)=\begin{cases}\mathbb{R} & \text{if } *=n\\ 0 & \text{otherwise}\end{cases}$$

The isom. $H^n_c(\mathbb{R}^n)\to\mathbb{R}$ is given by integration.

More generally $H^*_c(M)=H^{*+1}_c(M\times\mathbb{R})$.

Thm If M^n is a smooth manifold (w/o boundary) with a finite good cover, then

$$H^k(M)=H^{n-k}_c(M)^*.$$

Pf Define a map $\Phi:H^k(M)\longrightarrow H^{n-k}_c(M)^*$ as follows: if $[\omega]\in H^k(M)$ and $[\eta]\in H^{n-k}_c(M)$

then $\Phi[\omega]([\eta])=\int_M\omega\wedge\eta\in\mathbb{R}$

Stoke's thm: if $\alpha\in\Omega^{n-1}_c(M)$, then $\int_M d\alpha=0$

This implies that Φ is well-defined.

If $\omega-\omega'=d\alpha$, then

$$\int_M\omega\wedge\eta-\int_M\omega'\wedge\eta=\int_M(d\alpha)\wedge\eta\overset{\eta\text{ is closed}}{=}\int_M d(\alpha\wedge\eta)=0$$

similarly for η.

Show by induction that Φ is an isom.

True if $M = \mathbb{R}^n$ by exercise

Inductn step: Suppose true for $\mathcal{U}, V, \mathcal{U} \cap V$.

Show for $\mathcal{U} \cup V$.

Recall

$$0 \to \Omega^*(\mathcal{U} \cup V) \xrightarrow{\ominus} \Omega^*(\mathcal{U}) \oplus \Omega^*(V) \xrightarrow{\oplus} \Omega^*(\mathcal{U} \cap V) \to 0$$

is exact. Observe that (extending by 0 outside domains)

$$0 \to \Omega_c^*(\mathcal{U} \cap V) \xrightarrow{\oplus} \Omega_c^*(\mathcal{U}) \oplus \Omega_c^*(V) \xrightarrow{\ominus} \Omega_c^*(\mathcal{U} \cup V) \to 0$$

is exact. Proof: partition of unity. $\rho_{\mathcal{U}}, \rho_V$

Given $\alpha \in \Omega_c^*(\mathcal{U} \cup V)$, $(\rho_{\mathcal{U}}\alpha, -\rho_V\alpha) \longmapsto \alpha$

Apply $\mathrm{Hom}(-, \mathbb{R})$ to get an exact sequence.

$$\begin{tikzcd}
0 \arrow[r] & \Omega^*(\mathcal{U} \cup V) \arrow[r] \arrow[d, "\Phi"] & \Omega^*(\mathcal{U}) \oplus \Omega^*(V) \arrow[r] \arrow[d, "\Phi"] & \Omega^*(\mathcal{U} \cap V) \arrow[r] \arrow[d, "\Phi"] & 0 \\
0 \arrow[r] & \mathrm{Hom}(\Omega_c^{n-*}(\mathcal{U} \cup V), \mathbb{R}) \arrow[r] & \mathrm{Hom}(\Omega_c^{n-*}(\mathcal{U}) \oplus \Omega_c^{n-*}(V), \mathbb{R}) \arrow[r] & \mathrm{Hom}(\Omega_c^{n-*}(\mathcal{U} \cap V), \mathbb{R}) \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & H^{*-1}(\mathcal{U} \cap V) \arrow[r] \arrow[d, "\Phi"] & H^*(\mathcal{U} \cup V) \arrow[r] \arrow[d, "\Phi"] & H^*(\mathcal{U}) \oplus H^*(V) \arrow[r] \arrow[d, "\Phi"] & \cdots \\
\cdots \arrow[r] & H_c^{n-*+1}(\mathcal{U} \cap V)^* \arrow[r] & H_c^{n-*}(\mathcal{U} \cup V)^* \arrow[r] & H_c^{n-*}(\mathcal{U})^* \oplus H_c^{n-*}(V)^* \arrow[r] & \cdots
\end{tikzcd}$$

done by Five-lemma.

Compactly supported singular cohomology

If X is any top. space, define

$C_c^k(X; G) = \{$ singular cohain $\phi \in \mathrm{Hom}(C_k(X), G) \mid \exists K \subset X$ compact such that if $\sigma: \Delta_k \to X \setminus K$ then $\phi(\sigma) = 0\}$

$H_c^k(X; G) =$ homology of this

Thm If M^n is a manifold and M is oriented or $R = \mathbb{Z}/2$, then

$$H_k(M; R) = H_c^{n-k}(M; R).$$

12/10 Today: Čech cohomology

(Exotic cohomology theories: K-theory, bordism theory, elliptic cohomology, ...)

let X be a topological space

let $\mathcal{U} = \{\mathcal{U}_i \mid i \in I\}$ be an open cover of X.

(Assume I is ordered)

let G be an abelian group

For $n \geq 0$, define $$\check{C}^n(\mathcal{U}; G) := \prod_{\{i_0 < \cdots < i_n \mid \mathcal{U}_{i_0} \cap \cdots \cap \mathcal{U}_{i_n} \neq \phi\}} G$$

$\{\phi_{i_0, i_1, \ldots, i_n} \in G \mid i_0 < \cdots < i_n\}$

$\Uparrow$ $\phi_{i_0, \ldots, i_n} = 0$ if $\mathcal{U}_{i_0} \cap \cdots \cap \mathcal{U}_{i_n} = \phi$

Convention: if $\sigma: \{0, \ldots, n\} \circlearrowleft$ a permutn, then

$$\phi_{i_{\sigma(0)}, \ldots, i_{\sigma(n)}} = (-1)^{|\sigma|} \phi_{i_0, \ldots, i_n}$$

If $\phi \in \check{C}^n(\mathcal{U}; G)$, then

$$(\delta\phi)_{i_0, \ldots, i_n} = \sum_{j=0}^{n+1} (-1)^j \phi_{i_0, \ldots \hat{i_j}, \ldots, i_{n+1}}$$

eg $\phi \in \check{C}^0(\mathcal{U}; G)$, $(\delta\phi)_{ij} = \phi_j - \phi_i$

$\delta^2 = 0$ $\check{H}^n(\mathcal{U}; G) = \mathrm{Ker}(\delta: \check{C}^n \to) / \mathrm{Im}\, \delta$

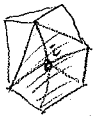
Example If X is connected, then $\check{H}^0(\mathcal{U}; G) = G$.

The set of open covers of X is partially ordered by refinement. $\{V_j \mid j \in J\}$ is a refinement of $\{\mathcal{U}_i \mid i \in I\}$ if $\exists \eta: J \to I$ $(j < j' \Rightarrow \eta(j) \leq \eta(j'))$ s.t. $V_j \subseteq \mathcal{U}_{\eta(j)}$

$$\check{C}^*(\mathcal{U}; G) \xrightarrow{r} \check{C}^*(V; G)$$

$$\overset{\psi}{\phi} \longmapsto r(\phi)_{j_0, \ldots, j_n} = \phi_{\eta(j_0), \ldots, \eta(j_n)}$$

Define $\check{H}^*(X; G) := \varinjlim \check{H}^*(\mathcal{U}; G)$



Example Suppose X is a simplical complex.

let $I := \{$ vertices in the simplical complex$\}$

For $i \in I$, let $\mathcal{U}_i = \bigcup_{\substack{\sigma \text{ is a} \\ \text{simplex containing } i}} \mathrm{int}(\sigma)$ (open star of i)

Lemma Given $i_0 < i_1 < \cdots < i_n$,

$\mathcal{U}_{i_0} \cap \cdots \cap \mathcal{U}_{i_n} \neq \phi \Leftrightarrow i_0, \ldots, i_n$ are the vertices of an n-simplex.

Cor $\check{C}^*(\mathcal{U}; G) = C^*(X; G)$ as chain complexes

$\Rightarrow \check{H}^*(\mathcal{U}; G) = H^*(X; G)$

$\| $ can show

$\check{H}^*(X; G)$

Remark Cech cohomology agrees with singular cohomology for a CW complex. But it is slightly different in general.

EX For any top. X,

$$\check{H}^0(X; G) = \prod_{\substack{\text{connected} \\ \text{components of } X}} G$$

But $$H^0(X; G) = \prod_{\substack{\text{path connected} \\ \text{components of } X}} G$$

A presheaf F of X assigns to each open set $U \subset X$ an abelian group $F(U)$, and to each inclusion $U \subset V$ a restriction map $r: f(V) \to f(U)$ s.t. $r: f(U) \to f(U)$ is the identity and if $U \subset V \subset W$, then

$$\begin{tikzcd} f(W) \arrow[r] \arrow[rr, bend right] & f(V) \arrow[r] & f(U) \end{tikzcd}$$

commutes

eg

- $F(U) = G$
- If X is a smooth manifold, $F(U) = \Omega^k(M)$

Define $\check{C}^n(\underset{\text{open cover}}{\mathcal{U}} ; F) = \prod_{i_0 < \dots < i_n} F(U_{i_0} \cap \dots \cap U_{i_n})$

$\delta : \check{C}^n(\mathcal{U}; F) \to \check{C}^{n+1}(\mathcal{U}; F)$ as before.

$\check{H}^*(X; F) = \varinjlim H^*(\mathcal{U}; F)$

Lemma For any finite open cover $\mathcal{U}$, the following sequence is exact

$$0 \to \Omega^k(X) \to \check{C}^0(\mathcal{U}; \Omega^k) \xrightarrow{\delta} \check{C}^1(\mathcal{U}; \Omega^k) \to \dots$$

Pf $\{ \rho_i \mid i \in I \}$ a partition of unity

For $n \geq 1$, define $L : \check{C}^k \to \check{C}^{k-1}$

$$L(\phi)_{i_0, \dots, i_{k-1}} = \sum_i \rho_i \phi_{i, i_0, \dots, i_{k-1}}$$

Check if $\delta\phi = 0$, then $\delta(L\phi) = \phi$. □

(Alternate) Proof of de Rham theorem (Bott-Tu)

(for a smooth mfd X w/ a finite good cover $\mathcal{U}$)

$$\begin{tikzcd}
\dots \arrow[r] & \check{C}^0(\mathcal{U};\Omega^2) \arrow[r, "\delta"] & \check{C}^1(\mathcal{U};\Omega^2) \arrow[r, "\delta"] & \check{\Omega}^2(\mathcal{U};\Omega^2) \arrow[r] & \dots \\
\dots \arrow[r] & \check{\Omega}^0(\mathcal{U};\Omega^1) \arrow[r, "\delta"] \arrow[u, "d"] & \check{C}^1(\mathcal{U};\Omega^1) \arrow[r, "\delta"] \arrow[u, "d"] & \check{\Omega}^2(\mathcal{U};\Omega^1) \arrow[r] \arrow[u, "d"] & \dots \\
\dots \arrow[r] & \check{\Omega}^0(\mathcal{U};\Omega^0) \arrow[r, "\delta"] \arrow[u, "d"] & \check{C}^1(\mathcal{U};\Omega^0) \arrow[r] \arrow[u, "d"] & \check{\Omega}^2(\mathcal{U};\Omega^0) \arrow[r] \arrow[u] & \dots
\end{tikzcd}$$

$$C^n = \bigoplus_{i+j=n} \check{C}^i(\mathcal{U}; \Omega^j)$$

$D : C^n \to C^{n+1}$ on $\check{C}^i(\mathcal{U}; \Omega^j)$, $D = \delta + (-1)^j d$

$$D^2 = \delta^2 + d^2 + (\delta d - d\delta) = 0$$

Key to the argument:

Above lemma (rows of diagram have cohom only in degree 0) implies

$H^n(C^*, D) = H^n_{dR}(X)$ refer to Bott-Tu

Poincaré lemma $\Rightarrow$ columns of diagram have cohom only in degree 0

$\Rightarrow H^n(C^*, D) = \check{H}^n(\mathcal{U}; \mathbb{R}) = H^n(X; \mathbb{R})$