Math 214 HW#3, due 3/3/15 at 12:40 PM

1. Lee 6.9.

2. Lee 7.2.

3. Lee 7.3.

4. Lee 7.13.

5. Let M be a smooth manifold. Show that the tangent bundle TM (regarded as a smooth manifold with twice the dimension of M) has a canonical orientation (even when M is not orientable!). *Hint:* Use ideas from the proof that a complex manifold has a canonical orientation.

6. Let M be a smooth manifold and let $f : M \to M$ be a smooth map.

   (a) Define the *diagonal*

   $$\Delta = \{(p, p) \mid p \in M\} \subset M \times M$$

   and the *graph*

   $$\Gamma(f) = \{(p, f(p)) \mid p \in M\} \subset M \times M.$$

   Check that Δ and $\Gamma(f)$ are submanifolds of $M \times M$ which are canonically diffeomorphic to M.

   (b) A *fixed point* of f is a point $p \in M$ with $f(p) = p$. A fixed point p is *nondegenerate* if $1 - df_p : T_pM \to T_pM$ is invertible. Show that all fixed points of f are nondegenerate if and only if $\Gamma(f)$ is transverse to Δ.

   (c) The *Lefschetz sign* of a nondegenerate fixed point p, denoted by $\epsilon(p) \in \{\pm 1\}$, is the sign of the determinant of $1 - df_p$. If all fixed points are nondegenerate, and if there are only finitely many fixed points, define the signed count of fixed points by

   $$\#\,\mathrm{Fix}(f) = \sum_{f(p)=p} \epsilon(p) \in \mathbb{Z}.$$

Show that if $\Gamma(f)$ is transverse to Δ, and if M is compact and oriented, then the intersection number[1]

$$\Gamma(f) \cdot \Delta = \#\,\mathrm{Fix}(f).$$

(d) Let A be a 2×2 integer matrix. The map $A : \mathbb{R}^2 \to \mathbb{R}^2$ descends to a map $f_A : T^2 \to T^2$, where $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. If A does not have 1 as an eigenvalue, show that all fixed points of f_A are nondegenerate, and compute $\#\,\mathrm{Fix}(f)$ in terms of A.

7. How difficult was this assignment?

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[1] For those of you who know what homology is, this can be used to prove the *Lefschetz fixed point theorem*

$$\#\,\mathrm{Fix}(f) = \sum_i (-1)^i \,\mathrm{Tr}(f_* : H_i(M;\mathbb{Q}) \to H_i(M;\mathbb{Q})).$$