Midterm #2 solutions

1. The series converges. We use the limit comparison test with

$$a_n = \frac{\sqrt{n + \cos n}}{\sqrt{n^3 + n^4}}, \qquad b_n = n^{-3/2}.$$

Then

$$\frac{a_n}{b_n} = \frac{\sqrt{n + \cos n}}{n^{-3/2}\sqrt{n^3 + n^4}} = \frac{n^{1/2}\sqrt{1 + \frac{\cos n}{n}}}{n^{-3/2} n^2 \sqrt{n^{-1} + 1}} = \frac{\sqrt{1 + \frac{\cos n}{n}}}{\sqrt{1 + n^{-1}}}.$$

Note that $\lim_{n\to\infty} \cos n/n = 0$ because $|\cos n| \le 1$. It follows that $\lim_{n\to\infty} a_n/b_n = 1$. We know that $\sum_{n=1}^{\infty} b_n$ converges by the p-test since $3/2 > 1$, so by the limit comparison test, $\sum_{n=1}^{\infty} a_n$ converges.

2. The series converges. If $n \ge 200$, then $n/100 \ge 2$, so

$$0 < \frac{1}{n^{(n/100)}} \le \frac{1}{n^2}.$$

We know that $\sum \frac{1}{n^2}$ converges by the p-test since $2 > 1$. So if we discard the first 199 terms from our series (which does not affect convergence), then the rest of the series converges by the comparison test. (One can also solve this problem using the root test.)

3. We know that the Maclaurin series for sin is

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \cdots .$$

Substituting $y = x^3$, we get

$$\sin(x^3) = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \cdots ,$$

Integrating from 0 to 1, we get

$$\int_0^1 \sin(x^3)\, dx = \frac{1}{4} - \frac{1}{10 \cdot 3!} + \frac{1}{16 \cdot 5!} - \frac{1}{22 \cdot 7!} + \cdots .$$

This is an alternating series in which the absolute values of the terms are decreasing and converging to zero. Therefore we can approximate the sum of the series by the sum of the first two terms

$$\frac{1}{4} - \frac{1}{10 \cdot 3!} = \frac{1}{4} - \frac{1}{60} = \frac{7}{30},$$

and the error is bounded from above by the third term

$$\frac{1}{16 \cdot 5!} = \frac{1}{16 \cdot 120} < 10^{-3}.$$

4. We first use the ratio test. Let

$$a_n = \frac{2^n (x+1)^n}{\ln n}.$$

Then

$$\frac{|a_{n+1}|}{|a_n|} = \frac{2^{n+1}|x+1|^{n+1} \ln n}{2^n |x+1|^n \ln(n+1)} = 2|x+1| \frac{\ln n}{\ln(n+1)}.$$

By l'Hospital's rule,

$$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \lim_{x\to\infty} \frac{\ln x}{\ln(x+1)} = \lim_{x\to\infty} \frac{x+1}{x} = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right) = 1.$$

Thefore

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 2|x+1|.$$

So the series converges if $|x+1| < 1/2$ and diverges if $|x+1| > 1/2$. We now check the cases where $|x+1| = 1/2$. If $x = -3/2$, then the series is $\sum_{n=2}^{\infty} (-1)^n / \ln n$ which converges by the alternating series test. If $x = -1/2$ then the series is $\sum_{n=2}^{\infty} 1/\ln n$ which diverges by comparison with the harmonic series. So the interval of convergence is $[-3/2, -1/2)$.

5. We first find the Maclaurin series for f. By the binomial theorem,

$$\begin{aligned}(1+y)^{-2} &= 1 - 2y + \frac{(-2)(-3)}{1 \cdot 2} y^2 + \frac{(-2)(-3)(-4)}{1 \cdot 2 \cdot 3} y^3 + \cdots \\ &= 1 - 2y + 3y^2 - 4y^3 + \cdots\end{aligned}$$

(One can also see this by squaring the geometric series formula for $(1+y)^{-1}$.) Substituting $y = x^3$ and multiplying by x^2, we get

$$f(x) = x^2 - 2x^5 + 3x^8 - 4x^{11} + 5x^{14} - 6x^{17} + \cdots$$

We know that the coefficient of x^{17} in the MacLaurin series is $f^{(17)}(0)/17!$. Therefore $f^{(17)}(0) = -6 \cdot 17!$.

6. We know that if $|x| < 1$ then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Differentiating this twice, we get

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2},$$

$$\sum_{n=2}^{\infty} n(n-1) x^{n-2} = \frac{2}{(1-x)^3}$$

if $|x| < 1$. Substituting $x = 1/2$, we conclude that

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{2^{n-2}} = \frac{2}{(1/2)^3} = 16.$$