Math 185 HW#2, due 2/9/12 at 8:10 AM

1. Find all values of i^i (not just the value coming from the principal branch of the logarithm).

2. Prove that there is no continuous logarithm function defined for all nonzero complex numbers. That is, there is no continuous function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that $e^{f(z)} = z$ for all $z \in \mathbb{C} \setminus \{0\}$. *Hint:* Modify the proof in the notes that there is no continuous square root function.

3. If $U \subset \mathbb{C}$ is an open set and $f : U \to \mathbb{C}$ is real differentiable, define
$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right),$$
$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$
   (a) Show that f is holomorphic if and only if $\partial f/\partial \overline{z} \equiv 0$, in which case $f'(z) = \partial f/\partial z$.

   (b) How do $\partial/\partial z$ and $\partial/\partial \overline{z}$ act on polynomials in z and $\overline{z}$?

   (c) Let $a, b, c \in \mathbb{C}$ constants. Under what conditions on a, b, c is the function $f(z) = az^2 + bz\overline{z} + c\overline{z}^2$ holomorphic on $\mathbb{C}$?

4. Gamelin, page 50, exercise 8.

5. Gamelin, page 53, exercise 3.

6. Prove uniqueness of a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ such that $f'(z) = f(z)$ and $f(0) = 1$. *Hint:* Let g be another such function and consider the function $h(z) = f(z)g(-z)$. What do you know about h?

7. Let log denote the principal branch of the logarithm, which is defined on the complement of the negative real axis and whose values have imaginary part in $(-\pi, \pi)$.

   (a) Show that $\log(zw) - \log(z) - \log(w) \in 2\pi i\mathbb{Z}$.

   (b) Let a, b, c be complex numbers which are not on the same line, and consider the triangle with vertices a, b, c. Give formulas in terms of the real and/or imaginary parts of log for the interior angles between the edges of the triangle.

(c) Use (a) and (b) to show that the sum of the interior angles in a triangle is π.

8. *Extra credit:* Let $a \in \mathbb{C}$ and assume that $|a| \neq 0, 1$. Show that all circles that pass through a and $1/\overline{a}$ intersect the circle $|z| = 1$ at right angles.