Math 185 Review problems

The following questions are from recent prelim exams for first-year grad students. Some of them would be too hard for our final exam, but trying to solve them should be a useful review. (If you can get more than half of these right, then you are in good shape for the complex analysis part of the prelim exam.)

1. Find the Laurent expansion of

$$f(z) = (1+z)^{-1} + (z^2-9)^{-1}$$

in the annulus $1 < |z| < 3$.

2. Compute

$$\int_0^\infty \frac{dx}{(x^2+1)^2}.$$

3. (a) Find the poles and residues of $1/(z^3\cos(z))$.

   (b) Show that the integral of the function above over a square contour centered at the origin with side $2\pi N$ tends to zero as the integer N tends to infinity.

   (c) Find the sum $1/1^3 - 1/3^3 + 1/5^3 - 1/7^3 + \cdots$.

4. Let $D \subset \mathbb{C}$ denote the unit disc. If $a, b \in D$, show that there exists a holomorphic bijection $f : D \to D$ with $f(a) = b$.

5. Calculate

$$\int_0^{2\pi} \frac{1}{1+\frac{1}{2}\sin\theta}\,d\theta.$$

6. If $0 < r < 1$, find

$$\sum_{k=0}^\infty r^k \cos(k\theta).$$

Your final answer should not involve any complex numbers.

7. Find a conformal map from the unit disk $|z| < 1$ to the sector $0 < \arg(z) < \pi/4$.

8. Compute

$$\int_{|z|=2} \frac{z^4}{z^5 - z - 1} dz.$$

9. Let $f, g : \mathbb{C} \to \mathbb{C}$ be holomorphic. Show that if $\mathrm{Re}(f(z)) > \mathrm{Re}(g(z))$ for all z with $|z| = 1$, then $\mathrm{Re}(f(z)) > \mathrm{Re}(g(z))$ for all z with $|z| < 1$.

10. Let $U \subset \mathbb{C}$ be simply connected, and suppose $f : U \to \mathbb{C}$ is holomorphic and never zero. Show that there is a holomorphic function $g : U \to \mathbb{C}$ with $e^g = f$. *Hint:* Use the fact that every closed 1-form on a simply connected domain is exact.