0. (optional, not for credit) If you want practice with proof by induction, read chapter 4 of the notes on proofs and do the exercises at the end.

1. Fraleigh, section 0, exercises 29–34.

2. In this exercise you will construct \( \mathbb{Q} \) starting from \( \mathbb{Z} \). Let \( S = \{(a, b) \mid a, b \in \mathbb{Z}, b \neq 0\} \). Define a relation \( \sim \) on \( S \) by

\[
(a, b) \sim (c, d) \iff ad = bc.
\]

(a) Show that \( \sim \) is an equivalence relation.

(b) Let \( \mathbb{Q} \) denote the set of equivalence classes. Denote the equivalence class of \( (a, b) \) by \([a, b]\). (Ordinarily we denote this by \( a/b \).) Show that the following operations of “addition” and “multiplication” on \( \mathbb{Q} \) are well defined:

\[
[a, b] + [c, d] = [ad + bc, bd],
\]

\[
[a, b][c, d] = [ac, bd].
\]

3. Recall the Division Theorem: if \( a \) and \( b \) are integers with \( b > 0 \), then there are unique integers \( q, r \) such that \( a = qb + r \) and \( 0 \leq r < b \).

(a) Show that \( \gcd(a, b) = \gcd(b, r) \). (This is the key step in proving that the euclidean algorithm works.)

(b) Prove that there exist integers \( x, y \) such that \( ax + by = \gcd(a, b) \).

*Hint:* use induction on \( \max(a, b) \) and part (a).

4. Find an integer solution \( x \), or explain why no solution exists:

(a) \( 83x \equiv 4 \pmod{157} \).

(b) \( 1001x \equiv 131 \pmod{611} \).

5. (a) Show that every positive integer \( n \) has a **binary expansion**, i.e. can be expressed as a sum of distinct powers of 2. For example, \( 2009 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^4 + 2^3 + 2^0 \). *Hint:* use the division theorem to write \( n = 2q + r \) with \( r \in \{0, 1\} \), and use induction.

(b) **Extra credit:** Show that the binary expansion of a given positive integer is unique.

6. How challenging did you find this assignment? How long did it take?