Math 113 Midterm #1 solutions

1. (a) To make an injective function $f : \{1,2,3\} \to \{4,5,6,7,8\}$ there are 5 possibilities for $f(1)$. Since $f(2)$ must be distinct from $f(1)$, there are only 4 possibilities for $f(2)$. Since $f(3)$ must be distinct from $f(1)$ and $f(2)$, there are 3 possibilities for $f(3)$. The total number of possibilities is $5 \cdot 4 \cdot 3 = 60$.

   (b) Using the Euclidean algorithm we find

$$\begin{aligned} 113 &= 103 + 10, \\ 103 &= 10 \cdot 10 + 3, \\ 10 &= 3 \cdot 3 + 1. \end{aligned}$$

   Thus $\gcd(113, 103) = 1$. Working backwards to express the gcd as a linear combination of 113 and 103, we have

$$\begin{aligned} 1 &= 10 - 3 \cdot 3, \\ 3 &= 103 - 10 \cdot 10, \\ 1 &= 10 - 3(103 - 10 \cdot 10) = 31 \cdot 10 - 3 \cdot 103, \\ 10 &= 113 - 103, \\ 1 &= 31(113 - 103) - 3 \cdot 103 = 31 \cdot 113 - 34 \cdot 103. \end{aligned}$$

   Thus a solution is $x = -34$, $y = 31$.

2. Reflexive: Given $x \in G$, to prove $x \sim x$ we must find $a \in G$ such that $axa^{-1} = x$. Take $a = e$. Then $axa^{-1} = exe^{-1} = exe = x$.

   Symmetric: if $x \sim y$ then there exists a such that $axa^{-1} = y$. We must prove $y \sim x$, i.e. we must find b such that $byb^{-1} = x$. Take $b = a^{-1}$. Then $byb^{-1} = a^{-1}y(a^{-1})^{-1} = a^{-1}axa^{-1}a = x$.

   Transitive: if $x \sim y$ and $y \sim z$ then we can write $axa^{-1} = y$ and $byb^{-1} = z$. We must prove that $x \sim z$, i.e. we must find c such that $cxc^{-1} = z$. Take $c = ba$. Then $cxc^{-1} = (ba)x(ba)^{-1} = baxa^{-1}b^{-1} = byb^{-1} = z$ so $x \sim z$.

3. (a) False. Q is not commutative since $ij = k$ but $ji = -k$. However $\mathbb{Z}_8$ is commutative, and commutativity is a structural property.

(b) False. Q has only two elements x with $x^2 = e$, namely 1 and -1. However D_4 has six such elements, namely the four reflections, the identity, and the 180 degree rotation. The cardinality of the set of elements with $x^2 = e$ is a structural property, since an isomorphism $G \to H$ induces an bijection from the set of elements $x \in G$ with $x^2 = e$ to the set of elements $x \in H$ with $x^2 = e$.

4. (a) $xy = (1\ 2)(3\ 5)(4\ 6\ 7\ 8)$.

   (b) Since disjoint cycles commute, $x^k = (1\ 2\ 3)^k(4\ 5\ 6\ 7\ 8)^k$. This is the identity whenever k is a multiple of both 3 and 5. The smallest positive integer k with this property is 15. Thus x has order 15.

5. (a) False. For example let $G = \mathbb{Z}$, $H_1 = \langle 2 \rangle$, $H_2 = \langle 3 \rangle$. Then $H_1 \cup H_2$ is not closed under the group operation since 2 and 3 are in $H_1 \cup H_2$ but $2 + 3 = 5 \notin H_1 \cup H_2$.

   (b) Claim: $\langle 498 \rangle = \langle 3 \rangle$. Proof: $\langle 498 \rangle \supset \langle 3 \rangle$ since $-2 \cdot 498 = 3$, and $\langle 498 \rangle \subset \langle 3 \rangle$ since $166 \cdot 3 = 498$.

   Since 3 is a divisor of 999, the subgroup generated by 3 has order $999/3 = 333$. Thus the subgroup generated by 498 has order 333.

   In general, if a and n are positive integers, then the subgroup of $\mathbb{Z}_n$ generated by $[a]$ has order $n/\gcd(a, n)$; see the proof in section 6 of the book.