The structure of area-minimizing double bubbles

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Abstract. We show that the least area required to enclose two volumes in $\mathbb{R}^n$ or S^n for $n \geq 3$ is a strictly concave function of the two volumes. We deduce that minimal double bubbles in $\mathbb{R}^n$ have no empty chambers, and we show that the enclosed regions are connected in some cases. We give consequences for the structure of minimal double bubbles in $\mathbb{R}^3$. We also prove a general symmetry theorem for minimal enclosures of m volumes in $\mathbb{R}^n$, based on an idea due to Brian White.

1 Introduction

1.1 The soap bubble problem

In nature, soap bubble clusters tend to enclose fixed volumes of air with the least possible surface area. This motivates us to ask the following mathematical question: given prescribed volumes $v_1, \ldots, v_m$, what is the set $B \subset \mathbb{R}^n$ of smallest area such that there exist disjoint sets $R_1, \ldots, R_m$, each a union of connected components of $\mathbb{R}^n - B$, with $\mathrm{vol}(R_i) = v_i$? By "area" and "volume" we mean $(n-1)$- and n-dimensional Hausdorff measure, respectively. We assume $n \geq 2$.

For $m = 1$, the answer is a sphere, by the classical isoperimetric theorem. J. Foisy, M. Alfaro, J, Brock, N. Hodges, and J. Zimba have proved in [10] that for $m = 2$ and $n = 2$, the unique solution is the "standard double bubble", consisting of three arcs of circles or line segments meeting at 120° angles. No further answers are known, except for recent progress in the case $m = 2$, $n = 3$ by Hass and Schlafly [12, 14, 13] using the results in this paper. It is natural to make the following conjecture, as Foisy [9] does for $n = 3$:

Double bubble conjecture. *The least-area way to enclose and separate two prescribed volumes in $\mathbb{R}^n$ is the "standard double bubble", consisting of three pieces of $(n-1)$-dimensional spheres intersecting in an $(n-2)$-dimensional sphere at 120° angles. (For the case of two equal volumes, the middle "sphere" is a flat disc.)*

(Frank Morgan has shown me a simple proof that there is a unique standard double bubble in $\mathbb{R}^n$ enclosing two given volumes. The idea is that if we fix the curvature of one of the spherical caps and vary the curvature of the other,

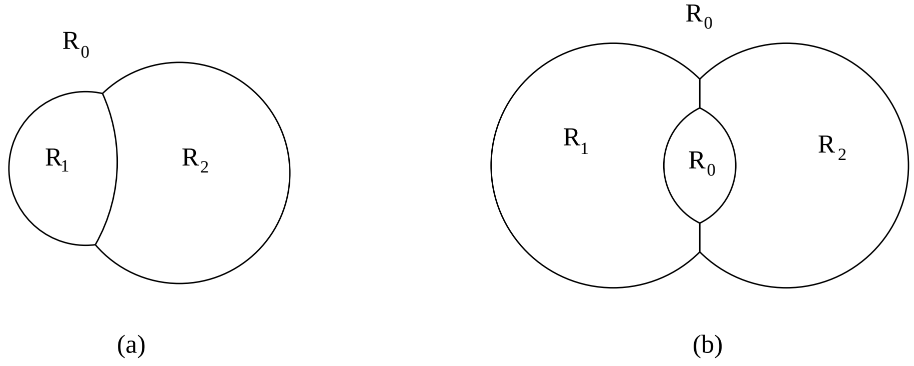

Figure 1: (a) The standard double bubble is the shortest enclosure of two prescribed volumes in $\mathbb{R}^2$. (b) A double bubble with an empty chamber.

while requiring the three pieces of spheres to meet at 120° angles, then we will achieve each ratio of volumes exactly once.)

In general, the soap bubble problem is complicated by the fact that the R_i's may be disconnected. Moreover, the "exterior" region

$$R_0 \equiv \mathbb{R}^n \setminus \left(B \cup \bigcup_{i=1}^{m} R_i \right)$$

can be disconnected. In other words, a minimal cluster may accidentally enclose some extra regions, called "empty chambers", which do not contribute to any of the volumes we are trying to enclose.

One might wish to first consider the simpler problem in which we require that the R_i's be connected. However, when $n \geq 3$, a solution to the latter problem exists only if some solution to the former problem has connected R_i's. This is because when $n \geq 3$, the area of a cluster with disconnected R_i's is approached by the areas of clusters in which the R_i's are connected by very thin tubes. Hence the infimum of area for clusters with connected regions equals the infimum of area for clusters with arbitrary regions. So an area minimizer for the connected regions problem, should it exist, is also a mimimizer for the arbitrary regions problem. Thus we have to allow disconnected regions from the outset. (When $n = 2$ the situation is different; in this case minimizers for the connected regions problem always exist, if we allow the R_i's to "bump" and count the boundaries with multiplicity (see [16]). Cox et al. show in [7] that for $m = 3$, no bumping occurs and the minimizer is, as had been expected, a complete graph on four vertices whose edges have constant curvature and meet at 120° angles.)

1.2 Concavity and applications to connectedness

In this paper we develop tools to prove connectedness in some cases. To state our main theorem, let $A_n(v_1, \ldots, v_m)$ denote the least area required to enclose and separate volumes $v_1, \ldots, v_m$ in $\mathbb{R}^n$. (For simplicity of notation we sometimes write $v = (v_1, v_2)$ and $A_n(v) = A_n(v_1, v_2)$.) Our central result is:

Theorem 3.2 (Concavity). *For each $n \geq 3$, the function $A_n(v_1, v_2)$ is strictly concave on every line in $[0, \infty)^2$. In other words, if $v, w \in [0, \infty)^2$ are two pairs of nonnegative volumes, and if $0 < t < 1$, then*

$$A_n(tv + (1-t)w) > tA_n(v) + (1-t)A_n(w).$$

We quickly deduce that $A_n(v_1, v_2)$ is strictly increasing in each v_i, and in turn that minimal double bubbles in $\mathbb{R}^n$ have no empty chambers (Theorem 3.4). It is also easy to deduce that if $v_1 > 2v_2$, then R_1 is connected (Theorem 3.5).

In some cases we can show that all R_i's are connected. For example, Corollary 4.4 shows that this is true for an area minimizing enclosure of two equal or almost equal volumes in $\mathbb{R}^3$. This is a corollary of Theorem 4.2, a general estimate which uses concavity to show that the connected components of R_i cannot be too small.

The basic idea behind the concavity theorem is as follows. Suppose B is a minimal cluster enclosing two prescribed volumes. Let H be a hyperplane through B. Let V and W be the open half spaces of $\mathbb{R}^n$ on either side of H, and let

$$v_i = 2\mathrm{vol}(R_i \cap V), \qquad w_i = 2\mathrm{vol}(R_i \cap W).$$

If we replace $B \cap W$ with the reflection of $B \cap V$ across H, we obtain a cluster enclosing volumes v_1 and v_2 whose area is $2\mathrm{area}(B \cap V) + \mathrm{area}(B \cap H)$, so

$$A_n(v_1, v_2) \leq 2\mathrm{area}(B \cap V) + \mathrm{area}(B \cap H).$$

Similarly,

$$A_n(w_1, w_2) \leq 2\mathrm{area}(B \cap W) + \mathrm{area}(B \cap H).$$

Adding these inequalities and dividing by two, we get

$$\frac{A_n(v_1, v_2) + A_n(w_1, w_2)}{2} \leq \mathrm{area}(B) = A_n\left(\frac{v_1 + w_1}{2}, \frac{v_2 + w_2}{2}\right).$$

If we could find hyperplanes dividing the volumes of the R_i's into any proportions we wish, or even any nearly equal proportions we wish, we would have a proof of (non-strict) concavity. This might not be possible. However, we show that if concavity fails, then there is a minimal double bubble that is symmetric about a point, i.e. a union of concentric spheres, and this gives a contradiction. A key technique is to study the topology of the "volume map", which sends a hyperplane to the pair of volumes into which it divides the R_i's.

Along the way, we flesh out an idea due to Brian White to prove another interesting result:

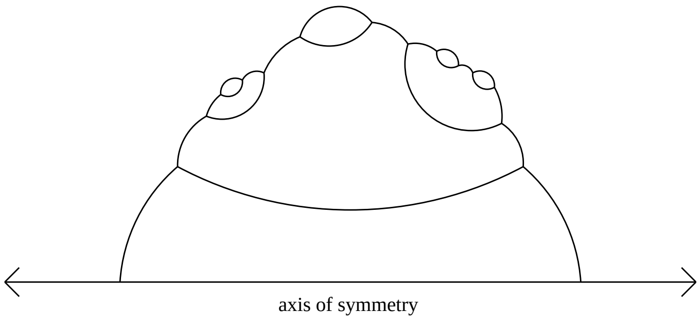

Figure 2: Example of a possible nonstandard double bubble.

Theorem 2.6 (Symmetry Theorem). *Let B be a minimal enclosure of m volumes in $\mathbb{R}^n$. Assume $m \leq n-1$. Then B is symmetric about some $(m-1)$-plane A (i.e. B is invariant under any isometry of $\mathbb{R}^n$ that fixes the points of A).*

For instance, if $m = 1$, we get a proof of the classical isoperimetric theorem, and if $m = 2$ (and $n \geq 3$), we find that every area minimizing double bubble in $\mathbb{R}^n$ is a hypersurface of revolution about some line. The case $n = 2$, $m = 3$ was originally written up by Joel Foisy in [9].

1.3 Outline and conclusions

In Section 2 we prove the symmetry theorem (2.6). This is the most technical part of the paper, because regularity (or lack thereof) of area-minimizing clusters is heavily involved. In Section 3 we use this result to prove the concavity theorem and the immediate corollaries described above. We also give generalizations to $m > 2$ and to clusters in spheres or hyperbolic space (3.6–3.10). In Section 4, we give more applications of concavity to connectedness of the enclosed regions.

In Section 5, we investigate the implications of symmetry and no empty chambers for the topological structure of minimal double bubbles in $\mathbb{R}^n$. We prove:

Theorem 5.1 (Structure Theorem). *Any minimal double bubble in $\mathbb{R}^n$ that is not the standard double bubble is a surface of revolution about some line, and consists of a topological sphere with a tree of annular bands attached. (See Figure 2.) The two caps are pieces of spheres, and the root of the tree has just one branch.*

This result, together with our topological complexity bounds, reduces the double bubble problem for a given n and a given pair of volumes to a finite dimensional optimization problem. For example, combining Theorem 5.1 with

Corollary 4.4, we see that the least-area enclosure of two equal or almost equal volumes in $\mathbb{R}^3$ is either the standard double bubble or a topological sphere with a single annular band attached. Computer experiments at the Geometry Center using a program by John Sullivan have convinced me that no cluster of the latter type in stable equilibrium exists, although this has not been proved. (There are families of unstable equilibria.)

More recently, Hass and Schlafly [14] (see also [13, 18]) have used our results, together with numerical computations, to complete the proof of the double bubble conjecture for equal volumes in $\mathbb{R}^3$. An interesting feature of their work is that a computer is used to rigorously prove a "continuous" result.

1.4 Existence and regularity

The work of Almgren [1] tells us that an area minimizing cluster B enclosing regions of volumes $v_1, \ldots, v_m$ in $\mathbb{R}^n$ exists. If we discard unnecessary points from B (and we will always do this), then B is compact and regular (locally a smooth hypersurface) on a dense subset B_{reg} whose complement has area zero. In fact, the regular set is real analytic since it has locally constant mean curvature. In a ball meeting only two R_i's, B is regular except on a set of Hausdorff dimension at most $n-8$. The boundary of each R_i is a cycle representable by integration; this means that ∂R_i is determined by the integrals of differential forms on it, and any exact form integrates to zero.

(When $n = 3$, regularity is stronger: Plateau observed experimentally, and J. Taylor finally proved in [19], that B consists of finitely many smooth surfaces which meet in threes at 120° angles along smooth curves and in sixes at vertices whose tangent cones are the cone over the regular tetrahedron. Our arguments in §5 give this improved regularity for arbitrary n when $m = 2$ (and the tetrahedral singularity is impossible in this case).)

1.5 Acknowledgments

I am extremely grateful to Frank Morgan for introducing me to the problem, for many helpful conversations, and for reading many drafts of this paper. I also wish to thank Joel Hass and John Sullivan for additional helpful conversations and editorial comments. This work began in the Geometry Group of the SMALL Undergraduate Research Project, Summer 1992, a National Science Foundation site for Research Experiences for Undergraduates (REU) (see [6, 7, 17]). I also wish to thank the Geometry Center for their hospitality. The conclusion of this project was supported by a National Science Foundation Graduate Fellowship.

2 Rotational symmetry

In this section we prove a general symmetry theorem for area minimizing clusters, which says in particular that any minimal double bubble is a hypersurface of revolution about some line. This fact is the starting point for the structure of double bubbles in $\mathbb{R}^n$ (Theorem 5.1), and we also need it to prove strict concavity of the least-area function (Theorem 3.2). For non-strict concavity, we do not need anything in this section past Lemma 2.3; see Theorem 3.6.

Lemma 2.1 (Bisectors Orthogonal) *Let B be a minimal enclosure of volumes $v_1, \ldots, v_m$ in $\mathbb{R}^n$, and let $H \subset \mathbb{R}^n$ be a hyperplane. Suppose H bisects each R_i. Then H and B_{reg} are orthogonal wherever they intersect.*

Proof. Let B_1 and B_2 be the halves of B on either side of H. We can make a new cluster B' enclosing regions R_i' with volumes $v_1, \ldots, v_m$ by replacing B_2 with the reflection of B_1 across H. Since B is minimal, $\text{area}(B') \geq \text{area}(B)$, so $\text{area}(B_1) \geq \text{area}(B_2)$. Similarly $\text{area}(B_2) \geq \text{area}(B_1)$, so $\text{area}(B') = \text{area}(B)$.

In particular, B' is minimal. Because of this, the tangent spaces to those regular points of B contained in H are either orthogonal or parallel to H; otherwise B' can be smoothed so as to decrease area. To complete the proof, we show that if B is tangent to H at a regular point p, then B' can be modified so as to decrease area.

Identify $\mathbb{R}^n$ with $H \times \mathbb{R}$. The cluster B is regular in $D \times [-\varepsilon, \varepsilon]$, where D is a closed ball of radius r about p in H. Since B is tangent to H at p, we can choose r sufficiently small that $B \cap \partial(D \times [-\varepsilon, \varepsilon]) \subset (\partial D) \times [-\varepsilon, \varepsilon]$, $\varepsilon << 1$, and the points of $D \times \{-\varepsilon\}$ and $D \times \{\varepsilon\}$ are in the same R_i'. Now remove from B' its entire intersection with $D \times [-\varepsilon, \varepsilon]$, and add back $(\partial D) \times [-\varepsilon, \varepsilon]$. This causes an area decrease on the order of r^{n-1}; the volumes of the R_i''s may change, but at most on the order of εr^{n-1}. By a standard first variation argument, we can restore the volumes by adjusting the cluster elsewhere, with an area increase on the order of εr^{n-1}; this is smaller than the initial decrease, if r is sufficiently small. □

Let $A \subset \mathbb{R}^n$ be an affine subspace. We can decompose the tangent bundle of $\mathbb{R}^n \setminus A$ as follows:

$$T(\mathbb{R}^n \setminus A) = A^{\|} \oplus A^{\nu} \oplus A^{r},$$

where $A^{\|}$ consists of vectors parallel to A, A^{ν} is the one-dimensional subbundle generated by rays radiating orthogonally from A, and A^r consists of "directions of rotation" about A; these are the vectors generated by one-parameter families of rotations of $\mathbb{R}^n$ that fix the points of A. We say that B is *symmetric about* A if B is invariant under all isometries of $\mathbb{R}^n$ that fix the points of A.

Lemma 2.2 (Infinitesimal Symmetry Implies Global Symmetry) *Let B be a minimal cluster in $\mathbb{R}^n$ and $A \subset \mathbb{R}^n$ an affine subspace with $\dim(A) \leq n-2$. Suppose that for almost every $p \in B_{reg} \setminus A$,*

$$A^r(p) \subset T_pB.$$

Then B is symmetric about A.

Proof. By the assumption on dimension, invariance under reflections about A follows from invariance under rotations about A.

Let ϕ be a rotation about A; we will show that $\phi_* \partial R_i = \partial R_i$ for each i. There is a one-parameter group $\{\phi_t\}$ of rotations about A with $\phi = \phi_1$. These induce a vector field X on $\mathbb{R}^n$. Let ω be an $(n-1)$-form on $\mathbb{R}^n$; we need to show that

$$(\partial R_i, \omega) = (\phi_* \partial R_i, \omega).$$

The right hand side is equal to $(\partial R_i, \phi^* \omega)$, so we want

$$\frac{d}{dt}\bigg|_{t=0} (\partial R_i, \phi_t^* \omega) = 0.$$

The left side of this is

$$(\partial R_i, (d\imath_X + \imath_X d)\omega) = (\partial R_i, \imath_X d\omega),$$

since ∂R_i is a cycle. By almost everywhere regularity, it is enough to show that $(\imath_X d\omega(p), \wedge^{n-1} T_p B) = 0$ for almost every $p \in B_{\text{reg}} \cap \partial R_i$. This vanishing certainly occurs if $X(p) \in T_p B$. We are done because $X(p) \in A^r(p)$ for all $p \notin A$, and by hypothesis $A^r(p) \subset T_p B$ for almost all $p \in B_{\text{reg}} \setminus A$. $\square$

Lemma 2.3 (Symmetry About Intersections) *Let B be a minimal enclosure of m volumes in $\mathbb{R}^n$. Assume $k > 1$, and let $H_1, \ldots, H_k$ be mutually orthogonal hyperplanes. Suppose B is symmetric about each H_i. Then B is symmetric about $A = \cap H_i$.*

Proof. B is invariant under the composition of the reflections across the H_i's, which is reflection across A. In particular, every hyperplane containing A bisects $R_1, \ldots, R_m$. By Lemma 2.1, B_{reg} is orthogonal to each such hyperplane. But the orthogonal directions to these hyperplanes are exactly the directions of rotation about A. Therefore B is symmetric about A, by Lemma 2.2. $\square$

We now want to understand how symmetries in subsets of a cluster may combine to give symmetries of the whole cluster.

Lemma 2.4 (Linear Algebra) *If $A_1, A_2 \subset \mathbb{R}^n$ are affine subspaces with nonempty intersection, then*

$$(A_1 \cap A_2)^r(p) = A_1^r(p) \oplus A_2^r(p)$$

whenever $p \notin span(A_1, A_2)$.

Proof. The right side of this equation is always contained in the left hand side, because any rotation fixing A_1 or A_2 fixes $A_1 \cap A_2$. Now suppose equality does not hold. Counting dimensions, we find that

$$\dim A_1^r(p) \cap A_2^r(p) > n - \dim \operatorname{span}(A_1, A_2) - 1.$$

Since $A_1^r(p) \cap A_2^r(p)$ is the orthogonal complement of $A_1^{\|}(p) \oplus A_2^{\|}(p) \oplus A_1^{\nu}(p) \oplus A_2^{\nu}(p)$ and $\dim(A_1^{\|}(p) \oplus A_2^{\|}(p)) = \dim \operatorname{span}(A_1, A_2)$, we have $A_1^{\nu}(p) \subset A_1^{\|}(p) \oplus A_2^{\|}(p)$. Since $A_1^{\nu}(p) \perp A_1^{\|}(p)$, $A_1^{\nu}(p) \subset A_2^{\|}(p)$, which implies $p \in \operatorname{span}(A_1, A_2)$. $\square$

Lemma 2.5 (Assembly) *Let B be an area minimizing enclosure of m volumes in $\mathbb{R}^n$. Let $H \subset \mathbb{R}^n$ be a hyperplane and let B_1, B_2 be the two symmetrizations of B about H. Suppose B_1 and B_2 minimize area for the volumes they enclose. (For instance, this is true when H bisects $R_1, \ldots, R_m$.) Let $A_1, A_2 \subset H$ be nonempty affine subspaces of dimension at most $n-2$. Suppose B_i is symmetric about A_i.*

Then $A_1 \cap A_2 \neq \emptyset$ and B is symmetric about $A_1 \cap A_2$.

Proof. First note that $B \cap H$ is symmetric about A_1 and A_2 in H. If $A_1 \cap A_2 = \emptyset$, then B is not compact, a contradiction. To prove symmetry, by Lemma 2.2 it is enough to show that for almost every $p \in B_{\mathrm{reg}} \setminus (A_1 \cap A_2)$, $(A_1 \cap A_2)^r(p) \subset T_pB$.

We begin by showing that for almost every $p \in B \cap H \setminus (A_1 \cup A_2)$ (with respect to $(n-2)$-dimensional Hausdorff measure), B, B_1, and B_2 are regular at p, and $(A_1 \cap A_2)^r(p) \subset T_pB$. (Note that $B \cap H$ cannot have dimension greater than $n-2$, or else B would have dimension greater than $n-1$, by our assumption of rotational symmetry.)

Since B_i is regular almost everywhere and symmetric about A_i, almost every point of $B_i \cap H \setminus A_i$ is regular. Hence almost every point of $B \cap H \setminus (A_1 \cup A_2)$ is a regular point of both B_1 and B_2. Let p be such a point. By Lemma 2.1, B_1 and B_2 are orthogonal to H at p, since H bisects the volumes enclosed by B_i. Thus a small ball around p meets only two of the R_i's for B, so B is regular in this ball except on a set of Hausdorff dimension at most $n-8$. It follows that almost every point of $B \cap H \setminus (A_1 \cup A_2)$ is a regular point of B_1, B_2, and B. If U is a regular neighborhood of such a point, then by uniqueness of analytic continuation, $U \cap B = U \cap B_1 = U \cap B_2$. It follows that $A_1^r(p), A_2^r(p) \subset T_pB$ for all $p \in U \setminus (A_1 \cup A_2)$. By Lemma 2.4, $(A_1 \cap A_2)^r(p) \subset T_pB$ for all $p \in U \setminus H$. By continuity, this is true for all $p \in U$.

We can now complete the proof except for sets $\tilde{A}_1$ and $\tilde{A}_2$ defined as follows. Let W_i be the closed half space from which B_i is formed. In W_2, let $\tilde{A}_1$ be the union of those orbits under rotation about A_2 whose intersection with H is contained in A_1. (Note that $\tilde{A}_1$ is half of a k-plane with $k \leq \dim(A_1)+1$.) Define $\tilde{A}_2$ similarly. By the previous paragraph, for almost every $p \in B \cap W_1 \setminus \tilde{A}_2$ (now with respect to $(n-1)$-dimensional measure), there exists a rotation ϕ about A_1 such that $\phi(p) \in H$, B and B_1 are regular at $\phi(p)$ and $(A_1 \cap A_2)^r(\phi(p)) \subset T_{\phi(p)}B$. Since B_1 is invariant under ϕ and ϕ is a rotation about $A_1 \cap A_2$, B is regular at p and $(A_1 \cap A_2)^r(p) \subset T_pB$. Likewise, this is true for almost every $p \in B \cap W_2 \setminus \tilde{A}_1$.

Finally, suppose $p \in B_{\mathrm{reg}} \cap \tilde{A}_i$. Either some regular neighborhood U of p in B is contained in $\tilde{A}_i$, or p is a limit of regular points of B not in $\tilde{A}_i$ whose tangent spaces contain the directions of rotation about $A_1 \cap A_2$. In the latter case it follows by continuity that $(A_1 \cap A_2)^r(p) \subset T_pB$. In the former case, using rotation invariance of both B_1 and B_2 we find that the points in $\mathbb{R}^n$ on either side of U lie in the same R_i, so U can be removed to decrease area, contradicting minimality of B. □

Theorem 2.6 (Symmetry Theorem) *Let B be a minimal enclosure of m volumes in $\mathbb{R}^n$. Assume $m \leq n-1$. Then B is symmetric about some $(m-1)$-plane.*

Proof. Since B is compact, we can apply the ham sandwich theorem to produce a hyperplane H_1 that bisects $R_1, \ldots, R_m$. By Lemma 2.5, it is enough to show that each of the symmetrizations of B across H_1 is symmetric about an $(m-1)$-plane in H_1. Let B_1 be one of the two symmetrizations. By the Borsuk-Ulam theorem, as used to prove the ham sandwich theorem in $\mathbb{R}^{n-1}$ (see e.g., [5]), there is a hyperplane H_2, orthogonal to H_1, that bisects $R_1, \ldots, R_m$. By Lemma 2.5 again, it is enough to show that each of the symmetrizations of B_1 across H_2 is symmetric about an $(m-1)$-plane in $H_1 \cap H_2$. Let B_2 be one of the two symmetrizations, and continue this process.

Eventually we obtain orthogonal hyperplanes $H_1, \ldots, H_{n-m+1}$, and a minimal cluster B_{n-m+1} symmetric about each H_i. We need to show that B_{n-m+1} is symmetric about the $(m-1)$-plane $A = \cap H_i$. We are done by Lemma 2.3. □

Remark 2.7 When $m = n$, the ham sandwich theorem and symmetrization show that for every set of m volumes, there exists a minimizer that is symmetric about a hyperplane. It is not clear whether all minimizers must have this property.

Corollary 2.8 (Classical Isoperimetric Theorem) *The unique area minimizing enclosure of a single volume in $\mathbb{R}^n$ is a sphere.*

Proof. By Almgren's work [1], a minimizer exists. By Theorem 2.6, any area minimizer is a union of concentric spheres. Suppose there is more than one sphere. The cluster has finite area (since a cluster consisting of a single sphere is a competitor), so the radii of the spheres are discrete. Hence we can move one of the spheres to violate symmetry, which is a contradiction. □

Corollary 2.9 (Axis of Symmetry) *If $n \geq 3$, any minimal double bubble in $\mathbb{R}^n$ is symmetric about some line.*

For $n = 2$ this is immediate from the main theorem in [10].

In §3 we need a slight variation on the proof of Theorem 2.6, so we state it here for convenience.

Lemma 2.10 (Axis in Hyperplane) *Let B be a minimal double bubble in $\mathbb{R}^n$, $n \geq 3$, and let $H \subset \mathbb{R}^n$ be a hyperplane. Suppose each symmetrization of B across H is area minimizing for the volumes it encloses. Then B is symmetric about a line in H.*

Proof. By Lemma 2.5, it is enough to show that each symmetrization is symmetric about a line in H. For each symmetrization, use the proof of Theorem 2.6, starting with $H_1 = H$. $\square$

Remarks. We personally prefer Knothe's proof of the isoperimetric theorem, which shows directly why a sphere has less area than any other smooth surface enclosing the same volume, without appealing to difficult existence and regularity theorems. See for instance the article by Gromov [11].

Similar symmetry arguments can be applied to other variational problems in $\mathbb{R}^n$, and examples abound in the literature. For example, the above arguments easily show that an area minimizing surface enclosing a given volume between two fixed parallel hyperplanes is symmetric about a line orthogonal to the hyperplanes, and an area minimizing surface enclosing a given volume inside the unit ball is symmetric about a diameter. In fact, work of Athanassenas [3] and Vogel [20] shows that the minimizer for the first problem is either a hemisphere or a cylinder. Bokowski and Sperner [4] and Almgren [2] have shown that the minimizer for the second problem is a piece of a sphere. For a survey of some other such problems, see [8, pp. 52-56].

3 Concavity of the least-area function

In this section we prove that the least area required to enclose two volumes in $\mathbb{R}^n$ is a strictly concave function of the volumes, and we give generalizations to $m > 2$ and to minimal clusters in round spheres or hyperbolic space. We give the simplest applications to connectivity of the R_i's; more examples are given in Section 4.

Lemma 3.1 (Continuity) *For any m and n, the least-area function $A_n(v_1, \ldots, v_m)$ is continuous.*

Proof. For simplicity, we just prove continuity along a line with $v_2, \ldots, v_m$ constant; the general proof is similar. Given an enclosure of volumes $v_1, \ldots, v_m$, we can increase v_1 by δ, with a controlled increase in area, by creating a sphere away from the cluster with volume δ and incorporating this volume into R_1. To decrease v_1, we can scale the entire cluster down so that R_1 has volume $v_1 - \delta$ and then add spheres to restore the volumes of $R_2, \ldots, R_m$ to $v_2, \ldots, v_m$. The resulting area increase will be controlled uniformly for v_1 in some neighborhood. We can cover the positive real line with such neighborhoods, and this implies continuity in v_1 for $v_1 > 0$. We also have continuity at $v_1 = 0$ because

$$A_n(0, \ldots, v_m) \le A_n(\delta, \ldots, v_m) \le A_n(0, \ldots, v_m) + A_n(\delta). \qquad \square$$

Theorem 3.2 (Strict Concavity) *If $n \ge 3$, if $v, w \in [0, \infty)^2$ are two pairs of nonnegative volumes, and if $0 < t < 1$, then*

$$A_n(tv + (1-t)w) > tA_n(v) + (1-t)A_n(w).$$

Proof. Suppose not. By continuity, the function

$$f(t) \equiv A_n(tv + (1-t)w) - tA_n(v) - (1-t)A_n(w)$$

takes its minimum on $[0,1]$ at some $t_0 \in (0,1)$. Let B be a minimal cluster enclosing volumes $t_0 v + (1-t_0)w$. By Corollary 2.9, B is symmetric about a line L.

We can parametrize the set of oriented hyperplanes in $\mathbb{R}^n$ by $S^{n-1} \times \mathbb{R}$; each oriented hyperplane is determined by a normal direction and a distance from the origin in that direction. Define the *volume map* $g : S^{n-1} \times \mathbb{R} \to \mathbb{R}^2$ by sending an oriented hyperplane to $(\mathrm{vol}(R_1 \cap U), \mathrm{vol}(R_2 \cap U))$, where U is the upper half space determined by the oriented hyperplane. Since B is compact, g is continuous.

Let us take the origin of $\mathbb{R}^n$ to lie in L. Choose $x \in S^{n-1}$ orthogonal to L, so that the hyperplane described by (x, r) contains L if $r = 0$ and is parallel to L otherwise. By symmetry of B we have

$$g(x,r) + g(x,-r) = t_0 v + (1-t_0)w$$

for all $r \in \mathbb{R}$. Consider the line segment in the volume plane

$$K = \left\{ \frac{tv + (1-t)w}{2} \;\middle|\; t, 2t_0 - t \in (0,1) \right\} \subset \mathbb{R}^2.$$

Either $g(x,r) \in K$ for some $r \neq 0$, or else by continuity and the symmetry observation above, $g(y \times \mathbb{R})$ hits K for every $y \in S^{n-1}$ close to x. In both cases, we can find a hyperplane H, not containing L, with $g(H) = (tv + (1-t)w)/2 \in K$.

Let U and V be the upper and lower half spaces determined by H; let $a_0 = \mathrm{area}(B \cap H)$, $a_1 = \mathrm{area}(B \cap U)$, and $a_2 = \mathrm{area}(B \cap V)$. Replacing $B \cap V$ with the reflection of $B \cap U$ across H, we find that

$$\begin{aligned} a_0 + 2a_1 &\geq A_n(tv + (1-t)w) \\ &= f(t) + tA_n(v) + (1-t)A_n(w) \\ &\geq f(t_0) + tA_n(v) + (1-t)A_n(w) \\ &= A_n(t_0 v + (1-t_0)w) + (t - t_0)A_n(v) + (t_0 - t)A_n(w). \end{aligned}$$

If we symmetrize in the other direction instead, we get

$$a_0 + 2a_2 \geq A_n(t_0 v + (1-t_0)w) + ((2t_0 - t) - t_0)A_n(v) + (t_0 - (2t_0 - t))A_n(w).$$

Adding, we obtain

$$2(a_0 + a_1 + a_2) \geq 2A_n(t_0 v + (1-t_0)w).$$

We know that this is an equality, so all the above inequalities are equalities. In particular, each symmetrization of B across H is area minimizing for the volumes it encloses.

By Lemma 2.10, B is symmetric about a line $L' \subset H$. Since B is compact, L and L' must intersect. But $L \neq L'$, so by applying Lemma 2.5 to a hyperplane containing L and L' (or by a simpler argument), we get that B is a union of concentric spheres. Then B must contain only one sphere, or else we can move one of the spheres and violate the symmetry we just established. So B encloses only one volume, and v and w both lie on one of the coordinate axes of $\mathbb{R}^2$. Now A_n is strictly concave along any line through the origin, since by scaling

$$A_n(\lambda v_1, \ldots, \lambda v_m) = \lambda^{(n-1)/n} A_n(v_1, \ldots, v_m).$$

But we assumed that A_n is not strictly concave along this line, which is a contradiction. □

Our methods also give non-strict concavity for $n = 2$; see Theorem 3.6. (Strict concavity in the case $n = 2$ holds by [10].)

Corollary 3.3 (Strictly Increasing) *For a fixed n, the function $A_n(v_1, v_2)$ is strictly increasing in each v_i.*

(Foisy et al. have proved this for $n = 2$ and $m = 2$ in [10]. For the connected regions problem in the plane, Cox et al. have obtained an analogous result when $m \leq 4$, in [6].)

Proof. Suppose $v_2 < v_2'$ and $A_n(v_1, v_2) \geq A_n(v_1, v_2')$. Then by concavity, $A_n(v_1, v_2') \geq A_n(v_1, w)$ for all $w > v_2'$. But by the isoperimetric theorem, $A_n(v_1, w) \geq A_n(w) \to \infty$ as $w \to \infty$, which is a contradiction, since there exist clusters enclosing volumes v_1 and v_2' with finite area. □

Theorem 3.4 (No Empty Chambers) *Minimal double bubbles in $\mathbb{R}^n$ never have empty chambers.*

Proof. If a minimal double bubble contains an empty chamber, we can declare the empty chamber to be part of R_1, thereby obtaining a bubble in which v_1 is larger but the total area is the same. This contradicts Corollary 3.3. □

Theorem 3.5 (Balancing) *If $v_1 > 2v_2$, then in any least-area enclosure of volumes v_1 and v_2 in $\mathbb{R}^n$, R_1 is connected.*

Proof. If $n \geq 3$, then since A_n is strictly concave along the line $v_1 + v_2 = c$ and $A_n(v_1, v_2) = A_n(v_2, v_1)$, we see that if $v_1 + v_2$ is held constant and v_1 and v_2 are brought closer together, then A_n increases. Now suppose that R_1 is disconnected in a minimal double bubble enclosing volumes v_1 and v_2. We can find a nonempty union Q of connected components of R_1 whose volume is at most $v_1/2 < v_1 - v_2$. If we declare Q to be part of R_2, we obtain a cluster with the same area whose volumes are more balanced, which is a contradiction.

When $n = 2$, we know by [16] that a minimal cluster is a union of finitely many arcs of circles and line segments meeting at $120°$ angles. Then we see that

Q must have an edge in common with R_2. (Otherwise ∂Q has no vertices and Q is floating in the middle of R_0. We can then move Q until it first touches the rest of the cluster, creating an illegal singularity.) If we remove this edge and declare Q to be part of R_2, then length decreases, contradicting concavity. $\square$

For $n = 2$, we already knew this by the main theorem in [10]. (But we obtain a new result along these lines in Corollary 3.10.)

We can also say something about the least area function for general m. The following theorem is equivalent to non-strict concavity for $m = 2$, and weaker than concavity for $m > 2$.

Theorem 3.6 (A Strong Minimum Principle) *Assume $m \leq n$. Let $T : \mathbb{R}^m \to \mathbb{R}$ be any linear function, and let $K \subset \mathbb{R}^m$ be any hyperplane; then $A_n + T$ satisfies the minimum principle on K. That is, for any $(m-1)$-dimensional flat disk $D \subset [0, \infty)^m$, the restriction of $A_n + T$ to D achieves its minimum on ∂D.*

Proof. Let $v \in D$ be a point in D on which $A_n + T$ takes its minimum, and which is as far from the center of D as possible. Suppose $v \in \mathrm{Int(D)}$. Let D' be a ball in D, centered at v. Let B be a minimal cluster enclosing volumes v. Since $m \leq n$, we can use the ham sandwich theorem to make B symmetric about orthogonal hyperplanes $H_1, \ldots, H_{n-m+1}$.

We have a continuous volume map $g : S^{n-1} \times \mathbb{R} \to \mathbb{R}^m$ as before. Now if $g(H) \in D'/2$, then $g(H) = v/2$. To see this, suppose $g(H) = w/2 \in D'/2$. Let $w' = 2v - w$. By considering the two symmetrizations of B about H and using the fact that B is minimal, we have as before that

$$2A_n(v) \geq A_n(w) + A_n(w').$$

Since T is linear, $2T(v) = T(w) + T(w')$, so

$$2[A_n + T](v) \geq [A_n + T](w) + [A_n + T](w').$$

Since $A_n + T$ takes its minimum on D at v, it must take that same value at w and w'. But if w and w' are distinct, then either w or w' is farther from the center of D than v is (since disks are convex!), contradicting our choice of v.

If $x \in S^{n-1}$ is orthogonal to $\cap H_i$, then by symmetry of B the path $g(x, \cdot)$ goes through and is symmetric about $v/2$. By basic topology (roughly, the principle that a threaded needle stays threaded), it follows that for every $y \in S^{n-1}$, the path $g(y, \cdot)$ hits $v/2$. In particular, there exists a hyperplane H, orthogonal to $H_1, \ldots, H_{n-m+1}$, that bisects R_1 and R_2.

Symmetrize B about H, and repeat this procedure until we get B to be symmetric about n orthogonal hyperplanes. By Lemma 2.3, B is a union of concentric spheres, so B is a single sphere and v lies on one of the coordinate axes. If $m > 2$ this is an immediate contradiction since $v \in \mathrm{Int(D)}$, and if $m = 2$ this is a contradiction as explained in the proof of Theorem 3.2. $\square$

Remark 3.7 [Dichotomy Between Concavity and Excess Symmetry] Conversations with John Sullivan suggest that by using more topology one might show that more generally, either $A_n + T$ satisfies the minimum principle on all $(m-k-1)$-dimensional disks in $\mathbb{R}^m$ for all T, or else there is a minimal enclosure of m nonzero volumes in $\mathbb{R}^n$ symmetric about a k-plane. For example, when $m = 3$, this would say that when $n \geq 3$, either the least-area function is concave, or else there is a minimal triple bubble with symmetry about a line (which seems unlikely). However we have not worked out the details. In any case we know extremely little about the soap bubble problem for $m > 2$.

Remark 3.8 [Hyperbolic Space] As Joel Hass points out, the above results and their proofs carry over, with minor rewording, to hyperbolic space $\mathbb{H}^n$. One needs to check that the least area function for one volume is concave; this is an easy exercise.

We also have an analogue of Theorem 3.2 in the round n-sphere S^n. First note that the results of §2 carry over from $\mathbb{R}^n$ to S^n, with essentially the same proofs, provided we interpret "hyperplane" to mean "equatorial S^{n-1}", etc. Then:

Theorem 3.9 (Concavity in Spheres) *For every $n \geq 3$, the least area required to partition the sphere S^n into three volumes is strictly concave on every line in the simplex*

$$\{v_1 + v_2 + v_3 = vol(S^n)\}.$$

For $n = 2$ there is concavity but it is not a priori strict.

Proof. For $n \geq 3$, this is a straightforward rewording of the proof of Theorem 3.2. A slight difference is that oriented hyperplanes in S^n are parametrized by S^n, and instead of looking at paths of the form $g(x \times \mathbb{R})$, we look at the images under g of geodesics. Note that we have strict concavity along an edge of the simplex because in a minimal sphere dividing S^n into two volumes, the pressure difference between the two regions decreases as the volumes become more equal. For $n = 2$ adapt the proof of Theorem 3.6. □

We immediately get an analogue of Theorem 3.5:

Corollary 3.10 (Balancing in Spheres) *If $v_1 > 2v_2$, then in any area-minimizing partition of S^n into volumes $v_1, v_2, vol(S^n) - v_1 - v_2$, we have R_1 connected.*

Proof. Same proof as Theorem 3.5. □

There is also an analogue of Theorem 3.6 in S^n.

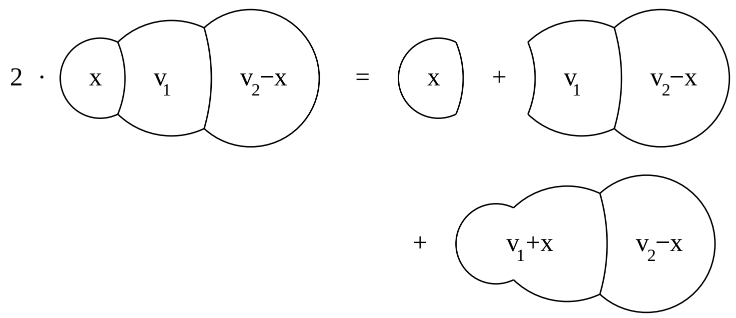

Figure 3: The proof of Lemma 4.1.

4 Examples of connectivity

We now give some slightly more involved examples of how concavity may be used to get at connectedness of the enclosed regions. We show for instance that a minimal enclosure of two equal or almost equal volumes in $\mathbb{R}^3$ has all regions connected. The idea, as in the proof of Theorem 3.5, is to remove various parts of the cluster and apply concavity to place lower bounds on the area of what remains. By combining these inequalities, we can bound the area of the entire cluster.

Our most useful decomposition is encapsulated in the following lemma, adapted from [6].

Lemma 4.1 (Decomposition) *Suppose that in a minimal enclosure of volumes v_1, v_2 in $\mathbb{R}^n$, R_2 has a connected component with volume x. Then*

$$2A_n(v_1, v_2) \geq A_n(x) + A_n(v_1, v_2 - x) + A_n(v_1 + x, v_2 - x).$$

Proof. We can think of this cluster as an enclosure of regions R_1, R_2, and R_3 with volumes v_1, x, and $v_2 - x$, respectively. Let $S_{ij} = \partial R_i \cap \partial R_j$, and let $a_{ij} = \text{area}(S_{ij})$. Note that $a_{23} = 0$; otherwise a neighborhood of a regular point in S_{23} can be removed to decrease area. Also any two S_{ij}'s intersect in a set of area zero, by almost everywhere regularity. We then have

$$\begin{aligned}
2A_n(v_1, v_2) &= (a_{02} + a_{12}) + (a_{01} + a_{12} + a_{13} + a_{03}) + (a_{01} + a_{02} + a_{03} + a_{13}) \\
&= \text{area}(\partial R_2) + \text{area}(\partial R_1 \cup \partial R_3) + \text{area}(\partial(\overline{R_1} \cup \overline{R_2}) \cup \partial R_3) \\
&\geq A_n(x) + A_n(v_1, v_2 - x) + A_n(v_1 + x, v_2 - x). \qquad \square
\end{aligned}$$

Figure 3 gives a schematic for the above proof.

This lower bound is well suited to ruling out small components because it is sharp when $x = 0$ and rapidly increasing when x is small. The simpler lower bound

$$2A_n(v_1, v_2) \geq A_n(v_1) + A_n(x) + A_n(v_2 - x) + A_n(v_1 + v_2)$$

(under the same hypotheses as above) is useful for ruling out large components, but we do not need it for our examples here. We mention, however, that this can be used to weaken the hypothesis in Theorem 3.5.

We now apply concavity to show that the last two terms in Lemma 4.1 do not decrease too fast when x is small. We have

$$\begin{aligned} A_n(v_1, v_2 - x) &\geq \frac{v_2 - x}{v_2} A_n(v_1, v_2) + \frac{x}{v_2} A_n(v_1), \\ A_n(v_1 + x, v_2 - x) &\geq \frac{v_2 - x}{v_2} A_n(v_1, v_2) + \frac{x}{v_2} A_n(v_1 + v_2). \end{aligned}$$

In each of these estimates we use concavity of A_n restricted to a line segment connecting the line $\{(\lambda v_1, \lambda v_2)\}$ to the axis (one volume zero). One can get slightly stronger bounds using different such segments (together with the fact that $A_n(\lambda v_1, \lambda v_2) = \lambda^{\frac{n-1}{n}} A_n(v_1, v_2)$ by scaling), but the estimates above are fairly good and easy to compute with. Putting these into Lemma 4.1, simplifying, and using the fact that $A_n(\lambda) = \lambda^{\frac{n-1}{n}} A_n(1)$, we obtain:

Theorem 4.2 (Basic Estimate) *Suppose that in a minimal enclosure of volumes v_1, v_2 in $\mathbb{R}^n$, R_2 has a connected component with volume $x > 0$. Then*

$$\frac{2A_n(v_1, v_2)}{A_n(1)} \geq v_2 x^{-1/n} + v_1^{\frac{n-1}{n}} + (v_1 + v_2)^{\frac{n-1}{n}}.$$

This lower bound goes to infinity as $x \to 0$. Hence:

Corollary 4.3 (Finiteness) *A minimal enclosure of two volumes in $\mathbb{R}^n$ separates $\mathbb{R}^n$ into finitely many components.*

Similar arguments show that this is also true in S^n.

Corollary 4.4 (Equal Volumes Connected) *In any least-area enclosure of two equal or almost equal volumes in $\mathbb{R}^3$, all R_i's are connected.*

Proof. By Theorem 3.4, R_0 is connected. Suppose that in a minimal enclosure of two unit volumes, one of the enclosed regions has a connected component with volume $0 < x \leq 1/2$. The basic estimate gives

$$\frac{2A_3(1,1)}{A_3(1)} \geq x^{-1/3} + 1 + \sqrt[3]{4}.$$

By calculating the volume of the standard double enclosing two equal volumes, we find that $2A_3(1,1)/A_3(1) \leq 3\sqrt[3]{2}$, so

$$x^{-1/3} \leq 3\sqrt[3]{2} - 1 - \sqrt[3]{4}.$$

This is false for $x = 1/2$, and hence for all smaller x, giving a contradiction. Since the estimates we used are continuous, and not sharp in this case, the result is also true when the volumes are almost equal. □

Many more calculations of this type can be carried out, although we do not know how to prove connectedness in all cases. We sketch two more examples below.

Example 4.5 [Extreme Double Bubbles in $\mathbb{R}^3$] Suppose $v > 1$. One can show by construction that

$$\frac{A_3(v,1)}{A_3(1)} \leq (v+1)^{2/3} + \sqrt[3]{5/16}.$$

If in a minimal enclosure of volumes v and 1, R_2 has a connected component with volume x, then combining this with the basic estimate we get a lower bound on x which tends to $2/5$ as $v \to \infty$. So in a minimal enclosure of two disparate volumes in $\mathbb{R}^3$, the exterior is connected, the region with larger volume is connected, and the region with smaller volume has at most two components. If the region with smaller volume is disconnected, Theorem 5.1 tells us that the cluster looks like a sphere with a narrow tube connecting the north and south poles and a thin band around some latitude line.

In fact Joel Hass [12] has used this result to prove the double bubble conjecture in $\mathbb{R}^3$ when the ratio between the two volumes is large.

Example 4.6 [Double/Triple Bubbles in S^2] The proof of [10, Lem. 2.4] carries over to S^2 to show that in a minimal partition of S^2 into three volumes, if any one R_i is connected then all three are. By Corollary 3.10, a minimal partition into a triple of volumes near the boundary of the 2-simplex has all regions connected. Since a minimal double bubble consists of curves meeting in threes, the graph consists of two vertices with three edges joining them. For three equal volumes the S^2 analogue of the basic estimate easily implies that all regions are connected. The edges have constant curvature and meet at 120° angles, and a simple argument using the Gauss-Bonnet theorem shows that they are longitude lines.

In fact Joe Masters has shown by computer [15] that our estimate rules out disconnected regions for all other triples of volumes. Thus the double bubble conjecture is proved for S^2.

5 The structure of minimal double bubbles in $\mathbb{R}^n$

Let B be a minimal double bubble in $\mathbb{R}^n$. Corollary 2.9 tells us that B is described by a subset of the upper half plane $\mathbb{H}$. We can use symmetry to enhance regularity. Area and volume in $\mathbb{R}^n$ correspond to area and volume in $\mathbb{H}$, multiplied by a smoothly varying "density" function on $\mathbb{H}$ that depends on the distance from the axis of symmetry. (The density at distance r from the axis is $r^{n-2}\mathrm{vol}(S^{n-2})$.) One can adapt the theory of soap bubbles in surfaces, as given in [16], to conclude that $B \cap \mathbb{H}$ consists of smooth curves meeting in threes at 120° angles. One might worry that there could be singularities along the axis, where the density degenerates to zero and the topology becomes infinitely complicated. However this cannot happen because of Corollary 4.3.

With axis singularities ruled out, we can show that any curve meeting the axis must be a circle or line orthogonal to the axis. This is because given a disc $D \subset \mathbb{R}^n$ and a real number $v \geq 0$, the least area hypersurface S with $\partial S = \partial D$ such that D and S together enclose volume v is a piece of a sphere (or D if $v = 0$). The case $v = 0$ is true because if $S \neq D$, orthogonal projection of S to the hyperplane of D is onto D and decreases area. If $v > 0$, there exists a piece S_0 of a sphere with $\partial S_0 = \partial D$ such that S_0 and D enclose volume v. Let S' be the rest of the sphere, and let v' be the volume enclosed by D and S'. Given any S satisfying the above requirements, S and S' together enclose volume $v + v'$, so by the isoperimetric theorem $\text{area}(S \cup S') \geq \text{area}(S_0 \cup S')$, i.e. $\text{area}(S) \geq \text{area}(S_0)$, with equality if and only if $S = S_0$.

An argument given by Foisy in [9] shows that B must intersect the axis. If not, we can contract the bubble towards the axis in a volume-preserving way so as to decrease area. To do this, choose a small $\varepsilon > 0$, and at each point of the cluster replace the distance r from the axis by $(r^{n-1} - \varepsilon)^{\frac{1}{n-1}}$. Areas of surfaces orthogonal to the direction of the axis are preserved, while areas of surfaces parallel to the axis decrease.

It is also easy to see that B is connected. If not we can slide two components along the axis until they first touch, creating an illegal singularity (surfaces meeting in fours), so that area can be decreased.

I claim next that if B is not a standard double bubble, then B intersects the axis exactly twice. Clearly B cannot intersect the axis only once, or else the surface intersecting the axis would have R_0 on either side and could be removed. Now suppose that B is nonstandard and intersects its axis more than twice. Since B is connected and has no empty chambers, some surface S_0 of B must intersect the axis with R_1 and R_2 on either side. Since B is connected and encloses more than one region, S_0 must meet two surfaces S_1 and S_2. Since B has no empty chambers, removal of part of either S_1 or S_2 would disconnect B. It follows that S_1 and S_2 are pieces of spheres. (A slight variation on an argument above shows that the least-area surface enclosing a given volume between two free discs is a piece of a sphere.) Since B is nonstandard, either S_1 or S_2, say S_1, must meet some hypersurface other than S_0, S_1, and S_2. One can then roll the two pieces of $B \setminus S_1$ around the sphere containing S_1 until they first touch. (See Figure 4.) The resulting singularity will be illegal; even if the surfaces somehow manage to meet along a curve in $\mathbb{R}^n$, they will meet in fours. Since this rolling process does not change area or volume, B is not minimal; this is the desired contradiction.

Since B has no empty chambers, we easily deduce:

Theorem 5.1 (Structure) *Any minimal double bubble in $\mathbb{R}^n$ that is not the standard double bubble is a surface of revolution about some line, and consists of a topological sphere with a tree of annular bands attached. (See Figure 2.) The two caps are pieces of spheres, and the root of the tree has just one branch.*

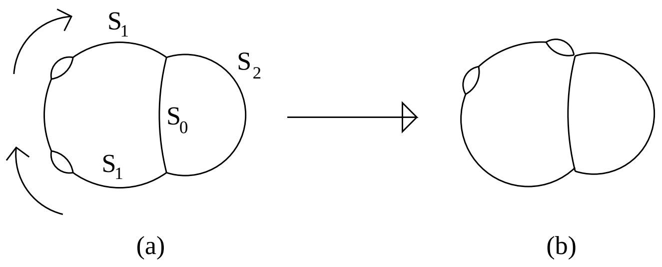

Figure 4: (a) A hypothetical minimal double bubble B in cross-section. S_1 is a piece of a sphere. (b) After rolling the left side of the bubble around this sphere, we obtain an illegal singularity, so B is not minimal. The example shown here does not have rotational symmetry after rolling.

The last clause of this theorem follows from the rolling argument above.

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