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INTEGRATION OF SINGULAR BRAID INVARIANTS AND GRAPH COHOMOLOGY

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Abstract. We prove necessary and sufficient conditions for an arbitrary invariant of braids with m double points to be the "m^{th} derivative" of a braid invariant. We show that the "primary obstruction to integration" is the only obstruction. This gives a slight generalization of the existence theorem for Vassiliev invariants of braids. We give a direct proof by induction on m which works for invariants with values in any abelian group.

We find that to prove our theorem, we must show that every relation among four-term relations satisfies a certain geometric condition. To find the relations among relations we show that H_1 of a variant of Kontsevich's graph complex vanishes. We discuss related open questions for invariants of links and other things.

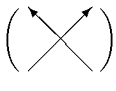
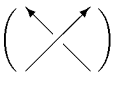
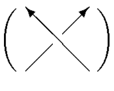
1. Introduction

1.1. **The mystery.** From a certain point of view, it is quite surprising that the Vassiliev link invariants exist, because the "primary obstruction" to constructing them is the only obstruction, for no clear topological reason.

The idea of Vassiliev invariants is to define a "differentiation" map from link invariants to invariants of "singular links", start with invariants of singular links, and "integrate them" to construct link invariants. A **singular link** is an immersion $\coprod_n S^1 \to S^3$ with m double points (at which the two tangent vectors are not parallel) and no other singularities. Let L_m denote the free $\mathbb{Z}$-module generated by isotopy classes of singular links with m double points. Let G be an abelian group, and let $v : L_0 \to G$ be a G-valued link invariant. The **derivative** of v,

$$\delta v : L_1 \to G,$$

is defined by

(1) δv  $:= v$  $- v$  .

Convention 1.1. In any such equation, all links are equal outside of the region drawn. The arrows indicate the orientations of the link components. (This equation makes sense in any oriented 3-manifold; we never use link projections.)

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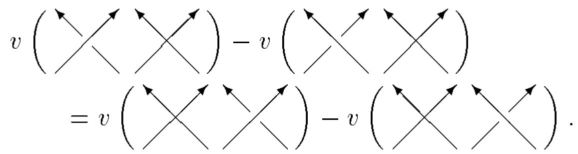
More generally we say that an invariant of links with m double points, $v : L_m \to G$, is **differentiable** if

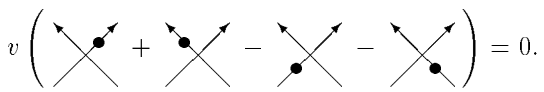
(DIFF*)



If v is differentiable, we define $\delta v : L_{m+1} \to G$ by applying (1) to one of the $m+1$ double points. By (DIFF*) this does not depend on the double point we choose. Also notice that δv is differentiable.

When is an invariant $v : L_m \to G$ of singular links expressible as δ^m of a link invariant? By the above remarks, v must be differentiable. There are two other necessary conditions. The first is the (topological) **four-term relation**:

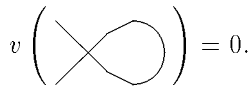
(T4T*)



Convention 1.2. A bold dot in a picture indicates a strand which is orthogonal to the paper and points towards the reader.

The second condition is **framing independence**:

(FI*)



Theorem 1.3 (Stanford [19]). *An invariant $v : L_m \to \mathbb{Q}$ is δ of a differentiable invariant $v' : L_{m-1} \to \mathbb{Q}$ if and only if v satisfies (DIFF*), (T4T*), and (FI*).*

We prove an analogous statement for braids in §2.2.3, which should give the idea of the proof. Mysteriously, these "primary obstructions" to integration may be the only obstructions. Namely:

Theorem 1.4 ([15, 6], etc.). *Suppose $v : L_m \to \mathbb{Q}$ satisfies (DIFF*), (T4T*), and (FI*). Suppose further that $\delta^k v = 0$ for some k. Then v is δ^m of a link invariant.*

The case $k = 1$ is the fundamental existence theorem for Vassiliev invariants, and the general case follows easily by induction on k. Three proofs of the fundamental theorem of Vassiliev invariants, due to Bar-Natan, Kontsevich, and several other authors, are surveyed in [6]. We would like to know if this result can be explained directly in terms of the topology of the stratification of $\mathrm{Maps}(\coprod_n S^1, \mathbb{R}^3)$. (Here we have been inspired by [2, 7, 22].) In this case we might expect the following more general statement to be true.

Conjecture 1.5. *An invariant $v : L_m \to \mathbb{Q}$ of singular links satisfying (DIFF*), (T4T*), and (FI*) is δ^m of a link invariant.*

This conjecture remains stubbornly open (although "half" of it is proved in [24]). This does not follow from Theorem 1.3 because it is not clear whether v^1 can be chosen to again satisfy the integrability conditions.

1.2. **The main result.** In this paper we prove the analogue of Conjecture 1.5 for braids.

Theorem 1.6 (Main result, proved in §2.2.3). *Let G be any abelian group. Let v be a G-valued invariant of singular braids with m double points. Then there exists a braid invariant w with $\delta^m w = v$ if and only if v satisfies (DIFF*) and (T4T*). In this case there is an explicit procedure for finding all such w (§5).*

Here by "braids" we mean pure braids with n strands for some fixed positive integer n. See §2.2 for precise definitions. (An analogue of Theorem 1.6 for non-pure braids follows trivially.)

An analogue of the fundamental theorem of Vassiliev invariants is known for braids, and this implies Theorem 1.6 when $\delta^k v = 0$ for some k. This is a large fraction of the cases, because Vassiliev invariants separate braids [4]. (In fact an analytic construction of the Vassiliev invariants of braids, and a proof that they separate braids, were known before Vassiliev invariants were invented, under a different name [13].)

We prove Theorem 1.6 by a direct construction, using induction on m. Although the theorem is only a slight generalization of what was already known, we feel that the proof clarifies the difficulties in this naive approach to integrating invariants. One might try to carry out this procedure in contexts outside of knot theory (cf. §§2.1, 3.1.3). In §§2.3.1, 2.3.2 we find that the key to proving Theorem 1.6 or Conjecture 1.5 this way is to show that every *relation among four-term relations* satisfies a certain geometric condition.

Some basic relations among four-term relations are described in §2.4; these have a natural interpretation in terms of the stratification of the space of immersions (see §3.2). To find the remaining relations among four-term relations, we are reduced to a difficult problem in the combinatorics of "chord diagrams". As Bar-Natan points out, the task can be interpreted as computing H_1 of an analogue of Kontsevich's graph complex. We can solve this problem for braids because there is a "sorting" process which gives us control over the combinatorics, explained in §4. We find that $H_1 = 0$; the geometric significance of this is discussed in §3.3. We do not know how to attack the corresponding combinatorial problem for links.

One more remark on the proof of Theorem 1.6: the construction works for invariants with values in an arbitrary abelian group G because the module of "weights" for braids is free. This is a known result which we end up re-proving (Theorem 2.10(4)). I think it is unknown whether the module of weights for links is free.

More related open questions and consequences of our results are discussed in §3.

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2. Integration theory

We want to analyze the obstructions to integrating a singular braid invariant to a braid invariant. In §§2.1, 2.3.1 we forget the geometry of braids and discuss the general (rather trivial) algebra underlying the integration process. This will clarify what we need to do to prove Theorem 1.6. The remaining subsections specialize to the case of braids.

2.1. **General notation.**

Definition 2.1. An **integration theory** is a sequence

$$\cdots \xrightarrow{\partial} \mathcal{O}_m \xrightarrow{\partial} \mathcal{O}_{m-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{O}_1 \xrightarrow{\partial} \mathcal{O}_0$$

of abelian groups. (We do not assume $\partial^2 = 0$.) Typically,

- $\mathcal{O}_0$ is the free $\mathbb{Z}$-module on a set of objects which we would like to study.
- $\mathcal{O}_m$ is the free $\mathbb{Z}$-module generated by "m-singular" objects (sometimes modulo some relations, or with some extra structure).
- ∂ of an m-singular object x is a difference between, or other combination of, some $(m-1)$-singular objects "near" x.

A more complicated algebraic structure might be interesting, for example for links with singularities other than double points, but this simple definition will suffice for this paper.

Fix an abelian group G, and let $\mathcal{O}_m^* := \mathrm{Hom}(\mathcal{O}_m, G)$. This is the module of invariants of m-singular objects. Let $\delta := \partial^t : \mathcal{O}_m^* \to \mathcal{O}_{m+1}^*$; this map is "differentiation" of invariants. We want to understand how to invert this process. For this purpose, some basic modules to understand are:

Definition 2.2. If $(\mathcal{O}_*, \partial)$ is an integration theory, define:

1. $C\mathcal{O}_m := \mathcal{O}_m / \partial \mathcal{O}_{m+1}$. This is the module of "integration **constants**". Let $\pi : \mathcal{O}_m \to C\mathcal{O}_m$ be the projection.
2. $P\mathcal{O}_m := \mathrm{Ker}(\partial | \mathcal{O}_m)$: the "**primary** integrability conditions" that an element of $\mathcal{O}_m^*$ must annihilate in order to "integrate one step" to (i.e. be δ of) an element of $\mathcal{O}_{m-1}^*$.
3. $S\mathcal{O}_m := \mathrm{Ker}(\partial^2 | \mathcal{O}_m)/P\mathcal{O}_m$: the **secondary** obstructions to integration, modulo the primary ones.
4. $W\mathcal{O}_m := C\mathcal{O}_m / \pi(P\mathcal{O}_m)$: the module of **weights**.
5. $F\mathcal{O}_m := \mathrm{Ker}(\delta^{m+1} | \mathcal{O}_0^*)$: the **finite type invariants** of order $\leq m$.

Exercise 2.3 (to get used to the notation)**.** There is a well-defined injection

$$\delta^m : F\mathcal{O}_m / F\mathcal{O}_{m-1} \to (W\mathcal{O}_m)^*,$$

which is an isomorphism (at least over $\mathbb{Q}$) if $S\mathcal{O}_m / \partial^k \mathcal{O}_{m+k} = 0$ for all m, k.

The concern of this paper is determining when $S\mathcal{O}_m$ vanishes completely.

2.2. **Integration theory for braids.**

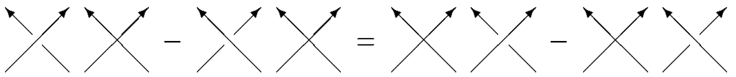
2.2.1. *Setup.* We define an integration theory for braids as follows. Fix a positive integer n and fix n distinct points $b_1, \ldots, b_n \in \mathbb{R}^2$.

Definition 2.4. A (pure) **singular braid** with m double points (and n strands) consists of n smooth maps $f_i : [0,1] \to \mathbb{R}^2$ such that

- $f_i(0) = f_i(1) = b_i$.
- The graphs of the f_i's in $[0,1] \times \mathbb{R}^2$ are disjoint, except for m double points.
- At each double point, the two tangent vectors are distinct.

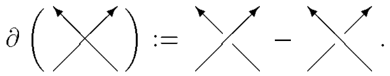
We draw a singular braid as the union of the graphs of the f_i's in $\mathbb{R}^2 \times [0,1]$. Later we sometimes use the term "singular braids" to refer to braids with more complicated singularities than double points, but Definition 2.4 is the default.

Definition 2.5. Let $\mathcal{B}_m$ be the free $\mathbb{Z}$-module generated by isotopy classes of singular braids with m double points, modulo the **differentiability relation**

(DIFF) 

This relation has four terms, each with two neighborhoods shown, and the two neighborhoods do not have to be at the same "height". The arrows indicate the orientations on the strands of the braid coming from the orientation on $[0,1]$.

Definition 2.6. Define $\partial : \mathcal{B}_m \to \mathcal{B}_{m-1}$ by



This means that to evaluate ∂ of a singular braid, we choose one of the m double points and subtract the two different ways of resolving it. The result does not depend on the choice of double point because we modded out by (DIFF).

2.2.2. *Diagrams and relations.* Having defined the integration theory $(\mathcal{B}_*, \partial)$, we will now describe the modules C, P, S, W of Definition 2.2 in this case. To state the result (Theorem 2.10), we need to introduce a certain algebra of diagrams.

Definition 2.7. Let $D_{*,*}$ be the bigraded algebra over $\mathbb{Z}$ generated by elements t_{ij} $(1 \le i \ne j \le n)$ of bidegree $(0,1)$ and r_{ij}^k $(i,j,k \in \{1,\dots,n\}$ distinct) of bidegree $(1,2)$ with the relations

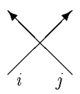
$$\begin{aligned} t_{ij} &= t_{ji}, \\ [t_{ij}, t_{kl}] &= 0 \qquad (i,j,k,l \text{ distinct}), \\ r_{ij}^k &= r_{ji}^k \\ [t_{ij}, r_{kl}^q] &= 0 \qquad (i,j,k,l,q \text{ distinct}). \end{aligned}$$

A monomial in the t_{ij}'s is called a **chord diagram**. A chord diagram $t_{i_1j_1}\cdots t_{i_mj_m}$ is **sorted** if $i_\alpha < j_\alpha$ and $j_1 \le j_2 \le \cdots \le j_m$. (The r_{ij}^k's will be needed in §2.3.2.)

Definition 2.8. Define $\pi' : \mathcal{B}_m \to D_{0,m}$ as follows. Let x be a singular braid with m double points, described by maps $f_i : [0,1] \to \mathbb{R}^2$. Suppose that at "heights" $y_1 < \cdots < y_m$, we have $f_{i_\alpha}(y_\alpha) = f_{j_\alpha}(y_\alpha)$ with $i_\alpha \ne j_\alpha$. (In general we perturb x to make the y_α's distinct.) Then

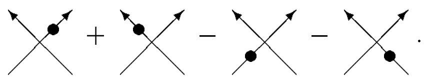
$$\pi'(x) := t_{i_1j_1}\cdots t_{i_mj_m}.$$

Pictorially,

π' :  $\longmapsto t_{ij}$,

where stacking of double points from bottom to top corresponds to multiplication of t_{ij}'s from left to right.

Definition 2.9. A **topological four-term relation** is an element of $\mathcal{B}_m$ of the form

(T4T) 

(Cf. Convention 1.2; $\mathbb{R}^2 \times [0,1]$ is rotated a bit here.)

2.2.3. *The main structure theorem.*

Theorem 2.10. 1. *There is an isomorphism $\imath : C\mathcal{B}_m \to D_{0,m}$ with $\imath \circ \pi = \pi'$. (Henceforth we identify $C\mathcal{B}_m$ with $D_{0,m}$ and π with π'.)*
2. *$P\mathcal{B}_m$ is the span of the (T4T) relations.*
3. *(proved in §2.4) $S\mathcal{B}_m = 0$.*
4. *$W\mathcal{B}_m$ is the free abelian group generated by sorted chord diagrams.*

Remark 2.11. Part (1) is well known and asserts that a basis for the integration constants for braids is given by chord diagrams. (2) is the braid analogue of Stanford's Theorem 1.3 and asserts that the primary obstructions to integrating a singular braid invariant are the (DIFF) and (T4T) relations. (3) is the hard part of this paper and asserts that the primary obstruction to integration is the only obstruction. (4) is nontrivial but known (see e.g. [9]), and a new proof will fall out in §4.2.

Proof of Theorem 1.6. By Definition 2.5 and Theorem 2.10(2), the hypothesis in Theorem 1.6 implies that v is well defined on $\mathcal{B}_m$ and annihilates $P\mathcal{B}_m$. By Theorem 2.10(3), $\partial : \mathcal{B}_m/P\mathcal{B}_m \to \mathcal{B}_{m-1}/P\mathcal{B}_{m-1}$ is well defined. The sequence

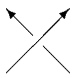
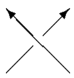
$$0 \longrightarrow \frac{\mathcal{B}_m}{P\mathcal{B}_m} \xrightarrow{\partial} \frac{\mathcal{B}_{m-1}}{P\mathcal{B}_{m-1}} \xrightarrow{\pi} W\mathcal{B}_{m-1} \longrightarrow 0$$

is exact. By the freeness assertion in Theorem 2.10(4), the dual sequence is also exact. Hence $v = \delta v'$, where $v' : \mathcal{B}_{m-1} \to G$ annihilates $P\mathcal{B}_{m-1}$. We are done by induction on m. $\square$

Proof of Theorem 2.10(1). It suffices to show:
1. π' is well defined modulo (DIFF).
2. $\pi' \circ \partial = 0$.
3. If x and y are singular braids with $\pi'(x) = \pi'(y)$, then $x - y$ is ∂ of something.

Parts (1) and (2) are easy. To prove (3), note that if $\pi'(x) = \pi'(y)$, then there is a path γ, in the space of singular braids, from x to y (since $\mathbb{R}^2$ is simply connected). If γ is generic then $\gamma(t)$ has only m double points, except for finitely many times t_i at which $\gamma(t_i)$ has $m+1$ double points. Let

$$\Phi(\gamma) := \sum_i \pm\gamma(t_i),$$

where the sign is $+$ if γ crosses from  to  and $-$ otherwise. Then $x - y = \partial\Phi(\gamma)$. $\square$

Proof of Theorem 2.10(2) (sketch). To see that a (T4T) relation is in $P\mathcal{B}_m$, i.e. that ∂ annihilates it, in each of the four terms apply Definition 2.6 to the double point involving the strand orthogonal to the paper. Everything cancels.

Conversely, $P\mathcal{B}_m$ is spanned by elements of the form $\Phi(\gamma)$ where γ is a loop. We need to show that $\Phi(\gamma)$ is a sum of (T4T) relations. Since $\mathbb{R}^2$ is contractible, we can homotope γ to a constant loop. If we choose this homotopy generically, then only the following "codimension $m+2$ events" can happen for an isolated intermediate path γ in our homotopy:
1. At some time t, $\gamma(t)$ has $m+2$ double points.
2. At some time t, $\gamma(t)$ has $m-1$ double points and one triple point.
3. At some time, γ arrives at a singular braid with $m+1$ double points but then "bounces back" instead of "passing through".

4. At some time t, one of the double points in the singular braid $\gamma(t)$ has both tangent vectors parallel.

In our homotopy of paths, an event of type (1) causes $\Phi(\gamma)$ to change by a (DIFF) relation. An event of type (2) causes $\Phi(\gamma)$ to change by a (T4T) relation. An event of type (3) does not change $\Phi(\gamma)$. An event of type (4) occurs when the "rotation" of a double point during the path γ (which is an element of $\pi_1(S^1)$) changes by ± 1 as we homotope γ. This causes $\Phi(\gamma)$ to change by a difference of two terms which, after a little twisting, are seen to be equal. At the end of the homotopy, $\Phi(\gamma) = 0$, so at the beginning of the homotopy, $\Phi(\gamma)$ is a sum of (T4T) relations. $\square$

2.3. **The secondary obstruction to integration.**

2.3.1. *Generalities.* Return to a general integration theory $(\mathcal{O}_*, \partial)$. When is the secondary obstruction $S\mathcal{O}_m = 0$? One can see by plugging through the definitions that

$$S\mathcal{O}_{m+1} = 0 \Leftrightarrow \operatorname{Ker}(\pi | P\mathcal{O}_m) = 0.$$

We need to understand the primary obstructions $P\mathcal{O}_m$. Often there will be a natural set of generators for $P\mathcal{O}_m$, although we might not know the relations. So if $G\mathcal{O}_m$ is the free abelian group on these generators, we have a surjection

$$G\mathcal{O}_m \xrightarrow{\psi} P\mathcal{O}_m \to 0.$$

The above observation now gives:

Lemma 2.12. $S\mathcal{O}_{m+1} = 0 \Leftrightarrow \operatorname{Ker}(\pi\psi) = \operatorname{Ker}(\psi)$ *in* $G\mathcal{O}_m$.

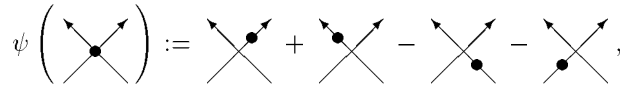
The Point 2.13. When integrating one step, to choose the integration constants (in $(C\mathcal{O}_m)^*$) so that the primary integrability conditions are again satisfied, we have to solve a system of *inhomogeneous* linear equations (parametrized by $G\mathcal{O}_m$). This can be solved (at least over a field) iff whenever a linear combination of the l.h.s.'s vanishes (i.e. whenever we have an element of $\operatorname{Ker}(\pi\psi)$), the corresponding combination of r.h.s.'s also vanishes (i.e. is in $\operatorname{Ker}(\psi)$).

2.3.2. *The case of braids.* With the preceding as a guide, we want to prove Theorem 2.10(3), asserting that the secondary obstruction $S\mathcal{B}_m$ vanishes. In the case of braids, our generators for $P\mathcal{B}_m$ are (T4T) relations.

Definition 2.14. Let $G\mathcal{B}_m$ be the free abelian group generated by singular braids such that:

- There are $m-2$ double points and one triple point.
- At each double point, the two tangent vectors are distinct.
- At the triple point, the three tangent vectors are linearly independent in $\mathbb{R}^2 \times [0,1]$.
- At the triple point, one of the three strands is distinguished.

Define $\psi : G\mathcal{B}_m \to P\mathcal{B}_m$ by



where the strand orthogonal to the paper is distinguished.

By Lemma 2.12, to prove Theorem 2.10(3), we need to prove $\operatorname{Ker}(\pi\psi) = \operatorname{Ker}(\psi)$. This condition, in words, says:

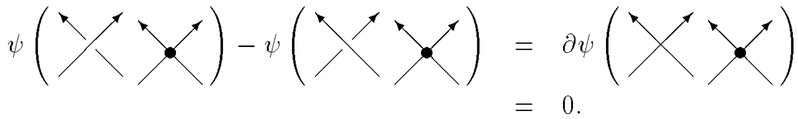
> *Every relation among four-term relations at the level of chord diagrams also holds at the level of geometry (modulo (DIFF)).*

There are some trivial relations among relations which we can dispense with immediately. We can define a map $\pi : G\mathcal{B}_m \to D_{1,m}$ by analogy with Definition 2.8, with the additional stipulation that a triple intersection of strands i, j, k, with the k^{th} strand distinguished, is sent to r_{ij}^k. (In other words, we extend π to an algebra map from singular braids with double points and marked triple points to $D_{*,*}$.)

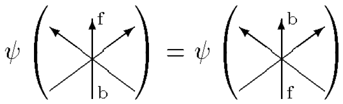
Lemma 2.15. *ψ descends to $D_{1,m}$, i.e. we have a commutative triangle*

$$\begin{tikzcd} G\mathcal{B}_m \arrow[r, "\psi"] \arrow[d, "\pi"'] & P\mathcal{B}_m \\ D_{1,m}. \arrow[ur, "\rho"'] & \end{tikzcd}$$

Proof. We need to check that if x and y are two generators of $G\mathcal{B}_m$ and $\pi(x) = \pi(y)$, then $\psi(x) = \psi(y)$. If $\pi(x) = \pi(y)$, there is a homotopy from x to y passing through braids with one extra double point. When we pass through such a braid the value of ψ is unchanged, because



When the three strands at the triple points in x and y are oriented differently, our homotopy must also pass through a singular braid with $m - 2$ double points and one triple point in which the tangent vectors to the three strands lie in a single plane. So we need to check that



in B_0^m. (Here the distinguished strand of the triple point has points on it labeled 'f' and 'b' for 'front' and 'back' of the page, and the other two strands are in the plane of the page.) This is an exercise using (DIFF). □

Conclusion. To prove Theorem 2.10(3), we must show that $\mathrm{Ker}(\pi\rho) = \mathrm{Ker}(\rho)$. (For a repackaging via the snake lemma of the algebra which led us here, see [6].)

2.4. **Relations among four-term relations are geometric.** We will now complete the proof of Theorem 2.10(3), modulo a combinatorial lemma whose proof is deferred to §4.2.

First observe that $\pi\rho : D_{1,*} \to D_{0,*}$ is the map obtained by sending

$$\begin{aligned} t_{ij} &\longmapsto t_{ij}, \\ r_{ij}^k &\longmapsto [t_{ij}, t_{ik} + t_{jk}], \end{aligned}$$

and extending multiplicatively. Here are some elements of $\mathrm{Ker}(\pi\rho)$.

3T: $x(r_{jk}^i + r_{ki}^j + r_{ij}^k)y$ with x, y arbitrary chord diagrams.

8T: $(\pi\rho x)y - x(\pi\rho y)$ with $x, y \in D_{1,*}$ arbitrary.

14T: $x([t_{ik} + t_{jk} + t_{kl}, r_{ij}^l] + [t_{ij}, r_{il}^k + r_{jl}^k] + [r_{ij}^k, t_{il} + t_{jl}])y$ with x, y arbitrary chord diagrams.

Notation 2.16. Let $D_{2,*} \subset D_{1,*}$ denote the span of the above elements (*not* the $(2,*)$ graded piece of the algebra $D_{*,*}$). Let $D_j := D_{j,*}$. Let $d : D_2 \to D_1$ be the inclusion. Let $d : D_1 \to D_0$ be the map $\pi\rho$. Clearly $d^2 = 0 : D_2 \to D_0$.

Lemma 2.17 (proved in §4.2)**.** *$H_1(D_*, d) = 0$.*

Proof of Theorem 2.10(3). By Lemmas 2.12, 2.15, and 2.17, it is enough to show that ρ annihilates (3T), (8T), and (14T). By Lemma 2.15, for each of these relations we just have to choose a convenient lift (under π) in $G\mathcal{B}_m$ and check that ψ annihilates it.

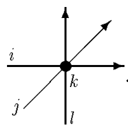
(3T) Let z be a braid with a triple point lifting the first term in (3T), i.e. $z \in G\mathcal{B}_*$ and $\pi(z) = xr^i_{jk}y$. We can lift the other two terms in (3T) by starting with z but distinguishing the other two strands of the triple point. When we apply ψ to the sum of these three lifts, we obtain a sum of twelve terms. From z one can make six different singular braids by sliding apart two of the strands of the triple point along the third strand. Each of these appears twice in the sum, with opposite sign. Hence ψ of our lift of (3T) vanishes.

(8T) Let $\tilde{x}, \tilde{y} \in G\mathcal{B}_*$ be lifts of x, y. Then $\psi(\tilde{x})\tilde{y} - \tilde{x}\psi(\tilde{y})$ is a lift of $(dx)y - x(dy)$, because $\pi\psi = d\pi$. But

$$\psi((\psi\tilde{x})\tilde{y} - \tilde{x}(\psi\tilde{y})) = \psi(\tilde{x})\psi(\tilde{y}) - \psi(\tilde{x})\psi(\tilde{y}) = 0.$$

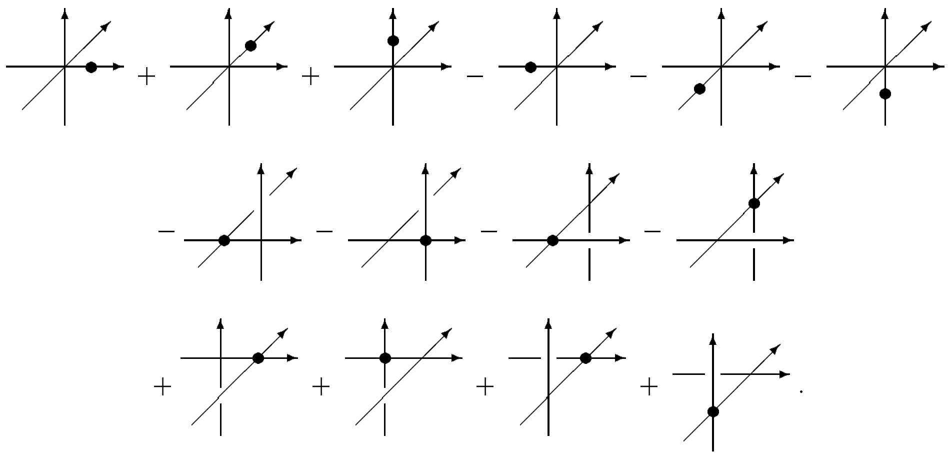
(14T) Let z be a singular braid with a quadruple point at which strands i, j, k, l intersect. For a suitable choice of z, we can represent each of the terms in (14T) by starting with z and sliding one strand of the quadruple point apart from two other strands along the fourth.

Let us choose z so that $\{v_i, v_j, v_l\}$ is a positively oriented basis for $\mathbb{R}^3$, and $v_k = v_i - v_j + v_l$. Then the quadruple point looks like

$z =$ 

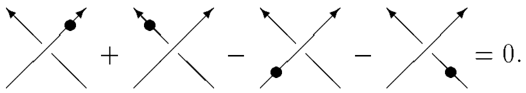
.

Our geometric representative of (14T) is



Strand l is distinguished in the first six terms, and strand k is distingushed in the remaining eight.

The reader may check that ψ of this is a sum of four (DIFF) relations, each of which looks something like



□

3. Discussion and questions

3.1. Other integration theories: a sampler.

3.1.1. *Knots and links.* One can set up an integration theory $(\mathcal{L}_*, \partial)$ for links in $\mathbb{R}^3$ with generic double points by analogy with Definitions 2.5, 2.6. The module $P\mathcal{L}_m$ was analyzed by Stanford [19] and is spanned by the T4T and FI relations. (The homotopy argument to prove this works only over $\mathbb{Q}$, or for based links.) The space $W\mathcal{L}_m$ has been extensively studied, see e.g. [3, 23]. As far as I know it is an open question whether $W\mathcal{L}_m$ is free.

Some new idea is needed to prove $S\mathcal{L}_m = 0$ (if it's true). It seems very difficult to prove a link analogue of Lemma 2.17 (cf. Open Question 3.6). One might attempt to avoid this difficulty by expanding the integration theory to include links with more complicated singularities, but I do not know how to circumvent the fundamental problem of determining relations among relations (to solve inhomogeneous linear equations); at best, one might replace the four-term relation on chord diagrams with the equally insidious IHX relation on Chinese characters (defined in [3]).

Open Question 3.1. Does Theorem 1.6 extend to *closed braids* (braids modulo conjugation)? These might be intermediate in difficulty.

3.1.2. *IHS's.* For other integration theories, the secondary obstructions might not be the most interesting thing to study. For example, following Ohtsuki's work [18] one can set up an integration theory $(\mathcal{I}_*, \partial)$ for integral homology 3-spheres (IHS's). $\mathcal{I}_m$ is generated by IHS's with m-component algebraically split framed links (ASL's), modulo some relations, and ∂ of a generator takes the difference between doing surgery on, or deleting, a link component. There is an interesting theory of finite type invariants [16, 5], even though the secondary obstructions $S\mathcal{I}_*$ are far from zero; $W\mathcal{I}_m$ is isomorphic to the free $\mathbb{Z}$-module generated by the *set* $(\wedge^3\mathbb{Z}^m)^*$ modulo signed permutations (the isomorphism sends an ASL to the set of its Milnor triple linking numbers), while $F\mathcal{I}_m/F\mathcal{I}_{m+1}$ is zero when m is not a multiple of 3, and a finite-dimensional vector space when $3|m$ [10, 11].

3.1.3. *Replacing the braid group with an arbitrary group.* The integration theory of braids in this paper has a generalization in which the braid group is replaced by an arbitrary group G. Let $I \subset \mathbb{Z}[G]$ be the augmentation ideal (the set of sums $\sum_g a_g \cdot g \in \mathbb{Z}[G]$ such that $\sum_g a_g = 0$).

Definition 3.2. For any group G, let $\mathcal{G}_0 := \mathbb{Z}[G]$, and

$$\mathcal{G}_m := \underbrace{I \otimes_{\mathbb{Z}[G]} \cdots \otimes_{\mathbb{Z}[G]} I}_{m\ I\text{'s}}$$

for $m > 0$. Define $\partial : \mathcal{G}_m \to \mathcal{G}_{m-1}$ by

$$\partial(x_1 \otimes \cdots \otimes x_m) := x_1 x_2 \otimes x_3 \otimes \cdots \otimes x_m.$$

(Note that to evaluate $\partial(x_1 \otimes \cdots \otimes x_m)$, we can multiply any two adjacent x_j's, by the definition of tensor product.)

If G is the group of pure braids on n strands, then $(\mathcal{G}_*, \partial)$ is almost the same as $(\mathcal{B}_*, \partial)$. More precisely:

Definition 3.3. A **noncommutative singular braid** is a singular braid in which no two double points are at the same height (i.e. the same $[0,1]$ coordinate). Let $(\tilde{\mathcal{B}}_*, \partial)$ be the free $\mathbb{Z}$-module generated by isotopy classes of noncommutative singular braids, modulo (DIFF), with ∂ as in Definition 2.6.

There is an isomorphism $i : (\tilde{\mathcal{B}}_*, \partial) \to (\mathcal{G}_*, \partial)$. If b is a singular braid with one double point, define $i(b) := \partial b \in I$ (where ∂ here is as in Definition 2.6). If b has m double points, write b as a product (i.e. a stacking) $b_1 \cdots b_m$ where each b_i has one double point, and define $i(b) := i(b_1) \otimes \cdots \otimes i(b_m)$. This does not depend on the choice of product decomposition by the definition of tensor product. The inverse map is well defined by the differentiability relation.

What is $P\mathcal{G}_m$? For any G, there is an obvious submodule $O\mathcal{G}_m \subset P\mathcal{G}_m$, spanned by expressions $a \otimes (b \otimes c - c \otimes b) \otimes d$ such that $b, c \in I$ commute and $a \in \mathcal{G}_j, d \in \mathcal{G}_{m-2-j}$. On the other hand, when G is the braid group, it is easy to see that (i of) the T4T relation, and also the relation that an invariant must annihilate in order to not change when the heights of two double points cross, are in $O\mathcal{G}_m$. Hence Theorem 2.10(2),(3) has the following corollary:

Theorem 3.4. *If G is the group of pure braids on n strands, then*

$$\mathrm{Ker}(\partial^m : \mathcal{G}_m \to \mathbb{Z}[G]) = O\mathcal{G}_m.$$

Open Question 3.5. For what other groups is this true? What is its significance?

3.2. **The role of the stratification.** Suppose we have an invariant of braids with m double points which we want to integrate. Then we have the following intuitive picture.

(a) The integrability conditions (DIFF*) and (T4T*) come from neighborhoods, in the closure of the space of immersions with m double points, of immersions with a codimension $m+1$ singularity.

The differentiability relation comes from a loop around a braid with $m+1$ double points. The (T4T*) relation comes from a loop around a braid with $m-2$ double points and one triple point.

(b) The relations among four-term relations, aside from the rather trivial (3T) relation, come from neighborhoods of immersions with codimension $m+2$ singularities.

The (8T) relation comes from a neighborhood of a braid with $m-4$ double points and two triple points, while (14T) comes from a neighborhood of a braid with $m-3$ double points and one quadruple point. (Note that a generic intersection of multiplicity k contributes $2k-3$ to the codimension.) All other relations among relations coming from quadruple points are generated by (14T) and (3T) relations, by Lemma 4.5. Other codimension $m+2$ singularities contribute relations among (T4T) relations which project to zero at the level of diagrams.

The (3T), (8T), (14T) relations have analogues for links, as should be clear from the geometric representatives we have given in §2.4.

Open Question 3.6. Do the analogues of the (3T), (8T), and (14T) relations generate all relations among four-term relations (at the level of chord diagrams) for links?

If so, then Theorem 1.6 holds, at least over $\mathbb{Q}$. (At first glance this only implies that an invariant v of singular links satisfying (T4T*), (DIFF*), and (FI*) can be integrated one step to an invariant v' satisfying (T4T*) and (DIFF*). But it is not hard to "correct" v', without changing its derivative, so that it satisfies (FI*) too; the (FI*) relation is easy to arrange because it only has one term.)

Generalizing in another direction, Stoimenow [20] has raised:

Open Question 3.7. For braids or links, is there a free resolution

$$\cdots \xrightarrow{d} E_2 \xrightarrow{d} E_1 \xrightarrow{d} E_0 = W$$

of the module of weights such that E_k comes from neighborhoods of codimension $m+k$ strata (in which at most $k+1$ double points from an object with m double points come together)?

3.3. **Connection with graph cohomology.** Let $(\tilde{D}_*, \tilde{d})$ be the following complex: $\tilde{D}_0, \tilde{D}_1$ are the link analogues of D_0, D_1; but $\tilde{D}_2$, instead of being a submodule of $\tilde{D}_1$, contains one generator for each (3T), (8T), or (14T) relation. The differential $\tilde{d} : \tilde{D}_1 \to \tilde{D}_1$ is the link anologue of d, and $\tilde{d} : \tilde{D}_2 \to \tilde{D}_1$ sends a generator to the corresponding (3T), (8T), or (14T) relation. If we can prove $H_1(\tilde{D}_*) = 0$, then Conjecture 1.5 follows. Bar-Natan has suggested that this complex should be related to Kontsevich's graph complex.

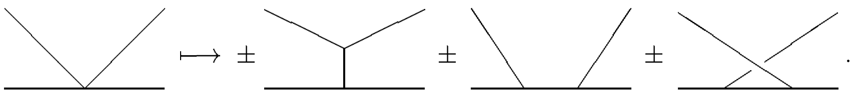
The relevant version of Kontsevich's complex is the following. Call a graph Γ, with vertices $V(\Gamma)$ and edges $E(\Gamma)$, **admissible** if: (a) Γ has one connected component; (b) there are n disjoint labeled oriented cycles; (c) there are no multiple edges or edges connecting a vertex with itself; (d) each vertex has degree ≥ 3; (e) there is an orientation on the space $\mathbb{R}^{E(\Gamma)} \oplus H_1(\Gamma;\mathbb{R})$. Let K^i be the free $\mathbb{Z}$-module generated by admissible graphs such that $\sum_{v \in V(\Gamma)}(\deg(v)-3) = i$. We impose the relation that switching the orientation on a graph is the same as multiplying it by -1. The coboundary $\delta_K : K^i \to K^{i+1}$ sends a graph Γ to the sum over all $e \in E(\Gamma)$ of the graph obtained by contracting the edge e, with an induced orientation. We throw out any inadmissible graphs that appear in this sum.

The interest of this complex is that there is a map from admissible graphs to differential forms on the space of links,

$$Q : K^* \to \Omega^* \left(\mathrm{Emb} \left(\coprod_n S^1, \mathbb{R}^3 \right) \right)$$

(defined by integration over configuration spaces) which, modulo possible "anomalies", is a chain map [14, 8]. In particular $H^0(K^*) = (W\mathcal{L})^*$, and the transpose of Q in degree zero contains the Vassiliev invariants of links [21, 1].

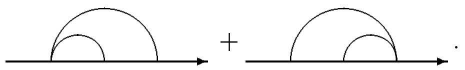
We have realized Bar-Natan's suggestion to the following extent. Let $K_i := K^i$ and let $d_K : K_i \to K_{i-1}$ be the adjoint of δ_K, where the inner product of two graphs is the number of isomorphisms between them. Thus $d_K(\Gamma)$ sums over all ways of splitting a high degree vertex of Γ into two lower degree vertices and adding an edge between them, e.g.



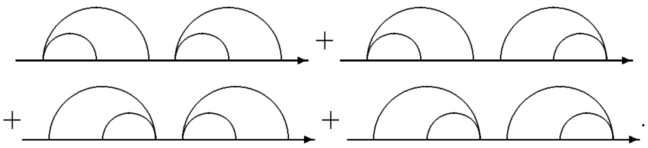
Theorem 3.8. *There is a (nontrivial) chain map $\tilde{D}_* \to K_*$.*

Proof. (suppressing orientations) We map $\tilde{D}_0 \to K_0$ in a standard fashion. The n distinguished cycles correspond to the link components, and for each double point we add an edge.

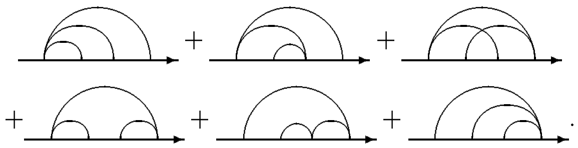
To show the higher degree maps, we restrict to the case $n = 1$ for artistic purposes, and draw a piece of the distinguished cycle as a horizontal line. In degree 1, we send a four-term relation to an expression of the form



In degree 2, we send a (3T) relation to zero. We send an (8T) relation to an expression of the form



We send a (14T) relation to an expression of the form



(This is the case where strands i, j, k, l appear in order from left to right on the horizontal line. There is not supposed to be a vertex where the two semicircles cross in the third term.) It is straightforward but tedious to check that this is a chain map, and to fill in the orientations. □

If there is actually an isomorphism in homology, then we have:

Dichotomy 3.9. *If $H_1 = 0$, then integration works well. If $H_1 \neq 0$, then there should be interesting invariants of 1-parameter families.*

Our result that $H_1(D_*) = 0$ for braids is perfectly consistent with this, because there are no 1-parameter families of braids: π_1 of the space of braids equals π_2 of the space of distinct n-tuples of points in $\mathbb{R}^2$, which is easily shown to vanish. Hatcher [12] has shown that for non-satellite knots, π_1 is generated by loops of reparametrizations. This suggests that for links, H_1 is not very large.

3.4. **Comparison with the Kontsevich integral.** The "universal" case of Theorem 1.6 is when $G = \mathcal{B}_m/P\mathcal{B}_m$ and $v : \mathcal{B}_m \to G$ is the projection. We can use Theorem 1.6 to integrate this to a (nonunique) braid invariant $Z^m : \mathcal{B}_0 \to \mathcal{B}_m/P\mathcal{B}_m$. Let

$$Z := \sum_m Z^m : \mathcal{B}_0 \to \overline{\bigoplus_m} \mathcal{B}_m/P\mathcal{B}_m,$$

where the bar indicates graded completion. The projection

$$\pi Z : \mathcal{B}_0 \to \overline{\bigoplus_m} W\mathcal{B}_m$$

is an example of a **universal Vassiliev invariant** for braids, which means that $\delta^m(\pi Z)$ equals π plus higher degree terms.

The Kontsevich integral (for a choice $b_1, \dots, b_n \in \mathbb{R}^2$) is another universal Vassiliev invariant. It is an algebra map, but it does not have integer coefficients. Although Z has integer coefficients, there does not exist any choice of Z^m's for which Z is an algebra map. (Proof: if Z is an algebra map then $Z^1(xy) = Z^1(x)y + xZ^1(y)$ for all braids x, y. Putting $x = y = 1$ (the trivial braid), we get $Z^1(1) = 0$. Since $Z^1\partial$ is the identity on $\mathcal{B}_1$, Z^1 of an arbitrary braid is the signed sum of the singular braids one crosses through in deforming it to the trivial braid. Then $Z^1(xy) = Z^1(x)y + Z^1(y)$, since we can deform xy to the trivial braid by first deforming x and then deforming y. So $xZ^1(y) = Z^1(y)$ for all braids x, y. This easily gives a contradiction.)

4. Proof that H_1 vanishes

We will now prove Lemma 2.17, giving the relations among four-term relations at the chord diagram level. Our list of generators for relations arises as follows. First, we show that aside from "trivial relations" and the 3-term relation, all relations among relations are generated "in degree 3". We can then find all degree 3 relations by a mechanical process.

4.1. An enlarged algebra of diagrams.

We begin by defining a complex T_* which is like D_* except that we drop the assumption that $[t_{ij}, t_{kl}] = 0$ for i, j, k, l distinct and put this commutativity relation on an equal footing with the four-term relation. We will determine all relations among four-term relations and commutativity relations, and then mod out by commutativity to recover Lemma 2.17.

The algebra $T_{*,*}$ is generated by symbols $t_{ij}, r_{ij}^k, c_{ij}^{kl}$ of bidegrees $(0,1), (1,2), (1,2)$ respectively, where $i, j, k, l \in \{1, \dots, n\}$ are distinct. We impose the relations

$$\begin{aligned} t_{ij} &= t_{ji}, \\ r_{ij}^k &= r_{ji}^k, \\ c_{ij}^{kl} &= c_{ji}^{kl} = -c_{kl}^{ij}. \end{aligned}$$

Let $T_0 := T_{0,*}, T_1 := T_{1,*}$, and define $d : T_1 \to T_0$ by sending

$$\begin{aligned} t_{ij} &\longmapsto t_{ij}, \\ r_{ij}^k &\longmapsto [t_{ij}, t_{ik} + t_{jk}], \\ c_{ij}^{kl} &\longmapsto [t_{ij}, t_{kl}], \end{aligned}$$

and extending multiplicatively.

Let T_2 be the submodule of T_1 generated by expressions xqy where $x, y \in T_0$ and q is one of the following relations among relations.

Trivial relation: $(dg)g' - gdg'$ with $g, g' \in T_1$ arbitrary.

3T: $r_{jk}^i + r_{ki}^j + r_{ij}^k$ with i, j, k distinct.

Jacobi: $[t_{ii'}, c_{jj'}^{kk'}] + [t_{jj'}, c_{kk'}^{ii'}] + [t_{kk'}, c_{ii'}^{jj'}]$ with i, i', j, j', k, k' distinct.

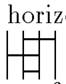
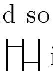
10T: $[t_{ij}, r_{kl}^q] - [t_{lq}, c_{ij}^{kl} + c_{ij}^{kq}] - [c_{ij}^{lq}, t_{kl} + t_{kq}]$ with i, j, k, l, q distinct.

22T: $[t_{ik} + t_{jk} + t_{kl}, r_{ij}^l] + [t_{ij}, r_{il}^k + r_{jl}^k - c_{ik}^{jl} - c_{jk}^{il}] + [r_{ij}^k + c_{ij}^{kl}, t_{il} + t_{jl}]$ with i, j, k, l distinct.

Observe that the 22-term relation is just the 14-term relation with commutativity relations c_{ij}^{kl} thrown in as necessary to make it work in the noncommutative case. In the same way the 10-term relation corresponds to the last relation of Definition 2.7.

Let $d : T_2 \to T_1$ be the inclusion.

Theorem 4.1. $H_1(T_*) = 0$.

4.2. **Sorting.** The strategy for the proof of Theorem 4.1, motivated by Theorem 2.10(4), is to sort everything in sight. To make the calculations more digestible, one can draw pictures of chord diagrams in the following standard manner. We draw n vertical lines corresponding to the strands. To represent the chord diagram $t_{i_1j_1}\cdots t_{i_mj_m}$, we draw a horizontal line between strands i_1 and j_1, above that we draw a horizontal line between strands i_2 and j_2, and so forth. For example, $t_{23}t_{13}t_{24} =$  is a typical sorted diagram, and $t_{23}t_{12} =$  is unsorted.

To prepare for sorting, we order the chord diagrams as follows. Define the **disorder** of a chord diagram $t_{i_1j_1}\cdots t_{i_mj_m}$ with $i_\alpha < j_\alpha$ to be the number of pairs (α,β) with $\alpha < \beta$ and $j_\alpha > j_\beta$. We define the **gravity** of the chord diagram to be $\sum_{\alpha=1}^m j_\alpha$. A diagram with disorder zero, or with the maximum gravity mn, is clearly sorted. If x and y are two chord diagrams, we say that x is **neater** than y, and y is **messier** than x, if either x has higher gravity than y, or x and y have the same gravity but y has higher disorder.

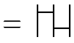
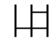
We say that a relation $xr^i_{jk}y$, where x and y are chord diagrams and $j < k$, is a **sorting relation** if:

- $i < k$.
- The chord diagram $t_{ij}y$ is sorted.

We say that a relation $xc^{ij}_{kl}y$, where x and y are chord diagrams, $i < j$, and $k < l$, is a sorting relation if:

- $j < l$.
- The chord diagram $t_{ij}y$ is sorted.

In d of each sorting relation, there is one term which is messier than the others. For example, dr^1_{23} relates the messy diagram $t_{23}t_{12} =$  to sorted diagrams:

$dr^1_{23} =$  $-$  $-$  $+$  .

More generally we have the following lemma, which already implies (by induction) the easy part of Theorem 2.10(4) (that sorted diagrams span H_0).

Lemma 4.2. *For each unsorted chord diagram x, there is a* unique *sorting relation y such that dy equals $\pm x$ plus a combination of neater chord diagrams.*

Proof. Write $x = t_{i_1j_1}\cdots t_{i_mj_m}$ with $i_\alpha < j_\alpha$. If x is unsorted, we can find α such that $j_\alpha > j_{\alpha+1}$. Let α be as large as possible. To see that there exists such a sorting relation, we check three cases:

Case 1. $i_\alpha, j_\alpha, i_{\alpha+1}, j_{\alpha+1}$ are distinct. Then

$$y = t_{i_1j_1}\cdots t_{i_{\alpha-1}j_{\alpha-1}} c^{i_{\alpha+1}j_{\alpha+1}}_{i_\alpha j_\alpha} t_{i_{\alpha+2}j_{\alpha+2}}\cdots t_{i_mj_m}$$

is a sorting relation that relates x to a chord diagram with the same gravity and with the disorder decreased by one.

Case 2. $i_\alpha = i_{\alpha+1}$. Then the sorting relation

$$y = t_{i_1j_1}\cdots t_{i_{\alpha-1}j_{\alpha-1}} r^{j_{\alpha+1}}_{i_\alpha j_\alpha} t_{i_{\alpha+2}j_{\alpha+2}}\cdots t_{i_mj_m}$$

relates x to one chord diagram with the same gravity and disorder one smaller, and two chord diagrams with higher gravity.

Case 3. $i_\alpha = j_{\alpha+1}$. Similarly to Case 2, this time we take

$$y = t_{i_1 j_1} \cdots t_{i_{\alpha-1} j_{\alpha-1}} r^{i_{\alpha+1}}_{i_\alpha j_\alpha} t_{i_{\alpha+2} j_{\alpha+2}} \cdots t_{i_m j_m}.$$

We leave the uniqueness as an exercise. □

Let S_0, S_1 denote the span of the sorted diagrams and sorting relations, respectively. Let U_0 denote the span of the unsorted diagrams.

Corollary 4.3. *If we restrict $d : T_1 \to T_0$ to S_1 and project to U_0, we get an isomorphism $S_1 \to U_0$.*

Proof. An upper triangular matrix with invertible diagonal entries is invertible. □

Lemma 4.4. *Relations of the form xgy, where $x \in T_0$, $g = r^i_{jk}$ or $g = c^{kl}_{ij}$ and $y \in S_0$, span T_1/dT_2.*

Proof. Let $x, z \in T_0$ and let $g = r^i_{jk}$ or $g = c^{kl}_{ij}$; we need to show that xgz is equivalent modulo dT_2 to relations of the desired type. By Corollary 4.3 we can write

$$z = dw + y,$$

where $w \in S_1$ and $y \in S_0$. Modulo d of a "trivial relation" in T_2,

$$xgz = xg(dw + y) \equiv x(dg)w + xgy.$$

The r.h.s. is a combination of relations of the required type. □

The proofs of Lemmas 4.2 and 4.4 are reminiscent of arguments in [17].

Lemma 4.5. *Sorting relations span $T_{1,3}/dT_{2,3}$.*

Proof. This is a tedious calculation which we merely summarize here.

First, we can use the (3T) relation to express any four-term relation r^i_{jk} with $i > \max(j,k)$ in terms of four-term relations with $i < \max(j,k)$. After doing this, there are five kinds of generators of $T_{2,3}$ that are not sorting relations.

(a) Four-term relation involving four strands, e.g. $r^1_{34}t_{12}$. The (22T) relation, with $i = 4, j = 3, k = 2, l = 1$, relates this to a combination of sorting relations. There are three other four-term relations on four strands that are not sorting relations, up to order-preserving reindexing, and appropriate (22T) relations relate these to sorting relations in the same way.

(b) Commutativity relation involving four strands, e.g. $c^{23}_{14}t_{12}$. The (22T) relation with $i = 1, j = 4, k = 2, l = 3$, plus the (22T) relation with $i = 1, j = 4, k = 3, l = 2$, together with two (3T) relations, relate this to sorting relations. There is one other commutativity relation on four strands that is not a sorting relation, namely $c^{13}_{24}t_{12}$, and we handle this the same way with $1, 2$ switched.

(c) Four-term relation involving five strands, e.g. $r^1_{45}t_{23}$. The (10T) relation, with $i = 2, j = 3, k = 1, l = 4, q = 5$, relates this to a combination of sorting relations. There are three other four-term relations on five strands that are not sorting relations, up to order-preserving reindexing, which we handle similarly using appropriate (10T) relations.

(d) Commutativity relations involving five strands, e.g. $c^{13}_{45}t_{12}$. The (10T) relation, with $i = 4, j = 5, k = 2, l = 1, q = 3$, relates this to sorting relations. The other relations of this type are handled by similar (10T) relations.

(e) Commutativity relations involving six strands, e.g. $c^{34}_{56}t_{12}$. The "Jacobi identity" relates a relation of this type to sorting relations. □

Lemma 4.6. *The sorting relations span T_1/dT_2.*

Proof. By Lemma 4.4 it is enough to show that if y is a sorted diagram and $g = r^i_{jk}$ or $g = c^{kl}_{ij}$, then gy is equivalent modulo dT_2 to a sum of sorting relations. We will do this by induction on $\deg(y)$.

The base case $y = 1$ is trivial, so assume $y = t_{ij}z$, $i < j$. By Lemma 4.5

$$gy = gt_{ij}z \equiv wz \pmod{dT_2},$$

where $w \in T_{1,3} \cap S_1$. There are two kinds of terms in wz. Terms of the form $t_{i'j'}g'z$ are equivalent modulo T_2 to elements of S_1 by the inductive hypothesis, since $\deg(z) = \deg(y) - 1$. Terms of the form $g't_{i'j'}z$ are already in S_1 except when $t_{i'j'}z$ is not sorted.

If $t_{i'j'}z$ is not sorted, and say $i' < j'$, then $j' > j$ (since $y = t_{ij}z$ is sorted), so $t_{i'j'}z$ has higher gravity than y. By Lemma 4.2 we can write $t_{i'j'}z = dw' + y'$, where $w' \in S_1$ and y' is a combination of sorted diagrams with higher gravity than y. Using the "trivial relations", as in Lemma 4.4 we have

$$g't_{i'j'}z = g'(dw' + y') \equiv (dg')w' + g'y' \pmod{dT_2}.$$

Now $(dg')w' \in S_1$, while in $g'y'$ every diagram in y' has higher gravity than y (and the same degree). A sub-induction on gravity takes care of these terms. □

Proof of Theorem 4.1. By Lemma 4.6, any homology class in $H_1(T_*)$ can be represented by an element $\alpha \in S_1$. By Corollary 4.3, the restriction of d to S_1 is injective. Therefore $\alpha = 0$. □

Proof of Lemma 2.17. There is a chain map $f : T_* \to D_*$ which sends the c's to zero. This map is surjective at the level of chains.

Suppose $\alpha \in D_1$ and $d\alpha = 0$. Choose $\beta \in T_1$ with $f(\beta) = \alpha$. Since $f(d\beta) = 0$ in D_0, we can arrange that $d\beta = 0$ in T_0 by adding some monomials containing c's to β. Then by Theorem 4.1, $\beta = d\gamma$ for some $\gamma \in T_2$, and $\alpha = df(\gamma)$. □

Proof of Theorem 2.10(4). By Corollary 4.3, any element of T_0 is homologous to an element of S_0, which is unique by Lemma 4.6 and Corollary 4.3. Thus $H_0(T_*) = S_0$. By Theorem 2.10(1–2), $W\mathcal{B} = H_0(D_*)$, which in turn equals $H_0(T_*)$. □

5. Reprise: the algorithm for integration

Given $v : \mathcal{B}_m/P\mathcal{B}_m \to G$, where G is any abelian group, here is how to find a (and all) $w : \mathcal{B}_0 \to G$ with $\delta^m w = v$, and why it works. The procedure is implicit in [17].

If $m = 0$, let $w := v$. Otherwise let $\tilde{\mathcal{B}}_*$ be as in Definition 3.3. π lifts to a map $\tilde{\pi} : \tilde{B}_m \to T_{0,m}$. Define $v' : \tilde{\mathcal{B}}_{m-1} \to G$ by induction on the messiness of chord diagrams as follows.

For each sorted diagram in $T_{0,m-1}$, choose an arbitrary geometric representative (i.e. lift under $\tilde{\pi}$) in $\tilde{\mathcal{B}}_{m-1}$ and define v' arbitrarily on this representative. (By Theorem 2.10(4), this set of choices is isomorphic to $W\mathcal{B}^*_{m-1}$, so we will get all possible v'.) Extend v' to all noncommutative singular braids with sorted chord diagrams by "integration along paths" in the space of singular braids. Since v satisfies (DIFF*) and (T4T*), this does not depend on the choice of path; cf. the proof of Theorem 2.10(2).

For an unsorted diagram x, choose a geometric representative of the sorting relation that relates x to neater diagrams. (A geometric representative of a commutativity relation consists of a difference of two singular braids near one that has two double points at the same height.) Define v' on the associated geometric representative of x so as to satisfy the geometric relation. Extend v' to all noncommutative singular braids with chord diagram x by integration along paths.

The map $v' : \tilde{\mathcal{B}}_{m-1} \to G$ satisfies commutativity, so that it descends to $\mathcal{B}_{m-1}$, and it satisfies (T4T*). This is because any sorting relation is satisfied by the previous paragraph and Lemma 2.15, but any other relation is a linear combination of sorting relations (by Lemma 4.6), even at the level of geometry (by §2.4).

Now $\delta v' = v$ and, in particular, v' is differentiable. Repeat this process with $v := v'$.

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