Embedded Contact Homology of 2-Torus Bundles over the Circle

by

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Abstract

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Embedded contact homology is an invariant of a contact 3-manifold Y, given by the homology of a chain complex generated by certain collections of embedded closed Reeb orbits with a differential that counts certain embedded pseudoholomorphic curves in $Y \times \mathbb{R}$. The main result of this dissertation computes the embedded contact homology of T^2 bundles over S^1 whose monodromy $A \in SL_2(\mathbb{Z})$ is $-\mathbb{1}$ or a hyperbolic matrix, equipped with certain standard contact forms. The form of the answer is nearly independent of A, essentially depending only on whether the eigenvalues are positive or negative.

To prove the main result, we first introduce a combinatorial formula for the ECH chain complex for the above contact manifolds, in terms of certain polygonal paths in the plane with vertices at lattice points. The bulk of this work is then to calculate the homology of the combinatorial chain complex.

This work extends the results and methods of Hutchings and Sullivan ("Rounding

corners of polygons and the embedded contact homology of T^3", *Geometry and Topology*, 2006) on the case $Y = T^3$.

Assistant Professor Michael Hutchings
Dissertation Committee Chair

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Chapter 1

Introduction

1.1 Embedded contact homology

Embedded contact homology (ECH), introduced in [5, §1.1 and §11], [6, §7] from ideas implicit in [3, 4], associates a graded Abelian group $ECH_*(Y, \lambda, \Gamma)$ to a 3-manifold Y, a contact form λ, and a homology class $\Gamma \in H_1(Y)$. A *contact form* on a 3-manifold is a 1-form λ such that $\lambda \wedge d\lambda > 0$. The contact form gives rise to the 2-plane field $\ker(\lambda)$, the *contact structure*, which is oriented by $d\lambda$. The contact form also gives rise to a vector field R characterized by $\lambda(R) = 1$ and $R \lrcorner d\lambda = 0$, the *Reeb vector field.* A *Reeb orbit* is a periodic orbit of the flow of the Reeb vector field.

The ECH of (Y, λ, Γ) is the homology of a chain complex generated by certain collections of embedded closed (nondegenerate) Reeb orbits in Y. Specifically, a generator of the chain complex is a collection (with multiplicity) of Reeb orbits, a hyperbolic Reeb orbit can only appear with multiplicity 1, and the sum (with multiplicity) of the homology of the Reeb orbits is constrained to be Γ. The matrix coefficient of the differential between two generators counts certain embedded J-holomorphic curves in $Y \times \mathbb{R}$, for a suitable almost complex structure J. (In addition to embedded curves, the count includes unions of embedded curves with multiple covers of "trivial cylinders" of the form $\beta \times \mathbb{R}$, where β is a Reeb orbit.) More precisely, the differential counts curves with ends at $\pm\infty$ at the appropriate collections of Reeb orbits, modulo the symmetry of translating a curve in the $\mathbb{R}$ direction of $Y \times \mathbb{R}$.

The embedded contact homology of Y is conjectured [5, §1.1] to be isomorphic to versions of the Seiberg-Witten Floer homology ($\check{HM}_*$) and the Ozsváth-Szabó Floer homology (HF^+_*) of $-Y$ (see [7, 9, 8]). The conjectural isomorphism with Seiberg-Witten Floer homology is a three-dimensional analogue of Taubes' $SW = Gr$ result [10, 11]. ECH is similar (but not isomorphic) to Symplectic Field Theory [1], which counts curves that need not be embedded.

Taubes has announced a proof ([13], following work in [12]) that ECH is isomorphic to Seiberg-Witten Floer homology. It follows that ECH is well-defined: ECH is independent of the extra choices involved in computing it. In particular, it does not depend on the contact form; it only depends on Y, Γ, and the contact structure (and in fact only on Y, Γ, and the Euler class of the contact structure).

The purpose of the present work is to compute ECH for a large family of T^2-bundles over S^1. This generalizes the T^3 case, which was computed by Hutchings and Sullivan [5]. The methods of the current work generalize those of [5]. Hutchings has used methods similar to the T^3 case in order to compute the ECH of $S^1 \times S^2$. (See [5, §12.2.1] for the statement of the result.) Other direct calculations of ECH include an easy calculation for S^3 by Hutchings [unpublished] and work in progress by David Farris on circle bundles.

1.2 Torus bundles over the circle

An element A of $SL_2(\mathbb{Z})$ gives a map $T^2 \to T^2$ and thus a mapping torus

$$Y_A = T^2 \times [0,1] \,/\, \left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), 0\right) \sim \left(A\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), 1\right). \tag{1.1}$$

(We will use coordinates (x, y) on the T^2 fibers and t on the base S^1.) We classify $A \in SL_2(\mathbb{Z})$, using the rank of $A - \mathbb{1}$. The rank 1 case is the *positive parabolic* case, and the full rank case may be further classified: A is *hyperbolic* if A has distinct real eigenvalues, *negative parabolic* if A has only one eigenvalue (necessarily -1), and *elliptic* if A has imaginary eigenvalues. We say A is *positive hyperbolic* if it is hyperbolic with positive eigenvalues, and similarly for *negative hyperbolic*.

We compute the ECH of (Y_A, λ_n, Γ), where:

- Y_A is the mapping torus of the action of $A \in SL_2(\mathbb{Z})$ on T^2, where A is hyperbolic or $-\mathbb{1}$

  and

- n is a positive integer, and the contact form λ_n (which we construct in Chapter 2) is of the form $a(t)\,dx + b(t)\,dy$. Its Reeb vector field is tangent to the T^2 fibers. As one travels once around the base S^1, the Reeb vector field turns counterclockwise through an angle $N\pi$ (as do the contact planes), where $N = 2n$ if A is positive hyperbolic and $N = 2n - 1$ if A is negative hyperbolic or $-\mathbb{1}$.

(In fact, λ_n is Morse-Bott, and we must perturb each circle of Reeb orbits into two nondegenerate orbits, one elliptic and one hyperbolic. See [5, §11.2.3].)

Note that in our situation the ECH vanishes unless the homology class Γ lies in the subspace of $H_1(Y_A)$ coming from the homology of a fiber, because all Reeb orbits lie in the fibers. Henceforth, assume Γ lies in that subspace. Identifying $H_1(T^2)$ with $\mathbb{Z}^2$, we then have $\Gamma \in \mathbb{Z}^2/\operatorname{Im}(\mathbb{1} - A)$, because there is a short exact sequence (coming from the long exact sequence of a mapping torus [2, p.151 Example 2.48, or p. 158 Exercise 30])

$$0 \to \frac{H_1(T^2)}{\mathbb{1} - A} \xrightarrow{\iota} H_1(Y_A) \to H_0(T^2) \to 0 \tag{1.2}$$

in which the map ι is induced by the inclusion of a fiber T^2 into Y_A.

In general, ECH has a non-canonical grading over $\mathbb{Z}/m$, where m is the divisibility of the image of $c_1(\xi) + 2PD(\Gamma)$ in $\operatorname{Hom}(H_2(Y), \mathbb{Z})$ (here ξ is the contact structure of λ). The grading is canonical if $\Gamma = 0$. In our case, $m = 0$, and thus ECH has a grading over $\mathbb{Z}$ if $\Gamma = 0$ and over a $\mathbb{Z}$-torsor if $\Gamma \neq 0$.

Theorem 1.1. *Let $A \in SL_2(\mathbb{Z})$ be hyperbolic or $-\mathbb{1}$, let n be a positive integer, and let $\Gamma \in \mathbb{Z}^2/\operatorname{Im}(\mathbb{1} - A)$. Then the ECH of the mapping torus Y_A with the contact form λ_n in the homology class Γ is given by:*

1. *If $\Gamma = 0$, then*

$$ECH_i(Y_A, \lambda_n, 0) \cong \begin{cases} \mathbb{Z} & \text{if } i > 0 \\ \mathbb{Z}^2 & \text{if } i = 0 \text{ and } A \text{ is positive hyperbolic} \\ \mathbb{Z} & \text{if } i = 0 \text{ and } A \text{ is negative hyperbolic or } -\mathbb{1} \\ 0 & \text{if } i < 0 \end{cases} \tag{1.3}$$

2. *If* $\Gamma \neq 0$, *then* $ECH_*(Y_A, \lambda_n, \Gamma)$ *is graded over a* $\mathbb{Z}$*-torsor which contains* i_0 *such that*

$$ECH_i(Y_A, \lambda_n, \Gamma) \cong \begin{cases} \mathbb{Z} & \textit{if } i \geq i_0 \\ 0 & \textit{if } i < i_0 \end{cases} \tag{1.4}$$

The result is independent of the contact form λ_n. This independence was expected, from the conjectured isomorphism with Seiberg-Witten Floer homology, and independence will follow from Taubes' proof of that conjecture [13]. In the present work, we prove directly that $ECH_*(Y_A, \lambda_n, \Gamma) \cong ECH_*(Y_A, \lambda_{n+1}, \Gamma)$ (Chapter 8).

The result is nearly independent of A, which was not expected.

1.3 The method of proof: Polygonal paths

To prove Theorem 1.1, following [5], we give a combinatorial description of a chain complex whose homology is ECH. The bulk of the present work is devoted to computing the homology of this combinatorial chain complex.

Fix A, n, and Γ as in Theorem 1.1. The combinatorial chain complex is generated by *labeled periodic paths*, certain polygonal paths in $\mathbb{R}^2$ with vertices in $\mathbb{Z}^2$, with some extra data. To wit, such a path turns counterclockwise, it satisfies the "periodicity" condition $(*)$ below, and each edge carries an 'e' or 'h' label. Each edge also carries an angle in $\mathbb{R}$, a lift from S^1 of the direction the edge is pointing in, and "counterclockwise" means that these angles are strictly increasing. The differential δ on the combinatorial chain complex is a signed sum of ways to "round a corner" while "locally losing one 'h'." Figure 1.1 shows an example of a labeled periodic path α, and Figure 1.2 shows the result of applying the differential to α. We will discuss both figures below.

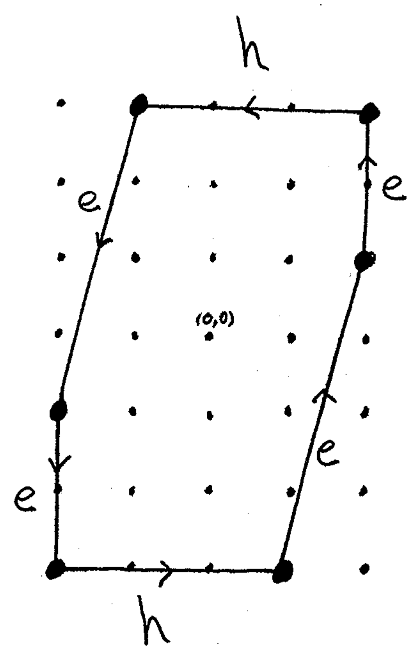

Figure 1.1: A labeled periodic path with $A = -\mathbb{1}$, $n = 1$, and $\Gamma = 0$.

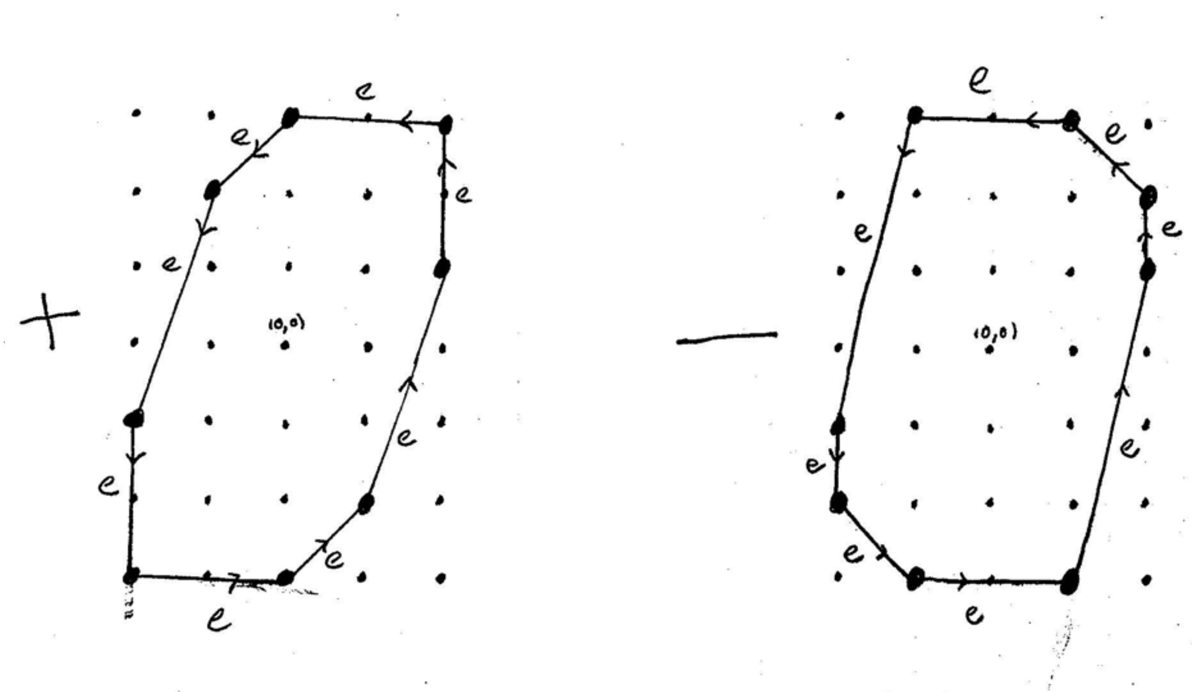

Figure 1.2: The differential: This linear combination of labeled periodic paths is $\delta\alpha$, where α is the labeled periodic path in Figure 1.1.

The periodicity condition is a relation between an edge at angle θ and an edge at approximately $\theta+N\pi$. (*Angles* will always be in $\mathbb{R}$, not S^1. Recall that the Reeb vector field turns through an angle $N\pi$.) More correctly, the condition uses the function $f_{A,n}\colon \mathbb{R} \to \mathbb{R}$, a lift to $\mathbb{R}$ of the natural action of A on $S^1 \subset \mathbb{R}^2$, and $f_{A,n}$ is the unique such lift satisfying $f_{A,n}(\theta_0) = \theta_0 + N\pi$ for $\binom{\cos\theta_0}{\sin\theta_0}$ an eigenvector of A.

Fix a representative $\gamma \in \mathbb{Z}^2$ of $\Gamma \in \mathbb{Z}^2/\operatorname{Im}(\mathbb{1} - A)$, so $\Gamma = [\gamma] = \gamma + \operatorname{Im}(\mathbb{1} - A)$. We choose $\gamma = 0$ if (and only if) $\Gamma = 0$. We can now state the periodicity condition:

($*$) If a path has an edge with angle θ starting at a point $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$, then the path has an edge with angle $f_{A,n}(\theta)$ starting at $A\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + \gamma$, with the same '$e$' or '$h$' label. If a path does not have an edge with angle θ then it does not have an edge with angle $f_{A,n}(\theta)$.

Thus, every time a path turns through an angle of approximately $N\pi$, it travels from some $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ to $A\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + \gamma$. (The choice of γ will not affect the homology–different representatives of the same Γ yield isomorphic combinatorial chain complexes. See Remark 3.9.)

Figure 1.1 shows an example of a labeled periodic path α, with $A = -\mathbb{1}$, $n = 1$, and $\Gamma = 0$. Thus, $N = 1$, $\gamma = 0$, and $f_{A,n}(\theta) = \theta + \pi$. In this case, we obtain a polygon symmetric under rotation by π about the origin. The polygonal path traverses the polygon infinitely many times. (This phenomenon is quite unlike the hyperbolic case, in which a path tends to a line parallel to an eigenvector of A as $\theta \to +\infty$, and tends to a line parallel to the other eigenvector as $\theta \to -\infty$.) Figure 1.2 shows $\delta\alpha$. Note that corners of α with both adjacent edges labeled 'e' do not contribute to $\delta\alpha$. Also, a family of corners identified by periodicity – opposite corners in the picture – are all rounded at once.

Each labeled periodic path encodes a collection of Reeb orbits. One edge (at angle

θ) can be thought of as describing the Reeb orbits in one fiber of the cover $T^2 \times \mathbb{R}$ of Y_A (namely the fiber $T^2 \times \{\theta\}$), and the periodicity condition ($*$) ensures that the resulting collection of Reeb orbits descends to Y_A. The 'e'/'h' label has to do with elliptic/hyperbolic Reeb orbits.[1] The condition ($*$) also enforces the total homology Γ of the collection of Reeb orbits. (See Appendix A or [5] for more details.)

The combinatorial chain complex is described in detail in Chapter 3. It was introduced in [5, §12.2.2], generalizing the chain complex used in the body of [5] to compute the ECH of T^3.

Chapters 4–8 compute the homology of the combinatorial chain complex. In particular, they (and Chapter 3) are purely combinatorial. Here is an outline of the body of this work.

- The contact form λ_n is constructed in Chapter 2.
- The combinatorial chain complex, denoted by $C_*(A, n, \gamma)$, is defined in Chapter 3.
- Chapter 4 defines subcomplexes, $C_*(B)$, of the combinatorial chain complex, involving paths enclosed by a *box* or *strip* B.
- *Flattened subcomplexes* of $C_*(B)$, with homology isomorphic to $H_*(B)$, are defined in Chapter 5. The flattened subcomplexes involve paths "maximal" with respect to B.
- Chapter 6 shows that the homology $H_*(B)$ coming from a strip B is isomorphic to $H_*(A, n, \gamma)$.
- We compute $H_*(A, 1, \gamma)$ in Chapter 7.

[1]Strictly speaking, the Reeb orbits of the Morse-Bott contact form λ_n lie in the T^2 fibers, and the Reeb orbits of perturbed contact forms are elliptic or hyperbolic.

- Chapter 8 shows that $H_*(A, n, \gamma)$ is independent of n.
- The isomorphism between the embedded contact homology $ECH_*(Y_A, \lambda_n, \Gamma)$ and the combinatorially defined homology $H_*(A, n, \gamma)$ is proved in Appendix A.

1.4 Future work

It seems that the ECH of the mapping torus Y_A can also be computed when A is elliptic or negative parabolic, and the result has the same form as that for negative hyperbolic A. The preliminary calculation was done by David Farris and the author in the elliptic case, and by the author in the negative parabolic case.

ECH in the positive parabolic case (other than $A = \mathbb{1}$) has not been computed.

Chapter 2

The manifolds Y_A and their contact structures

Fix a matrix $A \in SL_2(\mathbb{Z})$ which is hyperbolic or -1, and a positive integer n. The purpose of this chapter is to construct a contact form λ_n on the torus bundle Y_A, with the properties described in §1.2. The remaining chapters are combinatorial and independent of this chapter, except for Appendix A, which relates the combinatorics to the ECH of (Y_A, λ_n). However the notation $f_{A,n}$ introduced below in Lemma-Definition 2.13 will be needed in the combinatorial chapters.

In §2.1, we restate in coordinates the definition of the manifold Y_A. In §2.2, we describe the contact forms λ_n. The contact planes are "perpendicular" to the fibers, are constant on any fiber, and rotate by an amount $N\pi$ as one travels once around the base. We start with the contact forms λ_n for $A = -\mathbb{1}$, and with that motivation, we give the more complicated statement of the existence of λ_n for A hyperbolic. To make that statement precise, we define the map on lifted angles $f_{A,n}$. Finally, in §2.3, we prove Proposition 2.8, constructing λ_n for A hyperbolic. The construction follows the suggestion in [5, §12.2.2].

2.1 Y_A and $\Gamma \in H_1(Y_A)$

The smooth 3-manifolds Y_A we study are total spaces of 2-torus bundles over the circle (mapping tori).

Definition 2.1. Let $A \in SL_2(\mathbb{Z})$, choose coordinates x and y on T^2 in the usual way, and define

$$\phi_t(\begin{pmatrix} x \\ y \end{pmatrix}, t) = (A\begin{pmatrix} x \\ y \end{pmatrix}, t+1) \tag{2.1}$$

so that

$$Y_A = T^2 \times \mathbb{R}/\phi_t. \tag{2.2}$$

We also write $\binom{x}{y}$ as (x, y). (The subscript on ϕ_t is to distinguish it from ϕ_θ, to be introduced in a change of coordinates in §2.2. We suppress the dependence of ϕ_t on A.) The monodromy of the bundle is A^{-1}, and we call A the *inverse monodromy*. Conjugate matrices give the same bundle. The matrix A acts naturally on $\mathbb{R}^2$, its subset $\mathbb{Z}^2$, and its quotient T^2 (and we have identified $H_1(T^2)$ with $\mathbb{Z}^2$).

Definition 2.2. If $A \in SL_2(\mathbb{Z})$ is hyperbolic or $-\mathbb{1}$, denote the eigenvalues by a and a^{-1}, with $|a| \geq 1$.

2.2 The contact forms λ_n and the map $f_{A,n}$ on lifted angles

2.2.1 λ_n for $A = -\mathbb{1}$

Choose coordinates x and y on T^2 in the usual way (making $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$), and a coordinate θ on $\mathbb{R}$. (Eventually, S^1 will be the quotient of $\mathbb{R}$ by a certain map $f_{A,n}$.) We begin with $A = -\mathbb{1}$, where the form may be defined by a simple equation. Equation (2.3) below, though correct only for $A = -\mathbb{1}$, also gives the correct intuition for hyperbolic A. Tildes denote objects on the cover $T^2 \times \mathbb{R}$ of Y_A.

Lemma-Definition 2.3. Let

$$\widetilde{\lambda} = \cos\theta \, dx + \sin\theta \, dy \tag{2.3}$$

on $T^2 \times \mathbb{R}$. This is a contact form. The Reeb vector field of $\widetilde{\lambda}$ is

$$\widetilde{R} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}. \tag{2.4}$$

Proof. By direct calculation, the reader may verify that $\widetilde{\lambda} \wedge d\widetilde{\lambda} > 0$ if we choose the correct orientation (so $\widetilde{\lambda}$ is a contact form on $T^2 \times \mathbb{R}$) and that $\widetilde{\lambda}(\widetilde{R}) = 1$ and $\widetilde{R} \lrcorner d\widetilde{\lambda} = 0$ (so $\widetilde{R}$ is its Reeb vector field).

□

Lemma 2.4. *The manifold $Y_{-\mathbb{1}}$, as in Definition 2.1, can also be written as:*

$$Y_{-\mathbb{1}} \cong (T^2 \times \mathbb{R})/\phi_\theta \tag{2.5}$$

$$\phi_\theta((x,y),\theta) = ((-x,-y), \theta + (2n-1)\pi). \tag{2.6}$$

□

Proposition 2.5. *For all positive integers n, the form $\widetilde{\lambda}$ on $T^2 \times \mathbb{R}$ descends to a contact form λ_n on the quotient $Y_{-\mathbb{1}}$, and $\widetilde{R}$ descends to its Reeb vector field.*

Proof. We see that ϕ_θ respects $\widetilde{\lambda}$ and therefore $\widetilde{R}$. Thus $\widetilde{\lambda}$ and $\widetilde{R}$ descend to a contact form λ_n and its vector field R_n on $Y_{-\mathbb{1}}$. □

Note that the parameterization makes $Y_{-\mathbb{1}}$ a bundle over $S^1 = \mathbb{R}/(2n-1)\pi\mathbb{Z}$. In particular, λ_n depends indirectly on n, because the parameterization depends on n. (We suppress the dependence of ϕ_θ on n and A.) The contact planes turn through $(2n-1)\pi = N\pi$.

2.2.2 λ_n for hyperbolic A

Some generalizations of Theorem 2.5 are easy.

Example 2.6 (Elliptic A)**.** If A is the rotation matrix that rotates by $\pi/2$, we need only replace Equation (2.6) with

$$\phi_\theta((x,y),\theta) = (A(x,y), \theta + \frac{\pi}{2} + 2\pi(n-1)); \tag{2.7}$$

everything else remains the same. The map $\theta \mapsto \theta + \frac{\pi}{2} + 2\pi(n-1)$ accounts for how much the contact form turns as we travel once around the base S^1, and must be addition of $\pi/2$ (up to multiples of 2π) because A rotates by $\pi/2$. We call this map $f_{A,n}$.

Hyperbolic A have a more complicated $f_{A,n}$, and also change the lengths of vectors and covectors. For these reasons, we start with a form parameterized by t, not θ. We defer $f_{A,n}$, but we define N, because $f_{A,n}$ is *approximately* addition of $N\pi$. Instead of expressions such as $\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$, we will have a vector $(R_x(t), R_y(t))$ that rotates counterclockwise.

Definition 2.7. If A is positive hyperbolic, we write

$$N = 2n \tag{2.8}$$

If A is negative hyperbolic or $A = -\mathbb{1}$,

$$N = 2n - 1. \tag{2.9}$$

The quantity $N\pi$ is sometimes called the rotation number $f_{A,n}$, and the rotation number may be defined abstractly for all $A \in SL_2(\mathbb{Z})$. Note that n is the number of revolutions, rounded up to an integer:

$$n = \lceil \text{rotation number}/2\pi \rceil. \tag{2.10}$$

Recall that Y_A is the quotient of $T^2 \times \mathbb{R}$ by

$$\phi_t((x, y), t) = (A(x, y), t + 1). \tag{2.11}$$

Proposition 2.8. *For all hyperbolic $A \in SL_2(\mathbb{Z})$, and all positive integers n, there is a 1-form $\widetilde{\lambda} = \lambda_x(t)\, dx + \lambda_y(t)\, dy$ on $T^2 \times \mathbb{R}$ (with coordinates $((x, y), t)$) such that*

1. *The form $\widetilde{\lambda}$ is a contact form.*

2. *The form $\widetilde{\lambda}$ descends to a form, called λ_n, on the quotient Y_A.*

3. *The Reeb vector field of $\widetilde{\lambda}$ can be written*

$$\widetilde{R} = R_x(t)\frac{\partial}{\partial x} + R_y(t)\frac{\partial}{\partial y}, \tag{2.12}$$

where the argument θ of $(R_x(t), R_y(t)) \in \mathbb{R}^2$ satisfies $\theta'(t) > 0$.

4. *The argument θ of $(R_x(t), R_y(t))$ increases by $N\pi$ as t goes from 0 to 1. At $t = 0$ and $t = 1$, $(R_x(t), R_y(t))$ is an eigenvector of A.*

We defer the proof to §2.3. Proposition 2.8 should generalize in a straightforward way to all $A \in SL_2(\mathbb{Z})$, but we will work only with hyperbolic matrices and $-\mathbb{1}$.

Note (for well-definedness) that we have only made claims about $\theta(t)$ that are unchanged by adding some t-independent $2\pi k$.

It will turn out that θ increases from an appropriate θ_0 to $\theta_0 + N\pi$ as we go once around Y_A.

Since $\theta'(t) > 0$, we may reparameterize everything appearing in Proposition 2.8 as functions of θ instead of t. The result will be Proposition 2.16.

2.2.3 $f_{A,n}$ for $A \in SL_2(\mathbb{Z})$

We need to view the base S^1 of the bundle as the quotient of $\mathbb{R}$ by a map $f_{A,n}\colon \mathbb{R} \to \mathbb{R}$, so that $S^1 = \mathbb{R}/f_{A,n}$ (or more formally, we quotient by the infinite cyclic group $\langle f_{A,n}\rangle$). The map $f_{A,n}$ will satisfy $f_{A,n}(\theta) > \theta$ for all $\theta \in \mathbb{R}$, to make a good quotient, and $f_{A,n}$ will be defined in terms of the action of A on angles in $\mathbb{R}^2$, as suggested by our notation θ.

Notation Convention 2.9. Any matrix $A \in SL_2(\mathbb{Z})$ acts on $\mathbb{R}^2$ inducing a map $A_{S^1}\colon S^1 \to S^1$, obtained as the composition

$$S^1 \hookrightarrow \mathbb{R}^2 \setminus \{0\} \xrightarrow{A} \mathbb{R}^2 \setminus \{0\} \twoheadrightarrow S^1, \tag{2.13}$$

or more abstractly by viewing *this* S^1 as $(\mathbb{R}^2 \setminus \{0\})/\mathbb{R}_+$, the set of directions in $\mathbb{R}^2$.

See Figure 2.1. To obtain $f_{A,n}$, we will lift A_{S^1} to a map on the universal cover $\mathbb{R}$ of S^1.

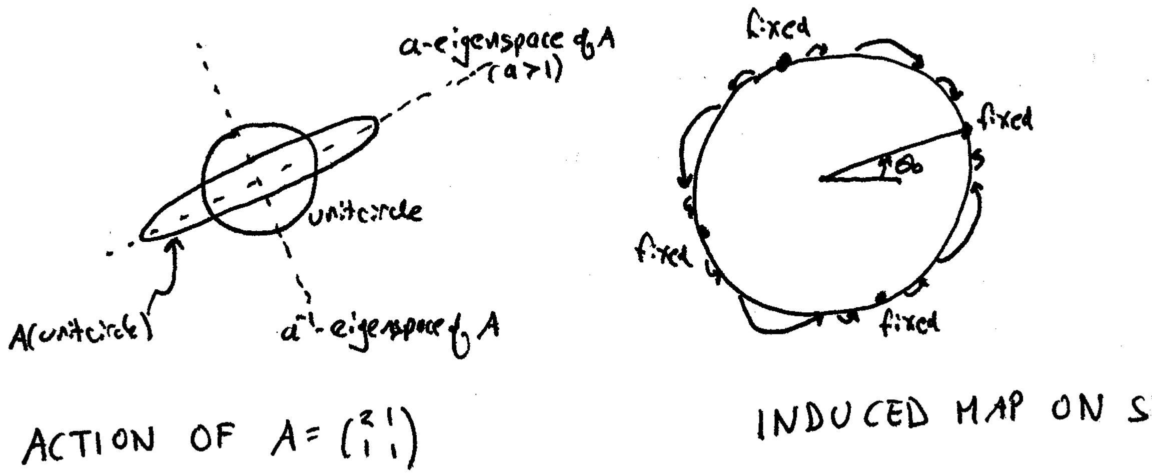

Figure 2.1: The action of A on $\mathbb{R}^2$ is illustrated by showing the ellipse A(unit circle). The induced map on S^1 is obtained by radially projecting the ellipse back to the circle.

Definition 2.10. The copy of $\mathbb{R}$ that is the universal cover of $S^1 = (\mathbb{R}^2 \setminus \{0\})/\mathbb{R}_+$ is called the space of *lifted angles*, or just *angles*.

Note that this $\mathbb{R}$ will have two different quotients:

- The circle S^1, the space of directions in $\mathbb{R}^2$, is the quotient $\mathbb{R}/2\pi\mathbb{Z}$.
- The S^1 that is the base of the bundle Y_A is the quotient $\mathbb{R}/f_{A,n}$.

Definition 2.11. A *lift* of A_{S^1} is a map $f\colon \mathbb{R} \to \mathbb{R}$ such that $\pi \circ f = A_{S^1} \circ \pi$ where π is the projection to the quotient, $\pi\colon \mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$.

In other words, $A(\cos\theta, \sin\theta)$ is a positive θ-dependent multiple of $(\cos f(\theta), \sin f(\theta))$.

Lemma 2.12. *Any two lifts $f, \hat{f}$ are related by $\hat{f} = f + 2\pi n$ for some $n \in \mathbb{Z}$, and each lift f of A_{S^1} satisfies*

$$f(\theta + 2\pi) = f(\theta) + 2\pi. \tag{2.14}$$

Proof. Lifts of A_{S^1} exist, adding $2\pi n$ ($n \in \mathbb{Z}$) yields another lift, and all lifts of A_{S^1} are of this form. To show (2.14), note that A_{S^1} and its lifts make sense not just for $A \in SL_2(\mathbb{Z})$, but also for $A \in SL_2(\mathbb{R})$, which is connected. For any $A \in SL_2(\mathbb{R})$ and any lift f of A_{S^1}, we may obtain a lifted homotopy from f to a lift g of $\mathbb{1}_{S^1}$, specifically $g(\theta) = \theta + 2\pi k$ for some $k \in \mathbb{Z}$. On that homotopy, for any $\theta \in \mathbb{R}$, $f(\theta + 2\pi) - f(\theta) \in 2\pi\mathbb{Z}$ does not change, reaching $g(\theta + 2\pi) - g(\theta) = 2\pi$. So $f(\theta + 2\pi) - f(\theta) = 2\pi$ as claimed. □

We will now define maps on lifted angles, and we will let $f_{A,n}$ be the "n'th smallest" of them.

Lemma-Definition 2.13. For any $A \in SL_2(\mathbb{Z})$:

- Definition: A *map on lifted angles* is a map $f\colon \mathbb{R} \to \mathbb{R}$ which is a lift of A_{S^1}, and which satisfies

$$f(\theta) > \theta \quad \text{for all } \theta \in \mathbb{R}. \tag{2.15}$$

- Claim: There is a smallest map on lifted angles (which we denote $f_{A,1}$) under the relation that f is smaller than $\hat{f}$ if and only if $f + 2\pi k = \hat{f}$ where $k > 0$.

- Definition: We let

$$f_{A,n}(\theta) = f_{A,1}(\theta) + 2\pi(n-1), \quad n = 1, 2, 3, \ldots. \tag{2.16}$$

- Claim: The maps on lifted angles are precisely the maps $f_{A,n}$ for n a positive integer.

Proof. Equation (2.14) shows that $f(\theta) - \theta$ is periodic, so bounded.

Among lifts f of A_{S^1}, the subset satisfying (2.15) is neither all lifts nor empty, because adding sufficiently negative $2\pi n$ to any given lift produces violations of (2.15), and adding sufficiently large positive $2\pi n$ produces f satisfying (2.15) (since $f(\theta) - \theta$ is bounded). Thus there is a smallest map on lifted angles. So all maps on lifted angles must be of the form (2.16), and these are all the maps on lifted angles since adding positive constants preserves (2.15). □

2.2.4 $f_{A,n}$ for hyperbolic A

Example 2.14. For any positive hyperbolic A, we show an example of a lift f_0 which is not a map on lifted angles. See Figure 2.1. The map A_{S^1} fixes the elements of S^1 on each eigenline (1-dimensional eigenspace). For any $\theta_0 \in \mathbb{R}$ "pointing along an eigenline" (i.e., mapping to one of the fixed elements of S^1, or in other words, with $(\cos\theta_0, \sin\theta_0)$ an eigenvector of A), there is a unique f_0 (shown) such that $f_0(\theta_0) = \theta_0$. Then this f_0 fixes all θ_0 that point along an eigenline. We see that there exist θ with $f_0(\theta) < \theta$.

We will show $f_{A,1}(\theta) = f_0(\theta) + 2\pi$ (and $f_{A,n}(\theta) = f_0(\theta) + 2\pi n$). We will have

$$f_{A,n}(\theta_0) = \theta_0 + 2\pi n \quad \text{for } \theta_0 \text{ pointing along an eigenline,} \tag{2.17}$$

even though $f_{A,n}$ has more complicated behavior for general θ. To show that $f_{A,1}$ is what we claim, we will need the fact that f_0 displaces any θ by less than π, i.e. $|f_0(\theta) - \theta| < \pi$ for all θ, which is evident from the fact that f_0 does not move any angle past any eigenline.

Lemma 2.15. *If A is hyperbolic or $A = -\mathbb{1}$, and $\binom{\cos\theta_0}{\sin\theta_0}$ is an eigenvector of A, then $f_{A,n}(\theta_0) = \theta_0 + N\pi$.*

Proof. If A is positive hyperbolic we have f_0 from Figure 2.1, a lift of A_{S^1} which cannot be a map on lifted angles. Clearly, $f_0 + 2\pi$ is a map on lifted angles, and is therefore $f_{A,1}$, giving

$$f_{A,n}(\theta) = f_0(\theta) + 2\pi n. \tag{2.18}$$

Thus, $f_{A,n}(\theta_0) = \theta_0 + 2\pi n$.

If A is negative hyperbolic or $-\mathbb{1}$, we may use the lift f_0 of $(-A)_{S^1}$. Then $f_0 + \pi$ is a lift of A_{S^1} and satisfies $f_0(\theta) + \pi > \theta$, so it is a map on lifted angles. However, $f_0(\theta) - \pi < \theta$ so $f_0 - \pi$ is not a map on lifted angles. Thus $f_{A,1} = f_0 + \pi$ and $f_{A,n} = f_{A,1} + 2\pi(n-1) = f_0 + (2n-1)\pi$. So $f_{A,n}(\theta_0) = \theta_0 + (2n-1)\pi$. □

Proposition 2.16. *Reparameterizing the $\mathbb{R}$ of $T^2 \times \mathbb{R}$ by θ (the argument of (R_x, R_y) from Proposition 2.8) instead of t yields*

$$Y_A = T^2 \times \mathbb{R}/\phi_\theta \tag{2.19}$$

$$\phi_\theta((x,y),\theta) = (A(x,y), f_{A,n}(\theta)), \tag{2.20}$$

and the Reeb vector field on $T^2 \times \mathbb{R}$ is $(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y})$ times a positive real function of θ.

Here we have made an (arbitrary) choice of a particular smooth function $\theta(t)$ giving the argument of (R_x, R_y), among the $\mathbb{Z}$-family of such choices.

Proof. We may reparameterize by Proposition 2.8, part 3, and part 4 insures that θ ranges over all of $\mathbb{R}$. That the Reeb field is proportional to the given field is just what it means to reparameterize by the argument θ. We must show that $t \mapsto t+1$ becomes $\theta \mapsto f_{A,n}(\theta)$. Define $\hat{f}$ by $\hat{f}(\theta(t)) = \theta(t+1)$. We first show that $\hat{f}$ is a lift of A_{S^1}, and then that $\hat{f}$ is the lift $f_{A,n}$.

We have $\phi_t^*\widetilde{\lambda} = \widetilde{\lambda}$, so $(\phi_t)_*\widetilde{R} = \widetilde{R}$, or in other words

$$A(R_x(t), R_y(t)) = (R_x(t+1), R_y(t+1)). \tag{2.21}$$

For each t, we have that $(R_x(t), R_y(t))$ is a positive real multiple of $(\cos\theta, \sin\theta)$, which we denote

$$(R_x(t), R_y(t)) \propto (\cos\theta(t), \sin\theta(t)). \tag{2.22}$$

Then

$$(R_x(t+1), R_y(t+1)) \propto (\cos\theta(t+1), \sin\theta(t+1)) = (\cos\hat{f}(\theta(t)), \sin\hat{f}(\theta(t))), \tag{2.23}$$

so

$$A(\cos\theta(t), \sin\theta(t)) \propto A(R_x(t), R_y(t)) = (R_x(t+1), R_y(t+1)) \propto (\cos\hat{f}(\theta(t)), \sin\hat{f}(\theta(t))), \tag{2.24}$$

i.e.

$$A_{S^1}(\cos\theta(t), \sin\theta(t)) = (\cos\hat{f}(\theta(t)), \sin\hat{f}(\theta(t))) \tag{2.25}$$

(see Notation Convention 2.9). But $\theta \mapsto (\cos\theta, \sin\theta)$ is exactly the projection $\pi\colon \mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$ from Definition 2.11, so $(A_{S^1} \circ \pi)(\theta(t)) = (\pi \circ \hat{f})(\theta(t))$ for all t. Thus $\hat{f}$ is a lift of A_{S^1}.

Recall that $f_{A,n}$ is defined to be the n'th-smallest lift of A_{S^1} satisfying $f(\theta) > \theta$. It is clear that $\hat{f}$ satisfies $f(\theta) > \theta$, because it was obtained by conjugating $t \mapsto t+1$ by the order-preserving map θ. So $\hat{f} = f_{A,k}$ for some positive integer k. Proposition 2.8, part 4 means that $\theta(0) = \theta_0$ for some θ_0 as in Lemma 2.15, and $\theta(1) = \theta_0 + N\pi$. But

$$\theta(1) = \hat{f}(\theta(0)) = f_{A,k}(\theta_0), \tag{2.26}$$

so $\theta_0 + N\pi = f_{A,k}(\theta_0)$. Applying Lemma 2.15 shows $n = k$ because different n's have disjoint possibilities for N. Thus $\hat{f} = f_{A,n}$, as desired. □

Remark 2.17. *Each closed Reeb orbit in $T^2 \times \mathbb{R}$ lies in some fiber $T^2 \times \{\theta\}$ where θ has rational slope, $\tan\theta \in \mathbb{Q} \cup \{\infty\}$, and each such T^2 fiber contains a family of Reeb orbits.*

We are in the Morse-Bott case, and in Appendix A, we will perturb the contact form in a neighborhood of such a fiber to have only one elliptic and one hyperbolic Reeb orbit.

2.3 Proof of Proposition 2.8

We prove Proposition 2.8 by constructing a curve $\boldsymbol{\lambda}(t) = (\lambda_x(t), \lambda_y(t))$ satisfying appropriate conditions. The conditions will include $\det(\boldsymbol{\lambda}, \boldsymbol{\lambda}') > 0$, which means the vector $\boldsymbol{\lambda}(t)$ "turns counterclockwise". Here, $'$ is just the t derivative, and $\det(\mathbf{a}, \mathbf{b}) = a_x b_y - a_y b_x$ is the familiar two-dimensional analog of the cross product.

Lemma 2.18. *The argument* $\arg(\mathbf{a})$ *of* $\mathbf{a}(t)$ *satisfies* $\arg(\mathbf{a})' > 0$ *if and only if* $\det(\mathbf{a}, \mathbf{a}') > 0$.

Proof. We have $\det(\mathbf{a}, \mathbf{a}') = |\mathbf{a}|^2 \arg(\mathbf{a})'$. □

Lemma 2.19. *To prove Proposition 2.8, it is sufficient that $\boldsymbol{\lambda}(t) = (\lambda_x(t), \lambda_y(t))$ be a smooth function $\mathbb{R} \to \mathbb{R}^2$ satisfying:*

1. *For all t, $\det(\boldsymbol{\lambda}(t), \boldsymbol{\lambda}'(t)) > 0$.*

2. *For all t,*

$$\boldsymbol{\lambda}(t+1) = (A^T)^{-1} \boldsymbol{\lambda}(t). \tag{2.27}$$

3. *For all t, $\det(\boldsymbol{\lambda}'(t), \boldsymbol{\lambda}''(t)) > 0$.*

4. *a) The argument of $\boldsymbol{\lambda}'(t)$ increases by the amount $N\pi$ as t goes from 0 to 1.*

b) The vector $\boldsymbol{\lambda}'(t)$ *is an eigenvector of* $(A^T)^{-1}$ *at* $t = 0$ *and* $t = 1$.

Note that we use A for periodicity of $(R_x(t), R_y(t)) \in \mathbb{R}^2$, but the inverse transpose $(A^T)^{-1}$ for $(\lambda_x(t), \lambda_y(t))$; if we were being more picky, we would say $\boldsymbol{\lambda}(t) \in (\mathbb{R}^2)^*$. Of course, A^T and $(A^T)^{-1}$ have the same eigenvectors.

Proof. We show part i of Proposition 2.8 from part i of Lemma 2.19, sometimes assuming earlier parts of Lemma 2.19.

1. We need $\widetilde{\lambda} \wedge d\widetilde{\lambda} > 0$. We have

$$\widetilde{\lambda} \wedge d\widetilde{\lambda} = (\lambda_x\, dx + \lambda_y\, dy) \wedge (\lambda_x'\, dt \wedge dx + \lambda_y'\, dt \wedge dy) \tag{2.28}$$

$$= \lambda_x \lambda_y'\, dt \wedge dy \wedge dx - \lambda_y \lambda_x'\, dt \wedge dy \wedge dx \tag{2.29}$$

$$= \det(\boldsymbol{\lambda}, \boldsymbol{\lambda}')\, \mathrm{vol} \tag{2.30}$$

   where we are using the same (unintuitive) orientation on $T^2 \times \mathbb{R}$ as we did for $Y_{-\mathbb{1}}$.

2. We need

$$\phi_t^* \widetilde{\lambda} = \widetilde{\lambda}. \tag{2.31}$$

   Converting this equation from $\widetilde{\lambda}$ to $\boldsymbol{\lambda}$ using the definition of pullback and the definition (2.11) of ϕ_t yields $A^T \boldsymbol{\lambda}(t+1) = \boldsymbol{\lambda}(t)$, which follows from (2.27).

3. We first determine the Reeb vector field in terms of $\boldsymbol{\lambda}(t)$ (a result also found in [5, §10.4]).

   Lemma 2.20. *If* $\lambda_x(t)\, dx + \lambda_y(t)\, dy$ *is a contact form on* $T^2 \times \mathbb{R}$*, its Reeb vector field is* $\widetilde{R} = R_x(t) \frac{\partial}{\partial x} + R_y(t) \frac{\partial}{\partial y}$ *where*

$$\mathbf{R}(t) = (R_x(t), R_y(t)) = \frac{1}{\det(\boldsymbol{\lambda}(t), \boldsymbol{\lambda}'(t))} (\lambda_y'(t), -\lambda_x'(t)). \tag{2.32}$$

Proof. The given curve $\mathbf{R}$ satisfies $1 = \boldsymbol{\lambda} \cdot \mathbf{R}$ and $0 = \mathbf{R} \cdot \boldsymbol{\lambda}'$, which means that the given vector field satisfies $1 = \widetilde{\lambda}(\widetilde{R})$ and $0 = \widetilde{R} \lrcorner d\widetilde{\lambda}$. □

We must show that $\mathbf{R}(t)$ has $\theta'(t) > 0$. Denote the argument of $\boldsymbol{\lambda}'$ by $\hat{\theta}$, retaining θ for the argument of $\mathbf{R}$, we have $\hat{\theta}'(t) > 0$ since $\det(\boldsymbol{\lambda}', \boldsymbol{\lambda}'') > 0$. (In particular, this det condition means $\boldsymbol{\lambda}'(t) \neq 0$, so we can speak of the argument of $\boldsymbol{\lambda}'$. Similarly, we have $\mathbf{R}(t) \neq 0$.)

Lemma 2.21.

$$\hat{\theta}(t) = \theta(t) + \pi/2 \tag{2.33}$$

is (one correct choice for) the argument of $\boldsymbol{\lambda}'(t)$.

We say "one correct choice" because adding $2\pi k$ produces other correct choices.

Proof. Part 1 of Lemma 2.19 ensures that the $1/\det(\boldsymbol{\lambda}(t), \boldsymbol{\lambda}'(t))$ factor of (2.32) is positive. Further, $(\lambda'_y, -\lambda'_x)$ is just (λ'_x, λ'_y) rotated by $-\pi/2$. □

Thus, $\theta'(t) = \hat{\theta}'(t) > 0$.

4. The argument of $\mathbf{R}$ increases by the same amount as that of $\boldsymbol{\lambda}'$ (Lemma 2.21). It remains only to show that if $\boldsymbol{\lambda}'(t)$ is an eigenvector of $(A^T)^{-1}$, then $\mathbf{R}(t)$ is an eigenvector of A. Recall that A has distinct real eigenvalues. The dual basis (in $(\mathbb{R}^2)^*$) of a basis of eigenvectors of A forms a basis of eigenvectors of A^T, and therefore of $(A^T)^{-1}$. If $\boldsymbol{\lambda}'(t)$ is an eigenvector of $(A^T)^{-1}$, then $\mathbf{R}(t)$, being nonzero and perpendicular to $\boldsymbol{\lambda}'(t)$, will be an eigenvector of A.

□

Lemma 2.22. *There exists a smooth map $\boldsymbol{\lambda}\colon \mathbb{R} \to \mathbb{R}^2$ satisfying the conditions given in Lemma 2.19.*

This lemma will complete the proof of Proposition 2.8.

Proof. The idea of this proof is to diagonalize $(A^T)^{-1}$ and then interpolate between C, the unit circle (in the new coordinates), and the ellipse $(A^T)^{-1}(C)$, thus satisfying condition 2 (and also 4b). The interpolation will be made to satisfy conditions 1 and 3, and we will spend the right amount of time on each curve to satisfy condition 4a.

We diagonalize $(A^T)^{-1}$. We let the X-axis be the a-eigenline (1-dimensional eigenspace) where $|a| > 1$, let the Y-axis be the a^{-1}-eigenline. (These coordinates resemble, but are not the same as, the eigencoordinates introduced in §3.1.) Thus, the main ellipse, $(A^T)^{-1}(C)$, meets the X-axis at $\pm a$ and the Y-axis at $\pm a^{-1}$.

Parameterize C by $(\cos\Theta, \sin\Theta)$ and $(A^T)^{-1}(C)$ by $(A^T)^{-1}(\cos\Theta, \sin\Theta)$. (So Θ is not the argument of some vector.) There exists an interpolation between C and $(A^T)^{-1}(C)$, a smooth curve $\boldsymbol{\lambda}_{int}\colon \mathbb{R} \to \mathbb{R}^2$ which agrees with C for $\Theta \in (-\infty, \epsilon)$ and agrees with $(A^T)^{-1}(C)$ for $\Theta \in (\pi/2 - \epsilon, \infty)$, and which has $\boldsymbol{\lambda}_{int}$ and $\boldsymbol{\lambda}'_{int}$ turning counterclockwise, i.e. it satisfies $\det(\boldsymbol{\lambda}_{int}, \boldsymbol{\lambda}'_{int}) > 0$ and $\det(\boldsymbol{\lambda}'_{int}, \boldsymbol{\lambda}''_{int}) > 0$. We may obtain this interpolation by smoothing a piecewise-smooth curve (see Figure 2.2). We would just connect $C(\epsilon)$ to $(A^T)^{-1}(C)(\pi/2 - \epsilon)$ by a line segment, and smooth the result, but in order to satisfy $\det(\boldsymbol{\lambda}'_{int}, \boldsymbol{\lambda}''_{int}) > 0$, i.e. that $\boldsymbol{\lambda}'_{int}$ turns strictly counterclockwise, we first bend the line segment out slightly to an arc of a very large circle.

Since $\boldsymbol{\lambda}'_{int}$ turns counterclockwise, its argument (in (X, Y)-coordinates) increases monotonically from $\pi/2$ to π as Θ goes from 0 to $\pi/2$. Thus as Θ increases from 0 to $N\pi$,

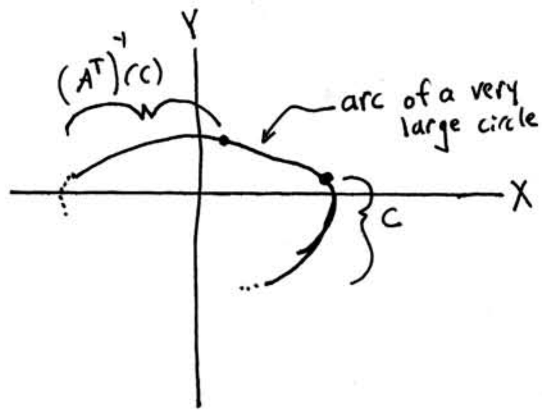

Figure 2.2: The piecewise-smooth curve which needs to be smoothed to produce $\boldsymbol{\lambda}_{int}$.

the argument of $\boldsymbol{\lambda}'_{int}$ increases by $N\pi$. The argument θ in (x, y)-coordinates also increases by $N\pi$.

We will make $\boldsymbol{\lambda}_{int}|[0, N\pi]$ into one period of $\boldsymbol{\lambda}$. Since the desired periodicity involves $t \mapsto t + 1$, we let $\Theta = N\pi t$, and define

$$\boldsymbol{\lambda}(t + k) = (A^T)^{-k}\boldsymbol{\lambda}_{int}(N\pi t), \quad k \in \mathbb{Z}, t \in [0, 1]. \tag{2.34}$$

This $\boldsymbol{\lambda}$ is smooth at $t = 0$, where a piece of C from $A^T(\boldsymbol{\lambda}_{int})$ meets a piece of C from $\boldsymbol{\lambda}_{int}$. Similar reasoning shows $\boldsymbol{\lambda}$ is smooth at all integer t. By construction, $\boldsymbol{\lambda}$ satisfies conditions 1, 2, 3, and 4a of Lemma 2.19. For 4b, note that $\boldsymbol{\lambda}'$ at $t = 0$ is (up to a rescaling) the tangent vector to C at $\Theta = 0$, which is parallel to the Y-axis, an eigenline of $(A^T)^{-1}$. Similarly, $\boldsymbol{\lambda}'(1)$ is a tangent vector to $(A^T)^{-1}(C)$ at $\Theta = N\pi$, which is also parallel to the Y-axis.

□

Chapter 3

The combinatorial chain complex $C_*(A, n, \gamma)$

This chapter introduces the combinatorial chain complex, $C_*(A, n, \gamma)$, depending on $A \in SL_2(\mathbb{Z})$, hyperbolic or $-\mathbb{1}$; n, a positive integer; and $\gamma \in \mathbb{Z}^2$. The homology $H_*(A, n, \gamma)$ is isomorphic to $ECH_*(Y_A, \lambda_n, \Gamma)$. (See Appendix A for details, including how a generator of $C_*(A, n, \gamma)$ encodes a generator of the analytical chain complex of ECH. Throughout this paper, as in [5], we pass from chain complexes to homology by changing notation from C to H.) In particular, the result is independent of the choice of representative γ of Γ (see Lemma 3.8 and Remark 3.9).

We start the chapter by changing coordinates on the plane to *eigencoordinates* of A in §3.1. We introduce the generators of the combinatorial chain complex, *labeled periodic paths*, in §3.2. We start with *polygonal paths* in §3.2.1, which are counterclockwise-turning polygonal paths in the plane with vertices in a lattice (the lattice $\mathbb{Z}^2$ in standard coordinates but denoted $L_{A,\Gamma}$ in eigencoordinates). We define (A, n, γ)*-periodic paths*, also called just *periodic paths*, in §3.2.2. We also introduce the auxiliary notion of *truncated path* in §3.2.2, essentially the result of taking just a part of a periodic path. In §3.2.3, we introduce *labeled* periodic and truncated paths, labeled with 'e' and 'h'.

The differential is introduced in §3.3, starting with its effect on unlabeled paths ("rounding a corner"), and then on 'e' and 'h' labels.

The grading is introduced in §3.4, with an existence/uniqueness proof deferred until §3.6. It follows the definition in §3.5 of the partial order $\leq$ on periodic and truncated paths, which is needed not only for §3.6 but also for the remainder of this work.

The chain complex in this chapter is the generalization (following [5, §12.2.2]) of the chain complex of [5, §2-3], with a different sign convention, among other minor details. (There is also the difference that we have chosen $\gamma \in \Gamma$, a choice which replaces a quotient operation by $\mathbb{Z}^2$ in defining $\overline{C}_*(A, n; \Gamma)$ in [5]. Remark 3.24 has more on this minor point.)

3.1 Change of coordinates

3.1.1 The lattice $L_{A,\Gamma}$ and coordinate systems (X, Y) and (x, y)

Recall that the eigenvalues of A are a and a^{-1}, where $|a| \geq 1$. Recall that $N = 2n$ if a A is positive hyperbolic ($a > 1$), and $N = 2n - 1$ if A is negative hyperbolic or $-\mathbb{1}$ ($a \leq -1$). Thus $N \geq 1$.

The generators of $C_*(A, n, \gamma)$ will be counterclockwise-turning polygonal paths in the plane, with vertices in a lattice, satisfying a periodicity condition, and with 'e' and 'h' labels on the edges. (It is only through the lattice and the periodicity condition that

(A, n, Γ) affect the chain complex. See Defintion 3.21 for the full statement of periodicity.) Writing down the lattice and the periodicity condition requires coordinates, and there are two convenient choices, related by an affine transformation:

- *Standard coordinates* (x, y), in which the lattice is $\mathbb{Z}^2$ and the periodicity condition involves
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \end{pmatrix} + \gamma, \tag{3.1}$$
which has a unique fixed point, $(\mathbb{1} - A)^{-1}\gamma$.

- *Eigencoordinates* (X, Y) in which the periodicity condition involves
$$\begin{pmatrix} X \\ Y \end{pmatrix} \mapsto \begin{pmatrix} aX \\ a^{-1}Y \end{pmatrix}, \tag{3.2}$$
and the lattice is not pretty. **We use eigencoordinates for the rest of this dissertation,** except when otherwise noted, and in particular, **we let** 0 **denote the origin** $(X, Y) = (0, 0)$**, which is in general different from** $(x, y) = (0, 0)$**. We represent angles by the polar coordinate** Θ **associated to** (X, Y). The point $(X, Y) = (0, 0)$ is the fixed point of the transformation (3.2), and corresponds to the point $(\mathbb{1} - A)^{-1}\gamma$ in standard coordinates, the fixed point of (3.1).

Although standard coordinates are close to the idea of Chapter 2, eigencoordinates are much more convenient for Chapter 4 and on, and they are just as convenient for the current chapter.

Construction of Eigencoordinates from Standard Coordinates.

- *Put the origin* $(X, Y) = 0$ *at* $(x, y) = (\mathbb{1} - A)^{-1}\gamma$.

- *If A is hyperbolic, make the X-axis parallel to an eigenvector of A with eigenvalue a ($|a| > 1$), and make the Y-axis parallel to an eigenvector with eigenvalue a^{-1}.*
- *If $A = -\mathbb{1}$, make the X-axis parallel to an eigenvector of $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ with eigenvalue a ($|a| > 1$), and make the Y-axis parallel to an eigenvector with eigenvalue a^{-1}. (See Remark 3.4.)*
- *Choose the orientation and scale of the X- and Y-axes so that the change of coordinates map, sending the pair of reals (x, y) to the (X, Y) that refers to the same point, preserves area and orientation.*

Definition 3.1. When being extra careful, we will denote the coordinates of a point p in the different systems by $x(p), y(p), X(p)$, and $Y(p)$. Normally, though, we will write $\mathbb{R}^2$ for the plane and $p = (X, Y)$.

We state the formal relation between (3.1) and (3.2):

Lemma 3.2. *By construction, for any $p = (X, Y)$, letting $q = (aX, a^{-1}Y)$, we have*

$$\begin{pmatrix} x(q) \\ y(q) \end{pmatrix} = A \begin{pmatrix} x(p) \\ y(p) \end{pmatrix} + \gamma. \tag{3.3}$$

□

Eigencoordinates are not quite unique. (For any nonzero $b \in \mathbb{R}$, we may compose with $(X, Y) \mapsto (bX, b^{-1}Y)$.)

Definition 3.3. For the rest of this work, for each (A, n, γ), fix one choice of eigencoordinate system.

The choice is just a choice of coordinates; different choices yield isomorphic chain complexes.

Remark 3.4. *We need eigencoordinates for* $-\mathbb{1}$ *to have some properties in common with hyperbolic* A*, most importantly, Proposition 3.6, Part 4. To achieve this, we have made the* X*- and* Y*-axes for* $A = -\mathbb{1}$ *a translation of those for some hyperbolic* A*, and to be definite, we chose* $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$*. (Our construction also helps with a technical issue appearing in Lemma 3.75.)*

We now consider how the lattice, $\mathbb{Z}^2$ in standard coordinates, appears in eigencoordinates.

Definition 3.5. Let the *lattice* be

$$L_{A,\Gamma} = \{p = (X, Y) \in \mathbb{R}^2 \mid (x(p), y(p)) \in \mathbb{Z}^2\}. \tag{3.4}$$

The notation $L_{A,\Gamma}$ (rather than $L_{A,\gamma}$) is justified by Lemma 3.8.

The point $(0, 0)$ referred to in the next proposition is $(X, Y) = (0, 0)$, by our conventions on coordinate systems. It is an important point, being fixed by the affine transformation (3.2), whereas $(x, y) = (0, 0)$ is unimportant.

Proposition 3.6. *1. A bounded region contains only finitely many points of* $L_{A,\Gamma}$*.*

2. The lattice is invariant under $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$*:*

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} (L_{A,\Gamma}) = L_{A,\Gamma}. \tag{3.5}$$

3. The point $(0, 0)$ *is in* $L_{A,\Gamma}$ *if and only if* $\Gamma = 0$*.*

4. *No two points in $L_{A,\Gamma}$ have the same X-coordinate and no two have the same Y-coordinate.*

5. *No point on the axes, $\{X = 0\} \cup \{Y = 0\}$, except possibly $(0, 0)$, is in $L_{A,\Gamma}$.*

Proof.

1. Discreteness is unaffected by a change of coordinates.

2. This claim is a restatement of the invariance of $\mathbb{Z}^2$ under (3.1), which in turn follows because $A \in SL_2(\mathbb{Z})$.

3. The point $(X, Y) = (0, 0)$ is the point $(x, y) = (\mathbb{1} - A)^{-1}\gamma$, the fixed point of (3.3). Let $CM = (\mathbb{1} - A)^{-1}\gamma$. ("$CM$" for center of mass.) We have $(0, 0) \in L_{A,\Gamma}$ if and only if $CM \in \mathbb{Z}^2$. Since $\gamma = (\mathbb{1} - A)(CM)$, $CM \in \mathbb{Z}^2$ if and only if $\gamma \in (\mathbb{1} - A)(\mathbb{Z}^2)$; since $\Gamma = [\gamma] \in \mathbb{Z}^2/\operatorname{Im}(\mathbb{1} - A)$, we have $\gamma \in (\mathbb{1} - A)(\mathbb{Z}^2)$ if and only if $\Gamma = 0$.

4. We state the proof only for the X-coordinate. The idea is that the X-axis, seen in (x, y)-coordinates, has irrational slope and therefore cannot contain two points of $\mathbb{Z}^2$.

   We must show no two distinct $p, q \in \mathbb{Z}^2$ lie on a line parallel to the X-axis. If they were on such a line, we would have $p - q$ parallel to the X-axis, so $p - q$ would be an eigenvector of A of eigenvalue a, by the construction of the X-axis. Then,

$$A^{-k}(p - q) = a^{-k}(p - q), \quad k = 1, 2, 3, \ldots \tag{3.6}$$

   would all lie in $\mathbb{Z}^2$, since A preserves $\mathbb{Z}^2$. This contradicts discreteness of $\mathbb{Z}^2$.

5. Again, the idea is the irrational slope of the axes. (We may consider just the X-axis.) An axis cannot contain both $(\mathbb{1} - A)^{-1}\gamma \in \mathbb{Q}^2$ and a lattice point. The formal proof

is easiest in eigencoordinates: If $(X, 0) \in L_{A,\Gamma}$, $X \neq 0$, then

$$\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)^{-k} \begin{pmatrix} X \\ \\ 0 \end{pmatrix} = a^{-k} X \in L_{A,\Gamma}, \tag{3.7}$$

contradicting discreteness of $L_{A,\Gamma}$.

□

Remark 3.7. *The details of what $L_{A,\Gamma}$ looks like (in (X, Y) coordinates) are very unimportant; for most of this work, it may as well be an irregular smattering of points, not even a lattice. It seems likely that the homology coming from any $L \subset \mathbb{R}^2$ satisfying a few properties is isomorphic to the homology $H_*(A, n, \gamma)$ we calculate in this work, with the condition $\Gamma = 0$ in the description of $H_*(A, n, \gamma)$ replaced by $0 \in L$. The properties needed are that L is a discrete subset of $\mathbb{R}^2$, satisfying Proposition 3.6, and a result that the points are "dense enough", Lemma 3.75. (There is an inessential use of the fact that $L_{A,\Gamma}$ is a lattice in Lemma-Definition 3.73.)*

Lemma 3.8. *The lattice $L_{A,\Gamma}$ (seen in eigencoordinates) is independent of the choice of representative γ of $\Gamma \in \mathbb{Z}^2/\operatorname{Im}(\mathbb{1} - A)$.*

Proof. If we have two representatives γ, γ' then $\gamma' = \gamma + (\mathbb{1} - A)v$ for some $v \in \mathbb{Z}^2$. Eigencoordinates for γ' will be centered on the fixed point of $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto A\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + \gamma'$, which is $(\mathbb{1} - A)^{-1}\gamma' = (\mathbb{1} - A)^{-1}\gamma + v$. The entire eigencoordinate system will be translated by v, as seen from standard coordinates; since $v \in \mathbb{Z}^2$, the translation leaves the coordinate system in the same place relative to the points of $\mathbb{Z}^2$. □

Remark 3.9. *By Lemma 3.8, the choice of representative γ of Γ does not affect the chain complex $C_*(A, n, \gamma)$.*

Seen from the point of view of eigencoordinates, different choices of γ yield the *same* chain complex, while from the point of view of standard coordinates, they yield isomorphic chain complexes related by a translation. (The only effect of (A, n, γ) on $C_*(A, n, \gamma)$ is through the lattice, the map $F_{A,n}$, and the affine transformation (3.1) or (3.2).) The name $C_*(A, n, \gamma)$ rather than $C_*(A, n, \Gamma)$ is chosen from the standard coordinates' point of view.

3.1.2 Angles in eigencoordinates

In eigencoordinates, we use the coordinate Θ for angles when making statements such as "$(X, Y) = r(\cos\Theta, \sin\Theta)$ with $r > 0$." This coordinate Θ parameterizes a certain line, abstractly the cover of the S^1 of directions in $\mathbb{R}^2$, the same line parameterized as θ in standard coordinates.

Recall that the term *angle* always refers to elements of $\mathbb{R}$, not S^1. We sometimes emphasize this convention by calling angles *lifted angles*.

The new coordinates have the virtue that an angle "pointing along an eigenline" of A (as in Chapter 2) points along an axis, i.e. $\Theta = k\pi/2$, $k \in \mathbb{Z}$. We have no need of θ in the rest of this work, once we describe the map on lifted angles $f_{A,n}(\theta)$ in eigencoordinates as $F_{A,n}(\Theta)$. The following proposition states all we need to know about $F_{A,n}$.

Proposition 3.10. *There exists a unique smooth $F_{A,n} \colon \mathbb{R} \to \mathbb{R}$ (for each A and n) such that*

1. $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} \cos\Theta \\ \sin\Theta \end{pmatrix}$ *is a positive real multiple of* $\begin{pmatrix} \cos F_{A,n}(\Theta) \\ \sin F_{A,n}(\Theta) \end{pmatrix}$.

2. $F_{A,n}(k\pi/2) = k\pi/2 + N\pi$ *for* $k \in \mathbb{Z}$.

3. $F_{A,n}(\Theta) > \Theta$.

4. $F'_{A,n}(\Theta) > 0$.

Proof. Condition 1 states that $F_{A,n}$ is a lift to $\mathbb{R} = \widetilde{S^1}$ of the map induced by $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ on $S^1 \subset \mathbb{R}^2$. (In the notation of §2.2, it is a lift of $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)_{S^1}$.) It is unique once Condition 2 chooses a value for, say, $F_{A,n}(0)$. For existence of $F_{A,n}$, we refer to the results on $f_{A,n}$ in §2.2, restating those results in eigencoordinates: Lemma-Definition 2.13 gives us Conditions 1 and 3, Lemma 2.15 gives us Condition 2, and Condition 4 states that $F_{A,n}$ preserves orientation on $\mathbb{R}$, which follows because the action of $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)_{S^1}$ on S^1 preserves orientation. □

Notation Convention 3.11. For $\Theta \in \mathbb{R}$, let $\vec{\Theta} = \left(\begin{smallmatrix} \cos\Theta \\ \sin\Theta \end{smallmatrix}\right)$.

Remark 3.12. *We describe the relation between these coordinate systems and* $Y_A \cong T^2 \times \mathbb{R}/\sim$ *from Chapter 2. The affine plane in which polygonal paths will live is* $\widetilde{T^2}$*, with its quotient* T^2 *being* $\mathbb{R}^2/\mathbb{Z}^2$ *in standard coordinates and* $\mathbb{R}^2/L_{A,\Gamma}$ *in eigencoordinates. Letting* V *be the vector space associated to the affine space* $\widetilde{T^2}$*, the set of directions in* V *forms* $(V \setminus \{0\})/\mathbb{R}_+$*; let* $\widetilde{\mathbb{P}_+}(V)$ *denote the universal cover of this circle, with coordinate* Θ *in eigencoordinates and* θ *in standard coordinates. Then* $Y_A \cong T^2 \times \widetilde{\mathbb{P}_+}(V)/\phi$ *where*

- *In standard coordinates,* ϕ *acts on* T^2 *by* $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto A\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$*, and acts on* $\widetilde{\mathbb{P}_+}(V)$ *by* $f_{A,n}$*.*
- *In eigencoordinates,* ϕ *acts on* T^2 *by* $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)$*, and acts on* $\widetilde{\mathbb{P}_+}(V)$ *by* $F_{A,n}$*.*

In particular, Y_A *is a bundle over the circle* $\mathbb{R}/f_{A,n}$ *or* $\mathbb{R}/F_{A,n}$*, which is* not *the circle* $(V \setminus \{0\})/\mathbb{R}_+$*. A polygonal path in the plane will encode data about a collection of Reeb orbits (see Appendix A for details), with the periodicity condition on paths involving* $F_{A,n}$ *and* $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$*, ensuring that the Reeb orbits lie in* Y_A*, not just* $T^2 \times \widetilde{\mathbb{P}_+}(V)$*.*

3.2 Labeled periodic paths, the generators of $C_*(A, n, \gamma)$

3.2.1 Polygonal paths

The kind of path we are really interested in is a *periodic path*. As a warm-up, we formalize the notion of a *polygonal path*, a counterclockwise-turning polygonal path in $\mathbb{R}^2$ with vertices in $L_{A,\Gamma}$. (We are using eigencoordinates (X, Y) on $\mathbb{R}^2$.)

A polygonal path is sequence of directed line segments, each associated with an angle $\Theta \in \mathbb{R}$. A directed line segment automatically has a direction $\Theta \in S^1$, but we choose a lift in $\mathbb{R}$.

To be sure of conventions, we define some fairly standard terms.

Definition 3.13. A *directed line segment* in $\mathbb{R}^2$ is a line segment such that one endpoint is declared the *beginning* and the other endpoint (which is a different point) is the *end*.

Definition 3.14. A subset $\mathcal{A}$ of $\mathbb{R}$ is *discrete* if $\mathcal{A} \cap B$ is finite for any bounded $B \subset \mathbb{R}$. If $\Theta_1, \Theta_2 \in \mathcal{A}$ with $\Theta_1 < \Theta_2$, we say Θ_2 is the *next* element of $\mathcal{A}$ after Θ_1, or equivalently Θ_1 is the *previous* element before Θ_2, if $\mathcal{A} \cap (\Theta_1, \Theta_2)$ is empty.

If $\mathcal{A}$ is discrete and $\Theta \in \mathcal{A}$ is not the largest element, there is a next element, and similarly for previous.

Definition 3.15. A *non-constant polygonal path* is a non-empty set Λ of pairs (σ, Θ), called *edges*, such that

1. Each σ is a directed line segment in $\mathbb{R}^2$, Θ is an angle in $\mathbb{R}$, and the vector from the beginning of σ to the end is a positive real multiple of $(\cos\Theta, \sin\Theta)$. We denote σ by $\mathrm{Edge}_\Lambda(\Theta)$.

2. The set of Θ appearing in Λ, denoted $\mathrm{Ang}(\Lambda)$, is discrete.

3. If $\Theta_1, \Theta_2 \in \mathrm{Ang}(\Lambda)$ and Θ_2 is the next element of $\mathrm{Ang}(\Lambda)$ after Θ_1, the end of $\mathrm{Edge}_\Lambda(\Theta_1)$ is the beginning of $\mathrm{Edge}_\Lambda(\Theta_2)$.

4. A *vertex* is the beginning or end of any σ appearing in Λ. We require each vertex to lie in the lattice $L_{A,\Gamma}$.

(One could regard Λ as a function from $\mathrm{Ang}(\Lambda)$ to edges, but we denote that function by $\mathrm{Edge}_\Lambda(\cdot)$ because we are reserving $\Lambda(\cdot)$ for Definition 3.18.)

The empty set would satisfy Definition 3.15 if we had not specifically excluded it.

Definition 3.16. A *polygonal path* Λ is either a non-constant polygonal path or is the *constant path at* p, for some $p \in L_{A,\Gamma}$:

The constant path at p, denoted const_p, is completely specified by the data of a point $p \in L_{A,\Gamma}$. We say p is the only *vertex* of const_p, and $\mathrm{Ang}(\mathrm{const}_p)$ is the empty set.

"Counterclockwise-turning" is the one part of the intuitive description of these paths not apparent from the definition. But Θ is strictly increasing as we travel along the path, because the definition joins the edge at Θ with the edge at the next-smallest Θ. Often successive Θ's differ by $\leq \pi$, and the path turns counterclockwise. Otherwise, we say the path has a *kink*:

Definition 3.17. A *corner* of a polygonal path Λ is a connected component of $\mathbb{R} \setminus \mathrm{Ang}(\Lambda)$. A *proper corner* is a corner of the form (Θ_1, Θ_2) with $\Theta_2 - \Theta_1 \leq \pi$. If $c = (\Theta_1, \Theta_2)$ with $\Theta_2 - \Theta_1 > \pi$, or if $c = \mathbb{R}$, we say c is a *kink*.

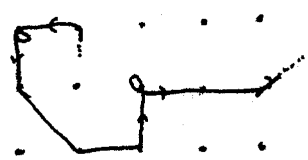

Figure 3.1: Example of a polygonal path. The picture shows a finite portion of an infinite path. Two kinks are shown. The picture is drawn with respect to standard (x, y)-coordinates rather than eigencoordinates (X, Y).

A corner lives in $\mathbb{R}$ and a vertex in $\mathbb{R}^2$. A corner of the form $(-\infty, \Theta)$ or (Θ, ∞) is not proper (and is not important). If Λ has a kink, we illustrate this by drawing a small loop. Figure 3.1 illustrates kinks with $\Theta_2 - \Theta_1 = 3\pi/2$ and $5\pi/2$; $\Theta_2 - \Theta_1$ can get arbitrarily large. The constant path has a kink.

Figure 3.1 also shows an edge that is not "primitive"–it has a lattice point on its interior. It is one edge of Λ, not two.

We may also think about a polygonal path as a piecewise-constant parameterized curve. We think of Θ as angle and as time; the path is "parameterized by angle". Imagine a point in the plane, usually staying at a lattice point, but sometimes jumping, at time Θ, in the direction specified by Θ.

A typical $\Theta \notin \operatorname{Ang}(\Lambda)$ lies in some (Θ_1, Θ_2) where Λ has edges at Θ_1 and Θ_2, the end of one edge being the same point as the beginning of the other; we let $\Lambda(\Theta)$ denote this point:

Definition 3.18. If Λ is a polygonal path, define a map, the *parameterization*, abusively also denoted Λ,

$$\Lambda\colon \mathbb{R} \setminus \operatorname{Ang}(\Lambda) \to L_{A,\Gamma}, \tag{3.8}$$

as follows: Suppose $\Theta \in \mathbb{R} \setminus \mathrm{Ang}(\Lambda)$, and c is the corner of Λ containing Θ.

1. If Λ is the constant path at p, $\Lambda(\Theta) = p$.

2. If Λ is non-constant, Λ has a least one edge at an endpoint Θ' of the interval c. If $\Theta' > \Theta$, let $\Lambda(\Theta)$ be the beginning of $\mathrm{Edge}_\Lambda(\Theta')$. If $\Theta' < \Theta$, let $\Lambda(\Theta)$ be the end of $\mathrm{Edge}_\Lambda(\Theta')$.

The point $\Lambda(\Theta)$ is also denoted $\Lambda(c)$ and called the *vertex of the corner*.

If, in part 2 of the above definition, $c = (\Theta_1, \Theta_2)$, we may use either Θ_1 or Θ_2 as Θ', obtaining the same point $\Lambda(\Theta)$.

Lemma 3.19. *A polygonal path* Λ *is determined by its parameterization.*

Proof. The complement of the domain of the parameterization (alternately, its discontinuities) gives $\mathrm{Ang}(\Lambda)$. If $\mathrm{Ang}(\Lambda) = \emptyset$, then $\Lambda = \mathrm{const}_p$, where $p = \Lambda(\Theta)$ (for any Θ). Otherwise, for each $\Theta \in \mathrm{Ang}(\Lambda)$, $\mathrm{Edge}_\Lambda(\Theta)$ is the line segment beginning at $\Lambda(\Theta - \epsilon)$ and ending at $\Lambda(\Theta + \epsilon)$ for a sufficiently small ϵ. □

Remark 3.20. *A polygonal path may be given a definition equivalent to the one we have given, as a map* $\mathbb{R} \setminus (\textit{discrete set}) \to L_{A,\Gamma}$*, which is constant on connected components and which jumps in the direction* $(\cos\Theta, \sin\Theta)$*.*

3.2.2 Periodic and truncated paths

We will now impose a "periodicity" condition. Recall that $\Lambda(\Theta)$ is defined if and only if $\Theta \notin \mathrm{Ang}(\Lambda)$.

Definition 3.21. We say that a polygonal path Λ is (A,n,γ)-*periodic*, and we call it a *periodic path* or a periodic polygonal path, if for any $\Theta \in \mathbb{R}$:

1. If $\Theta \in \mathrm{Ang}(\Lambda)$ then $F_{A,n}(\Theta) \in \mathrm{Ang}(\Lambda)$.

2. If $\Theta \notin \mathrm{Ang}(\Lambda)$ then $F_{A,n}(\Theta) \notin \mathrm{Ang}(\Lambda)$ and

$$\Lambda(F_{A,n}(\Theta)) = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \Lambda(\Theta). \tag{3.9}$$

Periodic paths generalize what are called closed admissible paths and periodic admissible paths in [5].

Abstractly, if Λ is periodic, the parameterization is equivariant under the action of the infinite cyclic group on $\mathbb{R} \setminus \mathrm{Ang}(\Lambda)$ by $F_{A,n}$ and on the lattice $L_{A,\Gamma}$ by $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$.

We see that the definition of (A,n,γ)-periodic path depends on A, n, and γ only through a, $F_{A,n}$, and $L_{A,\Gamma}$. In particular, only Γ matters, not its representative γ. It is an advantage of eigencoordinates (X,Y) that independence from γ is clear. The name "(A,n,γ)-periodic" has been chosen for compatibility with standard coordinates, in which the periodicity condition involves $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto A\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + \gamma$ in place of $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)$.

The definition implies that $\mathrm{Ang}(\Lambda)$ is invariant under $F_{A,n}$, and so is the set of corners.

Lemma 3.22. *An (A,n,γ)-periodic path is determined by the restriction of the parameterization $\Theta \mapsto \Lambda(\Theta)$ to $[0, N\pi)$, or more generally to any interval $[\Theta_0, F(\Theta_0))$. The path is also determined by the set of edges (σ, Θ) with $\Theta \in [\Theta_0, F_{A,n}(\Theta_0))$.*

We call that $F_{A,n}$ is monotonically increasing (Proposition 3.10, part 4).

Proof. We know $\mathbb{R}$ is the disjoint union of $(F_{A,n})^k[\Theta_0, F_{A,n}(\Theta_0))$, $k \in \mathbb{Z}$, so the restriction of the parameterization determines the parameterization. The restriction of the set of edges determines the restriction of the parameterization. □

It follows that a periodic polygonal path has an infinite or empty set of edges.

Remark 3.23. *We define* (A,n,γ)*-periodic paths only for* A *hyperbolic or* $-\mathbb{1}$*, but the definition generalizes straightforwardly for all* $A \in SL_2(\mathbb{Z})$*. The case* $A = \mathbb{1}$ *recovers the paths in the* T^3 *[5]: For instance, an* $(\mathbb{1},1,0)$*-periodic path is a convex polygon. An* $(\mathbb{1},2,0)$*-periodic path goes around* 4π *before closing up. A* $\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right),1,0\right)$*-periodic path satisfies*

$$\Lambda(\Theta + \pi/2) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \Lambda(\Theta). \tag{3.10}$$

Remark 3.24. *The straightforward generalization of Definition 3.21 to all* $A \in SL_2(\mathbb{Z})$*, combined with definitions we will make below, yields a chain complex* $C_*(A,n,\gamma)$ *whose homology is the ECH of the appropriate manifold* Y_A *provided* $A - \mathbb{1}$ *is a non-singular matrix. The definition is best stated in standard coordinates using* $A\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + \gamma$ *in place of* $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)$.

If $A - \mathbb{1}$ *is singular, the set of* (A,n,γ)*-periodic paths has a translation symmetry by which we must quotient. (In particular, note that* $A\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + \gamma$ *does not have a unique fixed point.) For all* $A \in SL_2(\mathbb{Z})$*, the ECH of* Y_A *is the homology of*

$$\overline{C}_*(A,n,\Gamma) = \left(\bigoplus_{\gamma\in\Gamma} C_*(A,n,\gamma)\right)/\mathbb{Z}^2, \tag{3.11}$$

where $\mathbb{Z}^2$ *acts by translating paths.*

Periodicity for $A = -\mathbb{1}$ is exactly $\Lambda(\Theta + N\pi) = -\Lambda(\Theta)$ (with $N = 2n-1$). One may think of the path as symmetric under "rotation by $N\pi$". For instance, a $(-\mathbb{1},1,\gamma)$-

periodic path is a convex polygon symmetric under rotation by π about $(0, 0)$. (The only effect of γ, or of Γ, is the relative placement of the lattice and the origin.)

Periodicity for A hyperbolic is harder to state concisely. One consequence is that as $\Theta \to +\infty$, a nonconstant periodic path expands in the X direction and shrinks in the Y direction, tending to the X-axis, and oppositely as $\Theta \to -\infty$. (We are using Proposition 3.6, Part 4 to say that a nonconstant path is nontrivial in both the X- and Y-directions.) The only constant periodic path is const_0, and only if $0 \in L_{A,\Gamma}$.

Definition 3.25. Any interval $[\Theta_0, F_{A,n}(\Theta_0))$ is a *fundamental domain.*

Example 3.26. The interval $[k\frac{\pi}{2}, k\frac{\pi}{2} + N\pi)$ is a fundamental domain for any $k \in \mathbb{Z}$. We will often refer to $[k\frac{\pi}{2}, k\frac{\pi}{2} + N\pi]$ as a fundamental domain–the distinction is irrelevant because no multiple of $\pi/2$ can belong to $\text{Ang}(\Lambda)$ (by Proposition 3.6, Part 4).

The discreteness of $\text{Ang}(\Lambda)$ implies that there are finitely many edges with lifted angle in $[0, N\pi]$, which is good–a path is determined by a finite amount of data. We will sometimes need to work with only the part of Λ in $[\Theta_0, F_{A,n}(\Theta_0))$, or sometimes some other interval. Thus, it is useful to have a notion, *truncated path*, with Θ restricted to an interval. (For convenience, we define truncated paths on closed intervals only.) Intervals appearing in this work will essentially always have endpoints at angles generic with respect to $L_{A,\Gamma}$, angles that cannot have an edge; for instance, $k\pi/2$.

Definition 3.27. A *truncated path*, also called a truncated polygonal path, is a pair $(\Lambda, [\Theta_0, \Theta_1])$, where Λ is a polygonal path and $\Theta_0 < \Theta_1$ are reals, such that $\text{Ang}(\Lambda) \subseteq (\Theta_0, \Theta_1)$. Abusively, we call the truncated path Λ.

We write $\overline{\mathrm{Dom}}(\Lambda) = [\Theta_0, \Theta_1]$, the *domain* of Λ, and we say that the truncated path Λ is *defined on* $\overline{\mathrm{Dom}}(\Lambda)$. Most definitions from polygonal paths carry over to truncated paths. In particular, *proper corners* of a truncated path are proper corners of the underlying polygonal path; those corners already lie in $[\Theta_0, \Theta_1]$. A *kink* of a truncated path is a kink of the underlying polygonal path that is contained in $[\Theta_0, \Theta_1]$. We do not care about corners with Θ_0 or Θ_1 as endpoints.

We define the *parameterization* of a truncated path to be the restriction to $\overline{\mathrm{Dom}}(\Lambda) \setminus \mathrm{Ang}(\Lambda)$ of the parameterization $\mathbb{R} \setminus \mathrm{Ang}(\Lambda) \to L_{A,\Gamma}$ of the underlying polygonal path. In particular, $\Lambda(\Theta_0)$ is the *beginning* of the truncated path and $\Lambda(\Theta_1)$ is the *end*. For concreteness, for a periodic path Λ, let $\overline{\mathrm{Dom}}(\Lambda) = \mathbb{R}$.

The interval $\overline{\mathrm{Dom}}(\Lambda)$ is not quite the domain of the parameterization Λ–it is the closure in $\mathbb{R}$ of the domain of

$$\Lambda\colon\ \overline{\mathrm{Dom}}(\Lambda) \setminus \mathrm{Ang}(\Lambda) \to L_{A,\Gamma}. \tag{3.12}$$

A periodic path is a kind of polygonal path and a truncated path is a polygonal path equipped with extra data.

Proposition 3.28. *A truncated path is equivalent to what is called an open admissible path in [5, Definition 2.1].*

We need this equivalence in order to make use of several results on open admissible paths.

Proof. The differences between the two definitions are as follows:

1. Open admissible paths are defined in terms of a parameterization. We have how to define a parameterization from the set of edges (Definition 3.18), and how to recover the set of edges (Lemma 3.19). (In particular, it is unnecessary to make a special provision for the constant path in the definition of open admissible path.)

2. Open admissible paths are defined in standard coordinates (x, y); in particular, it is part of the definition that $\tan\theta \in \mathbb{Q} \cup \{\infty\}$ (written "$T \subset \Theta$"). For truncated paths, this fact is a corollary.

3. The definition of an open admissible path includes a function $m\colon\ \mathrm{Ang}(\Lambda) \to \mathbb{Z}_{>0}$ measuring the size of the jump $\Lambda(\Theta+\epsilon)-\Lambda(\Theta-\epsilon)$ as a multiple of the smallest element of $\mathbb{Z}^2$ lying in that direction. This function may be recovered from the parameterization.

□

Definition 3.29. If Λ is a polygonal path and $\Theta_0, \Theta_1 \in \mathbb{R}$ with $\Theta_0 < \Theta_1$, let $\Lambda|[\Theta_0, \Theta_1]$ be the set of all edges (σ, Θ) of Λ such that $\Theta \in [\Theta_0, \Theta_1]$, together with the interval $[\Theta_0, \Theta_1]$ (forming a truncated path). If that set of edges is empty, the resulting constant path is $\mathrm{const}_{\Lambda(\Theta)}$ for any $\Theta \in [\Theta_0, \Theta_1]$.

Definition 3.30. If Λ is an (A, n, γ)-periodic path, we call $\Lambda|[\Theta_0, F_{A,n}(\Theta_0)]$ a *period* of Λ, provided $\Theta_0 \notin \mathrm{Ang}(\Lambda)$.

Then Λ is determined by any period of Λ.

3.2.3 Labeling paths with 'e' and 'h'

The next step is to give each edge of a period path the symbol 'e' or 'h' while respecting periodicity. (We have no need for a notion of labeled polygonal path.) These letters stand for "elliptic" and "hyperbolic" and describe different types of Reeb orbits (not with different types of matrices A). To get the signs right, we must also choose an ordering on the 'h' edges modulo periodicity, i.e. modulo the action of the infinite cyclic group generated by $F_{A,n}$. Here an ordering is an ordinary total ordering $\leq$ on the set. We draw the label on the edge, but more formally it is associated to the angle $\Theta \in \mathrm{Ang}(\Lambda)$, and to be entirely correct, the label is associated to an element of $\mathrm{Ang}(\Lambda)/F_{A,n}$.

Definition 3.31. A *labeled periodic path* (or *labeled* (A, n, γ)*-periodic path*) is a triple (Λ, ℓ, o), where

1. Λ is an (A, n, γ)-periodic path (the *underlying path*),
2. the *raw labeling* ℓ is a partition

$$\mathrm{Ang}(\Lambda)/F_{A,n} = \ell_e \cup \ell_h \tag{3.13}$$

   as a disjoint union and
3. o is an ordering on the set ℓ_h.

The pair (ℓ, o) is called the *labeling.* If $\Theta \in \mathrm{Ang}(\Lambda)$, we write $[\Theta]$ for its image in $\mathrm{Ang}(\Lambda)/F_{A,n}$, and say Θ *is labeled 'e'* or $[\Theta]$ *is labeled 'e'* if $[\Theta] \in \ell_e$, and similarly for 'h'.

We now define the underlying abelian group of the combinatorial chain complex. It is the free $\mathbb{Z}$-module generated by labeled periodic paths, modulo a sign.

Definition 3.32. Let

$$C_*(A, n, \gamma) = \mathbb{Z}\{\text{all labeled } (A, n, \gamma)\text{-periodic paths}\}/\sim, \tag{3.14}$$

where $(\Lambda, \ell, o) \sim (\Lambda, \ell, o')$ if o' is obtained from o by an even permutation of ℓ_h, and $(\Lambda, \ell, o) \sim -(\Lambda, \ell, o')$ if it is an odd permutation.

Once we introduce δ and a grading, we will be able to call $C_*(A, n, \gamma)$ a chain complex.

Definition 3.33. A *labeled truncated path* is a triple $(\Lambda_{\text{tr}}, \ell, o)$, where Λ_{tr} is a truncated path, the *raw labeling* ℓ is a partition

$$\text{Ang}(\Lambda_{\text{tr}}) = \ell_e \cup \ell_h \tag{3.15}$$

as a disjoint union, and o is an ordering on the set ℓ_h. The pair (ℓ, o) is called the *labeling*. We say Θ *is labeled 'e'* if $\Theta \in \ell_e$, and similarly for 'h'.

Definition 3.34. Let

$$C_*^{\text{tr}}(I) = \mathbb{Z}\{\text{all truncated paths with } \overline{\text{Dom}}(\Lambda_{\text{tr}}) = I\}/\sim, \tag{3.16}$$

where $(\Lambda_{\text{tr}}, \ell, o) \sim (\Lambda_{\text{tr}}, \ell, o')$ if o' is obtained from o by an even permutation of ℓ_h, and $(\Lambda_{\text{tr}}, \ell, o) \sim -(\Lambda_{\text{tr}}, \ell, o')$ if o' is obtained from o by an odd permutation.

We introduce $C_*^{\text{tr}}(I)$ for the sake of certain subcomplexes of $C_*^{\text{tr}}(I)$ in Definition 4.7.

Notation Convention 3.35 (Abusive)**.** We write const_p for any constant path at p (polygonal, truncated, periodic; labeled or unlabeled).

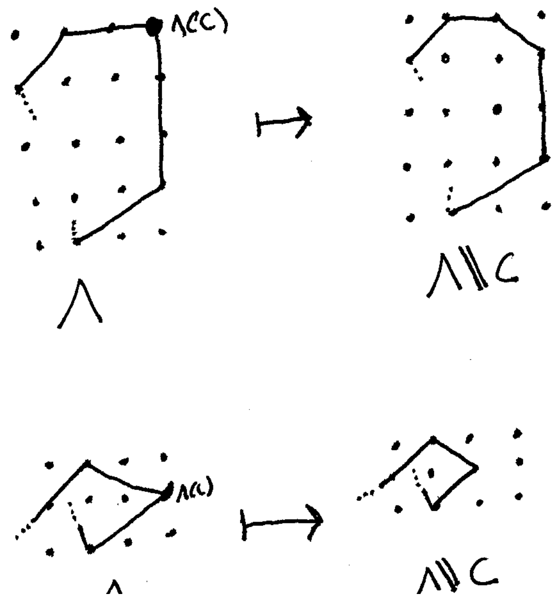

Figure 3.2: Two examples of rounding corner of a polygonal path.

3.3 The differential δ

In this section, we introduce the differential $\delta\colon\ C_*(A, n, \gamma) \to C_*(A, n, \gamma)$.

The action of δ on a labeled periodic path is best described by starting with its action on the underlying (unlabeled) path. To that end, §3.3.1 defines rounding a polygonal path at a corner, and then defines rounding a periodic path (not just at c but at all $(F_{A,n})^k(c)$). In §3.3.3 we define the effect on the labeling of rounding at c, which is to lose exactly one 'h' near c and leave the rest of the labeling untouched. The differential δ is then a sum over all corners.

3.3.1 Rounding a corner of a polygonal or truncated path

We introduce the operation of *rounding (without periodicity) a corner of a path*, which takes a polygonal path Λ and a proper corner c and produces another polygonal path denoted $\Lambda\backslash\backslash c$. It is helpful to begin the explanation with a picture. Figure 3.2 illustrates some paths Λ, distinguishes a vertex c for each, and shows the result of rounding Λ at c. (Someone walking along Λ has cut a corner.) The rounded path $\Lambda\backslash\backslash c$ "encloses" exactly one point fewer than Λ. A helpful image (told to me by Michael Hutchings) is to stretch a rubber band along the path Λ, after putting a nail in each lattice point, nails which hold Λ taut. Plucking out the nail at $\Lambda(c)$ causes the rubber band to snap into a new configuration, $\Lambda\backslash\backslash c$. (Small plastic boards, *geoboards*, with nonremovable nails, illustrate some of this; and Java applets exist.)

Recall that a corner c is an open interval of angles. We will need to use the closed interval, $\bar{c}$. Figure 3.2 illustrates that rounding at c only affects the two edges adjacent to c. (The path is unaffected outside $\bar{c}$.) The rounded path shortens those edges or removes them, and inserts any number of new edges.

The formal definition works by deleting $\Lambda(c)$ from the set of lattice points "enclosed" by the two edges adjacent to c. The points "enclosed" are those in the triangle with vertices $p = \Lambda(\Theta_1 - \epsilon)$, $\Lambda(c) = \Lambda(\Theta_1 + \epsilon) = \Lambda(\Theta_2 - \epsilon)$, and $q = \Lambda(\Theta_2 + \epsilon)$. This "triangle" will be a 2-gon if $p = q$ (though we always have $p \neq \Lambda(c)$, $\Lambda(c) \neq q$).

Definition 3.36. Let Λ be a polygonal path and $c = (\Theta_1, \Theta_2)$ a proper corner of Λ. We define $\Lambda\backslash\backslash c$, the result of *rounding (without periodicity)* Λ *at* c, as follows:

1. Let p be the beginning of $\mathrm{Edge}_\Lambda(\Theta_1)$ and q the end of $\mathrm{Edge}_\Lambda(\Theta_2)$.

2. Let $\mathcal{L}$ be the set of points of $L_{A,\Gamma}$ contained in the convex hull of p, $\Lambda(c)$, and q.

3. The convex hull of $\mathcal{L} \setminus \{\Lambda(c)\}$ is some polygon P. Traverse its boundary counterclockwise starting at p and ending at q, forming a set $\mathcal{E}$ of directed line segments.

   (If P is a point, this set $\mathcal{E}$ is empty. If P is a 2-gon and $p \neq q$, $\mathcal{E}$ is the edge from p to q. If P is a 2-gon and $p = q$, $\mathcal{E}$ is an edge from p to the other vertex of p, and an edge from there to q.)

4. Equip each directed line segment $\sigma \in \mathcal{E}$ with an angle Θ, so that the vector from the beginning of σ to the end is a positive real multiple of $(\cos\Theta, \sin\Theta)$ (see Definition 3.15, Part 1), and resolve the $2\pi\mathbb{Z}$ ambiguity by requiring $\Theta \in \bar{c} = [\Theta_1, \Theta_2]$.

5. We have formed a set of pairs (σ, Θ). Let $\Lambda\backslash\backslash c$ be the union of that set and $\{(\sigma, \Theta) \in \Lambda \mid \Theta \notin \bar{c}\}$.

   (If this construction results in an empty set of edges, let $\Lambda\backslash\backslash c = \mathrm{const}_p (= \mathrm{const}_q)$.)

The polygon P has an edge from q to p (if $q \neq p$) that is not included in $\Lambda\backslash\backslash c$.

Lemma 3.37. *For all* $\Theta \in c$, $\Lambda(\Theta) \neq (\Lambda\backslash\backslash c)(\Theta)$ *when both expressions are defined.*

Proof. By construction, $\Lambda\backslash\backslash c$ never touches the vertex $\Lambda(c)$, at least for $\Theta \in c$. (It may, of course, loop around and reach $\Lambda(c)$ for some Θ far from $\bar{c}$.) □

Lemma 3.38. *The parameterizations* $\Lambda(\Theta)$ *and* $(\Lambda\backslash\backslash c)(\Theta)$ *agree outside* $\bar{c}$.

Proof. This is true by construction, separately for nonconstant and constant $\Lambda\backslash\backslash c$. □

Rounding a truncated path Λ_{tr} works the same way as rounding a polygonal path. The term "proper corner" of Λ_{tr} rules out a corner at either end of the interval $\overline{\mathrm{Dom}}(\Lambda_{\mathrm{tr}})$.

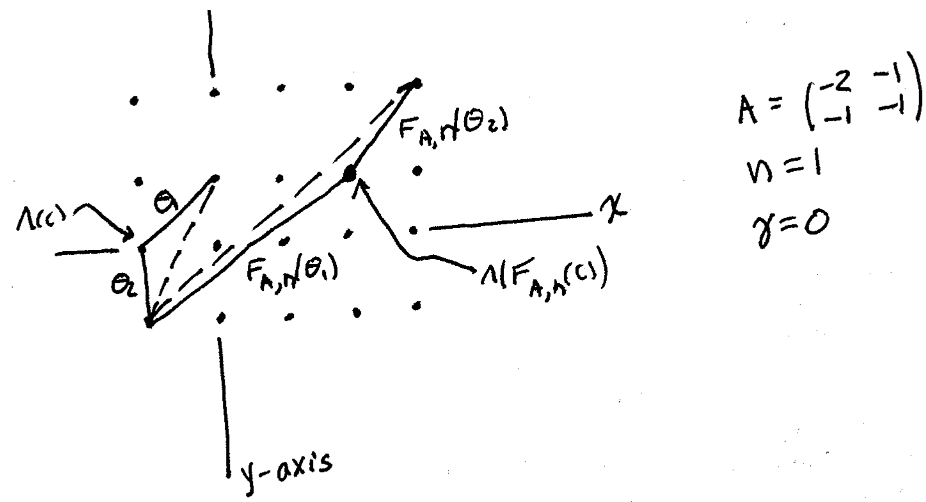

Figure 3.3: Example of rounding a corner of a periodic path. Drawn in standard coordinates (x, y). The solid lines are Λ and the dashed lines are $\Lambda \setminus [c]$. Only two periods of each path are shown. The paths continue infinitely in both directions.

Definition 3.39. Let $\Lambda_{\text{tr}} = (\Lambda, \overline{\text{Dom}}(\Lambda))$ be a truncated path, and c a proper corner of Λ_{tr}. We define $\Lambda_{\text{tr}} \setminus c$, the result of *rounding* Λ_{tr} *at* c, to be $(\Lambda \backslash\backslash c, \overline{\text{Dom}}(\Lambda))$.

Lemma 3.40. *The notion of rounding a truncated path matches [5]'s notion of rounding an open admissible path.* □

3.3.2 Rounding a corner of a periodic path

Now we consider rounding a corner of a periodic path. We would like the result to be periodic also. To that end, when we round at a corner c, we round at all of the corners $F_{A,n}^k(c)$, $k \in \mathbb{Z}$. See Figure 3.3. There is one potential confusion: we forbid *self-rounding*, which is when Λ has only one edge (modulo $F_{A,n}$), i.e. $\text{Ang}(\Lambda) = \{F_{A,n}^k(\Theta_0) \mid k \in \mathbb{Z}\}$.

Definition 3.41. A *self-rounding* corner is any corner of Λ if $\text{Ang}(\Lambda) = \{F_{A,n}^k(\Theta) \mid k \in \mathbb{Z}\}$ for some $\Theta \in \mathbb{R}$.

Self-rounding is only an issue if $N = 1$.

Lemma 3.42. *If $N > 1$, any self-rounding corner is a kink.*

Proof. Suppose $\mathrm{Ang}(\Lambda) = \{F^k_{A,n}(\Theta) \mid k \in \mathbb{Z}\}$. Since Λ cannot have an edge of the form πk, $\Theta \in (\pi k, \pi(k+1))$ for some k, and $F_{A,n}(\Theta) \in (\pi(k+N), \pi(k+N+1))$. In particular, $F_{A,n}(\Theta) > \pi(k+2)$ and $\Theta < \pi(k+1)$ so $c = (\Theta, F_{A,n}(\Theta))$ has width greater than π. □

Remark 3.43. *We forbid self-rounding to obtain a combinatorial chain complex that computes ECH, but in addition the result of self-rounding on a labeled path would be very difficult to define.*

Working with non-self-rounding corners, we may assume that c is not identified (by periodicity) with the adjacent corner, and the edge before c is not identified with the edge after.

For a periodic path, a proper corner is a corner that is not a kink (i.e., proper (Θ_1, Θ_2) has $\Theta_2 - \Theta_1 \geq \pi$). We define corner-mod-periodicity, and build in the conditions we need.

Definition 3.44. A *corner-mod-periodicity* of an (A, n, γ)-periodic path Λ is a set

$$[c] = \{F^k_{A,n}(c) \mid k \in \mathbb{Z}\}, \tag{3.17}$$

where c is a proper corner of Λ that is not a self-rounding corner. We call $[c]$ the *corner-mod-periodicity of c*.

We check a well-definedness condition regarding proper corners.

Lemma 3.45. *If c is a proper corner of an (A, n, γ)-periodic path Λ, then $F^k_{A,n}(c)$ is also a proper corner.*

Proof. The only thing to check is that if $c = (\Theta_1, \Theta_2)$ with $\Theta_2 - \Theta_1 \leq \pi$, then $F_{A,n}(\Theta_2) - F_{A,n}(\Theta_1) \leq \pi$. This follows from the monotonicity of $F_{A,n}$ (Proposition 3.10, Part 4) once we show that $F_{A,n}(\Theta_1 + \pi) - F_{A,n}(\Theta_1) = \pi$.

The action (denoted $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)_{S^1}$) of $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ on the circle in $\mathbb{R}^2$ sends opposite points to opposite points, and sends an arc going π around the circle to another such arc. The lift $F_{A,n}$ of $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)_{S^1}$ sends an interval of length π to another such interval, and preserves orientation. Thus, $F_{A,n}(\Theta_1 + \pi) = F_{A,n}(\Theta_1) + \pi$.

□

Definition 3.46. Let Λ be a periodic path and $[c] = \{F^k_{A,n}(c) \mid k \in \mathbb{Z}\}$ a corner-mod-periodicity. We define the result of *rounding (with periodicity)* Λ *at* $[c]$, $\Lambda \setminus [c]$, also called $\Lambda \setminus c$ or *rounding* Λ *at* $[c]$ or *at* c, to be the result of simultaneously rounding (without periodicity) Λ at $F^k_{A,n}(c)$ for all $k \in \mathbb{Z}$.

(If the result is $\mathrm{Ang}(\Lambda \setminus [c]) = \emptyset$, let $\Lambda \setminus [c]$ be the constant path at $\Lambda(\Theta)$ for $\Theta \notin \bigcup F^k_{A,n}(\bar{c})$.)

The last proviso enforces what is otherwise automatic, that rounding at $[c]$ only affects the parameterization inside $\bigcup F^k_{A,n}(\bar{c})$. Rounding at $[c] = [(\Theta_1, \Theta_2)]$ can only result in $\mathrm{Ang}(\Lambda \setminus c) = \emptyset$ if we already had $\Lambda(\Theta_1 - \epsilon) = \Lambda(\Theta_2 + \epsilon)$, in which case $\Lambda(\Theta)$ si the same for any $\Theta \in \bigcup F^k_{A,n}(\bar{c})$.

That the infinitely many rounding operations may be performed simultaneously without interfering with one another follows from Lemma 3.38, and is also illustrated in Figure 3.3, which illustrates a $\left(\left(\begin{smallmatrix} -2 & -1 \\ -1 & -1 \end{smallmatrix}\right), 1, 0\right)$-periodic path Λ with $\mathrm{Ang}(\Lambda) = \{\ldots, \Theta_1, \Theta_2, F_{A,n}(\Theta_1), F_{A,n}(\Theta_2),$ the smallest possible non-self-rounding $\mathrm{Ang}(\Lambda)/F_{A,n}$. Rounding (without periodicity) at

$c = (\Theta_1, \Theta_2)$ affects things before $q = \Lambda((\Theta_2, F_{A,n}(\Theta_1)))$ and rounding (without periodicity) at $F_{A,n}(c)$ affects things after q. The result $\Lambda \setminus c$ is again $\left(\left(\begin{smallmatrix} -2 & -1 \\ -1 & -1 \end{smallmatrix}\right), 1, 0\right)$-periodic.

As a consequence of our definitions, when we speak of rounding a periodic path, we always mean rounding it in the sense of periodic paths, and rounding a corner c is really rounding at the corner-mod-periodicity $[c]$.

Lemma 3.47. *The parameterizations* $\Lambda(\Theta)$ *and* $(\Lambda \setminus c)(\Theta)$ *agree outside* $\bigcup_{k \in \mathbb{Z}} F^k_{A,n}(\bar{c})$. □

Definition 3.48. If $\Lambda \setminus c$ is defined, and $\Theta \in c$, let $\Lambda \setminus \Theta = \Lambda \setminus c$.

An equivalent characterization of $\Lambda \setminus c$, using the notion of "to the left", is given in Proposition 3.69.

3.3.3 Losing one 'h' from the labeling

We can round a labeled periodic path or labeled truncated path α at a corner c by rounding the underlying unlabeled path at c and locally (in $\bar{c}$) decreasing the number of 'h' labels by one, while leaving the labels elsewhere (mod periodicity) unchanged. The result, $\delta_c \alpha$, is a signed sum of labeled paths. (All our labeled paths are periodic or truncated.)

The number of 'h' labels of Λ near c, i.e. in $\bar{c}$, is 0, 1, or 2. (If $c = (\Theta_1, \Theta_2)$, then Λ has edges at Θ_1 and Θ_2 and nowhere else in $\bar{c} = [\Theta_1, \Theta_2]$.) If that number is zero (i.e., if we round at a corner between two 'e' edges), then $\delta_c \alpha = 0$.

Since Λ and $\Lambda \setminus c$ already agree outside $\bar{c}$, or $[\bar{c}]$ if Λ is periodic, it makes sense to speak of the labelings agreeing outside $\bar{c}$ or $[\bar{c}]$. (In the periodic case, this means the labelings agree outside $F^k_{A,n}(\bar{c})$, $k \in \mathbb{Z}$.) Recall that we may write $\Lambda \setminus c$ when Λ is periodic, even though $\Lambda \setminus [c]$ is the more elegant description of where we are rounding.

Definition 3.49. Let $\alpha = (\Lambda, \ell, o)$ be a labeled periodic path or labeled truncated path in $C_*^{\mathrm{tr}}(I)$ or $C_*(A, n, \gamma)$. Let c be a proper corner of Λ if Λ is truncated, or a proper non-self-rounding corner of Λ if Λ is periodic. The result $\delta_c\alpha$ of *rounding* α *at* c is

$$\delta_c\alpha = \sum_{\ell'} \pm(\Lambda \setminus c, \ell', o'), \tag{3.18}$$

where the sum is over all raw labelings ℓ' of $\Lambda \setminus c$ such that

1. In $\bar{c}$, ℓ' has exactly one less 'h' than ℓ. (So $|\ell'_h \cap \bar{c}| = |\ell_h \cap \bar{c}| - 1$.)
2. The raw labeling ℓ' agrees with ℓ outside $\bar{c}$ if Λ is truncated, or outside $[\bar{c}] = \bigcup (F_{A,n})^k(\bar{c})$ if Λ is periodic.

and where o' and $\pm$ are determined by Sign Convention 3.50.

The sum has one term for each ℓ' satisfying the two conditions, which may be an empty sum.

The sign convention involves the order of 'h' edges. We may assume ℓ has at least one 'h' edge in $\bar{c}$ (otherwise $\delta_c\alpha = 0$). Assume that the *first* 'h' edge in the ordering is adjacent to c. This can always be arranged, possibly at the expense of a minus sign, by changing the order so that the desired 'h' edge is the first one. The idea is that this first 'h' edge is deleted from the ordering.

Recall that $[\Theta]$ is the equivalence class of Θ modulo $F_{A,n}$, if Λ is periodic.

Sign Convention 3.50. Suppose we have the hypotheses of Definiton 3.49, and an ℓ' satisfying the conditions given there. If Λ is truncated, let $[\Theta] = \Theta$. We may suppose, choosing if necessary a representative (Λ, ℓ, o) of the equivalence class in $C_* = \mathbb{Z}\{\text{paths}\}/\sim$, that

1. The first ‘h’-labeled angle in the ordering o, which we will call Θ_h, is an endpoint of $\bar{c}$. (Technically, it is $[\Theta_h]$ that has the first ‘h’ label, and Θ_h that we require to be an endpoint of $\bar{c}$.)

2. If Θ_h is the only ‘h’-labeled angle in $\bar{c}$, $\bar{c} \cap \ell_h = \{[\Theta_h]\}$, then o' is the restriction of the order o to $\ell'_h = \ell_h \setminus \{[\Theta_h]\}$.

3. If there are two ‘h’-labeled angles in $\bar{c}$, $\bar{c} \cap \ell_h = \{[\Theta_h], [\Theta_2]\}$, then o' is the restriction of the order o to ℓ'_h after identifying $[\Theta_2]$ with the ‘h’-labeled angle of ℓ'_h in $\bar{c}$, i.e. identifying $[\Theta_2]$ with $[\Theta']$ where $\bar{c} \cap \ell'_h = \{[\Theta']\}$.

Then (see Figure 3.4) the term appearing in the sum (3.18) is

$$+(\Lambda \setminus c, \ell', o') \tag{3.19}$$

if Θ_h is the *beginning* of c ($c = (\Theta_h, \Theta_2)$), and

$$-(\Lambda \setminus c, \ell', o') \tag{3.20}$$

if Θ_h is the *end* of c ($c = (\Theta_2, \Theta_h)$).

Once o satisfies Part 1 of Sign Convention 3.50, o' is determined by Part 2 or 3. We have to make a choice to get o to satisfy Part 1, and in particular, we may have to choose which of two ‘h’ edges to call Θ_h, but the choice does not matter:

Lemma 3.51. *Under the hypotheses of Definition 3.49, $\delta_c\alpha$ is well-defined as a element of $C^{\mathrm{tr}}_*(I)$ or $C_*(A,n,\gamma)$.*

Proof. In computing δ_c, we choose a representative $\pm(\Lambda, \ell, o)$ of an equivalence class in C_*, and in particular, if there are two ‘h’ labels in $\bar{c}$, we choose one to be first in o. We may

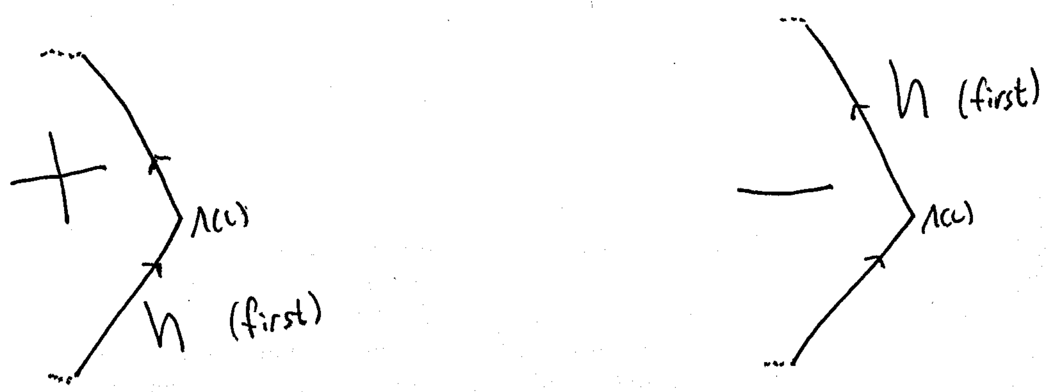

Figure 3.4: The sign convention for δ.

swap those two 'h''s with a transposition, which cancels the sign change between (3.19) and (3.20).

Any permutation fixing the first 'h' of o passes through to the corresponding permutation on o'. □

Definition 3.52. Let $\alpha = (\Lambda, \ell, o)$.

1. If c is a non-proper corner or self-rounding corner of Λ, let $\delta_c\alpha = 0$.

2. If $\Theta \in c$, let $\delta_\Theta\alpha = \delta_c\alpha$.

3. If Λ is periodic, let $\delta_{[c]}\alpha = \delta_c\alpha$.

Definition 3.53. 1. Let $\alpha = (\Lambda_{\text{tr}}, \ell, o)$ be a labeled truncated path. Let

$$\delta\alpha = \sum_c \delta_c\alpha \tag{3.21}$$

where the sum is over all proper corners c of Λ_{tr}.

2. Let $\alpha = (\Lambda, \ell, o)$ be a labeled (A, n, γ)-periodic path. Let

$$\delta\alpha = \sum_{[c]} \delta_{[c]}\alpha, \tag{3.22}$$

   where the sum is over all corners-mod-periodicity $[c]$ of Λ.

In other words, if Λ is periodic, we have one summand $\delta_c\alpha$ for the entire family $\{F^k_{A,n}(c)\}_{k\in\mathbb{Z}}$.

We now give two examples we will be using often.

Example 3.54. If Λ is a truncated or periodic path, we define two cycles E_Λ, H_Λ in $C^{\mathrm{tr}}_*(I)$ or $C_*(A,n,\gamma)$ as follows:

1. E_Λ is the labeled path consisting of the path Λ with edges labeled 'e'.

2. H_Λ is a sum of labeled paths: It is the sum over all ways to label one edge of Λ 'h' and all the others 'e'. The sum is empty ($H_\Lambda = 0$) if and only if Λ is constant.

In both cases, the ordering o is trivial.

Proof. We see $\delta E_\Lambda = 0$ because it cannot lose one 'h'. We see $\delta H_\Lambda = 0$ because, if $c = (\Theta_1, \Theta_2)$ (we may assume c is not a self-rounding corner), any contribution to $\delta_c H_\Lambda$ from the term of H_Λ where Θ_1 is labeled 'h' cancels the contribution from the term where Θ_2 is labeled 'h'. □

Definition 3.55. The number of 'h' labels, i.e. the number of elements of ℓ_h, is denoted $\#h(\ell)$.

Proposition 3.56. *The differential on labeled truncated paths is equivalent to the differential on open admissible paths from [5, Definitions 3.4, 3.5]; the isomorphism is given by reversing the ordering of 'h' labels and multiplying* (Λ_{tr}, ℓ, o) *by* $(-1)^{\#h(\ell)}$.

Proof. The definitions in [5] correspond to the ones here except for the sign convention there, which is + if the *last* 'h' in the ordering is just *after* c and − if it is before. □

Corollary 3.57 (by §3.2 of [5])**.** *On* $C_*^{\mathrm{tr}}(I)$*, we have* $\delta^2 = 0$. □

Thus $(C_*^{\mathrm{tr}}(I), \delta)$ is a chain complex. We have not given it a grading, but a relative grading is given in [5, §3.1.5].

Proposition 3.58. *On* $C_*(A,n,\gamma)$*, we have* $\delta^2 = 0$.

Proof. The proof in [5, §3.2] carries over to $C_*(A,n,\gamma)$ for paths Λ with at least three edges (mod periodicity), because whenever we consider $\delta_b\delta_a\alpha$ ($\alpha = (\Lambda, \ell, o)$, a a corner of Λ, b a corner of $\Lambda \setminus c$), there exists Θ outside $[\bar{a}]$ and $[\bar{b}]$, and all the action can be understood on the truncated path $\Lambda|[\Theta, F_{A,n}(\Theta)]$. (By the arguments in [5], either b is "created" by rounding at a ($b \subseteq a$) or else b comes from a corner of Λ, possible a corner smaller than the interval b, expanded by rounding at a. In either case, such Θ must exist. We choose Θ outside $\mathrm{Ang}(\Lambda)$.)

If Λ has at most one edge, we already have $\delta\alpha = 0$. (We use "no self-rounding"– see Definition 3.44 and Definition 3.53.) So, assume Λ has two edges: $a = (\Theta_1, \Theta_2)$ and $\mathrm{Ang}(\Lambda) = \{\ldots, \Theta_1, \Theta_2, F_{A,n}(\Theta_1), \ldots\}$. We may assume a is not a kink. Both edges of Λ participate in rounding at a and all edges of $\Lambda \setminus a$ receive labels by the rules for δ_a. Thus $\delta_a\alpha$ is one of the cycles $E_{\Lambda\setminus a}$ or $H_{\Lambda\setminus a}$, so $\delta\delta_a\alpha = 0$. □

Thus $(C_*(A,n,\gamma), \delta)$ is a chain complex. We will give it a grading in §3.4.

3.4 The index $I((\Lambda, \ell, o))$ gives a grading on $C_*(A, n, \gamma)$

At this point in building our chain complex $C_*(A, n, \gamma)$, we have a group and a differential δ on it. We need a grading. In this section, we will define the index $I(\alpha)$ of a labeled periodic path α, allowing us to write the group as $\bigoplus_i C_i(A, n, \gamma)$. There is only a relative index in general: $I(\alpha) \in \mathcal{Z}$ with $\mathcal{Z}$ a torsor over $\mathbb{Z}$, so $I(\alpha_1) - I(\alpha_2) \in \mathbb{Z}$. When $\Gamma = 0$, we choose a particular isomorphism $\mathcal{Z} \cong \mathbb{Z}$ by requiring $I(\mathrm{const}_0) = 0$.

The formula for the relative index, stated precisely in Definitions 3.63 and 3.64 as well as Lemma-Definition 3.65, is essentially

$$I((\Lambda, \ell, o)) = 2\,\mathrm{Size}(\Lambda) - \#h(\ell), \tag{3.23}$$

where $\#h$ is the number of 'h' labels, mod periodicity; and Size(Λ) is something like the number of lattice points enclosed by Λ, but only defined up to adding a constant. The number of lattice points enclosed by $\Lambda \setminus c$ is one less than the number enclosed by Λ, which will mean δ has degree -1 (Proposition 3.66).

We take an axiomatic approach, showing that there is essentially only one function Size.

Definition 3.59. A *relative size function* is a map

$$\mathrm{Size}\colon \{(A, n, \gamma)\text{-periodic paths}\} \to \mathbb{R} \tag{3.24}$$

such that

$$\mathrm{Size}(\Lambda \setminus c) = \mathrm{Size}(\Lambda) - 1 \tag{3.25}$$

where c is a proper corner-mod-periodicity of the (A, n, γ)-periodic path Λ.

A relative size function is, in effect, an embedding of the torsor $\mathcal{Z}$ into $\mathbb{R}$. We say $\mathbb{R}$ for generality, though we never need anything worse than $\mathbb{Q}$.

Proposition 3.60. *Given A, n, and γ, there exists a relative size function, and it is unique up to adding a constant.*

We will prove Proposition 3.60 in §3.6.

Example 3.61. Recall (though we normally work with just hyperbolic A and $-\mathbb{1}$) that $(\mathbb{1}, 1, 0)$-periodic paths are just convex polygons (with vertices in $L_{A,\Gamma}$). Let $\mathrm{Size}(\Lambda)$ be the number of points (nonstrictly) enclosed by Λ. This quantity is well-defined.

Example 3.62. A $(-\mathbb{1}, 1, \gamma)$-periodic path Λ is a convex polygon with $(X, Y) \mapsto (-X, -Y)$ symmetry. One period is $\Lambda|[0, \pi]$, and the polygon is $\Lambda|[0, 2\pi]$. Let $\mathrm{Size}(\Lambda)$ be one half of the number of points (nonstrictly) enclosed by the polygon.

When A is hyperbolic it is harder to define the number of points enclosed.

For the following definition, we write $I(\alpha, \alpha')$ for $I(\alpha) - I(\alpha')$, since we cannot yet define $I(\alpha)$.

Definition 3.63. If $\alpha = (\Lambda, \ell, o)$ and $\alpha' = (\Lambda', \ell', o')$ are labeled (A, n, γ)-periodic paths and Size is a relative size function, let

$$I(\alpha, \alpha') = [2\,\mathrm{Size}(\Lambda) - \#h(\ell)] - [2\,\mathrm{Size}(\Lambda') - \#h(\ell')], \tag{3.26}$$

where $\#h(\ell)$ is the cardinality of ℓ_h.

We see that $I(\alpha, \alpha')$ is the same for any relative size function.

When $\Gamma \neq 0$, we live with the ambiguity in $I(\alpha)$. When $\Gamma = 0$, we use the expression $\mathrm{Size}(\Lambda) - \mathrm{Size}(\mathrm{const}_0)$ to obtain $I(\mathrm{const}_0) = 0$.

Definition 3.64. If α is a labeled $(A, n, 0)$-periodic path, its *index* (or *absolute index*) is

$$I(\alpha) = 2(\mathrm{Size}(\Lambda) - \mathrm{Size}(\mathrm{const}_0)) - \#h(\ell). \tag{3.27}$$

Lemma-Definition 3.65. For any Γ, we may define a $\mathbb{Z}$-torsor $\mathcal{Z}$ and the *relative index*,

$$I\colon \{\text{Labeled } (A, n, \gamma)\text{-periodic paths}\} \to \mathcal{Z}, \tag{3.28}$$

such that $I(\alpha, \alpha') = I(\alpha) - I(\alpha')$.

If $\Gamma = 0$, $\mathcal{Z}$ is naturally $\mathbb{Z}$, and I is the absolute index.

Proof. The claim is straightforward but hard to make moderately formal. To be absurdly formal, let $\mathcal{Z}$ be the set

$$\{-\mathrm{Size} \mid \mathrm{Size} \text{ is a } \mathbb{Z}\text{-valued relative size function}\}, \tag{3.29}$$

and let $I(\Lambda, \ell, o)$ be the element $-\mathrm{Size}$ such that $2\,\mathrm{Size}(\Lambda) - \#h(\ell) = 0$. □

Proposition 3.66. *If α is a labeled (A, n, γ)-periodic path, $I(\delta\alpha) = I(\alpha) - 1$.*

Proof. We see that $I(\alpha, \delta\alpha) = 1$ using the definition of I, the fact that δ decreases Size by 1 (by the definition of relative size function), and decreases $\#h$ by 1 (by the definition of δ). □

3.5 The partial order $\leq$ ("to the left") on periodic paths

In the current section, we define a partial order on periodic paths. The relation is denoted $\Lambda \leq \Lambda'$ and called *to the left*, a term motivated by the point of view of bugs crawling along the paths. Another intuition for $\Lambda \leq \Lambda'$ is that Λ is smaller than Λ'. For instance, Λ will have smaller Size than Λ'.

Definition 3.67. 1. Given (A, n, γ)-periodic paths Λ, Λ', we say $\Lambda \leq \Lambda'$ if

$$\det(\Lambda'(\Theta) - \Lambda(\Theta), \vec{\Theta}) \geq 0 \tag{3.30}$$

for all Θ such that neither Λ nor Λ' has an edge at Θ. We say Λ is *to the left* of Λ'.

2. Given truncated paths Λ, Λ' with $\overline{\text{Dom}}(\Lambda) = \overline{\text{Dom}}(\Lambda')$, we say $\Lambda \leq \Lambda'$, or Λ is *to the left* of Λ', if (3.30) holds for all Θ where it makes sense, i.e., $\Theta \in \overline{\text{Dom}}(\Lambda)$ with $\Theta \notin \text{Ang}(\Lambda) \cup \text{Ang}(\Lambda')$.

3. We may compare truncated paths with different domains, or a truncated path with a periodic path, by restricting domains in the obvious way. For periodic paths specifically, saying $\Lambda \leq \Lambda'$ implies that Λ and Λ' have the *same* (A, n, γ).

In [5], truncated paths compared with $\leq$ are required to be defined on the same interval, and also required to have *matching endpoints*, i.e., $\Lambda(\Theta_i) = \Lambda'(\Theta_i)$, $i = 1, 2$, where Λ and Λ' are defined on $[\Theta_1, \Theta_2]$. We do not build either condition into the definition.

The restriction that Λ and Λ' have no edge at Θ is not really necessary because the expression $\det(\Lambda'(\Theta) - \Lambda(\Theta), \vec{\Theta})$ extends to a continuous function on all Θ.

We see, for (A, n, γ)-periodic Λ and Λ', that $\Lambda \leq \Lambda'$ iff $\Lambda|[0, N\pi] \leq \Lambda'|[0, N\pi]$ iff $\Lambda|I = \Lambda'|I$ for all intervals I.

Proposition 3.68. *The relation $\leq$ on (A, n, γ)-periodic paths is a partial order. The relation $\leq$ on truncated paths defined on a given interval is a partial order.*

Proof. We see that $\Lambda \leq \Lambda$. If

$$\Lambda \leq \Lambda' \leq \Lambda, \tag{3.31}$$

then $\det(\Lambda'(\Theta) - \Lambda(\Theta), \vec{\Theta}) = 0$. The function of Θ, $\Lambda'(\Theta) - \Lambda(\Theta)$, is piecewise constant, and each value it takes is a vector parallel to multiple Θ's (with linearly independent $\vec{\Theta}$'s). Thus, $\Lambda = \Lambda'$.

For transitivity, add determinants: $\det(\Lambda''(\Theta) - \Lambda'(\Theta), \vec{\Theta}) + \det(\Lambda'(\Theta) - \Lambda(\Theta), \vec{\Theta}) \geq 0$ so $\Lambda \leq \Lambda''$. □

Proposition 3.69 (Alternate characterization of rounding a corner)**.** *If* Λ *is a periodic or truncated path and* c *is a proper corner of* Λ*, then:*

1. $\Lambda \backslash c \leq \Lambda$.

2. $(\Lambda \setminus c)(\Theta)$ *disagrees with* $\Lambda(\Theta)$ *at every* $\Theta \in c$ *for which both expressions are defined.*

3. *If* $\Lambda' \leq \Lambda$ *is a path (periodic if* Λ *is periodic, or truncated and defined on the same interval as* Λ *if* Λ *is truncated) with* $\Lambda'(\Theta) \neq \Lambda(\Theta)$ *for at least one* $\Theta \in c$*, then* $\Lambda' \leq \Lambda \backslash c$.

Proof. Part 2 for periodic and truncated paths follows from the corresponding statement for polygonal paths, Lemma 3.37.

Parts 1 and 3 are essentially the same as [5, Proposition 2.10], and the proof given there works in this situation. Note that the proof in [5] does not use the requirement imposed there, that endpoints match. Note also (Lemma 3.47) that for periodic paths, Λ and $\Lambda \setminus c$ agree outside $\bigcup_{k \in \mathbb{Z}} F^k_{A,n}(\bar{c})$, reducing the problem to an operation local in Θ, taking place on each $F^k_{A,n}(\bar{c})$ separately.

□

It will turn out that for periodic paths (and for truncated paths with more disclaimers), $\Lambda \leq \Lambda'$ if and only if Λ may be obtained from Λ' by a sequence of roundings. (See the proof of Lemma 3.77, or [5, Proposition 2.13].)

Lemma 3.70. *1. If $\Lambda \leq \Lambda'$ and c is a kink of Λ', then $\Lambda(\Theta) = \Lambda'(\Theta)$ for all $\Theta \in c$. (We require the domain of Λ' to be strictly wider than c: If $c = (\Theta_1, \Theta_2)$ then $(\Theta_1 - \epsilon, \Theta_2 + \epsilon) \subset \overline{\text{Dom}}(\Lambda')$ for some $\epsilon > 0$.)*

2. If $\Lambda_{tr} \leq \Lambda'_{tr}$ are truncated paths, both defined on $[\Theta_1, \Theta_2]$ with matching endpoints, and c is a corner of Λ'_{tr} containing Θ_1 or Θ_2, then $\Lambda_{tr}(\Theta) = \Lambda'_{tr}(\Theta)$ for all $\Theta \in c$.

Proof. 1. This is [5, Proposition 2.12]. The proof given there does not use the stronger characterization of $\leq$ in that paper. The statement is local in Θ, so it follows for periodic paths as well as truncated ones.

2. A truncated path Λ_{tr} (respectively Λ'_{tr}) has an underlying polygonal path Λ (resp. Λ') defined on all of $\mathbb{R}$, with $\Lambda(\Theta) = \Lambda_{\text{tr}}(\Theta_1)$ for $\Theta \leq \Theta_1$ and $\Lambda(\Theta) = \Lambda_{\text{tr}}(\Theta_2)$ for $\Theta \geq \Theta_2$. We may apply Part 1 to the corner of Λ' containing Θ_1 or Θ_2.

□

Lemma 3.71. *There are only finitely many (A, n, γ)-periodic paths to the left of a given (A, n, γ)-periodic path.*

Proof. We will show that if Λ is fixed and $\lambda \leq \Lambda$, the points of $\lambda|[0, N\pi]$ must lie in a bounded rectangular region, depending on Λ. Since $L_{A,\Gamma}$ is discrete, it has only finitely many points in that region, so there are finitely many possible vertices of λ, finitely many possible angles $\Theta \in \text{Ang}(\lambda)$, finitely many possible sets $\text{Ang}(\Lambda)$, and finitely many paths λ.

The path λ can only travel up ($+Y$ direction) while $\Theta \in [0, \pi]$. Applying the definition of $\leq$ at $\Theta = 0$, $\Theta = \pi$ shows that

$$Y(\Lambda(0)) \leq Y(\lambda(0)), \qquad Y(\lambda(\pi)) \leq Y(\Lambda(\pi)), \tag{3.32}$$

with $Y(\lambda(\Theta))$ ($\Theta \in [0, \pi]$) in between,

$$Y(\lambda(\Theta)) \in [Y(\Lambda(0)), Y(\Lambda(\pi))]. \tag{3.33}$$

Similar reasoning bounds Y on each $\lambda|[k\pi, (k+1)\pi]$, and therefore on $\lambda|[0, N\pi]$. For the X-coordinate, use enough intervals of the form $[k\pi + \pi/2, (k+1)\pi + \pi/2]$. Thus $\lambda(\Theta)$ must lie in some bounded rectangle for $\Theta \in [0, N\pi]$, completing the proof. □

3.6 The proof of Proposition 3.60

Recall that the lattice $L_{A,\Gamma}$, which looks messy in our usual (X, Y)-coordinates, looks like $\mathbb{Z}^2$ in (x, y)-coordinates.

Definition 3.72. Given a polygonal path Λ and $\Theta \in \mathrm{Ang}(\Lambda)$, letting p be the beginning and q the end of $\mathrm{Edge}_\Lambda(\Theta)$, define the *multiplicity* $\mathrm{Mult}(\Lambda, \Theta)$ to be $\gcd(x(q) - x(p), y(q) - y(p))$.

The following definition integrates an area form over what is essentially $\Lambda|[\Theta_0, F_{A,n}(\Theta_0)]$, but because of our odd (piecewise-constant) parameterization of Λ, we write out $\int_{\Lambda \ldots}$ as an integral over each edge of Λ. Each edge (we recall) carries an orientation.

Lemma-Definition 3.73. For any $\Theta_0 \in \mathbb{R} \setminus \mathrm{Ang}(\Lambda)$, let

$$\mathrm{Size}_{\int}(\Lambda) = \sum_{\substack{\Theta \in \mathrm{Ang}(\Lambda) \\ \Theta_0 \leq \Theta < F_{A,n}(\Theta_0)}} \int_{\mathrm{Edge}_\Lambda(\Theta)} X\, dY + \frac{1}{2} \sum_{\substack{\Theta \in \mathrm{Ang}(\Lambda) \\ \Theta_0 \leq \Theta < F_{A,n}(\Theta_0)}} \mathrm{Mult}(\Lambda, \Theta). \tag{3.34}$$

The choice of Θ_0 does not affect $\mathrm{Size}_{\int}(\Lambda)$, and $\mathrm{Size}_{\int}$ is a relative size function.

This result, which we will prove shortly, establishes the existence portion of Proposition 3.60.

Incidentally, this size function has the pleasing property that $\text{Size}_\int(\text{const}_0) = 0$ when $\Gamma = 0$. It is one of at least two (inequivalent) ways to extend the idea $\text{Size}(\text{const}_0) = 0$ to the case $0 \notin L_{A,\Gamma}$, i.e. $\Gamma \neq 0$.

Remark 3.74. *We could obtain the same relative size function as in* (3.34) *by integrating any* 1*-form* ω *invariant under the pullback* $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)^* \omega = \omega$ *and satisfying* $d\omega = dX \wedge dY$*. The value of that integral may also be defined as the area (with signs and multiplicity) enclosed by the following path:* $\Lambda|[\Theta_0, F_{A,n}(\Theta_0)]$*, followed by a path* τ *to* 0*, followed by* $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)^{-1} \tau$ *traversed backwards to* $\Lambda(\Theta_0)$*.*

Proof of Lemma-Definition 3.73. We first show that $\text{Size}_\int(\Lambda)$ is independent of Θ_0. The integrals and multiplicities are both applied to $\text{Edge}_\Lambda(\Theta)$ for $\Theta \in \text{Ang}(\Lambda) \cap [\Theta_0, F_{A,n}(\Theta_0))$. So it is sufficient to show that removing a Θ from that set and adding $F_{A,n}(\Theta)$ does not change the result.

The integral is unchanged because $X\,dY$ is $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$-invariant. The multiplicity is unchanged because, in $\mathbb{Z}^2$, $\gcd(a, b)$ is A-invariant ($A \in SL_2(\mathbb{Z})$). Thus, $\text{Size}_\int(\Lambda)$ is independent of Θ_0.

It remains to show that $\text{Size}_\int(\Lambda \setminus c) = \text{Size}_\int(\Lambda) - 1$. Write $c = (\Theta_1, \Theta_2)$; we may assume $[\Theta_1, \Theta_2] \subset (\Theta_0, F_{A,n}(\Theta_0))$. The only differences between Λ and $\Lambda \setminus c$ occur on $[\Theta_1, \Theta_2]$, so the issue is a local one, and the proof in [5, Def. 3.3, Lemma 3.9] (using Pick's formula) applies here. □

The following lemma, that large rectangles contain many lattice points, is much

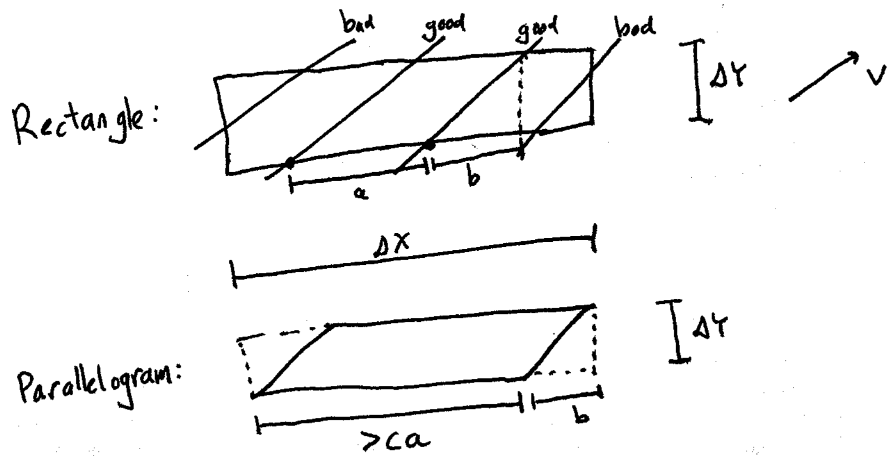

Figure 3.5: A rectangle with a lower bound on the number of lattice points enclosed.

stronger than we need for now, but will be necessary in later chapters.

Lemma 3.75. *For any real $\Delta Y > 0$ and any integer $c > 0$, there is a $\Delta X > 0$ such that, if $X_1 - X_0 \geq \Delta X$ and $Y_1 - Y_0 \geq \Delta Y$, then the number of lattice points contained in the rectangle $(X_0, X_1) \times (Y_0, Y_1)$ is at least c.*

Proof. The statement is a property of the lattice and eigencoordinates, not of A itself. It is also invariant under translation–it does not matter that the origin of eigencoordinates does not always agree with the origin of standard coordinates. Thus we may assume $\Gamma = 0$, and we may also assume that $A \neq -\mathbb{1}$ (since eigencoordinates for $A = -\mathbb{1}$ with $\Gamma = 0$ were chosen to match those of some hyperbolic A). We may assume that $X_1 - X_0$ is exactly ΔX.

Choose some nonzero element of the lattice. It will have $Y \neq 0$. Apply $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$

repeatedly to shrink its Y-coordinate until we obtain a new point with Y-coordinate is less than ΔY. Let v be the vector from origin to the new point. Consider the family of all lines parallel to v that go through lattice points. Such a line, if it passes through the top and bottom edge of our rectangle, must contain at least one lattice point in the rectangle. See Figure 3.5. Let a be the X-distance between adjacent lines in the family (i.e., translating a line by $(a, 0)$ yields the next line), and let b be such that $(b, \Delta Y)$ is parallel to v. (The lines cannot be parallel to either axis, so we may assume $a, b > 0$.) Choose

$$\Delta X > ca + b \tag{3.35}$$

which ensures that the rectangle contains a parallelogram (see the figure) which meets at least c of the lines, each with at least one lattice point, proving the claim. □

Lemma 3.76. *For any two (A, n, γ)-periodic paths Λ_0 and Λ, there exists an (A, n, γ)-periodic path Λ' so big that $\Lambda_0 \leq \Lambda'$ and $\Lambda \leq \Lambda'$.*

Proof. Construct Λ' by choosing a periodic sequence of points "very far" to the northeast, northwest, etc. of Λ and Λ_0, and connecting them, proceeding counterclockwise. □

Lemma 3.77. *For any two (A, n, γ)-periodic paths Λ_0 and Λ, there exists a sequence*

$$\Lambda_0, \Lambda_1, \Lambda_2, \ldots, \Lambda_m = \Lambda \tag{3.36}$$

such that for each $k = 0, \ldots, m-1$, one of Λ_k, Λ_{k+1} is obtained by rounding some corner of the other.

Proof. Given Λ_0 and Λ, let Λ' be as in Lemma 3.76. We will show that there is a sequence of roundings from Λ' to Λ (each path in a sequence being obtained by rounding the previous path at some corner). A similar argument will show there is a sequence of roundings from Λ' to Λ_0. We may then produce a sequence (3.36) from Λ_0 to Λ by joining the two sequences just constructed.

Suppose $\Lambda \leq \Lambda'$ and $\Lambda \neq \Lambda'$. The two paths must disagree at at least one Θ, which lies in some corner c of Λ'. By Lemma 3.70, Λ and Λ' agree on kinks, so c is not a kink. Let $\Lambda'_1 = \Lambda' \setminus c$. We have $\Lambda \leq \Lambda'_1 \leq \Lambda'$ by Proposition 3.69.

We may continue in this way: If $\Lambda'_1 \neq \Lambda$, we may round some corner of Λ'_1, obtaining a new path Λ'_2 with $\Lambda \leq \Lambda'_2 \leq \Lambda'_1 \leq \Lambda'$. This process can only terminate when we obtain a path equal to Λ. We must show it terminates after finitely many steps.

Each path in the sequence is to the left of the one before, so by transitivity of $\leq$ (Proposition 3.68), each path in the sequence is to the left of the original Λ'. We know there are only finitely many paths to the left of Λ' (Lemma 3.71), so provided each term of the sequence is distinct, the sequence must terminate after finitely many steps, and we are done. But $\Lambda'_{i+1} \leq \Lambda'_i$, and $\Lambda'_{i+1} \neq \Lambda'_i$. Since $\leq$ is a partial order, each term is distinct. □

Remark 3.78. *The symplectic action of a periodic path can be seen as the real reason the sequence of Λ'_i's terminates in the proof of Lemma 3.77. The symplectic action of a closed Reeb orbit is the integral of the contact form along the orbit. (See Appendix A for the correspondence between a periodic path and a collection of Reeb orbits.) Symplectic action must decrease under rounding, and the set of possible symplectic actions is discrete.*

Proof of Proposition 3.60. If remains to show the uniqueness portion of the Proposition. Let Size be a relative size function. Pick some path Λ_0 and let $\zeta = \mathrm{Size}(\Lambda_0) - \mathrm{Size}_\int(\Lambda_0)$. We have $\mathrm{Size}(\Lambda) = \mathrm{Size}_\int(\Lambda) + \zeta$ for all (A,n,γ)-periodic paths Λ, because Size and $\mathrm{Size}_\int$

increase or decrease together as we proceed along the sequence of Lemma 3.77 □

Chapter 4

Boxes and strips define subcomplexes of $C_*(A, n, \gamma)$

Assumption 4.1. Let $A \in SL_2(\mathbb{Z})$ be hyperbolic or $-\mathbb{1}$, let n be a positive integer, and let $\Gamma \in \mathbb{Z}^2/\operatorname{Im}(\mathbb{1} - A)$ with $\gamma \in \Gamma$.

In this chapter, we introduce two kinds of subcomplex of $C_*(A, n, \gamma)$, those induced by boxes and those induced by strips. Boxes and strips are more rectilinear analogues of periodic paths. Boxes have edges parallel to the X- and Y-axes, the axes of the eigencoordinate system. (The intermediate step in defining boxes is *off-lattice periodic paths* in §4.1, paths with vertices allowed to be anywhere in $\mathbb{R}^2$.) Strips have infinitely long edges parallel to the X-axis. The name "box" is intended to suggest a rectangular region in the plane, and "strip" is intended to suggest an infinitely long rectangle of the form $\mathbb{R} \times [Y_0, Y_1]$, but boxes and strips are slightly more complicated. A box or strip B gives rise to a family of actual rectangles and infinitely long rectangles, denoted $\square_{B,k\pi/2}$, respecting periodicity, and we have $\Lambda \leq B$ if and only if $\Lambda(k\pi/2) \in \Box_{B,k\pi/2}$ for all $k \in \mathbb{Z}$.

The subcomplexes we need are generated by labeled paths to the left of a box or strip. The main result is that we may use the homologies of boxes to recover the combinatorial homology $H_*(A, n, \gamma)$ by a direct limit.

See Propositions 4.18 and 4.19 for details.

The method of Chapter 5 allows us to compute $H_*(B)$ for B a box or strip. Strips are more useful than boxes, and we will show in Chapter 6 (using boxes in the proof) that $H_*(B) \cong H_*(A, n, \gamma)$ for a strip B.

We introduce off-lattice polygonal paths in §4.1. We define relations $\leq_\Theta$ and $<_\Theta$, essentially the relation $\leq$ from §3.5 considered at one Θ, and prove a few lemmas in §4.2. In §4.3, we define the chain complex generated by periodic paths to the left of a given off-lattice periodic path, a definition we will need for the case that the off-lattice path is a box. We also introduce the chain complex generated by truncated paths to the left of a given truncated path (with matching endpoints), and recall its homology from [5], which we will need in subsequent chapters. In §4.4, we define boxes and strips, including the chain complex of a strip, and characterize periodic paths to the left of a box or strip in terms of $\leq_\Theta$, and also in terms of $\Box_{B,\Theta}$. We then prove results about direct limits of boxes.

The concepts of this chapter are a mild generalization of [5, especially §3.1.5].

4.1 Off-lattice paths

Recall that the term *polygonal path* implies that the vertices lie in the lattice $L_{A,\Gamma}$.

Definition 4.2. An *off-lattice polygonal path* is defined by deleting from the definition of polygonal path (Definitions 3.15 and 3.16) the requirement that vertices lie in the lattice $L_{A,\Gamma}$. Instead, we allow vertices of the path to be arbitrary points in the plane. We carry over every definition from §3.2.1 and §3.2.2 concerning polygonal paths, including the definition of (A, n, γ)-periodic path, as well as Definition 3.67 of $\leq$.

Thus, a polygonal path is a kind of off-lattice polygonal path.

Every result of §3.2.1 and §3.2.2 carries over to off-lattice paths, except for the claims about the relation to the terminology of [5]. (Additionally, though the fact was only mentioned in passing, polygonal paths have edges only at certain Θ, constrained by the lattice $L_{A,\Gamma}$; off-lattice paths are not thus constrained.) We cannot round a corner of an off-lattice path. We still have the results on $\leq$ not dependent on rounding: Proposition 3.68 ($\leq$ is a partial order), and Lemma 3.70 ($\Lambda \leq \Lambda'$ agrees with Λ' at a kink of Λ'). Lemma 3.71 can be generalized to assert that there are finitely many polygonal paths (i.e., lattice polygonal paths) to the left of a given off-lattice polygonal path.

We are interested in two kinds of off-lattice periodic paths: periodic paths, and *boxes*, to be defined later.

4.2 Comparing points and paths with $\leq_\Theta$

Recall that the ordering $\leq$, originally defined on polygonal paths and now generalized to off-lattice polygonal paths, is: $\Lambda \leq \Lambda'$ if and only if

$$\det(\Lambda'(\Theta) - \Lambda(\Theta), \vec{\Theta}) \geq 0 \tag{4.1}$$

for all Θ for which $\Lambda'(\Theta)$ and $\Lambda(\Theta)$ are defined. We will often need to compare two points with respect to some fixed Θ or compare a point and a path, so this section defines a family

$\leq_\Theta$ of relations using (4.1).

Recall that the set of all angles at which Λ has an edge is denoted $\mathrm{Ang}(\Lambda)$, and $\mathrm{Edge}_\Lambda(\Theta)$ is the edge Λ has at Θ, i.e. the one of the directed line segments out of which Λ is built.

Definition 4.3. Suppose p and q are points in $\mathbb{R}^2$, $\Theta \in \mathbb{R}$, and Λ and Λ' are off-lattice periodic paths, or off-lattice truncated paths defined on an interval including Θ. Define:

1. $p \leq_\Theta q$ if $\det(q - p, \vec{\Theta}) \geq 0$.

2. $p <_\Theta q$ if $\det(q - p, \vec{\Theta}) > 0$.

3. $p \leq_\Theta \Lambda$ if $p \leq_\Theta \Lambda(\Theta)$, assuming $\Lambda(\Theta)$ is defined; if not, replace $\Lambda(\Theta)$ by some point $r \in \mathrm{Edge}_\Lambda(\Theta)$ (it does not matter which point we choose), so $p \leq_\Theta \Lambda$ if $p \leq_\Theta r$.

4. $\Lambda' \leq_\Theta \Lambda$ if $\Lambda'(\Theta) \leq_\Theta \Lambda(\Theta)$, assuming $\Lambda'(\Theta), \Lambda(\Theta)$ are defined; if not, replace whichever expressions are not defined by a choice of point on the appropriate edge, as in Part 3, above.

5. $p <_\Theta \Lambda$ if $p <_\Theta \Lambda(\Theta)$, and $\Lambda' <_\Theta \Lambda$ if $\Lambda'(\Theta) <_\Theta \Lambda(\Theta)$, with the same disclaimers as above if Θ is an edge of Λ or Λ'.

We now have a relation $\leq$ on paths, and relations $\leq_\Theta$ and $<_\Theta$ on points and paths. We do not write "$\Lambda < \Lambda'$".

For each $\Theta \in \mathbb{R}$, the relation $\leq_\Theta$ satisfies all forms of the transitivity property that make sense.

A truncated path defined on $[-\pi/2, \pi/2]$ has an overall motion in the $+X$ direction, so $\Lambda(-\pi/2) \leq_{\pi/2} \Lambda(\pi/2)$. We generalize this fact.

Lemma 4.4. *If Λ is an off-lattice periodic or truncated path, and $\Theta, \Theta' \in \overline{\mathrm{Dom}}(\Lambda)$ with $|\Theta' - \Theta| \leq \pi$, then $\Lambda(\Theta') \leq_\Theta \Lambda(\Theta)$.*

Proof. If $\Theta' \in [\Theta - \pi, \Theta]$, then each edge of Λ takes a step in some direction in that interval, which carries it rightwards, from the point of view of $\leq_\Theta$. Similar reasoning applies to $[\Theta, \Theta + \pi]$. □

Lemma 4.5. *If Λ is periodic and to the left of* const_0*, then* $\Lambda = \mathrm{const}_0$*. If Λ_{tr} is a truncated path, to the left of* const_p*, and with both endpoints at p, then* $\Lambda_{tr} = \mathrm{const}_p$*.*

Proof. For Λ periodic, apply Lemma 3.70, to show that Λ must agree with const_0 on its kink $\mathbb{R}$. For Λ_{tr} truncated, let Λ be the "underlying" polygonal path defined on $\mathbb{R}$ (from the definition of truncated path). We still have $\Lambda \leq \mathrm{const}_p$ so $\Lambda = \mathrm{const}_p$. □

4.3 Chain complexes of paths to the left of a fixed path

Since $\Lambda \setminus c \leq \Lambda$, the set of periodic paths Λ to the left of a given off-lattice periodic path Λ' is closed under rounding corners. Therefore, the subgroup of $C_*(A,n,\gamma)$ generated by (Λ, ℓ, o) with $\Lambda \leq \Lambda'$ is a subcomplex.

Definition 4.6. Given an off-lattice periodic path Λ', let $C_*(\Lambda')$, or $C_*(A,n,\gamma;\Lambda')$, denote the subcomplex of $C_*(A,n,\gamma)$ generated by (Λ,ℓ,o) with $\Lambda \leq \Lambda'$.

We will be very interested in certain subcomplexes of the form just described, namely the case where Λ' is a box, which we are now ready to define. The case that Λ' is a (lattice) periodic path is the one considered in [5]. We have introduced off-lattice paths not to get subcomplexes unobtainable from lattice paths, but rather to provide more convenient descriptions of some of those subcomplexes.

Now we turn to complexes involving truncated paths. We only use lattice truncated paths.

Definition 4.7. Given a truncated path Λ_{tr} defined on an interval I, let $C_*(\Lambda_{\text{tr}})$ denote the subgroup of $C_*^{\text{tr}}(I)$ generated by $(\Lambda'_{\text{tr}}, \ell, o)$ with $\Lambda'_{\text{tr}} \leq \Lambda_{\text{tr}}$, where we require $\Lambda'_{\text{tr}}, \Lambda_{\text{tr}}$ to have the same domain and matching endpoints.

The subgroup $C_*(\Lambda_{\text{tr}})$ is a subcomplex. We need the homology of the subcomplex in the case where I has width π. We especially need it for intervals of the form $[k\pi/2, k\pi/2+\pi]$, but we may easily state it for all $[\Theta_0, \Theta_0 + \pi]$, taking advantage of our convention that a truncated path may not have an edge at an angle forming an endpoint of its domain.

Proposition 4.8. *Suppose Λ_{tr} is defined on $I = [\Theta_0, \Theta_0 + \pi]$ for some $\Theta_0 \in \mathbb{R}$. If Λ_{tr} is not constant, then $H_*(\Lambda_{tr}) \cong \mathbb{Z}^2$, generated by $E_{\Lambda_{tr}}$ and $H_{\Lambda_{tr}}$. If Λ_{tr} is constant, $H_*(\Lambda_{tr}) \cong \mathbb{Z}$, generated by $E_{\Lambda_{tr}}$.*

In either case, we may say that $H_*(\Lambda_{\text{tr}})$ is generated by $E_{\Lambda_{\text{tr}}}$ and $H_{\Lambda_{\text{tr}}}$, since $H_{\text{const}_p} = 0$.

Proof. If $\Lambda_{\text{tr}} = \text{const}_p$, the only path that can appear in $C_*(\Lambda_{\text{tr}})$ is const_p, and the only way to label it is E_{const_p}. We have $\delta = 0$ and $H_*(\text{const}_p)$ is as claimed.

If Λ_{tr} is not constant, the result follows once we satisfy the hypotheses of [5, Proposition 5.6], which are that Λ_{tr} has distinct endpoints, and is *convex*: "it is parametrized by an interval of length $\leq 2\pi$ and it traverses a subset of the boundary of a convex polygon, possibly a 2-gon." The convex polygon is made by joining to Λ_{tr} an edge from $\Lambda_{\text{tr}}(\Theta_0+\pi)$ to $\Lambda_{\text{tr}}(\Theta_0)$. Each vertex of Λ_{tr} is to the left in the sense of $<_{\Theta_0}$ compared to the previous vertex. Thus, $\Lambda_{\text{tr}}(\Theta_0+\pi) <_{\Theta_0} \Lambda_{\text{tr}}(\Theta_0)$ (and in particular these points are distinct) and the extra edge can be given an angle in $(\Theta_0+\pi, \Theta_0+2\pi)$, making a convex polygon. □

4.4 Boxes and strips

4.4.1 Boxes

A box is essentially an off-lattice periodic path with edges parallel to the X and Y axes.

Definition 4.9. An (A,n,γ)-*box*, or just *box*, is an off-lattice (A,n,γ)-periodic path B that has an edge at each $k\frac{\pi}{2}$, $k\in\mathbb{Z}$, and has no other edges ($\text{Ang}(B)=\frac{\pi}{2}\mathbb{Z}$). We require $0 <_{k\pi/2} B$ for all k.

We are interested in B for the sake of $C_*(B)$, the subcomplex of $C_*(A,n,\gamma)$ generated by (Λ,ℓ,o) with $\Lambda\le B$, defined in Definition 4.6.

Proposition 4.10. *If B is an (A,n,γ)-box and Λ is an off-lattice (A,n,γ)-periodic path, then $\Lambda\le B$ if and only if $\Lambda \le_{k\pi/2} B$ for all $k\in\mathbb{Z}$.*

Proof. If $\Lambda\le B$, $\Lambda(k\frac{\pi}{2}+\epsilon) \le_{k\frac{\pi}{2}+\epsilon} q$ where q is the appropriate vertex of B ($q=B(c)$, $c=(k\frac{\pi}{2},(k+1)\frac{\pi}{2})$). Letting $\epsilon\to 0$, we get $\Lambda(k\frac{\pi}{2}) \le_{k\frac{\pi}{2}} q$ which means $\Lambda(k\frac{\pi}{2}) \le_{k\frac{\pi}{2}} B$.

Conversely, assume $\Lambda(k\frac{\pi}{2}) \le_{k\frac{\pi}{2}} q$ (where q is as in the previous paragraph), and similarly for $(k+1)\frac{\pi}{2}$. We will show $\Lambda \le_\Theta q$ when $\Theta\in(k\frac{\pi}{2},(k+1)\frac{\pi}{2})$. By Lemma 4.4, we have

$$\Lambda(\Theta) \ \leq_{k\frac{\pi}{2}} \ \Lambda(k\frac{\pi}{2}) \ \leq_{k\frac{\pi}{2}} \ q \tag{4.2}$$

and similarly for $(k+1)\frac{\pi}{2}$. Thus, $\Lambda(\Theta)$ lies in the quadrant-shaped region with vertex at q of points $\leq_{k\frac{\pi}{2}} \ q$ and $\leq_{(k+1)\frac{\pi}{2}} \ q$. All of that region is $\leq_\Theta \ q$, completing the proof. $\square$

Proposition 4.10 shows that we may compare boxes by comparing their edges. In particular, $C_*(B) \subseteq C_*(B')$ if and only if $B \leq B'$.

We will often need to say $\Lambda_{\text{tr}} \leq B$ where Λ_{tr} is defined on an interval such as $[0, \pi]$. Our existing conventions are unhelpful when the endpoints of the interval are angles at which B has edges, so we make a special definition. For the case of $[0, \pi]$, it will mean $\Lambda_{\text{tr}} \leq B$ if and only if $\Lambda_{\text{tr}} \ \leq_\Theta \ B$ for $\Theta = 0, \pi/2, \pi$.

Definition 4.11. If Λ_{tr} is a truncated path and B is a box, $\Lambda_{\text{tr}} \leq B$ if $\Lambda_{\text{tr}} \ \leq_{k\pi/2} \ B$ for all $k \in \mathbb{Z}$ such that $k\pi/2 \in \overline{\text{Dom}}(\Lambda_{\text{tr}})$.

Note that $\Lambda \leq B$ if and only if $\Lambda|[k\pi, (k+1)\pi] \leq B$ for all k.

4.4.2 Strips

A *strip* is an infinite analogue of a box. Imagine enlarging a box infinitely in the $\pm X$ directions, sending the vertical edges away to infinity and making the horizontal edges infinitely long. Thus, a strip is made of (horizontal) lines, whereas boxes and polygonal paths are made out of line segments. Recall that a polygonal path or an off-lattice polygonal path is a collection $\{(\text{directed line segment}, \text{angle})\}$.

Definition 4.12. An (A, n, γ)-*strip* or just *strip* B_∞ is a set

$$B_\infty = \{(\text{Edge}_{B_\infty}(s\pi), s\pi) \mid s \in \mathbb{Z}\}, \tag{4.3}$$

where:

1. For all s, $\mathrm{Edge}_{B_\infty}(s\pi)$ is an oriented line in $\mathbb{R}^2$, which is parallel to the X-axis and oriented in the direction of the vector $(\cos s\pi, \sin s\pi) = ((-1)^s, 0)$.

For $p \in \mathbb{R}^2$ we say $p \ \leq_{s\pi} \ B_\infty$ if $\det(q-p, \vec{\Theta}) \geq 0$ for one (and hence every) $q \in \mathrm{Edge}_{B_\infty}(s\pi)$; and similarly for $<_{s\pi}$.

2. We require $0 \ <_{s\pi} \ B_\infty$ for all $s \in \mathbb{Z}$.

3. For all $s \in \mathbb{Z}$, the line $\mathrm{Edge}_{B_\infty}((s+N)\pi)$ (ignoring orientation) is the image under $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ of $\mathrm{Edge}_{B_\infty}(s\pi)$.

The definitions of strip and $\leq_{s\pi} \ B_\infty$ are manifestly periodic: If Λ is periodic, $\Lambda(s\pi) \ \leq_{s\pi} \ B_\infty$ if and only if $\Lambda((s+N)\pi) \ \leq_{(s+N)\pi} \ B_\infty$.

A strip is determined by the Y-coordinates of its edges. Thus specifying a strip is equivalent to specifying a sequence of positive reals $\{Y_{s\pi}\}$ satisfying $Y_{(s+N)\pi} = a^{-1} Y_{s\pi}$. Similarly, a box is specified by a pair of sequences, specifying the X- and Y-coordinates of its edges, with suitable periodicity conditions.

Recall $\overline{\mathrm{Dom}}(\Lambda)$ is the domain of an off-lattice truncated path, and $\overline{\mathrm{Dom}}$ of an off-lattice periodic path is $\mathbb{R}$.

Definition 4.13. Suppose Λ is an off-lattice truncated or (A,n,γ)-periodic path. Given an (A,n,γ)-strip B_∞, we define $\Lambda \leq B_\infty$ if $\Lambda(s\pi) \leq_{s\pi} \ B_\infty$ for all $s \in \mathbb{Z}$ such that $s\pi \in \overline{\mathrm{Dom}}(\Lambda)$, provided Λ has no edge at $s\pi$; if it does have an edge, replace $\Lambda(s\pi)$ with some choice of $p \in \mathrm{Edge}_\Lambda(s\pi)$.

The appropriate transitivity properties of $\leq_{s\pi}$ are true.

Definition 4.14. Given an (A, n, γ)-strip B_∞, let $C_*(B_\infty)$, or $C_*(A, n, \gamma; B_\infty)$, denote the subgroup of $C_*(A, n, \gamma)$ generated by (Λ, ℓ, o) with $\Lambda \leq B_\infty$.

This subgroup is a subcomplex.

Remark 4.15. *A natural common generalization of the subcomplexes of $C_*(A, n, \gamma)$ associated to strips and off-lattice paths, possibly useful for the case of elliptic A, is the subcomplex generated by (Λ, ℓ, o) with $\Lambda(\Theta_s) \leq_{\Theta_s} L_{\Theta_s}$ for some periodic collections $\{\Theta_s\}$, $\{L_{\Theta_s}\}$ of angles and lines (with L_{Θ_s} oriented parallel to $(\cos\Theta_s, \sin\Theta_s)$).*

4.4.3 The region $\square_{B,\Theta}$

We introduce the notation $\square_{B,\Theta}$. This notation refers to a rectangle (an honest rectangle in the plane) with sides parallel to the axes, if B is a box; or a region of the form $\mathbb{R} \times [Y_{\min}, Y_{\max}]$, if B is a strip. (This notation further motivates the names "box" and "strip".) (One may think of $\square_{B,\Theta}$ as the points $p \leq B|[\Theta - \pi, \Theta + \pi]$.)

Definition 4.16. Given an (A, n, γ)-box B and $\Theta = k\pi/2$ for some $k \in \mathbb{Z}$, define

$$\square_{B,\Theta} = \{p \in \mathbb{R}^2 \mid p \leq_{(k-2)\pi/2} B,\ p \leq_{(k-1)\pi/2} B, \ldots, \text{ and } p \leq_{(k+2)\pi/2} B\}. \tag{4.4}$$

Given an (A, n, γ)-strip B and $\Theta = k\pi/2$ for some $k \in \mathbb{Z}$, define $\square_{B,\Theta}$ by (4.4), defining expressions of the form $p \leq_{s\pi+\pi/2} B$ to be trivially satisfied ($s \in \mathbb{Z}$).

Lemma 4.17. *Given an (A, n, γ)-periodic path Λ,*

1. *Given an (A, n, γ)-box B, $\Lambda \leq B$ if and only if $\Lambda(k\pi/2) \in \square_{B,\Theta}$ for all $k \in \mathbb{Z}$.*

2. *Given an (A, n, γ)-strip B, $\Lambda \leq B$ if and only if $\Lambda(s\pi) \in \square_{B,\Theta}$ for all $s \in \mathbb{Z}$.*

Proof. It is only necessary to verify "only if". In either case, assume $\Lambda \le B$ and Θ is one of the angles mentioned, a multiple of π or $\pi/2$. We have $\Lambda(\Theta) \le_\Theta B$, and must show $\Lambda(\Theta) \le_{\Theta'} B$ where Θ' is a nearby multiple of the same basic angle. But $\Lambda(\Theta) \le_{\Theta'} \Lambda(\Theta') \le_{\Theta'} B$, by Lemma 4.4, completing the proof. □

4.4.4 Direct limits of boxes

The point of the subcomplexes $C_*(B)$, where B is a box, is that $C_*(A,n,\gamma)$ is a direct limit of those subcomplexes. We could obtain $C_*(A,n,\gamma)$ as the direct limit over all boxes, but we will use the direct limit of a sequence.

Proposition 4.18. *Suppose $\{B_k\}$ is a sequence of (A,n,γ)-boxes (for $k=1,2,\dots$). Suppose the heights and widths tend to infinity: for each $i \in \mathbb{Z}$ the Y-coordinate of $\mathrm{Edge}_{B_k}(2\pi i)$ tends to $-\infty$ as $k \to \infty$, and similarly the X- or Y-coordinate at other $\Theta \in \frac{\pi}{2}\mathbb{Z}$ tends to $\pm\infty$, whichever direction takes the edge to the right (with respect to $\le_\Theta$). Then, letting Λ range over (A,n,γ)-periodic paths, we have*

$$\{\textit{all } \Lambda\} = \bigcup_k \{\Lambda \mid \Lambda \le B_k\} \tag{4.5}$$

and

$$C_*(A,n,\gamma) \cong \varinjlim C_*(B_k) \tag{4.6}$$

$$H_*(A,n,\gamma) \cong \varinjlim H_*(B_k). \tag{4.7}$$

Proof. Showing $\Lambda \le B_k$ is equivalent to showing $\Lambda|[0,N\pi] \le B_k$, which requires checking finitely many points of Λ (using Proposition 4.10). Therefore, given an (A,n,γ)-periodic path Λ, a sufficiently large k satisfies $\Lambda \le B_k$. Thus every generator (Λ,ℓ,o) of $C_*(A,n,\gamma)$

appears somewhere in the subcomplexes $\{C_*(B_k)\}$. Further, the homology of a direct limit is the direct limit of the homologies. $\square$

We do not need to relate $H_*(A, n, \gamma)$ to $H_*(B_\infty)$ for a strip B_∞ in the current chapter. We do need that $H_*(B_\infty)$ is a limit of $H_*(B_k)$ for appropriate boxes B_k. (This motivates the notation B_∞ for a strip.)

Proposition 4.19. *Suppose* $\{B_k\}$ *is a sequence of* (A, n, γ)*-boxes (for* $k = 1, 2, \dots$*). For each* $s \in \mathbb{Z}$*, suppose* $Y(\mathrm{Edge}_{B_k}(s\pi))$ *is independent of* k*, and let* B_∞ *be the strip with* $Y(\mathrm{Edge}_{B_\infty}(s\pi)) = Y(\mathrm{Edge}_{B_k}(s\pi))$*. Suppose* $X(\mathrm{Edge}_{B_k}(s\pi + \pi/2)) \to \pm\infty$ *(the sign being whichever choice takes the edge to the right, as in Proposition 4.18). Then, letting* Λ *range over* (A, n, γ)*-periodic paths, we have*

$$\{\Lambda \mid \Lambda \leq B_\infty\} = \bigcup_k \{\Lambda \mid \Lambda \leq B_k\} \tag{4.8}$$

and

$$C_*(B_\infty) \cong \varinjlim C_*(B_k) \tag{4.9}$$

$$H_*(B_\infty) \cong \varinjlim H_*(B_k). \tag{4.10}$$

Proof. The proof is similar to that of Proposition 4.18. Apply Proposition 4.10 and the definition of $\Lambda \leq B_\infty$ in Definition 4.13. $\square$

Chapter 5

The flattened subcomplex

Let $A \in SL_2(\mathbb{Z})$ be hyperbolic or $-\mathbb{1}$, let n be a positive integer, and let $\Gamma \in \mathbb{Z}^2/\operatorname{Im}(\mathbb{1} - A)$ with $\gamma \in \Gamma$. Recall that A is positive hyperbolic, we have $N = 2n$, and if A is negative hyperbolic or $-\mathbb{1}$, we have $N = 2n - 1$.

Recall that we work with periodic polygonal paths, satisfying a "periodicity" property involving A. In particular, all of a periodic path Λ is determined by periodicity from $\Lambda|[0, N\pi]$.

For any (A, n, γ)-box or strip B, we will define either one or two subcomplexes of $C_*(B)$. If B is a box, we may define two *flattened subcomplexes*, $C_*^{\text{flat}}(B, \Theta_0)$ where $\Theta_0 = \pi$ or $\Theta_0 = \pi/2$. These subcomplexes involve paths "maximal" on $[\Theta_0 + s\pi, \Theta_0 + (s+1)\pi]$ for each $s \in \mathbb{Z}$. If B is a strip, we can only define $C_*^{\text{flat}}(B, \pi/2)$.

Theorem 5.1. *Let B be an (A, n, γ)-box or strip and let Θ_0 be π or $\pi/2$, with $\Theta_0 = \pi/2$ if B is a strip. The inclusion*

$$C_*^{\text{flat}}(B, \Theta_0) \hookrightarrow C_*(B) \tag{5.1}$$

induces an isomorphism on homology.

This chapter generalizes [5, §6.1].

5.1 Description of the flattened subcomplexes

Recall that we use eigencoordinates (X, Y) on the plane, making the periodicity condition of an (A, n, γ)-periodic path use $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$. The corresponding polar coordinate is Θ, so (e.g.) $\Theta = 2\pi k$ means the $+X$-direction.

Definition 5.2. If B is an (A, n, γ)-box, a *B-flattenable angle* is π or $\pi/2$. If B is an (A, n, γ)-strip, the only B-flattenable angle is $\pi/2$. We say (B, Θ_0) is *flattenable* if B is an (A, n, γ)-box or strip and Θ_0 is a flattenable angle. For either Θ_0, let $\Theta_s = \Theta_0 + s\pi$.

Definition 5.3. Suppose (B, Θ_0) is flattenable. A (B, Θ_0)*-flattened path*, or just *flattened path*, is an (A, n, γ)-periodic path Λ with $\Lambda \leq B$ such that, for all $s \in \mathbb{Z}$, the restriction $\Lambda|[\Theta_s, \Theta_{s+1}]$ satisfies:

For any truncated path Λ' defined on $[\Theta_s, \Theta_{s+1}]$ with

$$\Lambda' \leq B, \tag{5.2}$$

$$\Lambda'(\Theta_s) = \Lambda(\Theta_s), \text{ and} \tag{5.3}$$

$$\Lambda'(\Theta_{s+1}) = \Lambda(\Theta_{s+1}), \tag{5.4}$$

we have $\Lambda' \leq \Lambda|[\Theta_s, \Theta_{s+1}]$.

The term (B, Θ_0)-flattened path will imply that (B, Θ_0) is flattenable. (Flattened paths would typically not exist if we tried to use $\Theta_0 = \pi$ with a strip.)

Recall that all of Λ repeats "after $N\pi$"; $\Lambda(F_{A,n}(\Theta))$ is determined by $\Lambda(\Theta)$ where $F_{A,n}$ is a distorted version of adding $N\pi$, satisfying $F_{A,n}(k\pi/2) = N\pi + k\pi/2$. In particular, the whole path is determined from $\Lambda|[0, N\pi]$, or more generally, $\Lambda|[\Theta_0, \Theta_N]$.

Definition 5.4. A *flattened labeling* is a sequence $\{\eta_s\}$ of symbols 'E' and 'H' (so $\eta_s \in \{\text{'}E\text{'}, \text{'}H\text{'}\}$ for all $s \in \mathbb{Z}$) satisfying $\eta_{s+N} = \eta_s$.

The idea is to associate η_s with $\Lambda|[\Theta_s, \Theta_{s+1}]$. The symbol '$E$' will mean that all edges are labeled 'e', and 'H' will mean that exactly one is 'h' (and the rest are 'e'). There is one compatibility requirement: If $\Lambda|[\Theta_s, \Theta_{s+1}]$ is a constant path, we should not label it with 'H' because it has no edge to label with 'h'.

Definition 5.5. A pair $(\Lambda, \{\eta_s\})$ consisting of a flattened path and a flattened labeling is *in range* if, for all s such that $\Lambda(\Theta_s) = \Lambda(\Theta_{s+1})$, we have $\eta_s = \text{'}E\text{'}$. Otherwise, the pair is *out of range*.

Recall that a labeled periodic path is a triple (Λ, ℓ, o) where Λ is a periodic path, ℓ assigns an 'e' or 'h' label to each edge of Λ, and o is an ordering on the 'h' labels. More formally, ℓ assigns a label not to an edge or to its angle $\Theta \in \text{Ang}(\Lambda)$, but to an equivalence class of angles modulo periodicity. We may view o as an ordering on the edges labeled 'h' with edge-angle in $[\Theta_0, \Theta_N]$. A flattened generator is a sum of labeled periodic paths, specifically a certain sum of ways of labeling a given flattened path Λ. Recall that Λ cannot have an edge at any angle $k\pi/2$ (in particular, at Θ_{s+1}).

Definition 5.6. Given a (B, Θ_0)-flattened path Λ and a flattened labeling $\{\eta_s\}$, if $(\Lambda, \{\eta_s\})$ is in range, the *flattened generator* $[\Lambda; \{\eta_s\}]$ is the sum of all labeled periodic paths (Λ, ℓ, o) where ℓ satisfies

- For all s with $\eta_s =$ 'E', ℓ assigns the label 'e' to all $\Theta \in \mathrm{Ang}(\Lambda) \cap [\Theta_s, \Theta_{s+1}]$.

- For all s with $\eta_s =$ 'H', ℓ assigns the label 'h' to exactly one $\Theta \in \mathrm{Ang}(\Lambda) \cap [\Theta_s, \Theta_{s+1}]$.

and o satisfies

- The 'h'-labeled edge at $\Theta \in [\Theta_0, \Theta_N]$ precedes (in o) the 'h'-labeled edge at $\Theta' \in [\Theta_0, \Theta_N]$ if and only if $\Theta < \Theta'$.

In other words, the first 'h' label in the ordering is the one in $[\Theta_0, \Theta_1]$, if any; the next is the one in $[\Theta_1, \Theta_2]$, if any; and so on through $[\Theta_{N-1}, \Theta_N]$.

We also call $[\Lambda; \{\eta_s\}]$ a (B, Θ_0)-*flattened generator*, which will imply that (B, Θ_0) is flattenable.

Notation Convention 5.7. If $(\Lambda, \{\eta_s\})$ is out of range, write $[\Lambda; \{\eta_s\}] = 0$, but do not call $[\Lambda; \{\eta_s\}]$ a flattened generator.

Strict application of the terms of Definition 5.6 to an out-of-range $(\Lambda, \{\eta_s\})$ would result in an empty sum, so the convention make sense.

Definition 5.8. Define $C_*^{\mathrm{flat}}(B, \Theta_0)$ to be the subgroup of $C_*(B)$ generated by all the flattened generators. Here, (B, Θ_0) is flattenable.

Later, we will see that $C_*^{\mathrm{flat}}(B, \Theta_0)$ is a subcomplex of $C_*(B)$, justifying the name:

Definition 5.9. We call $C_*^{\mathrm{flat}}(B, \Theta_0)$ the *flattened subcomplex* of $C_*(B)$ with flattenable angle Θ_0.

We will see below that $H_*^{\mathrm{flat}}(B, \Theta_0) \cong H_*(B)$.

Remark 5.10. *Many generalizations of this definition of flattening are potentially useful for the elliptic and parabolic cases. The definition given here partitions* $\mathbb{R}$ *as* $\bigcup[\Theta_s, \Theta_{s+1}]$. *(One could obtain a more honest "partition" using* $[\Theta_s, \Theta_{s+1})$*, but no edge at angle* Θ_{s+1} *can have its endpoints in the lattice* $L_{A,\Gamma}$*, so the difference does not matter.) Most generally, we may use any set of intervals (open, closed, or half-open) which partition* $\{$*angles compatible with* $L_{A,\Gamma}\}$*, and satisfy a periodicity condition and bounds on* $\Theta_{s+1} - \Theta_s$.

5.2 Flattened Λ is determined by $\{\Lambda(\Theta_s)\}$

A flattened path is determined by the sequence of points $\{\Lambda(\Theta_s)\}$.

Recall that we denote the X-coordinate of p in the plane by $X(p)$ and similarly for $Y(p)$ (these are eigencoordinates). We write $p \leq_{\Theta_s} p'$ if $\det\left(\begin{smallmatrix} X(p'-p) & \cos\Theta_s \\ Y(p'-p) & \sin\Theta_s \end{smallmatrix}\right) \geq 0$. Note that $\leq_{\Theta_s}$ is the same relation for any even value of s, and the opposite relation for any odd s. See §4.4.3 for $\square_{B,\Theta}$.

Lemma 5.11. *Suppose* (B, Θ_0) *is flattenable. The map* $\Lambda \mapsto \{\Lambda(\Theta_s)\}$ *is a bijection from* (B, Θ_0)*-flattened paths to sequences* $\{p^s\}_{s\in\mathbb{Z}}$ *of points in the lattice* $L_{A,\Gamma}$ *satisfying:*

1. $p^{s+N} = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)(p^s)$
2. $p^s \leq_{\Theta_{s+1}} p^{s+1}$
3. $p^s \in \square_{B,\Theta_s}$

If $\Lambda \leq B$ *is a periodic path (not necessarily flattened) the points* $p^s = \Lambda(\Theta_s)$ *still satisfy the above conditions.*

If $\Theta_0 = \pi/2$, Condition 2 means $X(p^s) \le X(p^{s+1})$ if s is odd and $X(p^s) \ge X(p^{s+1})$ if s is even.

The notation p^s has a superscript because it will later acquire a subscript as well.

Proof. First we show, given periodic $\Lambda \le B$, that $\{p^s\} = \{\Lambda(\Theta_s)\}$ satisfies Conditions 1-3. Periodicity of Λ is $\Lambda(F_{A,n}(\Theta)) = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \Lambda(\Theta)$, which means $\Lambda(\Theta_s + N\pi) = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \Lambda(\Theta_s)$. Condition 1 follows. As one travels from $\Lambda(\Theta_s)$ to $\Lambda(\Theta_{s+1})$ along edges of Λ, each step is in some direction in $[\Theta_s, \Theta_{s+1}]$. (See Lemma 4.4.) Therefore the sum of the edge vectors, $\Lambda(\Theta_{s+1}) - \Lambda(\Theta_s)$, must also lie in such a direction, so $\Lambda(\Theta_s) \le_{\Theta_{s+1}} \Lambda(\Theta_{s+1})$. This proves condition 2. Condition 3 follows from Lemma 4.17.

Now (to prove surjectivity) given $\{p^s\}$ satisfying conditions 1–3, we construct flattened Λ with $\Lambda(\Theta_s) = p^s$. We first construct $\Lambda|[\Theta_s, \Theta_{s+1}]$ for one s, then glue them together to obtain a periodic path Λ, then show Λ is a (B, Θ_0)-flattened path. (The construction is derived from the proof of [5, Proposition 2.10b].)

Let

$$\Pi = \{q \in \mathbb{R}^2 \mid p^s \le_{\Theta_{s+1}} q \le_{\Theta_{s+1}} p^{s+1} \text{ and } q \le_{\Theta_s + \frac{\pi}{2}} B\}, \tag{5.5}$$

a semi-infinite rectangular region. Let $\mathcal{H}$ be the convex hull of $\Pi \cap L_{A,\Gamma}$. Build $\Lambda|[\Theta_s, \Theta_{s+1}]$ by traversing $\partial\mathcal{H}$ counterclockwise from p^s to p^{s+1}. (If $p^s = p^{s+1}$, $\Lambda|[\Theta_s, \Theta_{s+1}] = \text{const}_{p^s}$.) The point p^s is the leftmost (with respect to $\le_{\Theta_{s+1}}$) point in all of $\mathcal{H}$ and p^{s+1} is the rightmost, so every edge we traversed of the convex hull takes a step to the right; thus every edge may be given an angle in $[\Theta_s, \Theta_{s+1}]$. Furthermore, since $\mathcal{H}$ is convex, these angles are increasing, and $\Lambda|[\Theta_s, \Theta_{s+1}]$ is a truncated path.

The resulting truncated paths, for each $s \in \mathbb{Z}$, may be glued together to form one path Λ because appropriate endpoints agree and the angles of the edges are increasing. This path is periodic because $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ sends p^s, p^{s+1}, B to p^{s+N}, p^{s+N+1}, B; it preserves the convex hull construction and thus sends $\Lambda|[\Theta_s, \Theta_{s+1}]$ to $\Lambda|[\Theta_{s+N}, \Theta_{s+N+1}]$.

By construction $\Lambda|[\Theta_s, \Theta_{s+1}] \leq B$, so $\Lambda \leq B$. It remains to check the maximality of Λ. Suppose $\Lambda' \leq B$ is defined on $[\Theta_s, \Theta_{s+1}]$. We have Λ' contained in Π. Any vertex $\Lambda'(\Theta)$ is in $\Pi \cap L_{A,\Gamma}$ and hence in $\mathcal{H}$. But $\Lambda(\Theta)$ (assuming $\Theta \notin \operatorname{Ang}(\Lambda)$) is on the boundary of the convex region $\mathcal{H}$ and all of $\mathcal{H}$ is $\leq_\Theta \Lambda(\Theta)$. Thus $\Lambda'(\Theta) \leq \Lambda(\Theta)$, proving that Λ is a flattened path.

Thus the image of $\Lambda \mapsto \{\Lambda(\Theta_s)\}$ is exactly the set specified in the statement of the lemma. To prove injectivity, suppose flattened paths Λ and Λ' satisfy $\Lambda(\Theta_s) = \Lambda'(\Theta_s)$ for all $s \in \mathbb{Z}$; then

$$\Lambda'|[\Theta_s, \Theta_{s+1}] \leq \Lambda|[\Theta_s, \Theta_{s+1}], \tag{5.6}$$

and vice versa. Thus Λ' agrees with Λ on $[\Theta_s, \Theta_{s+1}]$ for each s. □

The following result, a corollary of the proof of Lemma 5.11, is essentially a strengthening of the maximality condition of Definition 5.3.

Lemma 5.12. *A (B, Θ_0)-flattened path Λ satisfies: For any truncated path Λ' defined on $[\Theta_s, \Theta_{s+1}]$ with*

$$\Lambda' \leq B, \tag{5.7}$$

$$\Lambda'(\Theta_s) \leq_{\Theta_s} \Lambda(\Theta_s), \text{ and} \tag{5.8}$$

$$\Lambda'(\Theta_{s+1}) \leq_{\Theta_{s+1}} \Lambda(\Theta_{s+1}), \tag{5.9}$$

we have $\Lambda' \leq \Lambda|[\Theta_s, \Theta_{s+1}]$.

Proof. Since Lemma 5.11 gives a bijection, Λ may be built by the construction in the proof of that lemma. The proof given there that $\Lambda' \leq \Lambda|[\Theta_s, \Theta_{s+1}]$ applies to this case too. $\square$

We can now extend the definition of "in range" so that in-range $(\{p^s\}, \{\eta_s\})$ are in bijection with flattened generators.

Definition 5.13. A pair $(\{p^s\}, \{\eta_s\})$ consisting of a sequence $\{p^s\}$ of lattice points $(s \in \mathbb{Z})$ and a flattened labeling $\{\eta_s\}$ is *in range* if $\{p^s\}$ satisfies the conditions in Lemma 5.11 and, letting Λ be the path guaranteed by those conditions, $(\Lambda, \{\eta_s\})$ is in range. Otherwise, it is out of range.

5.3 Indexing the points $\{\Lambda(\Theta_s)\}$ by (half-)integers $\{b_s\}$

We now "combinatorialize" further, indexing the points p^s by integers if $\Gamma = 0$, or half-integers if $\Gamma \neq 0$. We do a special case as warm-up.

In the simplest case, B is a strip (so Θ_0 is $\pi/2$), $\Gamma = 0$ (so $0 \in L_{A,\Gamma}$), and $s \in \mathbb{Z}$ is even (so larger X-coordinates mean "bigger" Λ, i.e. $p \leq_{\Theta_s} q$ iff $X(p) \leq X(q)$). No two lattice points ever have the same X-coordinate, so the lattice points in $\square_{B,\Theta_s}$ are linearly ordered by their X-coordinate. Let $p^s_0 = 0$, let p^s_b (b a positive integer) be the lattice point in $\square_{B,\Theta_s}$ with the b'th-smallest positive X-coordinate, and let p^s_{-b} be the lattice point in $\square_{B,\Theta_s}$ with the b'th-largest negative X-coordinate. These conventions define a bijection between $\mathbb{Z}$ and the lattice points in $\square_{B,\Theta_s}$, because there are infinitely many points in $\square_{B,\Theta_s}$ with $X < 0$ and infinitely many with $X > 0$ (see Lemma 3.75).

We generalize the indexing $b \mapsto p^s_b$ in several directions. If $\Gamma \neq 0$, there is no good choice for p^s_0. Instead, let $p^s_{1/2}$ be the lattice point with smallest positive X-coordinate, and $p^s_{-1/2}$ with largest negative X-coordinate. Let $p^s_{b+1/2}$ and $p^s_{-b-1/2}$ be in order of X-coordinate, as before.

If B is a box (and $\Theta_0 = \pi/2$), we index by only part of $\mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$, namely some $\{b \mid b_{\min} \leq b \leq b_{\max}\}$. If s is odd, reverse the direction, ordering by $-X$. Finally, if $\Theta_0 = \pi$, order by Y instead of X.

We make all these considerations formal. Recall that all boxes and strips satisfy $\text{const}_0 \leq B$ (by definition), so if $\Gamma = 0$, we may always let $p^s_0 = 0$. (If $\Gamma = 0$ it is possible that B is too small for $p^s_{1/2}$ and $p^s_{-1/2}$, an unimportant case, but one we allow for.)

Definition 5.14. Let (B, Θ_0) be flattenable and let $s \in \mathbb{Z}$.

1. If $\Gamma = 0$, let $b \mapsto p^s_b$ be the map from a subset of $\mathbb{Z}$ to $\square_{B,\Theta_s} \cap L_{A,\Gamma}$ which satisfies $p^s_0 = 0$ and which respects the ordering $\leq_{\Theta_s}$, and respects "adjacency": If $p <_{\Theta_s} q$ with no points of $\square_{B,\Theta_s} \cap L_{A,\Gamma}$ in between (in the sense of $<_{\Theta_s}$), then $p = p^s_b$ and $q = p^s_{b+1}$ for some $b \in \mathbb{Z}$.

2. If $\Gamma \neq 0$, let $b \mapsto p^s_b$ be the map from a subset of $\mathbb{Z} + \frac{1}{2}$ to $\square_{B,\Theta_s} \cap L_{A,\Gamma}$ which respects $\leq_{\Theta_s}$ and adjacency, with $p^s_{1/2}$ is the leftmost $q \in \square_{B,\Theta_s} \cap L_{A,\Gamma}$ with $0 <_{\Theta_s} q$ (if such q exists), and $p^s_{-1/2}$ is the rightmost $q \in \square_{B,\Theta_s} \cap L_{A,\Gamma}$ such that $q <_{\Theta_s} 0$ (if such q exists).

We see that such a map exists and is unique. If B is a strip, the map is a bijection between $\mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$ and $\square_{B,\Theta_s} \cap L_{A,\Gamma}$. If B is a box, the map is a bijection with $\mathbb{Z} \cap [b^s_{\min}, b^s_{\max}]$ or $\mathbb{Z} + \frac{1}{2} \cap [b^s_{\min}, b^s_{\max}]$. We suppress the dependence of $[b^s_{\min}, b^s_{\max}]$, as well as $p^s_{b_s}$ itself, on B and Θ_0.

For a given flattened Λ, $\Lambda(\Theta_s)$ is in $\square_{B,\Theta_s}$ and may be written $p^s_{b_s}$ for some b_s in $\mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$, so Λ gives rise to a sequence $\{b_s\}$. We will turn the correspondence between Λ and $\{\Lambda(\Theta_s)\}$ (Lemma 5.11) into a correspondence between Λ and $\{b_s\}$. We first state the conditions $\{b_s\}$ satisfies.

Definition 5.15. A *size sequence* $\{b_s\}_{s\in\mathbb{Z}}$, also called an (A, n, γ)-size sequence, is a sequence with $b_s \in \mathbb{Z}$ if $\Gamma = 0$ and $b_s \in \mathbb{Z}+\frac{1}{2}$ if $\Gamma \neq 0$; and we require $b_{s+N} = b_s$ for all s.

The intent of the name size sequence is that b_s says how far from 0 the path goes, (at $\Theta = \Theta_s$).

Definition 5.16. A size sequence $\{b_s\}$ is *in range* if

1. $b_s \in [b^s_{\text{min}}, b^s_{\text{max}}]$ for all s, if B is a box.

2. $p^s_{b_s} \leq_{\Theta_{s+1}} p^{s+1}_{b_{s+1}}$ for all s.

If B is a strip, we only impose the second condition.

Proposition 5.17. *Let (B, Θ_0) be flattenable. Given a periodic path $\Lambda \leq B$, the sequence $\{b_s\}$ specified by*

$$\Lambda(\Theta_s) = p^s_{b_s} \tag{5.10}$$

is an in-range size sequence, and the map defined by (5.10) from (B, Θ_0)-flattened paths to in-range size sequences is a bijection.

Notation Convention 5.18. If $\{b_s\}$ is an in-range size sequence, let $\Lambda_{\{b_s\}}$ be the flattened path such that $\Lambda_{\{b_s\}}(\Theta_s) = p^s_{b_s}$.

Proof of Proposition 5.17. Comparing with Lemma 5.11 shows that the only thing to check is periodicity, that $p^{s+N}_{b_{s+N}} = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) p^s_{b_s}$ is equivalent to $b_{s+N} = b_s$. But since $\square_{B,\Theta_{s+N}} = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)(\square_{B,\Theta_s})$, and since $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ preserves $L_{A,\Gamma}$ and relates $\leq_{\Theta_s}$ to $\leq_{\Theta_{s+N}}$, we have $p^{s+N}_{b_s} = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) p^s_{b_s}$. □

The quantity b_s can be thought of as representing $\mathrm{Size}(\Lambda|[\Theta_s - \frac{\pi}{2}, \Theta_s + \frac{\pi}{2}])$. We will define Size of part of a path in Chapter 6.

Definition 5.19. A pair $(\{b_s\}, \{\eta_s\})$ consisting of a size sequence $\{b_s\}$ and a flattened labeling $\{\eta_s\}$ is in range if $\{b_s\}$ is in range and $(\Lambda_{\{b_s\}}, \{\eta_s\})$ is in range. Otherwise, it is out of range.

Then there is exactly one flattened generator for each in-range pair $(\{b_s\}, \{\eta_s\})$.

Notation Convention 5.20. If $(\{b_s\}, \{\eta_s\})$ is in range, let $[\{b_s\}; \{\eta_s\}]$ be the flattened generator $[\Lambda_{\{b_s\}}; \{\eta_s\}]$. If $(\{b_s\}, \{\eta_s\})$ is out of range, let $[\{b_s\}; \{\eta_s\}] = 0$.

Example 5.21. The constant path (assuming $\Gamma = 0$) is $[\{0\}; \{`E'\}]$, i.e. $b_s = 0$ and $\eta_s = `E'$ for all s.

Lemma 5.22. *Given in-range size sequences $\{b_s\}$, $\{b'_s\}$, we have $\Lambda_{\{b_s\}} \leq \Lambda_{\{b'_s\}}$ if and only if $b_s \leq b'_s$ for all $s \in \mathbb{Z}$.*

Proof. If $\Lambda_{\{b_s\}} \leq \Lambda_{\{b'_s\}}$ then $\Lambda_{\{b_s\}}(\Theta_t) \leq_{\Theta_t} \Lambda_{\{b'_s\}}(\Theta_t)$ for all $t \in \mathbb{Z}$; we use the order-preserving map $b \mapsto p^t_b$ and get $b_t \leq b'_t$ for all $t \in \mathbb{Z}$.

Running the above argument in reverse, the step that "$\leq_{\Theta_t}$" implies "$\leq$" is filled by Lemma 5.12. □

5.4 The differential in terms of $\{b_s\}$ and $\{\eta_s\}$

Lemma 5.23. *Given a (B, Θ_0)-flattened generator $[\{b_s\}; \{\eta_s\}]$ and a corner c of $\Lambda_{\{b_s\}}$ which does not contain any angles of the form Θ_t ($t \in \mathbb{Z}$), we have $\delta_c [\{b_s\}; \{\eta_s\}] = 0$.*

Proof. Such a corner c is in the interior of some $[\Theta_s, \Theta_{s+1}]$ and the rounding operation depends only on the truncated $\Lambda|[\Theta_s, \Theta_{s+1}]$ and its labelings, which are $E_{\Lambda|[\Theta_s,\Theta_{s+1}]}$ or $H_{\Lambda|[\Theta_s,\Theta_{s+1}]}$, which are cycles. □

Recall that Λ cannot have an edge at Θ_t, so Θ_t is in some corner.

Proposition 5.24. *The differential applied to a flattened generator breaks up as the sum*

$$\delta [\{b_s\}; \{\eta_s\}] = \sum_{t=0}^{N-1} \delta_{\Theta_t} [\{b_s\}; \{\eta_s\}] . \tag{5.11}$$

We write δ_{Θ_t} for δ_c.

Proof. The left-hand side is the sum of δ_c over corners-mod-periodicity $[c]$ where c contains at least one Θ_t, and a proper corner is too narrow to contain Θ_t and Θ_{t+1}. Thus each term matches one term of the right-hand side after discarding from the right all Θ_t contained in non-proper corners. □

We need to compute $\delta_{\Theta_t} [\{b_s\}; \{\eta_s\}]$. We start with the unlabeled path $\Lambda_{\{b_s\}}$. We will show that rounding it at Θ_t ($t \in \mathbb{Z}$) yields another flattened path, which is (therefore) $\Lambda_{\{b'_s\}}$ for some sequence $\{b'_s\}$. Essentially, this sequence is the result of decrementing b_t and leaving the other b_s alone, but respecting periodicity; we introduce the notation $\{b_s\}\backslash t$ to describe $\{b'_s\}$.

Definition 5.25. Given a size sequence $\{b_s\}$, let $\{b_s\}\backslash t$ be the sequence $\{b'_s\}$ where

$$b'_s = \begin{cases} b_s & \text{if } s \not\equiv t \pmod{N} \\ b_s - 1 & \text{if } s \equiv t \pmod{N}. \end{cases} \tag{5.12}$$

Lemma 5.26. *Suppose $\{b_s\}$ is an in-range size sequence and $N > 1$. Then Θ_t is in a proper corner of $\Lambda_{\{b_s\}}$ if and only if $\{b_s\} \setminus t$ is in range.*

Proof. Suppose $\{b_s\} \setminus t$ is in range, so we may form $\Lambda_{\{b_s\}\backslash t}$. Applying Lemma 5.22, we have $\Lambda_{\{b_s\}\backslash t} \leq \Lambda_{\{b_s\}}$. Any path to the left of $\Lambda_{\{b_s\}}$ must agree with $\Lambda_{\{b_s\}}$ on all kinks of $\Lambda_{\{b_s\}}$ (by Lemma 3.70), but $\Lambda_{\{b_s\}\backslash t} \neq \Lambda_{\{b_s\}}$ at Θ_t, so the corner containing Θ_t is not a kink. Since $N > 1$, a non-kink corner is not a self-rounding corner (Lemma 3.42).

Suppose Θ_t is in a corner-mod-periodicity of $\Lambda_{\{b_s\}}$. Then we may form the path $\Lambda_{\{b_s\}} \setminus \Theta_t$ and the point

$$p = \big(\Lambda_{\{b_s\}} \setminus \Theta_t\big)(\Theta_t). \tag{5.13}$$

We have $\Lambda_{\{b_s\}} \setminus \Theta_t \leq \Lambda_{\{b_s\}} \leq B$, so $p \in \square_{B,\Theta_t}$ by Lemma 4.17. Thus $p = p^t_{b'}$ for some b' in $\mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$, and $b' < b_t$. Then the size sequence formed from $\{b_s\}$ by replacing b_{t+kN} with b' satisfies the hypotheses needed to be in range. The sequence $\{b_s\} \setminus t$ (using $b_t - 1$, which satisfies $b' \leq b_t - 1 \leq b_t$) is also in range. $\square$

Proposition 5.27. *If Λ is a (B, Θ_0)-flattened path, $t \in \mathbb{Z}$, and Θ_t is in a corner-mod-periodicity of Λ, then $\Lambda\backslash\Theta_t$ is a flattened path. If $\Lambda = \Lambda_{\{b_s\}}$, then $\Lambda_{\{b_s\}}\backslash\Theta_t = \Lambda_{\{b_s\}\backslash t}$.*

Proof. We must show $\Lambda_{\{b_s\}} \setminus \Theta_t = \Lambda_{\{b_s\}\backslash t}$. (We know $\Lambda_{\{b_s\}} \setminus \Theta_t$ exists, and $\Lambda_{\{b_s\}\backslash t}$ exists by Lemma 5.26.) It is sufficient (by Proposition 3.69) to show that $\Lambda_{\{b_s\}\backslash t}$ is the rightmost element of

$$\mathcal{S} = \{\Lambda' \mid \Lambda' \leq \Lambda_{\{b_s\}}, \Lambda'(\Theta_t) \neq \Lambda_{\{b_s\}}(\Theta_t)\}. \tag{5.14}$$

We have $\Lambda_{\{b_s\}\backslash t} \leq \Lambda_{\{b_s\}}$ by Lemma 5.22, so $\Lambda_{\{b_s\}\backslash t} \in \mathcal{S}$. Let $\Lambda' \in \mathcal{S}$. We have $\Lambda'(\Theta_t) <_{\Theta_t} \Lambda_{\{b_s\}}(\Theta_t) = p^t_{b_t}$, but $\Lambda' \leq B$ so $\Lambda'(\Theta_t) \in \square_{B,\Theta_t}$. Since $p^t_{b_t-1}$ is the rightmost lattice point in $\square_{B,\Theta_t}$ that is $<_{\Theta_t} p^t_{b_t}$, we have $\Lambda'(\Theta_t) \leq_{\Theta_t} p^t_{b_t-1} = \Lambda_{\{b_s\}\backslash t}(\Theta_t)$.

Now we must compare Λ' and $\Lambda_{\{b_s\}\backslash t}$ at Θ_u, $u \not\equiv t \pmod{N}$. But at such Θ_u, $\Lambda_{\{b_s\}\backslash t}$ agrees with $\Lambda_{\{b_s\}}$, and $\Lambda' \leq \Lambda_{\{b_s\}}$. So $\Lambda'(\Theta_u) \leq_{\Theta_u} \Lambda_{\{b_s\}\backslash t}(\Theta_u)$ for all u. Then $\Lambda' \leq \Lambda_{\{b_s\}\backslash t}$, by applying Lemma 5.12 to each interval $[\Theta_u, \Theta_{u+1}]$. □

At this point, we know that $\delta_{\Theta_t}\left[\{b_s\};\{\eta_s\}\right]$ is some sum of labelings of $\Lambda_{\{b_s\}\backslash t}$ with 'e' and 'h'. It will turn out that those labels can be collected to form a sum of flattened generators.

We start with $N = 1$ (i.e. $n = 1$ and A is negative hyperbolic or $-\mathbb{1}$). The result is simple:

Lemma 5.28. *If $N = 1$, then $\delta\left[\{b_s\};\{\eta_s\}\right] = 0$, where $\left[\{b_s\};\{\eta_s\}\right]$ is a flattened generator.*

Proof. The condition $\eta_{s+N} = \eta_s$ means $\{\eta_s\}$ is determined by η_0. If η_0 is 'E' or 'H', then $\left[\{b_s\};\{\eta_s\}\right]$ is $E_{\Lambda_{\{b_s\}}}$ or $H_{\Lambda_{\{b_s\}}}$, respectively; both of these are closed under δ. □

(We now have $\delta = 0$ for $N = 1$. Thus

$$H_*^{\text{flat}}(B, \Theta_0) \cong C_*^{\text{flat}}(B, \Theta_0). \tag{5.15}$$

Computing the homology will be simple once we find the index of $\left[\{b_s\};\{\eta_s\}\right]$, below.)

We now consider $N > 1$. The sign change necessary to make η_t the first in the ordering will be denoted $(-1)^{\#_t\{\eta_s\}}$:

Notation Convention 5.29. If $t \in \{0, \ldots, N-1\}$ and $\{\eta_s\}$ is a flattened labeling, let $\#_t\{\eta_s\}$ be the number of integers s, $0 \le s < t$, with $\eta_s =$ 'H'. For any other $t \in \mathbb{Z}$, find $t + kN \in \{0, \ldots, N-1\}$ and use $t + kN$ in place of t.

Rounding at Θ_t affects only the two adjacent labels η_{t-1} and η_t. We introduce a notation to suppress the remaining, inactive, portion of $\{\eta_s\}$.

Notation Convention 5.30. Assume $N > 1$. Given a sequence

$$\boldsymbol{\eta} = (\eta_{t+1-N}, \ldots, \eta_{t-2}) \tag{5.16}$$

of 'E' and 'H' labels (and in particular, given $t \in \mathbb{Z}$), let $\boldsymbol{\eta}_{EH}$ be the flattened labeling $\{\eta_s\}_{s\in\mathbb{Z}}$ obtained by extending $\boldsymbol{\eta}$ with $\eta_{t-1} =$ 'E' and $\eta_t =$ 'H', and using $\eta_{s+N} = \eta_s$. Similarly define $\boldsymbol{\eta}_{EE}, \boldsymbol{\eta}_{HE}$, and $\boldsymbol{\eta}_{HH}$, letting $\boldsymbol{\eta}_{t-1}$ and $\boldsymbol{\eta}_t$ be the appropriate labels.

If $N = 2$, $\boldsymbol{\eta}$ is the empty sequence.

Proposition 5.31. *Suppose (B, Θ_0) is flattenable with $N > 1$. Suppose $\{b_s\}$ is an in-range size sequence, $t \in \mathbb{Z}$, and $\boldsymbol{\eta}$ is a sequence $(\eta_{t+1-N}, \ldots, \eta_{t-2})$ of 'E' and 'H' labels. Then δ_{Θ_t} acts as follows:*

$$\begin{aligned}
[\{b_s\}; \boldsymbol{\eta}_{EE}] &\mapsto && 0 \\
[\{b_s\}; \boldsymbol{\eta}_{EH}] &\mapsto && -(-1)^{\#_t\{\eta_s\}} [\{b_s\}\backslash t; \boldsymbol{\eta}_{EE}] \\
[\{b_s\}; \boldsymbol{\eta}_{HE}] &\mapsto && (-1)^{\#_{t-1}\{\eta_s\}} [\{b_s\}\backslash t; \boldsymbol{\eta}_{EE}] \\
[\{b_s\}; \boldsymbol{\eta}_{HH}] &\mapsto && (-1)^{\#_{t-1}\{\eta_s\}} [\{b_s\}\backslash t; \boldsymbol{\eta}_{EH}] - (-1)^{\#_t\{\eta_s\}} [\{b_s\}\backslash t; \boldsymbol{\eta}_{HE}]
\end{aligned} \tag{5.17}$$

where each of the four statements is true provided the $(\{b_s\}, \{\eta_s\})$ on the left-hand side is in range.

The expressions on the right hand side are allowed to be out of range, and Θ_t is allowed to be in a kink.

If $t \not\equiv 0 \pmod{N}$, then $\delta_{\Theta_t}\,[\{b_s\};\boldsymbol{\eta}_{HH}]$ may be written

$$(-1)^{\#_{t-1}\{\eta_s\}}\big(\,[\{b_s\}\setminus t;\boldsymbol{\eta}_{EH}] + [\{b_s\}\setminus t;\boldsymbol{\eta}_{HE}]\,\big). \tag{5.18}$$

Proof. Let c denote the corner containing Θ_t so that $\delta_c = \delta_{\Theta_t}$. If c is a kink, $\delta_{\Theta_t} = 0$, and $\{b_s\}\setminus t$ is out of range, so $[\{b_s\}\setminus t;\cdots] = 0$, and the claim is true. Otherwise, we may round at c, and we have some (possibly empty) sum of labelings of $\Lambda_{\{b_s\}\setminus t}$ (and $\{b_s\}\setminus t$ is in range).

Since c is not a kink, there is at least one edge with angle in $[\Theta_{t-1},\Theta_t]$ and one with angle in $[\Theta_t,\Theta_{t+1}]$. We need consider only the action taking place in $[\Theta_{t-1},\Theta_{t+1}]$.

If $\{\eta_s\} = \boldsymbol{\eta}_{EE}$, then both edges adjacent to c are labeled 'e', so $\delta_c\,[\{b_s\};\boldsymbol{\eta}_{EE}] = 0$.

If $\{\eta_s\} = \boldsymbol{\eta}_{EH}$, then $[\{b_s\};\boldsymbol{\eta}_{EH}]$ is a sum of terms with one edge of $\Lambda_{\{b_s\}}$ (with angle in $[\Theta_t,\Theta_{t+1}]$) labeled 'h' and all other edges in $[\Theta_{t-1},\Theta_{t+1}]$ labeled 'e'. The term with 'h' adjacent to c is sent to "all 'e'", namely $\pm[\{b_s\}\setminus t;\boldsymbol{\eta}_{EE}]$ (which is non-zero since $(\{b_s\}\setminus t,\boldsymbol{\eta}_{EE})$ is in range). The remaining terms are sent to zero. It costs us $(-1)^{\#_t\{\eta_s\}}$ to make the 'H' at $\boldsymbol{\eta}_t$ the first in the ordering, and then the extra -1 is because that 'H' follows the corner (see Sign Convention 3.50). The remaining 'H' labels are still ordered by increasing Θ as in the definition of flattened generator, Definition 5.6.

The $\boldsymbol{\eta}_{HE}$ case is similar to the $\boldsymbol{\eta}_{EH}$ case.

The $\boldsymbol{\eta}_{HH}$ case is a double sum, over edges α with angle in $[\Theta_{t-1},\Theta_t]$ and β in $[\Theta_t,\Theta_{t+1}]$. We have four kinds of terms in the sum:

1. Neither 'h' label is adjacent to c. These are sent to 0 by δ_{Θ_t}.

2. The edge α is adjacent to c and β is not. Each such term is sent to $(-1)^{\#_{t-1}\{\eta_s\}}$ times a labeling with one 'h', at β not adjacent to c. The resulting terms are some of the summands in $(-1)^{\#_{t-1}\{\eta_s\}}\left[\{b_s\} \setminus t; \boldsymbol{\eta}_{EH}\right]$. Omitted from $\pm\left[\{b_s\} \setminus t; \boldsymbol{\eta}_{EH}\right]$ are the terms with 'h' on one of the edges newly created or shortened by rounding, with angle in $[\Theta_t, \Theta_{t+1}]$.

3. The edge α is not adjacent to c, but β is. Each such term is sent to a summand in $-(-1)^{\#_t\{\eta_s\}}\left[\{b_s\}\backslash t; \boldsymbol{\eta}_{HE}\right]$, with an '$h$' label at α.

4. Both α and β are adjacent to c. This term is sent to a signed sum of terms with one 'h' label on a newly created or shortened edge. These are exactly the terms needed to complete the last right-hand side in (5.17), if the signs are correct. Note that regardless of whether $(\{b_s\} \setminus t, \boldsymbol{\eta}_{HE})$ is in range (and similarly for $\boldsymbol{\eta}_{EH}$), it is correct to view $\left[\{b_s\} \setminus t; \boldsymbol{\eta}_{HE}\right]$ as a sum of all ways to put one 'h' in $[\Theta_{t-1}, \Theta_t]$ (and appropriate labels elsewhere)–that sum is zero if and only if $(\{b_s\} \setminus t, \boldsymbol{\eta}_{HE})$ is out of range.

   Consider a term with the 'h' label on a newly created or shortened edge in $[\Theta_{t-1}, \Theta_t]$. We need this 'h' to have its place in the ordering of 'h' edges determined by its presence in $[\Theta_{t-1}, \Theta_t]$ (i.e. it comes from $\boldsymbol{\eta}_{t-1}$), so that it can be a term of $\left[\cdots; \boldsymbol{\eta}_{HE}\right]$. Therefore, let us perform the rounding of $\left[\{b_s\}; \boldsymbol{\eta}_{HH}\right]$ by making the 'h' coming from $\boldsymbol{\eta}_t$ into the first 'h' in the ordering, at the cost of $(-1)^{\#_t\{\eta_s\}}$ for reordering and -1 because it is after c. The remaining 'h' is where it should be in the ordering. Thus this sign is correct.

   Similarly, for a term of $\delta_{\Theta_t}\left[\{b_s\}; \boldsymbol{\eta}_{HH}\right]$ with 'h' on an edge in $[\Theta_t, \Theta_{t+1}]$, perform the rounding by losing the 'h' coming from $\boldsymbol{\eta}_{t-1}$.

□

Corollary 5.32. *If* (B, Θ_0) *is flattenable, then* $C_*^{\mathrm{flat}}(B, \Theta_0)$ *is a subcomplex of* $C_*(B)$. □

It is convenient for formal reasons to drop the hypothesis that $\{b_s\}$ is in range from Proposition 5.31, obtaining a statement that includes claims of the form $0 \mapsto 0$. This is possible provided B is strip.

Proposition 5.33. *Suppose* $N > 1$, B *is an* (A, n, γ)*-strip,* $\Theta_0 = \pi/2$, $\{b_s\}$ *is a size sequence,* $t \in \mathbb{Z}$, *and* $\boldsymbol{\eta}$ *is a sequence* $(\eta_{t+1-N}, \dots, \eta_{t-2})$ *of 'E' and 'H' labels. Then* δ_{Θ_t} *still acts as in* (5.17).

Proof. The new cases to check are those with $\{b_s\}$ out of range, so the left-hand side of (5.17) is 0. It is sufficient to show $\{b_s\} \setminus t$ is out of range. Since B is a strip, a size sequence is in range if and only if $p^s_{b_s} \leq_{\Theta_{s+1}} p^{s+1}_{b_{s+1}}$ for all s. This condition includes, in particular,

$$p^{t-1}_{b_{t-1}} \leq_{\Theta_t} p^t_{b_t} \tag{5.19}$$

$$p^{t+1}_{b_{t+1}} \leq_{\Theta_t} p^t_{b_t}. \tag{5.20}$$

(No other condition is affected by replacing $\{b_s\}$ with $\{b_s\} \setminus t$.) These relations can only get worse by replacing $p^t_{b_t}$ with $p^t_{b_t-1}$. (We are assuming $N > 1$ so $t - 1 \not\equiv t$.) Thus if $\{b_s\}$ is already out of range, $\{b_s\} \setminus t$ must be out of range. □

We now have δ completely described for all N.

5.5 The index $I(\,[\{b_s\};\{\eta_s\}]\,)$ of a flattened generator

Recall that we index $C_*(A,n,\gamma)$ and its various subcomplexes over a $\mathbb{Z}$-torsor $\mathcal{Z}$. If $\Gamma = 0$, we have $\mathcal{Z} \cong \mathbb{Z}$. Essentially, the index of a labeled periodic path is $I(\alpha) = 2(\mathrm{Size}(\Lambda) - \mathrm{Size}(\mathrm{const}_0)) - \#h(\ell)$; see §3.4 for details. We extend the notation $\#h$ to flattened labelings.

Definition 5.34. Let $\#h\{\eta_s\}$ denote the number of 'H' labels appearing in $\eta_0,\ldots,\eta_{N-1}$, where $\{\eta_s\}$ is a flattened labeling.

Proposition 5.35. *The index of a (B,Θ_0)-flattened generator $[\{b_s\};\{\eta_s\}]$ is*

$$I(\,[\{b_s\};\{\eta_s\}]\,) = 2\sum_{s=0}^{N-1} b_s - \#h(\{\eta_s\}) + i_0, \tag{5.21}$$

where $i_0 \in \mathcal{Z}$ is a constant (depending only on $A,n,\gamma,B,$ and Θ_0) with $i_0 = 0$ if $\Gamma = 0$.

In particular, a flattened generator is homogenous under the grading I, which descends from $C_*(B)$ to $C_*^{\mathrm{flat}}(B,\Theta_0)$.

Proof. By the definition of index and the fact that (5.21) is correct for the constant path $\{b_s\} = \{0\}$, it is sufficient to show that $\Lambda_{\{b_s\}} \mapsto \sum b_s$ agrees with some relative size function (restricted to flattened paths). The idea is that every flattened path is connected to every other by a sequence of rounding and "unrounding" at Θ_t's.

Suppose $\{b_s\}$ and $\{c_s\}$ are (A,n,γ)-size sequences with $\Lambda_{\{b_s\}} \leq \Lambda_{\{c_s\}}$. (Note that $c_s - b_s$ is not negative, and it is an integer, not a half-integer.) Either $\Lambda_{\{c_s\}} = \Lambda_{\{b_s\}}$ or there is a t at which $c_t > b_t$. In the latter case, the two paths disagree at Θ_t, so Θ_t is not in a kink of $\Lambda_{\{c_s\}}$. We may round at Θ_t, producing $\Lambda_{\{c_s\}\setminus t}$, with $\Lambda_{\{b_s\}} \leq \Lambda_{\{c_s\}\setminus t}$. We may continue this process recursively, terminating when we reach $\Lambda_{\{b_s\}}$, after $\sum_0^{N-1}(c_s - b_s)$ rounding operations. The number of operations is also the number by which a relative size function has decreased:

$$\mathrm{Size}(\Lambda_{\{c_s\}}) - \sum_0^{N-1} c_s = \mathrm{Size}(\Lambda_{\{b_s\}}) - \sum_0^{N-1} b_s. \tag{5.22}$$

Given (A, n, γ)-size sequences $\{b_s\}$ and $\{b'_s\}$, we may let $c_s = \max(b_s, b'_s)$ for all s, with $\Lambda_{\{b_s\}}, \Lambda_{\{b'_s\}} \leq \Lambda_{\{c_s\}}$, and then apply (5.22) twice. We have

$$\mathrm{Size}(\Lambda_{\{b_s\}}) - \sum_0^{N-1} b_s = \mathrm{Size}(\Lambda_{\{b'_s\}}) - \sum_0^{N-1} b'_s. \tag{5.23}$$

Subtracting a constant, namely $\mathrm{Size}(\Lambda_{\{b_s\}}) - \sum_0^{N-1} b_s$, from Size produces a new relative size function which agrees with $\Lambda_{\{b_s\}} \mapsto \sum b_s$ on flattened paths, completing the proof. □

5.6 The homology for $N = 1$

It is now easy enough to compute the homology in the case $N = 1$ (which is equivalent to $n = 1$, A negative hyperbolic or $-\mathbb{1}$).

Theorem 5.36. *Suppose A is negative hyperbolic or $-\mathbb{1}$, $n = 1$, $\Gamma \in \mathbb{Z}^2/\operatorname{Im}(\mathbb{1} - A)$ with $\gamma \in \Gamma$, B is an $(A, 1, \gamma)$-strip, and $\Theta_0 = \pi/2$. Then*

$$H_i^{\mathrm{flat}}(B, \Theta_0) \cong \begin{cases} \mathbb{Z} & \text{if } i \geq i_0 \\ 0 & \text{if } i < i_0, \end{cases} \tag{5.24}$$

where $i_0 \in \mathcal{Z}$ is as in Proposition 5.35.

Proof. We know $\delta = 0$, so $H_i^{\mathrm{flat}}(B, \Theta_0) \cong C_i^{\mathrm{flat}}(B, \Theta_0)$. But $C_*^{\mathrm{flat}}(B, \Theta_0)$ is freely generated over $\mathbb{Z}$ by one generator per in-range $(\{b_s\}, \{\eta_s\})$. Further, $\{b_s\}$ is entirely specified by b_0, with $b_0 \in \mathbb{Z}$ if $\Gamma = 0$ and $b_0 \in \mathbb{Z} + \frac{1}{2}$ if $\Gamma \neq 0$; $\{\eta_s\}$ is specified by $\eta_0 \in \{`E', `H'\}$. Identify $`E' = 0$, $`H' = 1$. The in-range requirement reduces to requiring $b_0 \geq 0$, with $\eta_0 = 0$ when $b_0 = 0$, i.e., $2b_0 - \eta_0 \geq 0$. The index is $2b_0 - \eta_0 + i_0$.

If $\Gamma = 0$, $2b_0$ takes on all even non-negative integer values, so $2b_0 - \eta_0$ takes on each non-negative integer value once. If $\Gamma \neq 0$, $2b_0$ takes on all odd positive integer values, so $2b_0 - \eta_0$ takes on each non-negative integer value once. Thus there is one generator of $H_*^{\text{flat}}(B, \Theta_0)$ for each non-negative integer. □

We consider only strips, because Chapter 6 will show that if B is a strip, $H_*(B, \Theta_0) \cong H_*(A, n, \gamma)$. We will have $H_*^{\text{flat}}(B, \Theta_0) \cong H_*(B, \Theta_0)$ by Theorem 5.1.

5.7 The proof of Theorem 5.1

We begin by proving Theorem 5.1 in the case that B is a box, a case which we restate here:

Proposition 5.37. *Let B be a box and let Θ_0 be π or $\pi/2$. The inclusion*

$$C_*^{\text{flat}}(B, \Theta_0) \hookrightarrow C_*(B) \tag{5.25}$$

induces an isomorphism on homology.

Proving Proposition 5.37 will take most of §5.7. A direct limit argument will then extend the result to strips, completing the proof of Theorem 5.1.

The idea of the proof is to use spectral sequences. We filter $C_*(B)$ by a quantity depending on the points $\Lambda(\Theta_s)$, namely $\sum_{s=0}^{N-1} b_s$, which makes sense on both $C_*(B)$ and $C_*^{\text{flat}}(B, \Theta_0)$. The map $C_*^{\text{flat}}(B, \Theta_0) \hookrightarrow C_*(B)$ induces a map on the E^1 sheets of the spectral sequences, $E^{1,\text{flat}}_{*,*} \to E^1_{*,*}$, which will turn out to be an isomorphism. It follows that $H^{\text{flat}}_*(B, \Theta_0) \to H_*(B)$ is also an isomorphism. (Roughly, each E^1 sheet is just $C^{\text{flat}}_*(B, \Theta_0)$.)

If $\Lambda \leq B$ is a path, which need not be a flattened path, we still have $\Lambda(\Theta_s) \in \square_{B,\Theta_s}$, and $\Lambda(\Theta_s)$ can still be uniquely expressed as $p^s_{b_s}$.

Definition 5.38. Given a periodic path Λ with $\Lambda \leq B$, let

$$\nu(\Lambda) = 2 \sum_{s=0}^{N-1} b_s, \tag{5.26}$$

where $\Lambda(\Theta_s) = p^s_{b_s}$.

The factor of 2 ensures we obtain integers. Here is the crucial property of ν:

Lemma 5.39. *If a corner c of Λ does not contain any Θ_s ($s \in \mathbb{Z}$), then $\nu(\Lambda \backslash c) = \nu(\Lambda)$. If a corner c of Λ contains some Θ_s, then $\nu(\Lambda \backslash c) < \nu(\Lambda)$.*

Proof. In the former case, no $\Lambda(\Theta_s)$ is affected. In the latter case, $(\Lambda \setminus c)(\Theta_s) <_{\Theta_s} \Lambda(\Theta_s)$ and $(\Lambda \setminus c)(\Theta_s) \in \square_{B,\Theta_s}$ so $(\Lambda \setminus c)(\Theta_s) = p^s_{b'}$ with $b' < b_s$. The terms b_t other than b_s are unaffected. □

The function ν defines a grading on $C_*(B)$ and $C^{\text{flat}}_*(B, \Theta_0)$. (It is a grading over $\mathbb{Z}$. Recall that the usual grading, the $*$ in C_*, is over the $\mathbb{Z}$-torsor $\mathcal{Z}$.) From the grading from ν, we may build an increasing filtration, making those chain complexes into filtered chain complexes. Formally:

Definition 5.40. Let $F_pC_i(B)$ be the subgroup of $C_i(B)$ ($i \in \mathcal{Z}$) generated by labeled periodic paths (Λ, ℓ, o) with $\nu(\Lambda) \leq p$. Let $F_pC^{\text{flat}}_i(B, \Theta_0)$ be the subgroup of $C^{\text{flat}}_i(B, \Theta_0)$ generated by flattened generators $[\Lambda; \{\eta_s\}]$ with $\nu(\Lambda) \leq p$.

Lemma 5.41. *We have $F_pC_i(B) \subseteq F_{p+1}C_i(B)$ and $\delta\colon F_pC_i(B) \to F_pC_{i-1}(B)$. Similar claims hold for $F_pC_i^{\text{flat}}(B,\Theta_0)$. The inclusion $C_*^{\text{flat}}(B,\Theta_0) \hookrightarrow C_*(B)$ sends $F_pC_i^{\text{flat}}(B,\Theta_0)$ to $F_pC_i(B)$.* □

Thus, we have two spectral sequences and a morphism between them.

Notation Convention 5.42. Let $E^*_{*,*}$ be the spectral sequence of $F_*C_*(B)$. Let $E^{*,\text{flat}}_{*,*}$ be the spectral sequence of $F_*C_*^{\text{flat}}(B,\Theta_0)$.

Since we are in a box, each b_s of a path $\Lambda \leq B$ is bounded above and below by $b^s_{\max}$ and $b^s_{\min}$, so $\nu(\Lambda)$ is bounded above and below in $C_*(B)$ (and $C_*^{\text{flat}}(B,\Theta_0)$). Thus the spectral sequences converge.

To prove an isomorphism on homology, it is sufficient to show, for some r, that the map $E^{r,\text{flat}}_{*,*} \to E^r_{*,*}$ is an isomorphism. We will use $r = 1$.

Notation Convention 5.43. Given an in-range size sequence $\{b_s\}$, abbreviate

$$\Lambda_{\{b_s\}}|[\Theta_s, \Theta_{s+1}] \tag{5.27}$$

as Λ_s.

Recall that E_Λ is the path Λ labeled with all 'e' labels and H_Λ is the sum of all ways to label the path Λ with exactly one 'h' label. This sum is 0 if Λ is the constant path. These labeled paths are cycles.

Lemma 5.44. *We have*

$$E^0_{p,*} \cong \bigoplus \bigotimes_{s=0}^{N-1} C_*(\Lambda_s) \tag{5.28}$$

$$E^{0,\text{flat}}_{p,*} \cong \bigoplus \bigotimes_{s=0}^{N-1} \mathbb{Z}\{E_{\Lambda_s}, H_{\Lambda_s}\} \tag{5.29}$$

where each sum is over in-range size sequences $\{b_s\}$ *such that* $\sum_{s=0}^{N-1} b_s = p$*. The induced map* $E^{0,\mathrm{flat}}_{p,*} \to E^0_{p,*}$ *is the sum of the tensor product of the inclusions* $\mathbb{Z}\{E_{\Lambda_s}, H_{\Lambda_s}\} \to C_*(\Lambda_s)$*.*

We already knew that $\mathbb{Z}\{E_{\Lambda_s}, H_{\Lambda_s}\}$ is a subcomplex of $C_*(\Lambda_s)$.

Though it is possible to make sense of the grading denoted by $*$ in Lemma 5.44, it is sufficient for our purposes to consider the isomorphism to be (for each p) an isomorphism of abelian groups that respects their differentials. (Definition 6.4 gives a grading on $C_*(\Lambda_s)$.)

Proof. The sheet $E^0_{*,*}$ has the same generators as $C_*(B)$, with a differential δ^0, where $\delta^0\alpha$ consists of the terms in $\delta\alpha$ that do not decrease ν. In other words, it is the sum of $\delta_c\alpha$ over corners c of the underlying path Λ of α where c does not contain any Θ_s.

For any in-range size sequence $\{b_s\}$, let $C_*(B; \{b_s\})$ be the subcomplex of $E^0_{*,*}$ generated by (Λ, ℓ, o) with $\Lambda(\Theta_s) = p^s_{b_s}$. (This new complex is not a subcomplex of $C_*(B)$. We may picture each $\Lambda(\Theta_s)$ as being pinned in place, not movable by δ^0.) (Recall, by Proposition 5.17, that the sequences possible as $\{\Lambda(\Theta_s\}$ for $\Lambda \leq B$ are precisely the in-range size sequences.) Thus we have

$$E^0_{p,*} \cong \bigoplus C_*(B; \{b_s\}), \tag{5.30}$$

the sum being over in-range $\{b_s\}$ with $\sum_{s=0}^{N-1} b_s = p$.

Similarly, $E^{0,\mathrm{flat}}_{*,*}$ has the same generators as $C^{\mathrm{flat}}_*(B, \Theta_0)$ and $\delta^0\alpha$ is the sum $\sum \delta_c\alpha$ over c not containing any Θ_s; thus $\delta^0 = 0$. We have subcomplexes of $E^{0,\mathrm{flat}}_{*,*}$, denoted $C^{\mathrm{flat}}_*(B, \Theta_0; \{b_s\})$, generated by flattened generators with underlying path Λ satisfying $\Lambda(\Theta_s) = p^s_{b_s}$. That is, the subcomplex is generated by $[\{b_s\}; \{\eta_s\}]$ with the given $\{b_s\}$

(and $\{\eta_s\}$ allowed to vary). As before, we have

$$E^{0,\mathrm{flat}}_{p,*} \cong \bigoplus C^{\mathrm{flat}}_*(B, \Theta_0; \{b_s\}), \tag{5.31}$$

the sum being over in-range $\{b_s\}$ with $\sum_{s=0}^{N-1} b_s = p$. We see that $C^{\mathrm{flat}}_*(B, \Theta_0; \{b_s\})$ is a subcomplex of $C_*(B; \{b_s\})$, and the induced map $E^{0,\mathrm{flat}}_{p,*} \to E^0_{p,*}$ (which just sends each generator to itself) is the direct sum of

$$C^{\mathrm{flat}}_*(B, \Theta_0; \{b_s\}) \hookrightarrow C_*(B; \{b_s\}). \tag{5.32}$$

We define a map of abelian groups

$$\bigotimes_{s=0}^{N-1} C_*(\Lambda_s) \to C_*(B; \{b_s\}) \tag{5.33}$$

by concatenating paths: Since a generator $(\Lambda_{\mathrm{tr}}, \ell, o)$ of $C_*(\Lambda_s)$ has Λ_{tr} with domain and endpoints matching those of $\Lambda_s = \Lambda_{\{b_s\}}|[\Theta_s, \Theta_{s+1}]$, we may concatenate to get a periodic path Λ with $\Lambda \leq \Lambda_{\{b_s\}} \leq B$, and we may label Λ according to the labeling of each Λ_{tr}, ordering the 'h' labels starting from the labels on $C_*(\Lambda_0)$, then those on $C_*(\Lambda_1)$, through $C_*(\Lambda_{N-1})$.

To see that this map is an isomorphism of abelian groups, the only thing to check is if Λ appears in $C_*(B; \{b_s\})$, then $\Lambda|[\Theta_s, \Theta_{s+1}]$ can appear in $C_*(\Lambda_s)$. But $\Lambda|[\Theta_s, \Theta_{s+1}] \leq \Lambda_{\{b_s\}}|[\Theta_s, \Theta_{s+1}]$ by the definition of $\Lambda_{\{b_s\}}$ being a flattened path.

We may define a differential making $\bigotimes_{s=0}^{N-1} C_*(\Lambda_s)$ the tensor product of chain complexes. The signs are determined by grading $C_*(\Lambda_s)$ by $\#h$, i.e.,

$$\alpha_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_{N-1} \mapsto \delta\alpha_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_{N-1} + (-1)^{\#h(\alpha_0)} \alpha_0 \otimes \delta\alpha_1 \otimes \cdots \otimes \alpha_{N-1} + \ldots, \tag{5.34}$$

which makes (5.33) an isomorphism of chain complexes. (In particular, δ^0 on $C_*(B; \{b_s\})$ includes only terms affecting just one α_s.)

This proves (5.28).

Similarly, concatenation yields an isomorphism

$$\bigotimes_{s=0}^{N-1} \mathbb{Z}\{E_{\Lambda_s}, H_{\Lambda_s}\} \xrightarrow{\approx} C_*^{\mathrm{flat}}(B, \Theta_0; \{b_s\}), \tag{5.35}$$

proving (5.29). The inclusions from (5.35) to (5.33) form a commutative square, completing the proof. □

Lemma 5.45. *The induced map $E^{1,\mathrm{flat}}_{*,*} \to E^1_{*,*}$ is an isomorphism.*

Proof. The sheet $E^1_{*,*}$ is the homology of $(E^0_{*,*}, \delta^0)$. The inclusion $\mathbb{Z}\{E_\Lambda, H_\Lambda\} \to C_*(\Lambda_s)$ induces an isomorphism on homology,

$$\mathbb{Z}\{E_\Lambda, H_\Lambda\} \xrightarrow{\approx} H_*(\Lambda_s), \tag{5.36}$$

by Proposition 4.8 (which applies to both constant and non-constant truncated paths, since $H_{\mathrm{const}} = 0$). Since $\mathbb{Z}\{E_\Lambda, H_\Lambda\}$ is torsion free, the tensor product of the homolgies in question is the homology of the tensor product. □

This completes the proof of Proposition 5.37.

We now finish the proof of Theorem 5.1 by proving the case of a strip.

Proposition 5.46. *Let B_∞ be an (A, n, γ)-strip and let $\Theta_0 = \pi/2$. The inclusion*

$$C_*^{\mathrm{flat}}(B_\infty, \Theta_0) \hookrightarrow C_*(B_\infty) \tag{5.37}$$

induces an isomorphism on homology.

Proof. Consider a sequence of boxes B_j, $j = 1, 2, \ldots$, with the horizontal edges matching B_∞, so $Y(\mathrm{Edge}_{B_j}(\Theta_s + \pi/2)) = Y(\mathrm{Edge}_{B_\infty}(\Theta_s + \pi/2))$, and with the vertical edges $B_j(\Theta_s)$

moving away from the origin (to the right with respect to Θ_s) without bound. This is the situation of Proposition 4.19, which tells us

$$C_*(B_\infty) \cong \varinjlim C_*(B_j). \tag{5.38}$$

The flattened paths similarly form a direct limit. (We know $\{\Lambda \mid \Lambda \leq B_\infty\} = \bigcup \{\Lambda \mid \Lambda \leq B_j\}$ for periodic paths Λ, and the same statement holds for flattened paths because the dependence of $\Lambda_{\{b_s\}}$ on B_j is only through horizontal edges. The location of vertical edges affects only $b^s_{\min}$ and $b^s_{\max}$.) Thus,

$$C^{\text{flat}}_*(B_\infty, \Theta_0) \cong \varinjlim C^{\text{flat}}_*(B_j, \Theta_0). \tag{5.39}$$

Since the homology of the direct limit is the direct limit of the homologies, $H_*(B_\infty) \cong \varinjlim H_*(B_j)$ and similarly for the flattened subcomplexes.

The diagram

$$\begin{array}{ccccccccc} \ldots & C^{\text{flat}}_*(B_j, \Theta_0) & \to & C^{\text{flat}}_*(B_{j+1}, \Theta_0) & \to & \ldots & \to & C^{\text{flat}}_*(B_\infty, \Theta_0) \\ & \downarrow & & \downarrow & & & & \downarrow \\ \ldots & C_*(B_j) & \to & C_*(B_{j+1}) & \to & \ldots & \to & C_*(B_\infty) \end{array} \tag{5.40}$$

commutes because all the arrows are just inclusions of subcomplexes. Applying the H_* functor yields a commutative diagram. Each vertical map $H^{\text{flat}}_*(B_j, \Theta_0) \to H_*(B_j)$ is an isomorphism by Proposition 5.37, so $H^{\text{flat}}_*(B_\infty, \Theta_0) \to H_*(B_\infty)$ is an isomorphism. □

Chapter 6

The homology of a strip is isomorphic to $H_*(A, n, \gamma)$

For this chapter, fix $A \in SL_2(\mathbb{Z})$ hyperbolic or $-\mathbb{1}$, $n \in \mathbb{Z}$ positive, and $\gamma \in \Gamma \subseteq \mathbb{Z}^2$.

Theorem 6.1. *For any (A, n, γ)-strip B_∞, we have*

$$H_*(B_\infty) \cong H_*(A, n, \gamma), \tag{6.1}$$

with the isomorphism on homology induced by the inclusion of chain complexes

$$C_*(B_\infty) \hookrightarrow C_*(A, n, \gamma). \tag{6.2}$$

The method of proof is to fix an index i and show that H_i of a box stabilizes once the box is big enough, i.e. $H_i(B) \cong H_i(B')$ when B is large enough. $B \leq B'$, by considering separately the case that B' is "wider" than B (the same height), and the case that B' is "taller" than B (same width). We use flattened subcomplexes with $\Theta_0 = \pi/2$ in the former case and $\Theta_0 = \pi$ in the latter case. More specifically, the way we prove an isomorphism of flattened complexes is to show that the "new" flattened generators, to the left of B' but not B, have index much larger than i, provided B is large enough. We bound the index of a (B, Θ_0)-flattened path using the relative size function Size_{Θ_0}, which we introduce below.

The results of this chapter (especially Lemma 6.14) play a role analogous to that of [5, Lemma 7.8] in the case $A = \mathbb{1}$, but the proof is different. In particular, the proof of Lemma 6.12 requires the eigenvalue a of A to satisfy $A \neq 1$. Furthermore, it turns out that the right way to bound the index of a (B, Θ_0)-flattened path is the expression $\mathrm{Size}_{\Theta_0}(\Lambda)$ we introduce below (§6.1), not the expressions involving $\sum b_s$ from Chapter 5, and this distinction has no analogue for $A = \mathbb{1}$.

6.1 The relative size function $\mathrm{Size}_{\Theta_0}(\Lambda)$

Recall that a relative size function is a function from the set of (A, n, γ)-periodic paths to $\frac{1}{2}\mathbb{Z}$. Roughly, it is something that counts the points enclosed by a periodic path, and given a relative size function Size, the index of a labeled periodic path is

$$I(\Lambda, \ell, o) = 2\,\mathrm{Size}(\Lambda) - \#h(\ell) + \mathrm{constant} \in \mathcal{Z}, \tag{6.3}$$

where $\mathcal{Z}$ is the $\mathbb{Z}$-torsor over which $C_*(A, n, \gamma)$ is graded. The constant depends on the relative size function (and A, n, γ). We will define two relative size functions, Size_π and $\mathrm{Size}_{\pi/2}$. (The two functions seem to agree, but we do not need that fact.) Below, we will put a lower bound on Size_{Θ_0} of a (B, Θ_0)-flattened path, and thereby bound the index of a (B, Θ_0)-flattened generator.

One may think of Size_{Θ_0} as comparing Λ with the X- or Y-axis, whichever is perpendicular to Θ_0. We begin by defining Size_{Θ_0} for a truncated path defined on

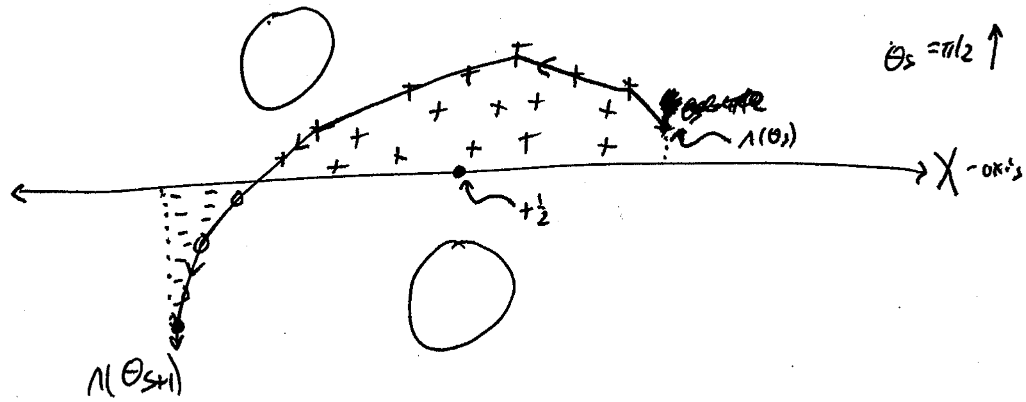

Figure 6.1: $\text{Size}_{\Theta_0}(\Lambda)$ is a signed count of points. Here, Λ is a truncated path defined on $[\Theta_s, \Theta_{s+1}] = [\pi/2, 3\pi/2]$.

$[\Theta_s, \Theta_{s+1}] = [\Theta_0 + s\pi, \Theta_0 + (s+1)\pi]$.

Remark 6.2. *It is easiest to picture the case* $\Theta_0 = \Theta_s = \pi/2$. *In that case,* $\text{Size}_{\pi/2}(\Lambda_{tr})$ *is a signed count of points "below"* Λ_{tr} *and "above" the* X*-axis, with signs analogous to an integral of a function of* X. *See Figure 6.1. We are only interested in points with* X*-coordinates between those of* $\Lambda_{tr}(\Theta_{s+1})$ *and* $\Lambda_{tr}(\Theta_s)$. *The case* $\Theta_s = 3\pi/2$ *counts points above* Λ_{tr} *and below the* X*-axis.*

Recall that const_p is an ambiguous notation in that it does not specify the domain. Therefore, we make a convention.

Notation Convention 6.3. The expression $\text{const}_p \leq \Lambda$ will mean $p \leq_\Theta \Lambda$ for all $\Theta \in \overline{\text{Dom}}(\Lambda)$. The expression $\text{const}_p|I \leq \Lambda$ (I an interval contained in $\overline{\text{Dom}}(\Lambda)$) will mean $p \leq_\Theta \Lambda$ for all $\Theta \in I$.

Definition 6.4. If Λ_{tr} is a truncated path defined on $[\Theta_s, \Theta_{s+1}]$, define $\mathrm{Size}_{\Theta_0}(\Lambda_{\mathrm{tr}})$ as the sum, for each $p \in L_{A,\Gamma}$ with

$$p \leq_{\Theta_s} \Lambda_{\mathrm{tr}} \text{ and } p \leq_{\Theta_{s+1}} \Lambda_{\mathrm{tr}}, \tag{6.4}$$

of the following:

- $+1$ if $0 <_{\Theta_s+\pi/2} p$ and $\mathrm{const}_p \leq \Lambda_{\mathrm{tr}}$.
- -1 if $p <_{\Theta_s+\pi/2} 0$ and $\mathrm{const}_p \not\leq \Lambda_{\mathrm{tr}}$.
- $+\frac{1}{2}$ if $p = 0$ and $\mathrm{const}_p \leq \Lambda_{\mathrm{tr}}$.
- $-\frac{1}{2}$ if $p = 0$ and $\mathrm{const}_p \not\leq \Lambda_{\mathrm{tr}}$.
- 0 otherwise.

One may imagine that the point 0 is bisected by the X-axis. Recall that $0 \in L_{A,\Gamma}$ if and only if $\Gamma = 0$, and no other $p \in L_{A,\Gamma}$ lies on the X- or Y-axis, so all other $p \in L_{A,\Gamma}$ satisfy $p <_{\Theta_s+\pi/2} 0$ or $0 <_{\Theta_s+\pi/2} p$.

We say p *contributes positively* to $\mathrm{Size}_{\Theta_0}(\Lambda_{\mathrm{tr}})$ if it appears in the sum with a positive sign, and similarly for *contributes negatively.*

The sum appearing in Definition 6.4 is finite, because a sufficiently large rectangular region contains all points that could be included in the sum.

Definition 6.5. If Λ is an (A, n, γ)-periodic path and $\Theta_0 \in \{\pi/2, \pi\}$, let

$$\mathrm{Size}_{\Theta_0}(\Lambda) = \sum_{s=0}^{N-1} \mathrm{Size}(\Lambda|[\Theta_s, \Theta_{s+1}]). \tag{6.5}$$

Also let

$$\mathrm{Size}_{\Theta_0}(\Lambda|[\Theta_t, \Theta_u]) = \sum_{s=t}^{u-1} \mathrm{Size}(\Lambda|[\Theta_s, \Theta_{s+1}]). \tag{6.6}$$

Lemma 6.6. *By periodicity,* $\mathrm{Size}_{\Theta_0}(\Lambda)$ *is also* $\mathrm{Size}(\Lambda|[\Theta_t, \Theta_{t+N}])$ *for any* t. □

Lemma 6.7. Size_{Θ_0} *is a relative size function.*

In other words, if Λ is an (A, n, γ)-periodic path and c is a proper non-self-rounding corner of Λ, $\mathrm{Size}_{\Theta_0}(\Lambda \setminus c) = \mathrm{Size}_{\Theta_0}(\Lambda) - 1$.

Proof. Case I: $c \subset (\Theta_s, \Theta_{s+1})$ *for some* s. The condition $\mathrm{const}_p \leq \Lambda|[\Theta_s, \Theta_{s+1}]$ agrees with the condition $\mathrm{const}_p \leq (\Lambda \setminus c)|[\Theta_s, \Theta_{s+1}]$ except for $p = \Lambda(c)$ (clear from a picture, or see Proposition 3.69). Passing from Λ to $\Lambda \setminus c$, the sum decreases by one, no matter where p is relative to 0.

Case II: $c \not\subset (\Theta_s, \Theta_{s+1})$ *for any* s *and* $N > 1$. We may assume c contains some Θ_s. Since the length of the interval c is less than π, we may assume $c \subset [\Theta_{s-1}, \Theta_{s+1}]$. By Lemma 6.6, we may assume that both $\Lambda|[\Theta_{s-1}, \Theta_s]$ and $\Lambda|[\Theta_s, \Theta_{s+1}]$ appear in the sum defining $\mathrm{Size}_{\Theta_0}(\Lambda)$, and they are the only way rounding at c affects Size.

There are no points "in between" $\Lambda|[\Theta_{s-1}, \Theta_{s+1}]$ and $(\Lambda \setminus c)|[\Theta_{s-1}, \Theta_{s+1}]$ except $\Lambda(\Theta_s)$. Any lattice point such that $p \leq_{\Theta_s} (\Lambda \setminus c)(\Theta_s)$ contributes the same to $\mathrm{Size}(\Lambda|[\Theta_{s-1}, \Theta_{s+1}])$ as to $\mathrm{Size}((\Lambda \setminus c)|[\Theta_{s-1}, \Theta_{s+1}])$.

Lattice points satisfying $p \not\leq_{\Theta_s} (\Lambda \setminus c)(\Theta_s)$ do not contribute to $\mathrm{Size}((\Lambda \setminus c)|[\Theta_{t-1}, \Theta_{t+1}])$. We will show that their total contribution to $\mathrm{Size}(\Lambda|[\Theta_{t-1}, \Theta_{t+1}])$ is 1.

The vertices of Λ immediately preceding and immediately following $\Lambda(c)$ are both $\leq_{\Theta_s} (\Lambda \setminus c)(c)$. The triangular region defined by $\mathrm{const}_p \leq \Lambda|[\Theta_{s-1}, \Theta_{s+1}]$, $p \not\leq_{\Theta_s} (\Lambda \setminus c)(c)$

contains only $\Lambda(c)$, which contributes $+1$ to $\text{Size}(\Lambda|[\Theta_{s-1}, \Theta_{s+1}])$. Any other point $\leq_{\Theta_s} \Lambda(c)$, $\not\leq_{\Theta_s} (\Lambda \setminus c)(c)$ has contribution totaling 0.

Case III: $c \not\subset (\Theta_s, \Theta_{s+1})$ for any s and $N = 1$.

The proof is similar to Case II, but rounding at c also means rounding at $F_{A,n}(c)$, so both $\Lambda(\Theta_s)$ and $\Lambda(\Theta_{s+1})$ are rounded. We have $(\Lambda \setminus c)(\Theta_{s+1}) \leq_{\Theta_s} (\Lambda \setminus c)(\Theta_s)$, so we view $\text{Size}_{\Theta_0}(\Lambda) = \text{Size}_{\Theta_0}(\Lambda|[\Theta_s, \Theta_{s+1}])$ as the sum of contributions to $\text{Size}_{\Theta_0}(\Lambda|[\Theta_s, \Theta_{s+1}])$ from $p \leq_{\Theta_{s+1}} (\Lambda \setminus c)(\Theta_{s+1})$ and contributions to $\text{Size}_{\Theta_0}(\Lambda|[\Theta_{s-1}, \Theta_s])$ from $p \not\leq_{\Theta_s} (\Lambda \setminus c)(\Theta_s)$. This reformulated count now proceeds the same way as in Case II for the region $p \not\leq_{\Theta_s} (\Lambda \setminus c)(\Theta_s)$. This sum is affected only by the rounding at Θ_s, not the rounding at Θ_{s+1}, and decreases by 1 by the reasoning from Case II.

□

6.2 Bounding Size_{Θ_0} of a flattened path

Lemma 6.8. *If Λ is a (B, Θ_0)-flattened path and $s \in \mathbb{Z}$, then no point contributes negatively to $\text{Size}_{\Theta_0}(\Lambda)$.*

Proof. We must show that any $p \in L_{A,\Gamma}$ with

$$p \leq_{\Theta_s} \Lambda, \tag{6.7}$$

$$p \leq_{\Theta_{s+1}} \Lambda, \tag{6.8}$$

$$p \leq_{\Theta_s + \pi/2} 0 \tag{6.9}$$

satisfies $\text{const}_p \leq \Lambda|[\Theta_s, \Theta_{s+1}]$.

The conditions ensure that $\text{const}_p|[\Theta_s, \Theta_{s+1}] \leq B$. Lemma 5.12 gives a maximality condition for flattened paths, from which $\mathrm{const}_p \leq \Lambda|[\Theta_s, \Theta_{s+1}]$ follows. □

Thus, if Λ is a (B, Θ_0)-flattened path, then $\mathrm{Size}_{\Theta_0}(\Lambda) \geq 0$. We now show that a certain kind of rectangular region contains points that contribute positively to $\mathrm{Size}_{\Theta_0}(\Lambda)$. Recall that a sufficiently large rectangular region contains as many lattice points as desired (Lemma 3.75).

Definition 6.9. Given a box or strip B, let $Y_{\min}(B)$ be the minimum of $|Y(\mathrm{Edge}_B(\Theta))|$ for horizontal edges (i.e. $\Theta = k\pi$) with $\Theta \in [0, 2N\pi]$. Similarly, given a box B, let $X_{\min}(B)$ be the minimum of $|X(\mathrm{Edge}_B(\Theta))|$ for vertical edges ($\Theta = k\pi + \pi/2$) with $\Theta \in [0, 2N\pi]$.

Recall that $X(\Lambda(\Theta_{s+N})) = aX(\Lambda(\Theta_s))$; and a, an eigenvalue of A, is not 1.

Lemma 6.10. *Given $s \in \{0, 1, \ldots, N-1\}$, and a (B, Θ_0)-flattened Λ with $\Lambda(\Theta_s) \neq 0$, there exists a rectangular region R with sides parallel to the (X, Y)-axes such that every lattice point in R contributes positively to* $\mathrm{Size}_{\Theta_0}(\Lambda)$. *The region R is the product of two open intervals, where:*

1. *If $\Theta_0 = \frac{\pi}{2}$, then R has width $|X(\Lambda(\Theta_s)) - X(\Lambda(\Theta_{s+N}))|$ and height $Y_{\min}(B)$.*
2. *If $\Theta_0 = \pi$, then R has height $|Y(\Lambda(\Theta_s)) - Y(\Lambda(\Theta_{s+N}))|$ and width $X_{\min}(B)$.*

Here "width" is the measure of the X interval and height is the measure of the Y interval.

Proof. We may assume $\Theta_0 = \pi/2$. (The other proof is similar.) Since $\Lambda(\Theta_s) \neq 0$ and 0 is the only possible lattice point on the Y-axis, $X(\Lambda(\Theta_s)) \neq 0$; so $X(\Lambda(\Theta_{s+N})) = aX(\Lambda(\Theta_s)) \neq X(\Lambda(\Theta_s))$. We assume

$$X(\Lambda(\Theta_s)) > X(\Lambda(\Theta_{s+N})) \tag{6.10}$$

and let

$$R = (X(\Lambda(\Theta_{s+N}), X(\Lambda(\Theta_s)) \times (0, Y_{\min}(B)). \quad (6.11)$$

The other case is similar, with $R = (X(\Lambda(\Theta_s), X(\Lambda(\Theta_{s+N})) \times (-Y_{\min}(B), 0)$.

Consider $p \in L_{A,\Gamma} \cap R$. We must show p contributes positively to $\mathrm{Size}_{\Theta_0}(\Lambda)$. Since Λ travels from $\Lambda(\Theta_s)$ to $\Lambda(\Theta_{s+N})$ and p has its X-coordinate in between, there is an edge σ of Λ (with angle $\Theta \in [\Theta_s, \Theta_{s+N}]$) that crosses the line $X = X(p)$, and we may assume it crosses going in the $-X$ direction. Therefore σ has angle $\Theta \in [\frac{\pi}{2} + 2\pi k, \frac{3\pi}{2} + 2\pi k] = [\Theta_{2k}, \Theta_{2k+1}]$ for some $k \in \mathbb{Z}$. We will show p contributes $+1$ to $\mathrm{Size}_{\Theta_0}(\Lambda|[\Theta_{2k}, \Theta_{2k+1}])$. We know $p \leq_\Theta B$ for $\Theta = \Theta_{2k}, \Theta_{2k+1}$ by comparing with σ. We know $p \leq_{\Theta_{2k}+\pi/2} B$ because $Y(p) < Y_{\min}(B)$. Thus $p \leq_\Theta B$ for all $\Theta \in [\Theta_{2k}, \Theta_{2k+1}]$. Then $\mathrm{const}_p \leq \Lambda|[\Theta_{2k}, \Theta_{2k+1}]$ since Λ is flattened and the truncated path const_p is to the left of B. Thus, since $Y(p) > 0$, we see that p contributes $+1$ to $\mathrm{Size}_{\Theta_0}(\Lambda)$. □

Corollary 6.11. *A (B, Θ_0)-flattened path has* $\mathrm{Size}_{\Theta_0}(\Lambda) \geq |R \cap L_{A,\Gamma}|$. □

There is no issue with counting the origin as $+\frac{1}{2}$ because $0 \notin R$.

6.3 Expanding the box

The following lemma will show $H_i(B) \cong H_i(B')$, for sufficiently large boxes B, B' with B' wider than B in the horizontal direction, using flattening with $\Theta_0 = \pi/2$.

Lemma 6.12. *Suppose $i \in \mathcal{Z}$ and $Y_i > 0$. There exists $X_i > 0$ such that for any (A, n, γ)-boxes B, B' with $X_{\min}(B) \geq X_i$ and $Y_{\min}(B) \geq Y_i$, and $B \leq B'$ and*

$$Y(\mathrm{Edge}_B(\Theta)) = Y(\mathrm{Edge}_{B'}(\Theta)) \quad (6.12)$$

for all horizontal edges, we have: $C_*^{\text{flat}}(B, \pi/2)$ is a subcomplex of $C_*^{\text{flat}}(B', \pi/2)$, and the inclusion induces an isomorphism

$$H_i^{\text{flat}}(B, \pi/2) \xrightarrow{\approx} H_i^{\text{flat}}(B', \pi/2). \tag{6.13}$$

Proof. To show $C_*^{\text{flat}}(B, \pi/2) \subseteq C_*^{\text{flat}}(B', \pi/2)$, note that the flattened path Λ satisfying $\Lambda(\Theta_s) = p_{b_s}^s$ depends only on the horizontal edges of B or B', not the vertical ones; and the in-range-ness of $\{p^s\}$ depends on B only by $p^s \in \Box_{B,\Theta_s}$, and we have $\Box_{B,\Theta_s} \subseteq \Box_{B',\Theta_s}$.

The inclusion of flattened chain complexes induces a map on homology which will be an isomorphism in degree i once we show that "new" flattened generators ($(B', \pi/2)$-flattened generators that are not $(B, \pi/2)$-flattened generators) have index $\geq i + 2$.

A $(B', \pi/2)$-flattened path Λ that is not a $(B, \pi/2)$-flattened path has $\Lambda(\Theta_s) = p^s \notin \Box_{B,\Theta_s}$ for some s. We may choose $s \in \{0, \ldots, N-1\}$ with $\Lambda(\Theta_s) \not\leq_{\Theta_s} \text{Edge}_B(\Theta_s)$. The path satisfies

$$\text{Size}_{\pi/2}(\Lambda) \geq |L_{A,\Gamma} \cap R| \tag{6.14}$$

where R is the rectangular region from Lemma 6.10. That region has height $Y_{\min}(B') \geq Y_i$. The width is $|X(\Lambda(\Theta_s)) - X(\Lambda(\Theta_{s+N}))|$, but $X(\Lambda(\Theta_{s+N})) = aX(\Lambda(\Theta_s))$ so the width is $|1-a||X(\Lambda(\Theta_s))|$. But $|X(\Lambda(\Theta_s))| \geq |X(\text{Edge}_B(\Theta_s))| \geq X_{\min}(B)$, and thus the width is $\geq |1-a|X_{\min}(B) \geq |1-a|X_i$.

Lemma 3.75 asserts that for fixed height, a rectangle may be made to contain as many points as desired by changing the width. In our case, for fixed Y_i, we may obtain $\text{Size}_{\pi/2}(\Lambda)$ as large as desired by choosing $|1-a|X_i$ large enough (note that $1-a \neq 0$). The index of a flattened generator is

$$I(\,[\Lambda; \{\eta_s\}]\,) = 2\,\text{Size}_{\pi/2}(\Lambda) - \#h\{\eta_s\} + i_{\pi/2} \tag{6.15}$$

where $i_{\pi/2}$ (which depends on (A, n, γ)) is independent of B and B'. Since we may make $\mathrm{Size}_{\pi/2}(\Lambda)$ as large as desired, we may make $I(\,[\Lambda; \{\eta_s\}]\,)$ larger than $i + 2$. (Note that $\#h\{\eta_s\} \leq N$. Since (A, n, γ) are fixed throughout, so is N.)

□

Lemma 6.13. *Suppose $i \in \mathcal{Z}$ and $Y_i > 0$. There exists $X_i' > 0$ such that for any (A, n, γ)-boxes B and B' with $X_{\min}(B) \geq X_i'$ and $Y_{\min}(B) \geq Y_i$ and $B \leq B'$, and*

$$X(\mathrm{Edge}_B(\Theta)) = X(\mathrm{Edge}_{B'}(\Theta)) \tag{6.16}$$

for all vertical edges (i.e. for $\Theta = \pi/2 + s\pi$ for all $s \in \mathbb{Z}$), we have: $C_*^{\mathrm{flat}}(B, \pi)$ is a subcomplex of $C_*^{\mathrm{flat}}(B', \pi)$, and the inclusion induces an isomorphism

$$H_i^{\mathrm{flat}}(B, \pi) \xrightarrow{\approx} H_i^{\mathrm{flat}}(B', \pi). \tag{6.17}$$

The above claim is similar to Lemma 6.12, with the roles of X and Y *mostly* reversed. We still fix Y_i, because our goal involves strips parallel to the X-axis.

Proof. The proof is similar to that of Lemma 6.12. For each Y_i, we must choose X_i' so as to get enough points in a rectangle with width $\geq X_i'$ and height $|1 - a^{-1}||Y(\Lambda(\Theta_s))| \geq |1 - a^{-1}|Y_i$ (and $1 - a^{-1} \neq 0$). This can be done, by Lemma 3.75. □

Lemma 6.14. *Suppose $i \in \mathcal{Z}$ and $Y_i > 0$. If B and B'' are (A, n, γ)-boxes with $B \leq B''$, $X_{\min}(B) \geq \max(X_i, X_i')$ (with X_i, X_i' as in Lemmas 6.12 and 6.13), and $Y_{\min}(B) \geq Y_i$, then*

$$C_*(B) \hookrightarrow C_*(B'') \tag{6.18}$$

induces an isomorphism

$$H_i(B) \xrightarrow{\approx} H_i(B''). \tag{6.19}$$

Proof. We may pass from B to B'' by first expanding B horizontally to B', then vertically to B'', applying Lemmas 6.12 and 6.13. We have $H_i(B) \cong H_i^{\text{flat}}(B,\pi/2) \cong H_i^{\text{flat}}(B',\pi/2) \cong H_i(B')$ and similarly $H_i(B') \cong H_i(B'')$ (using π in place of $\pi/2$). Thus $H_i(B)$ and $H_i(B'')$ are isomorphic. We must show that the isomorphism $H_i(B) \xrightarrow{\cong} H_i(B')$ is induced by $C_*(B) \to C_*(B')$ (and similarly for B' to B''). But the inclusions

$$\begin{array}{ccc} C_i^{\text{flat}}(B,\pi/2) & \longrightarrow & C_i^{\text{flat}}(B',\pi/2) \\ \downarrow & & \downarrow \\ C_i(B) & \longrightarrow & C_i(B') \end{array} \tag{6.20}$$

commute and induce commuting maps on H_i. We already knew that three of those maps are isomorphisms, so $H_i(B) \to H_i(B')$ is also an isomorphism. □

6.4 Taking the direct limit

We restate Theorem 6.1.

Theorem 6.15. *Given an* (A,n,γ)*-strip* B_∞*, the inclusion* $C_*(B_\infty) \hookrightarrow C_*(A,n,\gamma)$ *induces an isomorphism* $H_*(B_\infty) \xrightarrow{\approx} H_*(A,n,\gamma)$.

Proof. Fix $i \in \mathcal{Z}$. Let $Y_i = Y_{\min}(B_\infty)$. Let B be formed by shortening the horizontal edges of B_∞ (so $Y(\text{Edge}_B(\Theta)) = Y(\text{Edge}_{B_\infty}(\Theta))$ for horizontal edges) and putting in vertical edges sufficiently far from the origin, so $X_{\min}(B)$ is bigger than both X_i, X_i' from Lemma 6.12 and Lemma 6.13. (These finitely many constrains on vertical edges may be satisfied while respecting periodicity.)

Then for any B' with $B \leq B'$ we have that $C_*(B) \hookrightarrow C_*(B')$ induces an isomorphism on H_i. Forming a sequence $\{B'_k\}$ that exhausts the plane,

$$H_i(B) \cong \varinjlim H_i(B'_k) \cong H_i(A, n, \gamma), \tag{6.21}$$

with $H_i(B) \xrightarrow{\approx} H_i(A, n, \gamma)$ induced by the inclusion of chain complexes. Form a sequence $\{B''_k\}$ with horizontal edges matching B_∞ (and B), exhausting B_∞, we have

$$H_i(B) \cong \varinjlim H_i(B''_k) \cong H_i(B_\infty), \tag{6.22}$$

with $H_i(B) \xrightarrow{\approx} H_i(B_\infty)$ induced by the inclusion of chain complexes. Furthermore the commutative diagram of inclusions

$$\begin{array}{ccc} C_*(B) & \longrightarrow & C_*(B_\infty) \\ & \searrow & \downarrow \\ & & C_*(A, n, \gamma) \end{array} \tag{6.23}$$

yields a commutative diagram of isomorphisms of H_i, so $C_*(B_\infty) \hookrightarrow C_*(A, n, \gamma)$ induces an isomorphism on H_i. This holds for all i. □

Chapter 7

The $n = 1$ case

This chapter computes $H_*(A, 1, \gamma)$. We already have everything needed for some matrices A:

Theorem 7.1. *Suppose A is negative hyperbolic or $-\mathbb{1}$, $n = 1$, and $\Gamma \in \mathbb{Z}^2/\operatorname{Im}(\mathbb{1} - A)$ with $\gamma \in \Gamma$. Then*

$$H_i(A, n, \gamma) \cong \begin{cases} \mathbb{Z} & \textit{if } i \geq i_0 \\ 0 & \textit{if } i < i_0, \end{cases} \tag{7.1}$$

where $i_0 \in \mathcal{Z}$ is as in Proposition 5.35.

Proof. Combine Theorem 5.36 and Theorem 6.1. □

In the current chapter, we calculate $H_i(A, 1, \gamma)$ for positive hyperbolic A.

For the remainder of this chapter, fix A positive hyperbolic, let $n = 1$, and fix $\Gamma \in \mathbb{Z}^2/\operatorname{Im}(\mathbb{1} - A)$ with $\gamma \in \Gamma$. As usual, $\gamma = 0$ if and only if $\Gamma = 0$. Note that $N = 2$.

Theorem 7.2. *1. If $\gamma = 0$, then*

$$H_i(A,1,0) \cong \begin{cases} \mathbb{Z} & \text{if } i > 0 \\ \mathbb{Z}^2 & \text{if } i = 0 \\ 0 & \text{if } i < 0. \end{cases} \tag{7.2}$$

2. If $\Gamma \neq 0$ with $\gamma \in \Gamma$, then

$$H_i(A,1,\gamma) \cong \begin{cases} \mathbb{Z} & \text{if } i \geq i_0 \\ 0 & \text{if } i < i_0 \end{cases} \tag{7.3}$$

where $i_0 \in \mathcal{Z}$ is as in Proposition 5.35.

This theorem will follow from a calculation of the homology of a strip (using Theorem 6.1). Fix an $(A,1,\gamma)$-strip B_∞.

Proposition 7.3. *1. If $\gamma = 0$, then*

$$H_i^{\text{flat}}(B_\infty,\pi/2) \cong \begin{cases} \mathbb{Z} & \text{if } i > 0 \\ \mathbb{Z}^2 & \text{if } i = 0 \\ 0 & \text{if } i < 0. \end{cases} \tag{7.4}$$

2. If $\Gamma \neq 0$ with $\gamma \in \Gamma$, then

$$H_i^{\text{flat}}(B_\infty,\pi/2) \cong \begin{cases} \mathbb{Z} & \text{if } i \geq i_0 \\ 0 & \text{if } i < i_0 \end{cases} \tag{7.5}$$

where $i_0 \in \mathcal{Z}$ is as in Proposition 5.35.

Proving Proposition 7.3 will take the rest of the chapter.

7.1 Description of the chain complex

7.1.1 Flattened generators for $N = 2$

In this subsection, we will give a description of the structure of the chain complex $C_*^{\text{flat}}(B_\infty)$. The main result of this section is Proposition 7.22 below.

We recall (and summarize) the results of the previous chapters, specializing to $N = 2$.

Recall that $C_*^{\text{flat}}(B_\infty)$ is generated (over $\mathbb{Z}$) by flattened generators $[b_0, b_1; \eta_0, \eta_1]$ where $b_0, b_1 \in \mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$ index flattening points and $\eta_0, \eta_1 \in \{$'E', 'H'$\}$ describe labels. We may say that the chain complex is freely generated by those flattened generators which satisfy certain inequalities, or that it is a quotient, setting some $[b_0, b_1; \eta_0, \eta_1] = 0$. Viewing the chain complex as a quotient will allow δ to be more concisely stated.

Here, the pair (b_0, b_1) (or equivalently the 2-periodic sequence $\{\ldots, b_0, b_1, b_0, b_1, \ldots\}$) describes the shape of the flattened path and (η_0, η_1) (or the 2-periodic sequence) describes its labeling. Formally:

1. The abelian group $C_*^{\text{flat}}(B_\infty)$ is generated by all formal expressions $[b_0, b_1; \eta_0, \eta_1]$ where $b_0, b_1 \in \mathbb{Z}$ if $\Gamma = 0$, or $\mathbb{Z} + \frac{1}{2}$ if $\Gamma \neq 0$, and $\eta_0, \eta_1 \in \{$'E', 'H'$\}$; modulo

$$[b_0, b_1; \eta_0, \eta_1] = 0 \tag{7.6}$$

unless the following conditions hold:

$$X(p^1_{b_1}) \leq X(p^0_{b_0}), \quad X(p^1_{b_1}) \leq X(p^2_{b_0}) \tag{7.7}$$

$$p^0_{b_0} = p^1_{b_1} \text{ implies } \eta_0 = \text{'}E\text{'} \tag{7.8}$$

$$p^1_{b_1} = p^2_{b_0} \text{ implies } \eta_1 = \text{'}E\text{'}. \tag{7.9}$$

Here, $p^s_{b_s}$ is a certain lattice point indexed by b_s, as in Chapter 5. Of course, $X(p)$ is the X-coordinate (using eigencoordinates) of p. Also, periodicity allows us to write $p^2_{b_2}$ as $p^2_{b_0}$. If the conditions (7.7)–(7.9) hold, we say $((b_0, b_1), (\eta_0, \eta_1))$ is *in range* and $[b_0, b_1; \eta_0, \eta_1]$ is a *flattened generator*. (Otherwise, it is 0, which is not a flattened generator.)

2. The index of $[b_0, b_1; \eta_0, \eta_1]$ is $2(b_0 + b_1) + i_0 - \#H$, where $i_0 \in \mathcal{Z}$ is a constant, and $\#H$ denotes the number of 'H' labels. If $\Gamma = 0$, then $i_0 = 0$.

3. The differential, δ, on $C^{\text{flat}}_*(B_\infty)$ is as follows:

$$\begin{aligned}
[b_0, b_1; EE] &\mapsto 0 \\
[b_0, b_1; EH] &\mapsto [b_0 - 1, b_1; EE] - [b_0, b_1 - 1; EE] \\
[b_0, b_1; HE] &\mapsto -[b_0 - 1, b_1; EE] + [b_0, b_1 - 1; EE] \\
[b_0, b_1; HH] &\mapsto -[b_0 - 1, b_1; EH] - [b_0 - 1, b_1; HE] \\
&\qquad + [b_0, b_1 - 1; EH] + [b_0, b_1 - 1; HE],
\end{aligned} \tag{7.10}$$

where we have dropped unnecessary punctuation from $[b_0, b_1; \text{'}E\text{'}, \text{'}E\text{'}]$, and we are taking advantage of the quotient definition of $C^{\text{flat}}_*(B_\infty)$ to describe δ concisely. The expressions on the right-hand side are sometimes 0. Furthermore, the signs stated take advantage of our convention on the ordering of 'H' labels for flattened generators (η_0 before η_1).

We frequently work with the pairs (b_0, b_1) which encode just the flattened path and not its labeling. If (7.7) holds, we say (b_0, b_1) is *in range*, which is equivalent to $((b_0, b_1), (\text{'}E\text{'}, \text{'}E\text{'}))$ being in range, i.e. $[b_0, b_1; EE] \neq 0$.

Notation Convention 7.4. When there is a pair named (b_0, b_1), we let

$$X_0 = X(p^0_{b_0}) \tag{7.11}$$

$$X_1 = X(p^1_{b_1}) \tag{7.12}$$

$$X_2 = X(p^2_{b_0}). \tag{7.13}$$

By periodicity, X_2 is also $X(p^2_{b_2}) = aX_0$, where a is the eigenvalue of A with $a > 1$ (here A is positive hyperbolic). We can rewrite (7.7)–(7.9) as

$$X_1 \le X_0, \quad X_1 \le X_2 \tag{7.14}$$

$$X_0 = X_1 \text{ implies } \eta_0 = \text{`}E\text{'} \tag{7.15}$$

$$X_1 = X_2 \text{ implies } \eta_1 = \text{`}E\text{'}. \tag{7.16}$$

Recall that no two lattice points have the same X-coordinate, allowing us to replace "$p = p$" with "$X = X$".

7.1.2 "EH-maximal", etc.

Our first goal is to figure out exactly which data $\big((b_0, b_1), (\eta_0, \eta_1)\big)$ are in range, i.e. which define flattened generators $[b_0, b_1; \eta_0, \eta_1]$. Our convention is that out-of-range data result in $[b_0, b_1; \eta_0, \eta_1] = 0$, which we do not call a "flattened generator".

For fixed (b_0, b_1), the labeling with the most 'H''s determines what labelings are allowed.

Lemma-Definition 7.5. For any (b_0, b_1) (with $b_s \in \mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$ depending on Γ), one of the following is the case regarding the four expressions $[b_0, b_1; \eta_0, \eta_1]$ for $\eta_0, \eta_1 \in \{\text{`}E\text{'}, \text{`}H\text{'}\}$:

0. All $[b_0, b_1; \eta_0, \eta_1]$ are 0. (Recall that we say (b_0, b_1) is *out of range* in this case.)

1. Only $[b_0, b_1; EE]$ is nonzero. We say that (b_0, b_1) is *EE-maximal.*

2. Both $[b_0, b_1; EE]$ and $[b_0, b_1; EH]$ are nonzero, and the other two are 0. We say that (b_0, b_1) is *EH-maximal.*

3. Both $[b_0, b_1; EE]$ and $[b_0, b_1; HE]$ are nonzero, and the other two are 0. We say that (b_0, b_1) is *HE-maximal.*

4. All four $[b_0, b_1; \eta_0, \eta_1]$'s are nonzero; we say that (b_0, b_1) is *HH-maximal.*

Proof. The conditions on 'E' and 'H' labels either allow both labels, or only allow 'E', in any given spot. □

Lemma 7.6. *The pair (b_0, b_1) is EE-maximal if and only if $(b_0, b_1) = (0, 0)$, in which case $\gamma = 0$.*

Proof. We need to check that having only $[b_0, b_1; EE]$ be nonzero is equivalent to $(b_0, b_1) = (0, 0)$ and $\gamma = 0$. It is clear from (7.8) and (7.9) that having only $[b_0, b_1; EE]$ be nonzero is equivalent to $p^0_{b_0} = p^1_{b_1} = p^2_{b_0}$ (which takes care of (7.7)). But $p^0_{b_0} = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) p^0_{b_0}$ means $p^0_{b_0}$ is at 0, and so is $p^1_{b_1}$. These points can only be at 0 when $0 \in L_{A,\Gamma}$, which is equivalent to $\Gamma = 0$ and to $\gamma = 0$. In this case, $b_0 = 0$ and $b_1 = 0$. □

Recall that we refer to (b_0, b_1) as being in range if $\big((b_0, b_1), (E, E)\big)$ is in range. It is clear from the conditions defining "in range" that if $\big((b_0, b_1), (\eta_0, \eta_1)\big)$ is in range – i.e., $[b_0, b_1; \eta_0, \eta_1]$ is nonzero – then (b_0, b_1) is in range. Thus, examining in-range (b_0, b_1)'s is equivalent to examining the flattened generators of the form $[b_0, b_1; EE]$. Flattened generators with other labels (other $\{\eta_s\}$) will occur only for in-range pairs (b_0, b_1). The index of $[b_0, b_1; EE]$ is $2(b_0 + b_1)$ if $\Gamma = 0$, and $2(b_0 + b_1) + i_0$ for some constant i_0 if $\Gamma \neq 0$.

We call the set of in-range pairs with $b_0 + b_1$ fixed a *level* because we imagine drawing a picture with $b_0 + b_1$ increasing as one goes up. (The b_0 and b_1 axes are diagonal.) The diagram will turn out to be an infinite $\vee$, with ragged edges. Note that $b_0 + b_1 \in \mathbb{Z}$ even if $b_s \in \mathbb{Z} + \frac{1}{2}$. For each level, there is a "segment" of in-range pairs (see Definition 7.16), and the segments get strictly wider as one goes up, i.e., as $b_0 + b_1$ increases.

Lemma 7.7. *When $b_0, b_1 \geq 0$, with $(b_0, b_1) \neq (0, 0)$, then (b_0, b_1) is HH-maximal. When $b_0, b_1 \leq 0$, with $(b_0, b_1) \neq (0, 0)$, then (b_0, b_1) is out of range.*

In other words, $[b_0, b_1; \eta_0, \eta_1]$ is always nonzero in the first case and $[b_0, b_1; \eta_0, \eta_1]$ is always zero in the second case.

Proof. In the first case, b_0 or b_1 is positive; say $b_0 > 0$, $b_1 \geq 0$. Then $X_0 > 0$, $X_1 \leq 0$, $X_2 > 0$. This gives $X_1 < X_0, X_2$ (which we also get if $b_0 \geq 0$, $b_1 > 0$). Then (7.14)–(7.16) are satisfied and $[b_0, b_1; \eta_0, \eta_1] \neq 0$. If $b_0 < 0$, $b_1 \leq 0$, then $X_0 < 0$, $X_1 \geq 0$, contradicting (7.14), so $[b_0, b_1; \eta_0, \eta_1] = 0$, and similarly if $b_0 \leq 0$, $b_1 < 0$. $\square$

If (b_0, b_1) is in range, then the entire $\vee$-shaped region "above" (b_0, b_1) is HH-maximal.

Lemma 7.8. *If (b_0, b_1) is in range, and if $b_0' \geq b_0$, $b_1' \geq b_1$ with $(b_0', b_1') \neq (b_0, b_1)$, then (b_0', b_1') is HH-maximal.*

In particular, $(b_0 + 1, b_1)$ and $(b_0, b_1 + 1)$ are in range.

Proof. Increasing b_0 or b_1 moves X_0 and X_2, or X_1, in the "good" direction, making it easier to satisfy (7.7), so (b_0', b_1') is in range. Further, we now have strict inequalities,

$$X_1 < X_0, \quad X_1 < X_2, \tag{7.17}$$

which make (7.15) and (7.16) trivial, allowing all 'H' labels. □

7.1.3 Neighbors of an in-range pair

We now examine the other two "quadrants", starting with the case $b_0 > 0$, $b_1 < 0$. (In these quadrants, the points that b_0 and b_1 index are on the same side of the Y axis and have a chance of violating the definition of in range).

Remark 7.9. *Condition* (7.14) *is sensitive to* $\min\{X_0, X_2\}$. *Since* $X_2 = aX_0$ *with* $a > 1$, *that means it is* X_0 *(which is positive) that matters when* $b_0 > 0$ *and* X_2 *when* $b_0 < 0$.

We begin by considering one pair (b_0, b_1) at a time.

Lemma 7.10. *We consider two cases.*

1. *When* $b_0 > 0$, $b_1 < 0$ *we have:*
   - *If* $X_1 < X_0$ *then* (b_0, b_1) *is* HH*-maximal.*
   - *If* $X_1 = X_0$ *then* (b_0, b_1) *is* EH*-maximal.*
   - *If* $X_1 > X_0$ *then* (b_0, b_1) *is out of range.*
2. *When* $b_0 < 0$, $b_1 > 0$ *we have:*
   - *If* $X_1 < X_2$ *then* (b_0, b_1) *is* HH*-maximal.*
   - *If* $X_1 = X_2$ *then* (b_0, b_1) *is* HE*-maximal.*
   - *If* $X_1 > X_2$ *then* (b_0, b_1) *is out of range.*

Proof. We will only show part 1; the proof of part 2 is similar. To do that, we need only verify that $X_1 < X_2$ in the first two cases, the ones with $X_1 \leq X_0$. (Everything else is a straightforward application of (7.14)–(7.16).) But

$$X_1 \leq X_0 < aX_0 = X_2. \tag{7.18}$$

□

Now we relate (b_0, b_1) to its neighbors. The crucial fact is that the points indexed as $p^1_{b_1}$ are a subset of those indexed as $p^0_{b_0}$. In turn, the points indexed as $p^2_{b_0}$ are a subset of those indexed as $p^1_{b_1}$.

Lemma 7.11.

$$\begin{aligned} \{p^2_b \mid b \in \mathbb{Z} \text{ or } \mathbb{Z}+\tfrac{1}{2}\} &\subset \{p^1_b \mid b \in \mathbb{Z} \text{ or } \mathbb{Z}+\tfrac{1}{2}\} \\ &\subset \{p^0_b \mid b \in \mathbb{Z} \text{ or } \mathbb{Z}+\tfrac{1}{2}\}. \end{aligned} \tag{7.19}$$

Furthermore, there are infinitely many points p^0_b with $b > 0$ that are not of the form $p^1_{b'}$, and infinitely many with $b < 0$. A similar statement is true relating p^2 and p^1.

As elsewhere in this subsection, we are working with A positive hyperbolic. A similar result is true for A negative hyperbolic, but not for $A = -\mathbb{1}$.

Proof. The idea is that each successive set lies in a narrower region $\square_{B,\Theta_s}$. Each set mentioned is the set of lattice points between two lines bounding the strip B_∞, and the lines get closer together as s (in p^s) increases. More formally, if points p^0_b have Y-coordinate in an interval $[-Y_0, Y_1]$, then points p^1_b lie in $[-\frac{1}{a}Y_0, Y_1]$, and points p^2_b in $[-\frac{1}{a}Y_0, \frac{1}{a}Y_1]$.

The infinitely many points p^0_b ($b > 0$) not of the form $p^1_{\cdots}$ are the infinitely many lattice points in the region defined by $X > 0$, $Y \in [-Y_0, -\frac{1}{a}Y_0)$ (i.e. in the region for p^0 minus the region for p^1). Here, $a > 1$. This region has infinitely many points by Lemma 3.75 on the number of points in an open rectangle, such as $(X, Y) \in (0, \Delta X) \times (-Y_0, -\frac{1}{a}Y_0)$;

the lemma shows that the number of lattice points may be made as large as desired by a suitable choice of ΔX. The analogous results for $b_0 < 0$ and for points $p^2_{\cdots}$ not of the form $p^1_{\cdots}$ follow similarly. □

From Lemma 7.11, we will deduce several things. We must first treat the case $b_s = 1/2$ with $b_0 + b_1 < 0$ by itself.

Lemma 7.12. *Assume $\Gamma \neq 0$ and $b_0 + b_1 < 0$. If $b_0 = 1/2$, or $b_1 = 1/2$, then (b_0, b_1) is out of range.*

Proof. As usual, we prove only one half of the statement: Assume $b_0 = 1/2$. Then $b_1 < -1/2$, so since $b_1 \in \mathbb{Z} + \frac{1}{2}$, we have $b_1 \leq -3/2$. Then $p^0_{b_0}$ is the lattice point (with Y in the appropriate interval) with the smallest positive X coordinate. But $p^1_{-\frac{1}{2}}$ is the lattice point with the smallest positive X in a smaller Y-interval, so it is at least as big: $X(p^0_{b_0}) \leq X(p^1_{-\frac{1}{2}})$ so $X(p^0_{b_0}) < X(p^1_{-\frac{3}{2}}) \leq X(p^1_{b_1})$. Thus (7.7) is violated and (b_0, b_1) is out of range. □

Now we treat the remaining part of the quadrant $b_0 > 0$, $b_1 < 0$ (as well as the quadrant $b_1 > 0$, $b_0 < 0$). Assuming $b_0 > 0$ and ignoring $b_0 = 1/2$, we may assume $b_0 \geq 1$. Furthermore, it turns out not to be necessary to assume $b_1 < 0$.

Lemma 7.13. *If (b_0, b_1) is in range with $b_0 \geq 1$, then $(b_0 - 1, b_1 + 1)$ is also in range. If (b_0, b_1) is in range with $b_1 \geq 1$, then $(b_0 + 1, b_1 - 1)$ is also in range.*

Proof. We prove only the first statement. The reason it is true is that while we have moved $p^1_{b_1}$ "one step" left, we have moved $p^0_{b_0}$ left by a possibly smaller amount, since p^0 indexes a larger set of points. If $p^0_{b_0}$ was on the correct side of $p^1_{b_1}$ before the move, the resulting points after the move are certainly on the correct sides. Formally, count the points p^0_c, $c \in \mathbb{Z}$ or

$\mathbb{Z} + \frac{1}{2}$, "between" $p^1_{b_1+1}$ and $p^0_{b_0-1}$ in the sense of having X-coordinate greater than $X(p^1_{b_1+1})$ and less than $X(p^0_{b_0-1})$. That number of points will be at least as great as the number between $p^1_{b_1}$ and $p^0_{b_0}$ in the same sense of "between". We impose $b_0 \geq 1$ to get the condition $b_0 - 1 \geq 0$ which ensures $X(p^2_{b_0-1}) \geq 0$ and therefore $X(p^2_{b_0-1}) = aX(p^0_{b_0-1}) \geq X(p^0_{b_0-1})$, so that p^2 cannot cause a problem. □

It follows from the above lemmas that the set of in-range (b_0, b_1) with some fixed $b_0 + b_1$ is "connected" in the obvious sense. (Lemma 7.17 will show that that connected set is finite.)

Definition 7.14. For $l \in \mathbb{Z}$, call the set of in-range (b_0, b_1) with $b_0 + b_1 = l$ *connected* if there is an interval $\mathcal{I}$ (possibly empty or unbounded) such that (b_0, b_1) with $b_0 + b_1 = l$ (and with $b_0, b_1 \in \mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$ as usual) is in range if and only if $b_0 \in \mathcal{I}$.

Lemma 7.15. *The set of in-range (b_0, b_1) with $b_0 + b_1 = l$ is always connected. It is nonempty if $l > 0$ or if $l = 0$ and $\Gamma = 0$. It is empty if $l < 0$.*

Proof. For $l = b_0 + b_1 > 0$ (any Γ), or for $l = 0$ ($\Gamma = 0$), we apply Lemma 7.13 repeatedly, moving from (b_0, b_1) "inwards" to $(b_0 \mp 1, b_1 \pm 1)$, towards the pairs known to be in range, those with $b_0, b_1 \geq 0$. (These are known to be in range by Lemma 7.7 when $(b_0, b_1) \neq (0, 0)$ and Lemma 7.6 when $(b_0, b_1) = (0, 0)$.) More precisely, suppose (b_0, b_1) is in range, with $b_0 + b_1 = l$; then b_0 and b_1 are both nonnegative, which is the known case, or one of them (say b_1) is negative. Then $b_1 < 0$ means $b_0 = l - b_1 > l \geq 0$, so $b_0 > 0$. In fact $b_0 \geq 1$, because if $\Gamma = 0$ then $b_0 \in \mathbb{Z}$, and if $\Gamma \neq 0$ we are assuming $l > 0$. Thus we can apply Lemma 7.13 and move "left", showing each pair is in range until we have reached the $b_0, b_1 \geq 0$ region.

Similar considerations for $\Gamma \neq 0$ show that level $l = 0$ is connected (possibly empty). In this case, we consider the regions $b_0 > 0$ and $b_1 > 0$ (here, $b_1 = -b_0$) separately. In each, we may apply Lemma 7.13 repeatedly, moving toward the center until we reach the last point, $(b_0, b_1) = \pm\left(\frac{1}{2}, -\frac{1}{2}\right)$; we cannot apply Lemma 7.13 to that point. Thus the in-range pairs with $b_0 + b_1 = 0$ and $b_0 > 0$ form either the empty set or a connected set containing $\left(\frac{1}{2}, -\frac{1}{2}\right)$; and similarly for those with $b_1 > 0$. There are now four cases, depending on which of the points $\pm\left(\frac{1}{2}, -\frac{1}{2}\right)$ are in range. But these two points are "adjacent", so we have that level $l = 0$ is connected (possibly empty).

For $l < 0$, we already have that some pairs are out of range. Specifically, those with $b_0, b_1 \leq 0$, for all Γ. For $\Gamma \neq 0$, we may expand this region (using Lemma 7.12) to: If $b_0, b_1 \leq \frac{1}{2}$ with $b_0 + b_1 < 0$, then (b_0, b_1) is out of range. For any (b_0, b_1) outside of this region, Lemma 7.13 allows us to move towards the region known to be out of range, so (b_0, b_1) cannot be in range. Thus, there are no in-range pairs with $b_0 + b_1 < 0$.

□

We define a *segment* to be a finite, nonempty, connected set of pairs.

Definition 7.16. A *segment* in level l is a set of ordered pairs $(b_0, b_1) \in \mathbb{Z}^2$ or $(\mathbb{Z} + \frac{1}{2})^2$ of the following form: Fix *endpoints* (c_0, c_1) and (d_0, d_1) with $c_0 + c_1 = d_0 + d_1 = l$. We may assume $c_0 \leq d_0$ (and therefore $c_1 \geq d_1$). The segment with these endpoints is

$$\{(b_0, b_1) \mid b_0 + b_1 = l \text{ and } c_0 \leq b_0 \leq d_0\}. \tag{7.20}$$

Of course, we may equivalently require $c_1 \geq b_1 \geq d_1$.

Lemma 7.17. *For any $l \in \mathbb{Z}$, there are only finitely many in-range pairs (b_0, b_1) with $b_0 + b_1 = l$.*

This statement is close to some results of Chapter 6, but we will prove it directly.

Proof. We will show that $(k+l, -k)$ is out of range for a sufficiently large positive choice of $k \in \mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$. (A similar argument which we will not state applies to $(-k, l+k)$, completing the proof.) Specifically, we will show that $X(p^1_{-k}) > X(p^0_{k+l})$. There are infinitely many points of the form p^0_b ($b > 0$, i.e., $X > 0$) not of the form $p^1_{b'}$, so consider the $(l+1)$'th such point and call it $p^0_{k'}$ (so $k' > 0$). There being infinitely many points p^1_{-k} ($k > 0$), and these points being a discrete set, there is some p^1_{-k} "after" $p^0_{k'}$, i.e., $X(p^0_{k'}) < X(p^1_{-k})$. Our numbering conventions mean that p^1_{-k} is the k'th point (or $(k+\frac{1}{2})$'th point) in the $X > 0$ part of the p^1 region defined by the strip B_∞. But it must be at least the $(k+l+1)$'th point (or $(k+l+\frac{3}{2})$'th point) in the p^0 region, and in particular, p^0_{k+l+1} must come "sooner" (not strictly):

$$X(p^0_{k+l+1}) \leq X(p^1_{-k}). \tag{7.21}$$

Then, taking another point, one "sooner" than before,

$$X(p^0_{k+l}) < X(p^1_{-k}), \tag{7.22}$$

as desired. □

We now have sufficient information about which (b_0, b_1) are in range.

Proposition 7.18. *Let A be positive hyperbolic, $n = 1$, and $\Gamma \in \mathbb{Z}^2/\operatorname{Im}(A - \mathbb{1})$ with $\gamma \in \Gamma$. For any $l \in \mathbb{Z}$, the set of in-range pairs (b_0, b_1) with $b_0 + b_1 = l$ is a segment if $l > 0$ and empty if $l < 0$. If $\Gamma = 0$ and $l = 0$, the set is a segment. If $\Gamma \neq 0$ and $l = 0$, the set is either empty or a segment. The segment at index $l+1$ is wider than the one at index l (whenever both are nonempty); formally, if (b_0, b_1) is in range, then so are $(b_0 + 1, b_1)$ and $(b_0, b_1 + 1)$.*

Proof. Most of the statement of this proposition just applies the finiteness result of Lemma 7.17 to the statement of Corollary 7.15. The last sentence follows from Lemma 7.8, which implies that each level is wider than the one below it. □

We will not describe the set of in-range pairs any more precisely–we will not specify the shape of the ragged edges of the $\vee$. To do so would require considering the details of where lattice points fall in the strip in question, which depends on A. However, just knowing Proposition 7.18 and some information about labels will be sufficient to determine the homology. That is the reason that the homology for positive hyperbolic A is essentially independent of A.

7.1.4 Return to EH-maximal, etc.

To finish describing the generators of $C_*^{\text{flat}}(B_\infty)$, we must determine which labels are possible. That is, for each in-range (b_0, b_1), we find out which (η_0, η_1) make $[b_0, b_1; \eta_0, \eta_1] \neq 0$. Lemma 7.6 already shows that only $(0, 0)$ is EE-maximal. It will turn out that the distinction between EH-maximal and HE-maximal is not important. We must find which in-range pairs are HH-maximal. Lemma 7.8 shows that within any segment of in-range pairs, the set of HH-maximal pairs is at least as big as needed to "cover" the segment below. We will soon show that it is exactly big enough. First, we have a preparatory lemma.

Lemma 7.19. *Suppose (b_0, b_1) is HH-maximal. If $b_0 > 0$, $b_1 < 0$, then $(b_0 - 1, b_1)$ is in range. If $b_0 < 0$, $b_1 > 0$, then $(b_0, b_1 - 1)$ is in range.*

Proof. We prove the first case, assuming $X_1 < X_0$, $X_1 < X_2$ by HH-maximality, and that each X_s is positive by the inequalities on the b_s's. Intuitively, the worst that $p^0_{b_0-1}$ could be (the smallest X) is $p^1_{b_1}$. Formally, since the p^1's are a subset of the p^0's,

$$p^1_{b_1} = p^0_c \text{ for some } c, \tag{7.23}$$

and since $0 < X_1 < X_0$, i.e. $0 < X(p^0_c) < X(p^0_{b_0})$, we must have $0 < c < b_0$. (Incidentally, this fact already means $b_0 \geq 3/2$.) We need $b_0 - 1$ and its lattice point $p^0_{b_0-1}$; but $c \leq b_0 - 1$ so $X(p^0_c) \leq X(p^0_{b_0-1})$, i.e.

$$X(p^1_{b_1}) \leq X(p^0_{b_0-1}). \tag{7.24}$$

We still have $b_0 - 1 > 0$, so p^2 cannot cause a problem:

$$X(p^1_{b_1}) \leq X(p^2_{b_0-1}), \tag{7.25}$$

by reasoning as in Lemma 7.10. Thus, $(b_0 - 1, b_1)$ is in range. □

Corollary 7.20. *An in-range pair (b_0, b_1) other than $(\frac{1}{2}, \frac{1}{2})$ is HH-maximal if and only if at least one of the pairs "below" it, $(b_0 - 1, b_1)$ and $(b_0, b_1 - 1)$, is in range.*

Proof. "If" follows from Lemma 7.8. "Only if" follows by cases:

1. If $b_0 > 0$, $b_1 < 0$, or vice versa, apply Lemma 7.19.

2. If $b_0, b_1 \leq 0$, then Lemma 7.7 tells us that (b_0, b_1) is out of range (so the claim is trivially true) unless $(b_0, b_1) = (0, 0)$ (in which case the claim follows because $(-1, 0)$ and $(0, -1)$ are out of range).

3. If $b_0, b_1 \geq 0$, $(b_0, b_1) \neq (0, 0)$, then Lemma 7.7 tells us that (b_0, b_1) is HH-maximal. But if we also have $(b_0, b_1) \neq (\frac{1}{2}, \frac{1}{2})$, then one of $(b_0 - 1, b_1)$ and $(b_0, b_1 - 1)$ is also in this same region or is $(0, 0)$, and in either case, is in range.

□

We now have the tools to establish our result. Informally, each segment, except at level 0, contains zero or more HE-maximal cells, then one or more HH-maximal cells "covering" the segment below, and then zero or more EH-maximal cells. The only EE-maximal cell appears in level $l = 0$, at the pair $(0, 0)$, when $\Gamma = 0$. More precisely, level 0 contains zero or more HE-maximal cells, then one EE-maximal cell if $\Gamma = 0$, then zero or more EH-maximal cells.

Definition 7.21. Given a segment S of ordered pairs with endpoints (c_0, c_1) and (d_0, d_1) (we assume $c_0 \leq d_0$), the segment S' *covering* S is the segment with endpoints $(c_0, c_1 + 1)$ and $(d_0 + 1, d_1)$.

Note that S' is in level $l + 1$ if S is in level l, and that S' has one more element than S. (It is "one wider" than S.)

Proposition 7.22. *Let A be positive hyperbolic, $n = 1$, and $\Gamma \in \mathbb{Z}^2/\operatorname{Im}(A - \mathbb{1})$ with $\gamma \in \Gamma$. For each $l \in \mathbb{Z}$, the flattened generators with $l = b_0 + b_1$ are as follows.*

1. *If $l < 0$, there are no generators.*

2. *If $l = 0$, there are no HH-maximal pairs. There is an EE-maximal pair (at $(0, 0)$) if and only if $\Gamma = 0$. The part of $l = 0$ with $b_1 > 0$ has zero or more HE-maximal pairs. The part of $l = 0$ with $b_0 > 0$ has zero or more EH-maximal pairs.*

3. *If $l > 0$, the set of pairs (b_0, b_1) which are HH-maximal are exactly the segment S' covering the segment S of in-range pairs in level $l - 1$, if S is nonempty. Otherwise, $S = \emptyset$ and there is a single HH-maximal pair, $(\frac{1}{2}, \frac{1}{2})$ in level 1; let $S' = \{(\frac{1}{2}, \frac{1}{2})\}$.*

"Left" of S' (i.e., larger b_1), we have zero or more HE-maximal pairs. "Right" of S' (i.e., larger b_0), we have zero or more EH-maximal pairs.

Proof. 1. Already shown (see Lemma 7.15).

2. In this case, there are no HH-maximal pairs by Corollary 7.20. Most of the rest of the claim follows from Lemma-Definition 7.5. To distinguish HE- from EH-maximal pairs, note that b_0 and b_1 have opposite signs in level 0, and apply the two halves of Lemma 7.10.

3. The statement that S' covers nonempty S follows from Corollary 7.20 if $l > 1$, or if $l = 1$ and $\Gamma = 0$. It also follows if $l = 1$, $\Gamma \neq 0$, and level 0 is nonempty (then the claim of Corollary 7.20 is true even of $(\frac{1}{2}, \frac{1}{2})$, which is HH-maximal by Lemma 7.7). If level 0 is empty, Corollary 7.20 prevents any HH-maximal pairs other than $(\frac{1}{2}, \frac{1}{2})$, proving the claim about $S' = \emptyset$. The remaining pairs must be HE- or EH-maximal and cannot lie in Lemma 7.7's region $b_0, b_1 \geq 0$, so they must fall under one of the two halves of Lemma 7.10, which makes the ones with $b_1 > 0$ HE-maximal, and vice versa.

□

7.2 The homology

Now that we have determined what is in the chain complex $C_*^{\text{flat}}(B_\infty)$, we will compute its homology.

We will describe the action of the boundary map δ, following a change of basis.

Definition 7.23. Let

$$[b_0, b_1; H] \;=\; [b_0, b_1; EH] \;+\; [b_0, b_1; HE]\,. \tag{7.26}$$

This notation is useful even when one (or both!) of the expressions on the right-hand side are zero.

If (b_0, b_1) is EH-maximal or HE-maximal, it has two generators, $[b_0, b_1; EE]$ and $[b_0, b_1; H]$, in the new basis.

If (b_0, b_1) is HH-maximal, we still need $[b_0, b_1; EH]$, and we introduce notation:

Definition 7.24. Let

$$[b_0, b_1; EH'] \;=\; [b_0, b_1; EH] \tag{7.27}$$

if (b_0, b_1) is HH-maximal, and

$$[b_0, b_1; EH'] \;= 0 \tag{7.28}$$

otherwise.

Lemma 7.25. *The differential δ on $C^{\text{flat}}_*(B_\infty)$ acts as follows:*

$$[b_0, b_1; EE] \;\mapsto 0 \tag{7.29}$$

$$[b_0, b_1; H] \;\mapsto 0 \tag{7.30}$$

$$[b_0, b_1; EH'] \;\mapsto -[b_0 - 1, b_1; EE] \;+\; [b_0, b_1 - 1; EE] \tag{7.31}$$

$$[b_0, b_1; HH] \;\mapsto -[b_0 - 1, b_1; H] \;+\; [b_0, b_1 - 1; H]\,. \tag{7.32}$$

Recall that we use, as always, the convention that $[b_0, b_1; \eta_0, \eta_1] \;= 0$ when $((b_0, b_1), (\eta_0, \eta_1))$ is out of range.

Proof. The result follows directly from item 3 on page 125 and our new definitions.

□

Now we have found the structure of the chain complex. We may now compute the homology, proving Proposition 7.3 and thus Theorem 7.2. We restate Proposition 7.3:

Proposition 7.26. *1. If $\gamma = 0$, then*

$$H_i^{\text{flat}}(B_\infty, \pi/2) \cong \begin{cases} \mathbb{Z} & \text{if } i > 0 \\ \mathbb{Z}^2 & \text{if } i = 0 \\ 0 & \text{if } i < 0. \end{cases} \tag{7.33}$$

2. If $\Gamma \neq 0$ with $\gamma \in \Gamma$, then

$$H_i^{\text{flat}}(B_\infty, \pi/2) \cong \begin{cases} \mathbb{Z} & \text{if } i \geq i_0 \\ 0 & \text{if } i < i_0 \end{cases} \tag{7.34}$$

where $i_0 \in \mathcal{Z}$ is as in Proposition 5.35.

Remark 7.27. *The proof of the theorem will show us the generators of the homology. For convenience, let $i_0 = 0$ if $\gamma = 0$ (equivalently, if $\Gamma = 0$). The homology in index $2l + 1 + i_0$ ($l \geq 0$) is generated by*

$$\sum_{\substack{b_0, b_1 \in \mathbb{Z} \text{ or } \mathbb{Z}+\frac{1}{2} \\ b_0+b_1=l+1}} [b_0, b_1; EH]\,. \tag{7.35}$$

The homology in index $2l + i_0$ ($l \geq 0$) is generated by

$$\sum_{\substack{b_0, b_1 \in \mathbb{Z} \text{ or } \mathbb{Z}+\frac{1}{2} \\ b_0+b_1=l+1}} [b_0, b_1; HH] \tag{7.36}$$

with an additional generator

$$\sum_{\substack{b_0,b_1 \in \mathbb{Z} \text{ or } \mathbb{Z}+\frac{1}{2} \\ b_0+b_1=l+1 \\ b_0>0}} [b_0, b_1; HH] \tag{7.37}$$

in index 0 *if* $\Gamma = 0$.

Proof (of Theorem and Remark). We see from Lemma 7.25 that the chain complex splits as a direct sum, one summand being generated by EE's and EH''s, the other by H's and HH's.

We begin with EE's and EH''s. We see that the EE's are cycles, only EH''s map to EE's, and nothing maps to EH''s. The EE's form segments at least in level $l > 0$, and sometimes in $l = 0$. Let us assume for now that $l \geq 0$ and that there is at least one EE in level l. The EH''s form a segment in level $l + 1$, mapping to the EE's in level l. We call this pattern a *trapezoid*: The EH''s form the segment covering the segment of EE's, and each EH' generator maps to the two EE's below it, except the two endpoints, which map to the only EE below them. The homology of a trapezoid is $\mathbb{Z}$: Each EE is a boundary and the only combination of EH''s to be a cycle is the sum of all the EH''s, which is not a boundary. Since the EH''s are in level $l + 1$, the contribution to the total homology is $\mathbb{Z}$ in index $2(l + 1) + i_0 - \#H = 2l + 1 + i_0$, generated by

$$\sum_{b_0+b_1=l+1} [b_0, b_1; EH']\,. \tag{7.38}$$

The sum may be taken over all (b_0, b_1) with $b_0+b_1 = l+1$ because we define $[b_0, b_1; EH'] = 0$ except for HH-maximal pairs (b_0, b_1). Below, we will show we can drop the $'$ from EH'.

To finish the contribution to homology of the EE's and EH''s, we must examine the case that $l = 0$ and level 0 has no EE's (so $\Gamma \neq 0$). (There are no EE's below level

0 and no EH''s below level 1.) Then, the EE's in level 0 and EH''s in level 1 form a degenerate version of the same story as before. There is an EH', at $(\frac{1}{2}, \frac{1}{2})$, and no EE below it; the homology is $\mathbb{Z}$ generated by $[\frac{1}{2}, \frac{1}{2}; EH']$. Since that is the only EH' in level $1 = l + 1$, it can also be written as in (7.38).

Now we have that the contribution of the EE's and EH''s is $\mathbb{Z}$ in each index $2l + 1 + i_0$, $l \geq 0$. This is exactly the "odd" portion of (7.33) and (7.34). The generators are as given in (7.35) except for the discrepancy between EH and EH'.

For $l > 0$, the H's in level l and HH's in level $l + 1$ form the same trapezoid pattern as the EE's and EH''s, with the homology of each trapezoid given by the sum of its HH's, resulting in a contribution of $\mathbb{Z}$ to homology in index $2(l+1)+i_0-\#H = 2l+i_0$. In particular, each H is a cycle and is a boundary, a fact we will need for EH' vs. EH. The generator of $H^{\text{flat}}_{2l+i_0}(B_\infty)$ is as given in (7.36) (though only for $l > 0$ so far).

We must consider the remaining generators: H's in level $l = 0$ and HH's in level $l + 1 = 1$. (There are no generators "below" these.) If $\Gamma \neq 0$, there are zero or more H's in level 0, and exactly one more HH in level 1 than H in level 0. These generators form a (possibly degenerate) trapezoid with homology $\mathbb{Z}$ generated by the sum of the HH's, as in (7.36) (for $l = 0$). If $\Gamma = 0$, the trapezoid region occupied by EE's and EH''s is "broken" by the lack of $[0, 0; EH]$ into two trapezoids of H's and HH's, each trapezoid contributing one $\mathbb{Z}$ in index 0. More precisely, the H's and HH's with $b_0 > 0$ form one trapezoid and those with $b_0 \leq 0$ form another. The contribution to homology is $\mathbb{Z}^2$, generated by $\sum_{b_0>0} [b_0, b_1; HH]$ and $\sum_{b_0\leq 0} [b_0, b_1; HH]$. We may equivalently say that $\mathbb{Z}^2$ is generated by $\sum_{b_0>0} [b_0, b_1; HH]$ and

$$\sum_{b_0>0} [b_0, b_1; HH] + \sum_{b_0\leq 0} [b_0, b_1; HH], \tag{7.39}$$

which is just the $l = 0$ version of (7.36).

Now we have found the contribution of the H's and the HH's, namely $\mathbb{Z}$ in index $2l + i_0$ ($l \geq 0$) except for $\Gamma = 0$, index 0, in which case the contribution is $\mathbb{Z}^2$. This matches the homology claimed above, as well as the generators.

It only remains to relate EH' and EH. Recall that $EH' = EH$ in HH-maximal pairs and $EH' = 0$ otherwise. The sum (7.38) is, in effect, over all HH-maximal pairs (b_0, b_1) (in some level). The sum in (7.35) is over those as well as the additional EH's with large b_0 (as in Proposition 7.22), in EH-maximal pairs. But in such pairs, $[b_0, b_1; EH] = [b_0, b_1; H]$. As shown above, the $[b_0, b_1; H]$'s are boundaries, and may be added without affecting the homology class. So (7.35) and (7.38) are homologous, allowing us to use whichever one we find more elegant. □

Chapter 8

The splicing isomorphism from n to $n + 1$

Thus far, we have calculated the embedded contact homology, $H_*(A, n, \gamma)$, only for $n = 1$. The purpose of this chapter is to show the following theorem.

Theorem 8.1. *For any $A \in SL_2(\mathbb{Z})$ with A hyperbolic or $A = -\mathbb{1}$, and any $\gamma \in \mathbb{Z}^2$, $H_*(A, n, \gamma)$ is independent of n (for all positive $n \in \mathbb{Z}$).*

We fix a convention.

Assumption 8.2. For the remainder of this chapter, fix (A, n, γ) with $A \in SL_2(\mathbb{Z})$ hyperbolic or $A = -\mathbb{1}$, $n \in \mathbb{Z}$ positive, and $\gamma \in \Gamma \subseteq \mathbb{Z}^2$.

We will prove that

$$H_*(A, n, \gamma) \cong H_*(A, n + 1, \gamma), \tag{8.1}$$

and the theorem follows. Recall that for any (A, n, γ)-strip $B_\infty^{(n)}$, we have

$$H_*(A, n, \gamma) \cong H_*^{\text{flat}}(B_\infty^{(n)}), \tag{8.2}$$

and similarly with $n+1$ in place of n. We will choose a strip $B_\infty^{(n+1)} = \text{Splice}(B_\infty^{(n)})$, and construct a map

$$S : C_*^{\text{flat}}(B_\infty^{(n)}) \to C_*^{\text{flat}}(B_\infty^{(n+1)}) \tag{8.3}$$

which induces an isomorphism on homology, from which (8.1) follows. The map S is called *splicing*; it sums over all ways cut a flattened generator open at Θ_0 and "splice" two H arcs in, while shortening the path on $[\Theta_{-1}, \Theta_0]$ and $[\Theta_0, \Theta_1]$

We often abreviate $C_*^{\text{flat}}(B_\infty, \Theta_0) = C_*^{\text{flat}}(B_\infty, \pi/2)$ as $C_*^{\text{flat}}(B_\infty)$.

The argument in this chapter is a generalization of the splicing argument from the T^3 case [5].

8.1 A concise formula for the differential

It will be useful to have a concise formula for the differential. To that end, the effect of δ on $\{\eta_s\}$ will be described with a notation $\{\eta_s\}\backslash t$ ($t \in \mathbb{Z}$), analogous to $\{b_s\}\backslash t$. In fact, we can make $\{\eta_s\}$ work exactly like that for $\{b_s\}$, if we encode 'E' and 'H' as

$$\begin{aligned} 0 &= \text{`}E\text{'} \\ 1 &= \text{`}H\text{'}. \end{aligned} \tag{8.4}$$

(Mnemonic: $\eta_s \in \{0, 1\}$ gives the number of 'h' labels.) Rounding 'H' yields 'E', which is just subtracting 1, and rounding 'E' yields $[\{b_s\}; \{\eta_s\}] = 0$, so we define labelings including -1 and define them to be out of range.

Definition 8.3. An *extended flattened labeling* is a sequence $\{\eta_s\}$ with

$$\eta_s \in \{1, 0, -1, -2, \ldots\} \tag{8.5}$$

for $s \in \mathbb{Z}$, satisfying $\eta_{s+N} = \eta_s$.

Definition 8.4. Given a sequence $\{\eta_s\}$, let $\{\eta_s\}\backslash t$ be defined as in Definition 5.25, but with "η" in place of "b".

Definition 8.5. If $\eta_s \in \{0, 1\}$ for all $s \in \mathbb{Z}$, we define *in range*, etc., by regarding η_s as equivalent to the corresponding sequence of 'E''s and 'H''s, using (8.4). If any $\eta_s \leq -1$, then we define everything involving η_s to be out of range, and in particular, $[\{b_s\}; \{\eta_s\}] = 0$. We define $\#_t\{\eta_s\}$ to count the number of s's with $\eta_s = 1$.

Remark 8.6. *A pair $(\{b_s\}, \{\eta_s\})$, where $\{b_s\}$ is a size sequence and $\{\eta_s\}$ is an extended flattened labeling, is in range if and only if it satisfies all of the following conditions:*

1. $p^s_{b_s} \leq_{\Theta_{s+1}} p^{s+1}_{b_{s+1}}$.

2. $\eta_s \in \{0, 1\}$ *for all* $s \in \mathbb{Z}$.

3. *For all* $s \in \mathbb{Z}$, *if* $p^s_{b_s} = p^{s+1}_{b_{s+1}}$, *then* $\eta_s = 0$.

With this notation in hand, we can state $\delta\,[\{b_s\}; \{\eta_s\}]$ in full, complete with signs. Roughly, we have a signed sum of terms, in each of which one b_t and one η_u are decremented, with u "adjacent" to t, so η_u labels $[\Theta_{t-1}, \Theta_t]$ or $[\Theta_t, \Theta_{t+1}]$. The proposition is carefully stated so as not to assume the $(\{b_s\}, \{\eta_s\})$ given are in range. In the event these data are out of range, the proposition reduces to $0 \mapsto 0$, but it is useful that the same formula applies in this case.

Proposition 8.7. *Let B be an (A,n,γ)-strip and $\Theta_0 = \pi/2$. Suppose $\{b_s\}$ is a size sequence, $\{\eta_s\}$ is an extended flattened labeling, and $t \in \mathbb{Z}$. Then*

$$\begin{aligned}\delta_{\Theta_t}\left[\{b_s\};\{\eta_s\}\right] &= (-1)^{\#_{t-1}\{\eta_s\}}\left[\{b_s\}\backslash t;\{\eta_s\}\backslash(t-1)\right] \\ &\quad - (-1)^{\#_t\{\eta_s\}}\left[\{b_s\}\backslash t;\{\eta_s\}\backslash t\right].\end{aligned} \tag{8.6}$$

Proof. If $N = 1$, $t \equiv t-1$ and the terms on the right hand side cancel, agreeing with Lemma 5.28. Henceforth, assume $N > 1$. If any η_s is negative, both sides of (8.6) are 0. Otherwise, we may apply Proposition 5.33

□

In Proposition 8.7, the sum is arranged in terms of which b_s is decremented. We may rearrange in terms of which η_s is decremented.

Corollary 8.8. *The sum may be rearranged so as to obtain $\delta = \sum_{u=0}^{N-1}\delta^{(u)}$ where*

$$\begin{aligned}\delta^{(u)}\left[\{b_s\};\{\eta_s\}\right] &= (-1)^{\#_u\{\eta_s\}}\Big(\left[\{b_s\}\backslash(u+1);\{\eta_s\}\backslash u\right] \\ &\qquad - \left[\{b_s\}\backslash u;\{\eta_s\}\backslash u\right]\Big).\end{aligned} \tag{8.7}$$

□

8.2 The maps Splice and S

Given an (A,m,γ)-strip $B_\infty^{(m)}$, let $Y_s = (-1)^{s+1}Y(\mathrm{Edge}_{B_\infty^{(m)}}(s\pi))$, so each edge of the strip lies along the line $Y = (-1)^{s+1}Y_s$, where Y_s is strictly positive. The sequence $\{Y_s\}$, the *widths* of $B_\infty^{(m)}$, has a "periodicity" property. The period, denoted M, is $2m$ if a is positive hyperbolic and $2m-1$ if a is negative hyperbolic or $-\mathbb{1}$. We have

$$Y_{s+M} = \frac{1}{a}Y_s. \tag{8.8}$$

The strip $B_\infty^{(m)}$ is determined by its widths, and the widths are determined by any consecutive M widths using periodicity. We will sometimes denote the infinite sequence $\{Y_s\}_{s\in\mathbb{Z}}$ by $(Y_0,\ldots,Y_{M-1})$.

Recall that a point $\Lambda(\Theta_s)$ of $\Lambda \leq B_\infty^{(m)}$ must lie in a certain region – in our new notation, it must have Y-coordinate in $[-Y_s, Y_{s+1}]$ if s is even, or $[-Y_{s+1}, Y_s]$ if s is odd.

We may construct Splice($B_\infty^{(n)}$) by specifying its widths, which we call Y'_s.

Definition 8.9. Given an (A, n, γ)-strip $B_\infty^{(n)}$, with widths

$$\{Y_s\} = (Y_0, \ldots, Y_{N-1}), \tag{8.9}$$

we define Splice($B_\infty^{(n)}$) to be the $(A, n+1, \gamma)$-strip with widths

$$\{Y'_s\} = (Y'_0, \ldots, Y'_{N+1}) = (Y_0, Y_1, Y_0, Y_1, \ldots, Y_{N-2}, Y_{N-1}). \tag{8.10}$$

In other words, $Y'_s = Y_{s-2}$ for $3 \leq s \leq N+2$, $Y'_1 = Y_1$, and $Y'_2 = Y_0$. This definition specifies Y'_s for $1 \leq s \leq N+2$, and the infinite sequence $\{Y'_s\}$ is determined by periodicity (of period $N+2$). In particular, $Y'_0 = Y_0$.

Intuitively, we have spliced in two edges (Y'_1 and Y'_2) to the strip, creating room for two H arcs to be spliced in by S, and just reindexed the remaining edges. For the rest of this section, let p^s denote $\Lambda(\Theta_s)$ for a $(B_\infty^{(n)}, \pi/2)$-flattened path, and let p'^s denote $\Lambda(\Theta_s)$ for a $(B_\infty^{(n+1)}, \pi/2)$-flattened path.

From now on, an expression of the form "$\mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$" will be understood to mean $\mathbb{Z}$ if $\Gamma = 0$ and $\mathbb{Z}+\frac{1}{2}$ if $\Gamma \neq 0$.

The result of Splice is that the flattening points p'^0, p'^1, p'^2, and p^0 all lie in the same region, the region with Y-coordinate in $[-Y_0, Y_1]$. Therefore, for any $b \in \mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$, we have

$$p'^{0}_{b} = p'^{1}_{-b} = p'^{2}_{b} = p^{0}_{b}. \tag{8.11}$$

This fact is very convenient for S. For example, $p'^{1}_{c_1} <_{\Theta_2} p'^{2}_{c_2}$ is equivalent to $-c_1 < c_2$. Also note that

$$p'^{s+2}_{b} = p^{s}_{b} \text{ for } 3 \leq s \leq N+2. \tag{8.12}$$

As we did with $\{Y_s\}$, we introduce a notation

$$\{b_s\}_{s\in\mathbb{Z}} = (b_0, \ldots, b_{N-1}); \tag{8.13}$$

and similarly

$$\{\eta_s\}_{s\in\mathbb{Z}} = (\eta_0, \ldots, \eta_{N-1}). \tag{8.14}$$

Define a *sequence pair* to be a pair $(\{b_s\}, \{\eta_s\})$ of N-periodic sequences ($b_s \in \mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$, and $\eta_s \in \mathbb{Z}$ with $\eta_s \leq 1$).

In these terms, there is one generator of $C^{\text{flat}}_{*}(B^{(n)}_{\infty})$, denoted

$$[b_0, \ldots, b_{N-1}; \eta_0, \ldots, \eta_{N-1}] = [\{b_s\}; \{\eta_s\}], \tag{8.15}$$

for each sequence pair such that

$$\eta_s \in \{0, 1\} \tag{8.16}$$

$$p^{s}_{b_s} \leq_{\Theta_{s+1}} p^{s+1}_{b_{s+1}} \tag{8.17}$$

and

$$p^{s}_{b_s} = p^{s+1}_{b_{s+1}} \text{ implies } \eta_s = 0 \tag{8.18}$$

for all s. For convenience in working with S, we define $[\{b_s\};\{\eta_s\}] = 0$ if $\eta_s < 0$, or if (8.17) or (8.18) is not satisfied, for any s. In this case, we say that $(\{b_s\},\{\eta_s\})$ is *out of range*. We proceed similarly for $B_\infty^{(n+1)}$, indexing generators by $\{b'_s\}$ which define points $p'^s_{b'_s}$ and by $\{\eta'_s\}$, and by requiring similar conditions on these data. We denote a generator by $[\{b_s\};\{\eta_s\}] = [b_0,\ldots,b_{N+1};\eta_0,\ldots,\eta_{N+1}]$. In the second notation, the number of arguments inside the angle brackets distinguishes this generator of $C_*^{\text{flat}}(B_\infty^{(n+1)})$ from a generator of $C_*^{\text{flat}}(B_\infty^{(n)})$. The first notation is technically ambiguous, but it will never be unclear where a generator lives. As before, $[\{b_s\};\{\eta_s\}] = 0$ if $(\{b_s\},\{\eta_s\})$ is out of range.

Intuitively, S sums over all ways to splice two 'H' arcs into a generator while shortening the adjacent arcs, and preserving the index. Formally:

Definition 8.10. Given an (A,n,γ)-strip $B_\infty^{(n)}$, let $B_\infty^{(n+1)} = \text{Splice}(B_\infty^{(n)})$. Given a sequence pair $(\{b_s\},\{\eta_s\})$, we will define $S[b_0,\ldots,b_{N-1};\eta_0,\ldots,\eta_{N-1}]$, which we abbreviate as $S[b_0,\mathbf{b};\boldsymbol{\eta}]$. Here $\mathbf{b}$ denotes the "word" (finite sequence) $b_1,\ldots,b_{N-1}$ (of length ≥ 0) and $\boldsymbol{\eta}$ denotes $\eta_0,\ldots,\eta_{N-1}$ (length ≥ 1). Define

$$S[b_0,\mathbf{b};\boldsymbol{\eta}] = \sum_{\substack{c_0,c_1,c_2\in\mathbb{Z} \text{ or } \mathbb{Z}+\frac{1}{2} \\ c_0+c_1+c_2-1=b_0}} [c_0,c_1,c_2,\mathbf{b};1,1,\boldsymbol{\eta}]. \tag{8.19}$$

Note that $[b_0,\mathbf{b};\boldsymbol{\eta}] \in C_*^{\text{flat}}(B_\infty^{(n)})$ (it is either 0 or a generator), and the right-hand side is an element of $C_*^{\text{flat}}(B_\infty^{(n+1)})$.

The "1, 1" on the right hand side encodes the two 'H' arcs. We will show below that S "shortens" the path. The condition $c_0+c_1+c_2-1 = b_0$ serves to force S to preserve the index. Note that the definition makes sense for all $N \geq 1$.

Lemma 8.11. *With hypotheses as in Definition 8.10,*

1. *All nonzero terms appearing in the right-hand side of the definition of S satisfy $c_0 \leq b_0$ and $c_2 \leq b_0$.*

2. *The sum is finite (only finitely many terms are nonzero).*

3. *$S(0) = 0$. In other words, if the $\{b_s\}, \{\eta_s\}$ on the left-hand side are out of range, then the terms on the right-hand side are also out of range.*

(We allow S to act on out-of-range data, because it will make it much easier to prove $S\delta = \delta S$.)

Proof. We must examine the conditions under which the summands on the right hand side are nonzero. We must have

$$p'^0_{c_0} \leq_{\Theta_1} p'^1_{c_1} \leq_{\Theta_2} p'^2_{c_2}; \tag{8.20}$$

but the new arcs are both 'H', $\eta'_0 = \eta'_1 = 1$, so

$$p'^0_{c_0} <_{\Theta_1} p'^1_{c_1} <_{\Theta_2} p'^2_{c_2}. \tag{8.21}$$

Therefore,

$$c_0 > -c_1 < c_2. \tag{8.22}$$

In particular, using $c_0 + c_1 + c_2 - 1 = b_0$, we have

$$c_0 \leq b_0 \text{ and } c_2 \leq b_0. \tag{8.23}$$

This proves part (1), and is an upper bound for c_0 and c_2 in the sum defining S. Since $p'^{-1} = p^{-1}$ and $p'^3 = p^1$, the relations $p'^{-1}_{b_{-1}} \leq_{\Theta_0} p'^0_{c_0}$ and $p'^2_{c_2} \leq_{\Theta_3} p'^3_{b_1}$ provide lower bounds on c_0 and c_2 even though the bounds are hard to state. Thus, we have proved part (2), because c_1 is uniquely determined by c_0 and c_2.

To prove part (3), note that $(\{b_s\}, \{\eta_s\})$ becomes 0 by failure of any of the conditions (8.16)–(8.18). Since we are assuming $\eta_s \leq 1$, failure of (8.16) means $\eta_s < 0$ for some s, and so $\eta'_{s+2} < 0$. If (8.17) or (8.18) fail, some point is on the wrong side of another point, or too close. This relation carries over to the output of S if it does not involve $s = 0$. If it involves $s = 0$, then it becomes even worse among the terms of the output that have any hope of being nonzero, the ones satisfying part (1).

□

Lemma 8.12. *S defines a map of graded abelian groups*

$$S : C_*^{\text{flat}}(B_\infty^{(n)}) \to C_*^{\text{flat}}(\text{Splice}(B_\infty^{(n)})). \tag{8.24}$$

If $\Gamma = 0$, then S is a degree 0 map. If $\Gamma \neq 0$, there is only a relative grading on the chain complexes, but S is still homogeneous.

Proof. We may extend S by linearity from generators to the whole chain complex, obtaining

$$S : C_*^{\text{flat}}(B_\infty^{(n)}) \to C_*^{\text{flat}}(B_\infty^{(n+1)}), \tag{8.25}$$

by Lemma 8.11. The grading claim follows from the index formula

$$I\big(\,[\{b_s\};\{\eta_s\}]\,\big) = 2\sum_{s=0}^{N-1} b_s - \sum_{s=0}^{N-1} \eta_s + i_0^{(n)} \tag{8.26}$$

for $C_*^{\text{flat}}(B_\infty^{(n)})$ and the analogous formula for $C_*^{\text{flat}}(B_\infty^{(n+1)})$, where $i_0^{(n)} = i_0^{(n+1)} = 0$ if $\Gamma = 0$.

□

8.3 S is a chain map

Our goal is to show that S induces an isomorphism on H_*. To that end, we show

Lemma 8.13. *The map S commutes with δ, and therefore induces a map*

$$H_*^{\text{flat}}(B_\infty^{(n)}) \xrightarrow{S_*} H_*^{\text{flat}}(B_\infty^{(n+1)}). \tag{8.27}$$

The proof is deferred until we rephrase Lemma 8.13 as Lemma 8.14.

Recall that, on $C_*^{\text{flat}}(B_\infty^{(n)})$,

$$\delta = \sum_{t=0}^{N-1} \delta^{(t)}, \tag{8.28}$$

where $\delta^{(t)}$ is the component of δ that removes an 'H' label at position t ($\eta_t = 1 \mapsto \eta_t = 0$).

We have

$$\begin{aligned} \delta^{(t)}\left[\{b_s\};\{\eta_s\}\right] = (-1)^{\#_t\{\eta_s\}}\Big[&\left[\{b_s\}\backslash(t+1);\{\eta_s\}\backslash t\right] \\ &- \left[\{b_s\}\backslash t;\{\eta_s\}\backslash t\right]\Big] \end{aligned} \tag{8.29}$$

for any sequence pair, not necessarily in range.

A similar formula is true for $B_\infty^{(n+1)}$ with $N+2$ in place of N. We will write

$$\delta = \delta^{(0)} + \delta^{(1)} + \sum_{t=0}^{N-1} \delta^{(t+2)}. \tag{8.30}$$

To prove that $S\delta = \delta S$, it is sufficient to show the following.

Lemma 8.14. *Given a strip $B_\infty^{(n)}$ compatible with (A,n,γ), $B_\infty^{(n+1)} = \text{Splice}(B_\infty^{(n)})$, and $\alpha \in C_*^{\text{flat}}(B_\infty^{(n)})$,*

1. $\delta^{(0)}S\alpha = \delta^{(1)}S\alpha = 0.$

2. $\delta^{(t+2)} S\alpha = S\delta^{(t)}\alpha$ *for* $0 \le t \le N-1$.

It is sufficient to prove this lemma for α a generator, $\alpha = [b_0, \mathbf{b}; \boldsymbol{\eta}]$.

Proof of part (1). We will show $\delta^{(0)} S\alpha = 0$.

$$\begin{aligned}
&\delta^{(0)} S\,[b_0, \mathbf{b}; \boldsymbol{\eta}] \\
&= \delta^{(0)} \sum_{\substack{c_0,c_1,c_2 \\ c_0+c_1+c_2-1=b_0}} [c_0, c_1, c_2, \mathbf{b}; 1, 1, \boldsymbol{\eta}] \\
&= \sum_{\substack{c_0,c_1,c_2 \\ c_0+c_1+c_2-1=b_0}} (-1)^{\#_0(1,1,\boldsymbol{\eta})} \Big[[c_0, c_1-1, c_2, \mathbf{b}; 0, 1, \boldsymbol{\eta}] \\
&\qquad - [c_0-1, c_1, c_2, \mathbf{b}; 0, 1, \boldsymbol{\eta}] \Big] \\
&= (-1)^0 \Big[\sum_{\substack{c_0,c_1,c_2 \\ c_0+c_1+c_2-1=b_0}} [c_0, c_1-1, c_2, \mathbf{b}; 0, 1, \boldsymbol{\eta}] \\
&\quad - \sum_{\substack{c_0',c_1',c_2 \\ (c_0'+1)+(c_1'-1)+c_2-1=b_0}} [c_0', c_1'-1, c_2, \mathbf{b}; 0, 1, \boldsymbol{\eta}] \Big] \\
&= 0.
\end{aligned} \tag{8.31}$$

Note that we have reindexed in the second-to-last line.

Here, we may start with an honest generator $((\{b_s\}, \{\eta_s\})$ in range), but application of S results in a sum of both honest generators and 0's. We can apply $\delta^{(0)}$ to both types; we need not sort out which is which. Similar claims will apply below. We may apply S and δ to any sum of $[\ldots;\ldots]$'s, provided $\eta_s \le 1$, and S and δ will never result in $\eta_s > 1$.

The proof for $\delta^{(1)}$ is similar. The only significant difference is that $\#_1(1, 1, \boldsymbol{\eta}) = 1$, resulting in an overall factor of -1. □

Proof of part (2). If $N = 1$, the claim reduces to $\delta^{(2)} S\alpha = 0$, which may be verified by reindexing similarly to (8.31). So assume $N \ge 2$.

If $1 \le t \le N-2$, then $\delta^{(t)}$ and S do not "interact". S only affects $b'_0, b'_1, b'_2, \eta'_1, \eta'_2$ and $\delta^{(t)}$ only affects other b_s's and η_s's. The signs also agree because $\#_{t+2}(1,1,\boldsymbol{\eta}) = 2 + \#_t(\boldsymbol{\eta})$. It remains to check $t=0$ and $t=N-1$. We will verify $t=0$ as $t=N-1$ is similar. (The reader may verify that if $N=1$, a similar proof works.)

For $t=0$, unlike the convention above, we let $\mathbf{b} = (b_2, \ldots, b_{N-1})$, so that $\{b_s\} = (b_0, b_1, \mathbf{b})$. We make an analogous change in the definition of $\boldsymbol{\eta}$, so that $\{\eta_s\} = (\eta_0, \boldsymbol{\eta})$.

$$\begin{aligned}
\delta^{(t+2)} S\alpha &= \delta^{(2)} S\,[b_0, b_1, \mathbf{b}; \eta_0, \boldsymbol{\eta}] \\
&= \delta^{(2)} \sum_{\substack{c_0,c_1,c_2 \\ c_0+c_1+c_2-1=b_0}} [c_0, c_1, c_2, b_1, \mathbf{b}; 1, 1, \eta_0, \boldsymbol{\eta}] \\
&= (-1)^{\#_2(1,1,\eta_0,\boldsymbol{\eta})} \Big[\sum_{\substack{c_0,c_1,c_2 \\ c_0+c_1+c_2-1=b_0}} [c_0, c_1, c_2, b_1-1, \mathbf{b}; 1, 1, \eta_0-1, \boldsymbol{\eta}] \\
&\qquad - \sum_{\substack{c_0,c_1,c_2 \\ c_0+c_1+c_2-1=b_0}} [c_0, c_1, c_2-1, b_1, \mathbf{b}; 1, 1, \eta_0-1, \boldsymbol{\eta}] \Big] \qquad (8.32) \\
&= (-1)^{\#_0(\eta_0,\boldsymbol{\eta})} \Big[S\,[b_0, b_1-1, \mathbf{b}; \eta_0-1, \boldsymbol{\eta}] \\
&\qquad - S\,[b_0-1, b_1, \mathbf{b}; \eta_0-1, \boldsymbol{\eta}] \Big] \\
&= S\delta^{(0)}\,[b_0, b_1, \mathbf{b}; \eta_0, \boldsymbol{\eta}] \\
&= S\delta^{(t)}\alpha.
\end{aligned}$$

For the signs to agree, we have used $\#_2(1,1,\eta_0,\boldsymbol{\eta}) = 2 + \#_0(\eta_0,\boldsymbol{\eta})$. A similar argument, including a similar matching of signs, works for $t = N-1$. □

This completes the proof of Lemma 8.13. We now have a map $H_*(A,n,\gamma) \longrightarrow H_*(A,n+1,\gamma)$ and it remains only to show that it is an isomorphism.

8.4 S induces an isomorphism

In this section, we will show that S induces an isomorphism on homology, completing the proof of Theorem 8.1 and completing the calculation of $H_*(A, n, \gamma)$. The plan is to define filtrations on $C_*^{\text{flat}}(B_\infty^{(n)})$ and $C_*^{\text{flat}}(B_\infty^{(n+1)})$ compatible with S and consider the spectral sequences, ${}^nE^*_{*,*}$ and ${}^{n+1}E^*_{*,*}$, of these filtered complexes. S will induce an isomorphism on the $E^1_{*,*}$ sheets of the spectral sequences, and therefore an isomorphism on homology.

Definition 8.15. Given a $(B_\infty^{(n)}, \pi/2)$-flattened generator $[b_0, b_1; \eta_0, \eta_1]$, let

$$\mu(\,[b_0, b_1; \eta_0, \eta_1]\,) = 2\sum_{s=0}^{N-1} b_s. \tag{8.33}$$

Given a $(B_\infty^{(n+1)}, \pi/2)$-flattened generator $[\{b'_s\}; \{\eta'_s\}]$, let

$$\mu'(\,[\{b'_s\}; \{\eta'_s\}]\,) = b'_0 + b'_2 + 2\sum_{s=3}^{N+1} b'_s. \tag{8.34}$$

The crucial property of μ is that μ strictly decreases when we round a flattened generator; whereas μ' strictly decreases under rounding at any vertex except b'_1.

The functions μ and μ' define gradings on $C_*^{\text{flat}}(B_\infty^{(n)}, \Theta_0)$ and $C_*^{\text{flat}}(B_\infty^{(n+1)}, \Theta_0)$. (They are gradings over $\mathbb{Z}$. Recall that the usual grading, the $*$ in C_*, is over the $\mathbb{Z}$-torsor $\mathcal{Z}$.) From the gradings, we may build increasing filtrations, denoted by $F_pC_*^{\text{flat}}(B_\infty^{(n)}, \Theta_0)$ and $F_pC_*^{\text{flat}}(B_\infty^{(n+1)}, \Theta_0)$, making those chain complexes into filtered chain complexes.

The differentials on the two complexes respect the filtrations. Thus, we have two spectral sequences.

Lemma 8.16. *Suppose B_∞ is an (A, n, γ)-strip, and $i \in \mathcal{Z}$. Then there exists $X_i > 0$ such that, for any $(B_\infty, \pi/2)$-flattened generator $[\{b_s\}; \{\eta_s\}]$ with $I(\,[\{b_s\}; \{\eta_s\}]\,) = i$, we have $|X(p^s_{b_s})| \leq X_i$ for $s \in \{0, \ldots, N-1\}$.*

Proof. Essentially the same proof as Lemma 6.12. □

It follows that, for a given $i \in \mathcal{Z}$, there are upper and lower bounds on $b_0, \ldots, b_{N-1}$ of flattened generators with index i. Therefore, there are upper and lower bounds on $\mu(\,[\{b_s\};\{\eta_s\}]\,)$, and similarly for μ'. Thus the spectral sequences converge.

Lemma 8.17. *S respects the filtrations.*

Proof. We have $\mu(\,[b_0,\mathbf{b};\boldsymbol{\eta}]\,) = 2b_0 + \sum_{s=1}^{N-1} b_s$ and

$$\mu'(S\,[b_0,\mathbf{b};\boldsymbol{\eta}]\,) = c_0 + c_2 + 2\sum_{s=1}^{N-1} b_s. \tag{8.35}$$

Recall that $c_0, c_2 \le b_0$ in any nonzero summand of S. Thus $\mu'(S\alpha) \le \mu(\alpha)$. □

Thus we have a morphism of spectral sequences.

To prove an isomorphism on homology, it is sufficient to show, for some r, that the map $E^{r,\mathrm{flat}}_{*,*} \to E^r_{*,*}$ is an isomorphism. We will use $r = 1$.

Definition 8.18. Let $C_*(b'_0, b'_2)$ be the chain complex generated by "truncated flattened generators" defined on $[\Theta_0, \Theta_2]$ with underlying path Λ' satisfying $\Lambda'(\Theta_0) = p'^0_{b'_0}$ and $\Lambda'(\Theta_2) = p'^2_{b'_2}$. By truncated flattened generators, we mean that Λ' satisfies the usual maximality condition, and is labeled according to 'E' or 'H', on $[\Theta_0, \Theta_1]$ and on $[\Theta_1, \Theta_2]$.

The complex $C_*(b'_0, b'_2)$ is isomorphic to the analogous complex in the proof of [5, Proposition 7.5]. Its homology is 0 if $b'_0 \neq b'_2$ and $\mathbb{Z}$ if $b'_0 = b'_2$. Let $HH_{b'_0}$ denote the truncated flattened generator with $-b'_1 = b'_0 - 1$ and $b'_2 = b'_0$, and with both labels 'H'. Then $H_*(b'_0, b'_0)$ is generated by $HH_{b'_0}$.

Given $\{b_s\}$, we abbreviate $\Lambda_{\{b_s\}}|[\Theta_s, \Theta_{s+1}]$ as Λ_s. Given $\{b'_s\}$, define Λ'_s similarly.

Lemma 8.19. *We have isomorphisms of abelian groups, respecting the differentials,*

$$ {}^{n}E^{1}_{*,*} \cong \bigoplus \bigotimes_{s=0}^{N-1} \mathbb{Z}\{E_{\Lambda_s}, H_{\Lambda_s}\} \tag{8.36}$$

$$ {}^{n+1}E^{1}_{*,*} \cong \bigoplus \mathbb{Z}\{HH_{b'_0}\} \otimes \bigotimes_{s=2}^{N+1} \mathbb{Z}\{E_{\Lambda'_s}, H_{\Lambda'_s}\} \tag{8.37}$$

where the first sum is over in-range size sequences $\{b_s\}$ *and the second sum is over* $\{b'_s\}$ *of the form* $b'_0 = b_0$, $b'_2 = b_0$, *and* $b'_3 = b_1, \ldots, b'_{N+1} = b_N$, *for an in-range size sequence* $\{b_s\}$. *The map* $S\colon {}^{n}E^{1}_{*,*} \to {}^{n+1}E^{1}_{*,*}$ *is an isomorphism.*

Proof. Similar to [5, Proposition 7.5]. □

Corollary 8.20. *S induces an isomorphism* $H^{\text{flat}}_*(B^{(n)}_\infty) \to H^{\text{flat}}_*(B^{(n+1)}_\infty)$. □

This completes the proof of Theorem 8.1.

Appendix A

Combinatorial homology and ECH

The goal of this appendix is to prove that $H_i(A, n, \gamma) \cong ECH_*(Y_A, \lambda_n, \Gamma)$, i.e., the combinatorially defined homology is isomorphic to the analytically defined homology. The argument is very similar to the T^3 case [5, §10-11], and we mostly describe the differences.

A.1 Periodic paths and Morse-Bott orbit sets

Throughout this appendix, we will just say *Reeb orbit* for a closed embedded Reeb orbit, and *Reeb orbit family* for a component of the space of embedded closed Reeb orbits of (Y_A, λ_n).

Definition A.1. A *Morse-Bott orbit set* of (Y_A, λ_n, Γ) is a finite set of pairs $\alpha = \{(\alpha_i, m_i)\}$ where each α_i is a Reeb orbit family and each m_i is a positive integer, and the homology classes of the Reeb orbits satisfy $\sum_i m_i[\alpha_i] = \Gamma$.

Our first step is to show that a Morse-Bott orbit set is equivalent to a collection of angles and integer vectors; this is the data of a periodic path, modulo a $\mathbb{Z}^2$-translation ambiguity.

Lemma A.2. *A Morse-Bott orbit set of* (Y_A, λ_n, Γ) *is equivalent to a discrete* $f_{A,n}$*-invariant set* $\mathrm{Ang} \subset \mathbb{R}$ *together with a function* $\overrightarrow{\mathrm{Edge}} : \mathrm{Ang} \to \mathbb{Z}^2 \setminus \{0\}$ *satisfying*

$$\overrightarrow{\mathrm{Edge}}(f_{A,n}(\theta)) = A(\overrightarrow{\mathrm{Edge}}(\theta)) \tag{A.1}$$

$$\sum_{\theta \in \mathrm{Ang} \cap (\theta_0, f_{A,n}(\theta_0))} \overrightarrow{\mathrm{Edge}}(\theta) \in \Gamma. \tag{A.2}$$

Here, the choice of $\theta_0 \in \mathbb{R} \setminus \mathrm{Ang}$ does not matter. By "equivalent" we mean we construct a bijection.

Proof. A Reeb orbit family is equivalent to an element $[\theta] \in \mathbb{R}/f_{A,n}$, the image of $\theta \in \mathbb{R}$ with $\tan\theta \in \mathbb{Q} \cup \{\infty\}$, because the (closed) Reeb orbits in Y_A form S^1-families in the corresponding fibers of the bundle Y_A, a bundle over $\mathbb{R}/f_{A,n}$. Thus a finite set of Reeb orbit families is equivalent to a discrete $f_{A,n}$-invariant set.

Given $\theta \in \mathbb{R}$ with $\tan\theta \in \mathbb{Q} \cup \{\infty\}$, let $\binom{x_\theta}{y_\theta} \in \mathbb{Z}^2 \setminus \{0\}$ be the unique positive multiple of $\binom{\cos\theta}{\sin\theta}$ with $\gcd(x_\theta, y_\theta) = 1$. Given (α_i, m_i) with α_i corresponding to $[\theta_i]$, let $\overrightarrow{\mathrm{Edge}}(\theta_i) = m_i \binom{x_{\theta_i}}{y_{\theta_i}}$. □

Lemma A.3. *A Morse-Bott orbit set of* (Y_A, λ_n, Γ) *is equivalent to an* (A, n, γ)*-periodic path.*

Proof. A periodic path Λ yields Ang and $\overrightarrow{\mathrm{Edge}}$ as in Lemma A.2. For an angle θ_0 at which Λ has no edge,

$$\Lambda(f_{A,n}(\theta_0)) = A\Lambda(\theta_0) + \gamma \tag{A.3}$$

$$\Lambda(f_{A,n}(\theta_0)) - \Lambda(\theta_0) = \sum_{\theta \in \mathrm{Ang}(\Lambda) \cap (\theta_0, f_{A,n}(\theta_0))} \overrightarrow{\mathrm{Edge}_\Lambda}(\theta), \tag{A.4}$$

where $\overrightarrow{\mathrm{Edge}_\Lambda(\theta)}$ is the vector from the beginning to the end of that edge. It follows that

$$\Lambda(\theta_0) = (A - \mathbb{1})^{-1} \left(-\gamma + \sum_{\theta \in \mathrm{Ang}(\Lambda) \cap (\theta_0, f_{A,n}(\theta_0))} \overrightarrow{\mathrm{Edge}_\Lambda(\theta)} \right). \tag{A.5}$$

Thus we may obtain Λ uniquely, given Ang and $\overrightarrow{\mathrm{Edge}}$. As we move θ_0 past some θ', observe that Λ defined by (A.5) jumps by $\overrightarrow{\mathrm{Edge}}(\theta')$. □

It follows that we may think of a labeled periodic path as a Morse-Bott orbit set with each Reeb orbit family α_i labeled 'e' or 'h' (and with an ordering on the 'h' labels). We may perturb the contact form in a neighborhood of the T^2-fiber containing α_i, replacing the family of degenerate Reeb orbits with one elliptic and one hyperbolic Reeb orbit. Then:

- (α_i, m_i) with an 'e' label corresponds to the elliptic Reeb orbit with multiplicity m_i.
- (α_i, m_i) with an 'h' label corresponds to the elliptic Reeb orbit with multiplicity $m_i - 1$, together with the hyperbolic Reeb orbit with multiplicity 1.

Given a positive real L, we may perform the above small perturbations simultaneously on all fibers containing Reeb orbits with symplectic action less than L, obtaining a contact form $\lambda^{<L}$. The subcomplex of the embedded contact homology chain complex for $(Y_A, \lambda^{<L}, \Gamma)$ generated by collections α of Reeb orbits with the symplectic action of α less than L is isomorphic (as an abelian group) to a subcomplex of $C_*(A, n, \gamma)$, and a direct limit yields a chain complex isomorphic (as an abelian group) to $C_*(A, n, \gamma)$.

The combinatorially defined index using $\mathrm{Size}_\int$ from §3.6 is a straightforward generalization from the T^3 case, and agrees with the analytically defined index.

A.2 The differential

We will not prove that the combinatorial and analytical differentials agree; we will prove a weaker statement sufficient to show that the homologies are isomorphic. Denote the combinatorial differential by δ_{com} and the analytical differential by δ_{an}.

Definition A.4. If a labeled periodic path (Λ, ℓ, o) has 'h'-labeled edges at θ_1, θ_2, and θ_3 (with no edges in $(\theta_1, \theta_2) \cup (\theta_2, \theta_3)$), we say (Λ', ℓ', o') is obtained from (Λ, ℓ, o) by *double rounding* if

1. Λ' is obtained from Λ by rounding at (θ_1, θ_2) and at (θ_2, θ_3), i.e., $\Lambda' = \Lambda \setminus (\theta_1, \theta_2) \setminus (\theta_2, \theta_3)$.

2. The labeling ℓ' has only 'e' labels in $[\theta_1, \theta_3]$.

3. Outside of $[\theta_1, \theta_3]$ and its periodic images $(f_{A,n})^k[\theta_1, \theta_3]$, the labeling ℓ' and its ordering o' are the restriction of ℓ and o.

Lemma A.5. *Suppose α and β are labeled periodic paths.*

1. *If $\langle \delta_{\mathrm{an}} \alpha, \beta \rangle \neq 0$, then $\langle \delta_{\mathrm{com}} \alpha, \beta \rangle \neq 0$ or β is obtained from α by double rounding.*

2. *If $\langle \delta_{\mathrm{com}} \alpha, \beta \rangle \neq 0$, then $\langle \delta_{\mathrm{com}} \alpha, \beta \rangle = \langle \delta_{\mathrm{an}} \alpha, \beta \rangle$.*

The proof is the same as in the T^3 case, because the situation in Y_A is locally the same as T^3.

Lemma A.6. *The number of 'h' labels of a generator of $H_*(A, n, \gamma)$ is either N or $N-1$.*

Proof. For the $N = 1$ case, see the proof of Theorem 5.36. For the $N = 2$ case, see the explicit generators given at the end of Chapter 7. For $N > 2$, each application of the splicing isomorphism increases both N and $\#h$ by 2. □

The spectral sequence argument given in [4] now shows that the chain complex with δ_{an} has the same homology as the chain complex with δ_{com}. That argument requires the number of 'h' labels of generators of the homology $H_*(A, n, \gamma)$ to lie within a range of 3, and $\#h$ lies within a range of 2.

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