Weyl laws and dense periodic orbits

Michael Hutchings

Berkeley Mathematics Happy Hour
June 18, 2020

Outline

1. Existence results for periodic orbits in Hamiltonian mechanics
2. Technology: the ECH spectrum
3. A Weyl law for the ECH spectrum, and applications

Hamiltonian mechanics

- In the simplest version of Hamiltonian mechanics, the state of a physical system is described by a point in $\mathbb{R}^{2n}$ with "position" variables $x_1, \ldots, x_n$ and "momentum" variables $y_1, \ldots, y_n$.
- Given a "Hamiltonian" or "energy" function

$$H : \mathbb{R}^{2n} \longrightarrow \mathbb{R},$$

the time evolution of the system is given by Hamilton's equations

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}.$$

That is, we follow trajectories of the Hamiltonian vector field

$$X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial y_i} \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial}{\partial y_i} \right).$$

Periodic orbits

- It follows from the previous equations that $dH/dt = 0$ (conservation of energy). This means that we stay on a level set

$$Y = \left\{ z \in \mathbb{R}^{2n} \mid H(z) = k \right\}.$$

- We would like to understand the dynamics of the vector field X_H on Y.
- A basic dynamical question is to understand the periodic orbits (loops in Y which are trajectories of X_H). These correspond to physical behavior which repeats in time.
- If k is a regular value of H, then up to scaling, the vector field X_H on Y does not depend on the Hamiltonian H having Y as a regular level set.
- It follows that up to reparametrization, periodic orbits of X_H depend only on Y. These are also called closed characteristics of Y.

Star-shaped hypersurfaces

- A hypersurface Y in $\mathbb{R}^{2n}$ is called star-shaped if it is tranverse to the radial vector field. This includes the case when Y is convex (the boundary of a convex domain X) as long as $0 \in \mathrm{int}(X)$.
- If Y is star-shaped, then there is a canonical choice of scaling of the Hamiltonian vector field, called the Reeb vector field, and denoted by R.
- The scaling is such that $\lambda(R) = 1$, where

$$\lambda = \frac{1}{2}\sum_{i=1}^{n}(x_i\,dy_i - y_i\,dx_i).$$

  (The star-shaped condition implies that the Hamiltonian vector field is not in the kernel of λ.)
- Periodic orbits of R are called Reeb orbits for short. (We'll use this term just to refer to simple, i.e. embedded orbits.)

Existence of periodic orbits

Let Y be a compact hypersurface in $\mathbb{R}^{2n}$.

Theorem (Weinstein, 1970s)

If Y is convex then it has a Reeb orbit.

Theorem (Rabinowitz, 1970s)

If Y is star-shaped then it has a Reeb orbit.

Old conjecture

Every compact star-shaped hypersurface in $\mathbb{R}^{2n}$ has at least n Reeb orbits.

- Trivial for $n = 1$. In this case Y is a circle which is itself a Reeb orbit.
- $n = 2$ case proved by myself and (at the time) graduate student Dan Cristofaro-Gardiner in 2012.
- Open for $n > 2$ (various partial results).

Example: ellipsoids

Let $a_1, \ldots, a_n > 0$. Identify $\mathbb{R}^{2n} = \mathbb{C}^n$. Define the ellipsoid

$$\partial E(a_1, \ldots, a_n) = \left\{ z \in \mathbb{C}^n \,\middle|\, \pi \sum_{i=1}^{n} \frac{|z_i|^2}{a_i} = 1 \right\}.$$

This is a star-shaped hypersurface. The Reeb vector field is

$$R = 2\pi \sum_{i=1}^{n} \frac{1}{a_i} \frac{\partial}{\partial \theta_i}.$$

- There is a Reeb orbit γ_i of period a_i where $z_j = 0$ for $j \neq i$.
- If the ratios a_i/a_j for $i \neq j$ are irrational, then there are no other Reeb orbits.
- If $a_1 = \cdots = a_n$ (a sphere) then every point is on a Reeb orbit of the same period. These are the fibers of the Hopf fibration.

Remark: Some vector fields have no periodic orbits

- The “Seifert conjecture” is false: there exist vector fields on S^3 with no periodic orbit. (Schweizer, Harrison, K. Kuperberg, G. Kuperberg, 1970s-1990s)
- There exist compact hypersurfaces in $\mathbb{R}^{2n}$, $n > 1$, with no periodic orbit of the Hamiltonian vector field. (Hermann, Ginzburg-Gurel, 1980s-2000s)
- Reeb vector fields on star-shaped hypersurfaces are special because they fit into the general framework of contact geometry and the Weinstein conjecture (beyond the scope of this talk).
- Unknown in general what exactly one needs to assume about a vector field to ensure that a periodic orbit exists.

Stronger existence results in three dimensions

Theorem (Hofer-Wysocki-Zehnder 2003, improved by CG-H-Pomerleano 2017)

A nondegenerate compact star-shaped hypersurface in $\mathbb{R}^4$ has either two or infinitely many Reeb orbits.

Here "nondegenerate" means that the Reeb orbits are cut out transversely in a suitable sense; this holds for C^∞ generic hypersurfaces. For example the ellipsoid $\partial E(a_1, a_2)$ is nondegenerate iff a_1/a_2 is irrational.
"Generic" means in a countable intersection of open dense sets.

Theorem (Irie 2015)

If Y is a C^∞ generic compact star-shaped hypersurface in $\mathbb{R}^4$, then the set of Reeb orbits is dense in Y.

The above are special cases of more general results about Reeb vector fields on compact three-manifolds which we will not state here.

Technology for proving existence of Reeb orbits

- The basic strategy is to define some kind of topological invariant which "counts" Reeb orbits.
- Simply counting Reeb orbits with signs won't work, because typically there are infinitely many of them.
- Instead one usually defines some kind of "contact homology" which is the homology of some chain complex built out of Reeb orbits, and whose differential counts pseudoholomorphic curves.
- When this homology is nontrivial, Reeb orbits must exist!
- Original versions of contact homology in "symplectic field theory" of Eliashberg-Givental-Hofer (2000).
- The version of contact homology relevant for this talk is called "embedded contact homology" (ECH).

ECH generators

Let Y be a nondegenerate compact star-shaped hypersurface in $\mathbb{R}^4$.

Definition

An ECH generator is a finite set of pairs $\alpha = \{(\alpha_i, m_i)\}$ where:

- The α_i are distinct Reeb orbits.
- The m_i are positive integers.
- If α_i is hyperbolic (i.e. its linearized return map has real eigenvalues) then $m_i = 1$.

Sometimes we use the multiplicative notation $\alpha = \prod_i \alpha_i^{m_i}$.

Definition

The symplectic action of α as above is defined by

$$I(\alpha) = \sum_i m_i \mathcal{A}(\alpha_i)$$

where $\mathcal{A}(\alpha_i)$ denotes the period of the Reeb orbit α_i.

Embedded contact homology

- The embedded contact homology of Y is the homology of a chain complex (over $\mathbb{Z}$) which is generated by the ECH generators.
- The chain complex has a $\mathbb{Z}$ grading (too complicated to define here).
- The differential counts certain embedded pseudoholomorphic curves in $\mathbb{R} \times Y$.
- In general, ECH is a topological invariant of compact three-manifolds (by a theorem of Taubes relating ECH to Seiberg-Witten Floer homology).
- For our case where Y is a compact star-shaped hypersurface in $\mathbb{R}^4$, it is known that

$$ECH_*(Y) \simeq \begin{cases} \mathbb{Z}, & * = 0, 2, 4, \ldots \\ 0, & \text{else} \end{cases}$$

The ECH spectrum

The ECH spectrum of Y is a sequence of real numbers

$$0 = c_0(Y) < c_1(Y) \leq c_2(Y) \leq \cdots < \infty$$

defined as follows:

- $c_k(Y)$ is the minimum L such that the grading $2k$ class in ECH can be represented in the ECH chain complex by a linear combination of ECH generators each having symplectic action $\leq L$.
- It follows from the definition that $c_k(Y)$ equals the symplectic action of some ECH generator α which has grading $2k$.
- Intuitively, the ECH spectrum measures the symplectic action of certain specially selected, "homologically essential" ECH generators.
- The fact that $c_k(Y) \leq c_{k+1}(Y)$ follows using some additional structure on ECH (the U map) beyond the scope of this talk.
- $c_k(Y)$ is also defined for Y degenerate by taking a limit.

Example: ECH spectrum of an ellipsoid

Let $a_1, a_2 > 0$ with a_1/a_2 irrational, and let $Y = \partial E(a_1, a_2)$.

- Recall that there are just two Reeb orbits γ_1 and γ_2 of period a_1 and a_2 respectively.
- The ECH generators have the form $\alpha = \gamma_1^{m_1}\gamma_2^{m_2}$ where m_1, m_2 are nonnegative integers. The symplectic action of such a generator is $\mathcal{A}(\alpha) = m_1 a_1 + m_2 a_2$.
- It turns out that these generators all have even grading so the differential vanishes.
- The ECH spectrum then consists of the nonnegative integer linear combinations of a_1 and a_2, arranged in increasing order with repetitions.
- Equivalent statement: $c_k(Y)$ is the smallest L such that the triangle in the plane bounded by the x and y axes and the line $a_1 x + a_2 y = L$ contains $k + 1$ lattice points.

A Weyl law for the ECH spectrum

Let Y be a compact star-shaped hypersurface in $\mathbb{R}^4$, and let $X \subset \mathbb{R}^4$ be the domain with $\partial X = Y$.

Theorem

(H. 2010, generalization to compact three-manifolds with graduate students Dan C-G and Vinicius Ramos, 2012)

$$\lim_{k\to\infty} \frac{c_k(Y)^2}{k} = 4\,\mathrm{vol}(X).$$

Compare to the 4d case of the classical Weyl law: if λ_k is the k^{th} eigenvalue of the Laplacian on X with Dirichlet boundary conditions, then

$$\lim_{k\to\infty} \frac{\lambda_k^2}{k} = \frac{32\pi^2}{\mathrm{vol}(X)}.$$

Example: the ellipsoid again

Let $Y = \partial E(a_1, a_2)$ with a_1/a_2 irrational, and $X = E(a_1, a_2)$.

- We have seen that the triangle bounded by the x and y axes and the line $a_1 x + a_2 y = c_k(Y)$ encloses $k + 1$ lattice points.
- The area of this triangle is $c_k(Y)^2/2a_1 a_2$. Thus

$$\frac{c_k(Y)^2}{k} \approx 2a_1 a_2.$$

- On the other hand a calculus exercise shows that

$$\mathrm{vol}(X) = \frac{a_1 a_2}{2}.$$

- Thus

$$\lim k \to \infty \frac{c_k(Y)^2}{k} = 4\,\mathrm{vol}(X)$$

  in this example.

Proof of existence of two Reeb orbits (with Dan C-G)

Let Y be a compact star-shaped hypersurface in $\mathbb{R}^4$.

Lemma

If there are only finitely many Reeb orbits, then $c_k(Y) < c_{k+1}(Y)$ for all k.

(Proof uses the U map, beyond the scope of this talk.)

- Now suppose there is only one Reeb orbit, of period T.
- Then $c_k(Y) = m_k T$ where $\{m_k\}$ is a strictly increasing sequence of nonnegative integers.
- Thus $c_k(Y) \geq kT$.
- Then
$$\lim_{k\to\infty} \frac{c_k(Y)^2}{k} \geq \lim_{k\to\infty} kT = +\infty.$$
- This contradicts the fact that $\lim_{k\to\infty} \frac{c_k(Y)^2}{k} = 4\,\mathrm{vol}(X) < \infty$.

Proof of generic density of Reeb orbits (Irie)

We want to prove that for a generic compact star-shaped hypersurface Y in $\mathbb{R}^4$, the set of Reeb orbits is dense in Y.
The key ingredient is the following "C^∞ closing lemma". (The rest is a standard argument.)

Lemma (Irie)

Let Y be a compact star-shaped hypersurface in $\mathbb{R}^4$. Let $U \subset Y$ be a nonempty open set. Then there is a smooth family $\{Y_t\}_{t\in\mathbb{R}}$ of compact star-shaped hypersurfaces such that:

- $Y_0 = Y$.
- *Y_t agrees with Y outside of U.*
- *For all $\epsilon > 0$, there exists t with $|t| < \epsilon$ such that Y_t has a Reeb orbit passing through (the perturbation of) U.*

Compare C^1 closing lemma by Pugh (1980s).

Proof of Irie's closing lemma

- We can make a family $\{Y_t\}$ such that Y_t agrees with Y outside of U, and $\text{vol}(X_t)$ is a strictly increasing function of t, where $\partial X_t = Y_t$. (Push out a subset whose closure is in U using a cutoff function.) This is all we need!
- Suppose that there exists $\epsilon > 0$ such that if $|t| < \epsilon$ then there is no Reeb orbit through (the perturbation) of U.
- Then the Reeb orbits for Y_t with $|t| < \epsilon$, and their symplectic actions, are independent of t.
- It follows from continuity properties of the ECH spectrum that $c_k(Y_t)$ does not depend on t when $|t| < \epsilon$.
- Then by the Weyl law, $\text{vol}(X_t) = \frac{1}{4}\lim_{k\to\infty}\frac{c_k(Y_t)^2}{k}$ does not depend on t either when $|t| < \epsilon$.
- Contradiction!

Related results

- C^∞ closing lemma, and generic density of periodic orbits, for Hamiltonian diffeomorphisms of closed surfaces (Asaoka-Irie 2015)
- Density of minimal hypersurfaces for generic Riemannian metrics on compact manifolds of dimension $3, \ldots, 7$. (Weyl law by Liokumovich-Marques-Neves 2016, then similar argument to above by Irie-Marques-Neves 2017)
- Equidistribition of minimal hypersurfaces for generic metrics in dimension $3, \ldots, 7$ (Marques-Neves-Song 2017)
- C^∞ generic equidistribution of Reeb orbits in 3d (Irie 2018)
- Higher order asymptotics in the Weyl law in terms of the Ruelle invariant in some cases (H. 2019)