A NOTE ON EINSTEIN MANIFOLDS WITH BOUNDARY

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Abstract. In this note, we discuss the generalization of the compactness result of hyperkähler 4 manifolds with boundary in [2] to Einstein 4-manifolds with boundary.

Notations:

- B_r: Euclidean ball of radius r.
- S: Second fundamental form
- H: Mean curvature

Let (M, g) be a complete Riemannian manifold with compact boundary,

- Let $p \in M$, then $q \in \partial M$ a foot point of p if $d(p, q) = d(p, \partial M)$.
- i_b is the supreme of $t > 0$ such that the boundary exponential map $\exp^\perp : N_{\partial M} \to M$ is a diffeomorphism onto its image for any $\|v\| < t$.
- If B is an open subset of ∂M, then $C(B, t_1, t_2) = \{\exp^\perp(x, v) | x \in B, \|v\| \in [t_1, t_2)\}$.
- $N_r(\partial M, g) = \{x \in M | d(x, \partial M) \le r\}$

1. Einstein Manifolds with boundary

Theorem 1.1. *Let (M, g) be a compact Einstein 4-manifold with boundary, $|Ric| \le 3$. Suppose $\mathbb{RP}^3$ cannot be smoothly embedded in $M \backslash \partial M$, $vol(\partial M) \le C$,*

$$|S| \le C, |Rm_{\partial M}| \le C, |H|_{Lip(\partial M)} \le C, inj_{\partial M} \ge i_0, \chi(M) \le C, H \ge H_0 > 0,$$

Then there exists i'_0 such that $i_b \ge i'_0$. If in addition $Ric \ge 0$, then one can replace $H \ge H_0 > 0$ by $H > 0$.

Proof. The Chern-Gauss-Bonnet formula together with Gauss-Codazzi equations imply $\int_M |Rm|^2 \le C$. Suppose the conclusion is not true, then we can find a sequence $(M_i, \tilde{g}_i)$ such that $i_{b,\tilde{g}_i} \to 0$. In particular, $i_{b,\tilde{g}_i} < -\frac{3}{2} \ln \left|\frac{H_0 - 3}{H_0 + 3}\right|$, by Proposition 3.6 in [2], there exists focal points $p_i \in M_i$ whose distance to ∂M_i is equal to $i_{b,\tilde{g}_i}$. Rescale the metric $g_i = i_{b,\tilde{g}_i}^{-2} \tilde{g}_i$ and from now on we only use metric g_i.

Let q''_i be any point on ∂M_i, then by Lemma 1.2 below and Theorem 3.1 in [1], $C(B_{\partial M}(q''_i, 1), 0, 0.9999)$ converges in $C^{1,\alpha}$ sense to flat $B_1 \times [0, 0.9999)$. It follows that $vol(B(q'_i, 0.01)) \ge 0.4 vol(B_{0.01})$ for any q'_i with $d(q'_i, \partial M_i) \in [0.99, 1.01]$. Hence, $\sup_{B(p_i, 0.002)} |Rm| \le C$. Let q_i be a foot point of p_i. It follows that $C(B_{\partial M}(q_i, 0.0001), 0, 1.0001)$ converges in $C^{1,\alpha}$ sense to flat $B_{0.0001} \times [0, 1.0001)$, which contradicts that p_i is a focal point of ∂M_i.

□

Lemma 1.2. *Let (M_i, g_i) be a sequence of complete Einstein 4-manifold with compact boundary, such that $\mathbb{RP}^3$ does not smootly embed in $M_i \backslash \partial M_i$. Suppose*

$|Ric_{g_i}| \to 0$. *Suppose* $|S_i| \to 0$, $|Rm_{\partial M_i}| \to 0, |H_i|_{Lip(\partial M_i)} \to 0, inj_{\partial M_i} \to \infty$, $i_{b,g_i} \geq i_0$, $\int_{M_i} |Rm_{g_i}|^2 \leq C$. *Then for any* $r_1 < i_0$, *there exists* $C' > 0$, *depending on* C, i_0, r_1, *such that* $\sup_{N_{r_1}(\partial M, g_i)} |Rm_{g_i}| \to 0$.

Proof. One only needs to show $\sup_{N_{r_1}(\partial M, g_i)} |Rm_{g_i}| \leq C'$.

Without loss of generality, assume $i_0 = 1$ and $r_1 > 0.9999$. Denote $\alpha = r_1$, $\beta = \frac{1}{4}(1-\alpha)$. Suppose the conclusion is not true, then by passing to a subsequence we may assume

$$\sup_{N_\alpha(\partial M_i, g_i)} |Rm_{g_i}| \to \infty.$$

Let $p_i \in N_{r_1}(\partial M_i, g_i)$ achieves this supremum.

Claim 1 There exists a subsequence such that

$$d_{g_i}(p_i, \partial M_i)^2 |Rm_{g_i}(p_i)| \to \infty. \tag{1}$$

If this is not true, we have $\sup_i d_{g_i}(p_i, \partial M_i)^2 |Rm_{g_i}|(p_i) < \infty$. Rescale $\tilde{g}_i = |Rm_{g_i}(p_i)| g_i$, then $|Rm_{\tilde{g}_i}(p_i)| = 1$, and $|Rm_{\tilde{g}_i}| \leq 1$ in $N_{\alpha |Rm(p_i)|^{\frac{1}{2}}}(\partial M_i, \tilde{g}_i)$, and $\sup_i d_{\tilde{g}_i}(p_i, \partial M_i) < \infty$, $i_{b,\tilde{g}_i} \geq |Rm_{g_i}(p_i)|^{\frac{1}{2}}$ for all i. Hence (M_i, g_i, p_i) subconverges in pointed $C^{1,\alpha}$ sense to $(M_\infty, \tilde{g}_\infty, p_\infty)$, which is a complete Ricci-flat 4-manifold with flat, totally geodesic boundary, hence must be flat. This contradicts that $|Rm_{g_\infty}(p_\infty)| = 1$ and proves Claim 1.

Now rescale g_i in another way, let $g_i' = d_{g_i}(p_i, \partial M_i)^{-2} g_i$, so $d_{g_i'}(p_i, \partial M) = 1$. Since $d_{g_i}(p_i, \partial M_i) \leq \alpha$, the rescaled metric g_i' satisfies $i_{b,g_i'} \geq \alpha^{-1}$ as well as all other conditions of the assumptions of the theorem, but with different bounds, regardness of whether $d_{g_i}(p_i, \partial M_i)$ is uniformly bounded from below or not. Moreover, (1) is equivalent to $|Rm_{g_i'}(p_i)| \to \infty$.

By the ϵ-regularity Theorem of Cheeger-Tian, there exists a universal constant ϵ_0 such that for sufficiently large i,

$$\int_{B_{g_i'}(p_i, \beta)} |Rm_{g_i'}|^2 \geq \epsilon_0.$$

Claim 2 There exists a subsequence such that

$$\sup_{N_\alpha(\partial M_i, g_i')} |Rm_{g_i'}| \to \infty,$$

If not, we have $\sup_{N_\alpha(\partial M_i, g_i')} |Rm_{g_i'}| \leq C$. Hence $\sup_{N_{\alpha-\beta}(\partial M_i, g_i')} |Rm_{g_i'}| \to 0$. By Bishop-Gromov volume comparison, we have $\mathrm{vol}(B_{g_i'}(q_i, \beta)) \geq 0.45 \mathrm{vol}(B_\beta)$ for any q_i with $d_{g_i'}(q_i, \partial M_i) \in [1 - 5\beta, 1 + 5\beta]$. Note that $B_{g_i'}(p_i, \beta) \subset N_{\alpha^{-1}(\alpha+\beta)}(\partial M_i, g_i') \subset N_{\alpha+\beta}(\partial M_i, g_i)$. By Proposition 1.3, $|Rm_{g_i'}(p_i)|$ is bounded, which is a contradiction to Claim 1 and finishes the proof of Claim 2.

Now Claim 2 enables us to get by induction, for each fixed positive integer N, N sequences of metrics $g_i^{(0)} = g_i, g_i^{(1)} = g_i', \cdots, g_i^{(N)}$, and points $p_i^{(j)} \in N_\alpha(\partial M_i, g_i^{(j)})$ for $0 \leq j \leq N-1$, $p_i^{(0)} = p_i$, such that for $0 \leq j \leq N-1$, $p_i^{(j)}$ achieves the supremum of $|Rm_{g_i^{(j)}}|$ in $N_\alpha(\partial M_i, g_i^{(j)})$, and

$$|Rm_{g_i^{(j)}}(p_i)| \to \infty,$$

$$d_{g_i^{(j+1)}}(p_i^{(j)}, \partial M_i) = 1,$$

$$\int_{B_{g_i^{(j+1)}}(p_i^{(j)},\beta)} |Rm_{g_i^{(j+1)}}|^2 \geq \epsilon_0,$$

$$B_{g_i^{(j+1)}}(p_i^{(j)},\beta) \subset N_{\alpha^{-1}(\alpha+\beta)}(\partial M_i, g_i^{(j+1)}) \subset N_{\alpha+\beta}(\partial M_i, g_i^{(j)}),$$

$$B_{g_i^{(j+1)}}(p_i^{(j)},\beta) \cap N_{\alpha+\beta}(\partial M_i, g_i^{(j+1)}) = \emptyset.$$

It follows that for each fixed i, $B_{g_i^{(j+1)}}(p_i^{(j)},\beta)$ does not interect each other for different j. Since $\int_{M_i} |Rm_{g_i}|^2 \leq C$, we have $N\epsilon_0 \leq C$. This is a contradiction, since N can be any positive integer. □

Proposition 1.3. *Let (M, g) be a Riemannian 4-manifold with $|Ric| \leq 3$. Suppose $B(p, 5)$ has compact closure, $\mathbb{RP}^3$ cannot be smoothly embedded in $B(p, 5)$, and for any $q \in B(p, 2)$,*

$$vol(B(q,1)) \geq (\frac{1}{3}+\delta)vol(B_1)$$

Then there exists a constant $C = C(\delta)$ such that

$$\sup_{B(p,1)} |Rm| \leq C.$$

Proof. Suppose the conclusion is not true, then we have a sequence (M_i, g_i, p_i) satisfies the conditions, but there exists $q_i' \in B(p_i, 1)$ with $|Rm_{g_i}(q_i')| \to \infty$. By the following Lemma 1.4, we can find points $q_i \in B(p_i, 2)$ such that $|Rm_{g_i}(q_i)| \geq |Rm_{g_i}(q_i')|$, and

$$\sup_{B_{g_i}(q_i, |Rm_{g_i}(q_i')|^{\frac{1}{2}}|Rm_{g_i}(q_i)|^{-\frac{1}{2}})} |Rm_{g_i}| \leq 4|Rm_{g_i}(q_i)|.$$

Rescale the metric $\tilde{g}_i = |Rm_{g_i}(q_i)|g_i$. Then we have for large i,

$$\sup_{B_{\tilde{g}_i}(q_i, |Rm_{g_i}(q_i')|^{\frac{1}{2}})} |Rm_{\tilde{g}_i}| \leq 4,$$

$$|Rm_{\tilde{g}_i}(q_i)| = 1,$$

$$\mathrm{vol}_{\tilde{g}_i}(B_{\tilde{g}_i}(q_i, r)) \geq (\frac{1}{3}+\frac{\delta}{2})\mathrm{vol}(B_r), \forall r \leq |Rm_{g_i}(q_i)|^{\frac{1}{2}},$$

Hence, for a subsequence, $(M_i, \tilde{g}_i, q_i)$ converges in pointed $C^{1,\alpha}$ topology to a complete non-flat Ricci-flat 4-manifold $(M_\infty, g_\infty, q_\infty)$ with maximum volume growth. By Cheeger-Naber and Bando-Kasue-Nakajima, M_∞ is a Ricci-flat ALE space whose tangent cone at infinity is $\mathbb{R}^4/\mathbb{Z}_2$, hence $\mathbb{RP}^3$ can be smoothly embedded into $B_{g_i}(p_i, 5)$ for large i, which contradicts our assumption.

□

The following point selection lemma is well-known and elementary.

Lemma 1.4. *Let (M, g) be a Riemannian manifold. Suppose $\sup_{B(p,2)} |Rm| < \infty$, $|Rm(p)| \neq 0$. Then there exists a point $q \in B(p, 2)$ such that $|Rm(q)| \geq |Rm(p)|$, and* $\sup_{B(q, |Rm(p)|^{\frac{1}{2}}|Rm(q)|^{-\frac{1}{2}})} |Rm| \leq 4|Rm(q)|.$

Proof. If this is not true, let $A = |Rm(p)|^{\frac{1}{2}}$, then there exist $q_1 \in B(p,2)$ such that $d(q_1, p) < 1$ and $|Rm(q_1)| > 4|Rm(p)| = 4A^2$. By induction, we can find a sequence of points $q_0 = p, q_1, q_2, \cdots$ with $d(q_{j+1}, q_j) < |Rm(q_j)|^{-\frac{1}{2}}A$, $d(q_{j+1}, p) < 2-2^{-j}$ and $|Rm(q_{j+1})| > 4^{j+1}A^2$. This is an obvious contradiction since $\sup_{B(p,2)} |Rm| < \infty$. □

Question 1.5. *(Ruling out interior bubbles within boundary injectivity radius)*

Let (M, g) be a complete Einstein 4-manifold with compact boundary, $|Ric| \leq 3$. Suppose $|S| \leq C$, $|Rm_{\partial M}| \leq C, |H|_{Lip(\partial M)} \leq C, inj_{\partial M} \geq i_0$, $i_b \geq i_0$, suppose also $\sup_{N_{r_1}(\partial M,g)} |Rm| \leq C$ for some $r_1 < i_0$. Under what topological assumption can we conclude $\sup_{N_l(\partial M,g)} |Rm| \leq C'$ for some $l \in (r_1, i_0)$? Do we actually need a topological assumption to achieve this?

Note that under the assumptions $C^{1,\alpha}$ geometry is well controlled with $N_{r_1}(\partial M, g)$, so one only worries about interior curvature blow ups.

The point is that we only have volume noncollapsing near the blow up points, but not a precise volume lower bound, so after rescaling we get a Ricci-flat ALE space, but we need to rule out the case that the tangent cone at infinity is $\mathbb{R}^4/Q_8$ or $\mathbb{R}^4/\Gamma$, Γ is a perfect group. Note that in hyperkahler case we do not have such concern, since there is no -2 class in a collar neighborhood of ∂M due to triviality of cap product.

A positive answer to this question together with the arguments in Lemma 1.2 imply the following conjecture:

Conjecture 1.6. *Let (M, g) be a complete Einstein 4-manifold with compact boundary, such that $\mathbb{RP}^3$ does not smoothly embed in $M\backslash\partial M$,(possibly with other topological assumptions), $|Ric| \leq 3$. Suppose $|S| \leq C$, $|Rm_{\partial M}| \leq C, |H|_{Lip(\partial M)} \leq C, inj_{\partial M} \geq i_0$, $i_b \geq i_0$, $\int_M |Rm|^2 \leq C$. Then for any $r_1 < i_0$, there exists $C' > 0$, depending on C, i_0, r_1, such that $\sup_{N_{r_1}(\partial M,g)} |Rm| \leq C'$.*

This conjecture is true under the additional assumption that ∂M is diffeomorphic to S^3. This can be proved by the arguments in Lemma 1.2 and the following proposition.

Proposition 1.7. *Let (M, g) be a Riemannian 4-manifold with $|Ric| \leq 3$. Suppose $B(p, 5)$ has compact closure and embeds smoothly as an open subset of $S^3 \times (0, 1)$, and for any $q \in B(p, 2)$,*

$$vol(B(q, 1)) \geq v_0$$

Then there exists a constant C such that

$$\sup_{B(p,1)} |Rm| \leq C.$$

Proof. If the curvature blows up, the same arguments as in proposition 1.3 implies there exists a non-flat 4d Ricci-flat ALE space E that has an open embedding in $S^3 \times (-1, 1) \subset S^4$. Hence by a topological result of Crisp-Hillman, the boundary at infinity of E is S^3/Γ, where Γ is Q_8 or the perfect group. That contradicts with Chern-Gauss-Bonnet formula and signature formula applied to E since $H_2(E, \mathbb{R}) = 0$. See details in [3]. □

Hence,

Theorem 1.8. *Let $\mathcal{M}$ be the set of pointed Riemannian manifolds (M, g, p) such that (M, g) is a compact Einstein 4-manifold with boundary, $|Ric| \leq 3$, $\mathbb{RP}^3$ cannot be smoothly embedded in $M \backslash \partial M$, ∂M is diffeomorphic to S^3. $vol(\partial M) \leq C$,*

$$|S| \leq C, |Rm_{\partial M}| \leq C, |H|_{Lip(\partial M)} \leq C, inj_{\partial M} \geq i_0, \chi(M) \leq C, H \geq H_0 > 0,$$
$$d(p, \partial M) \leq K$$

Then $\mathcal{M}$ is precompact in pointed Gromov-Hausdorff topology, and an element in $\partial\mathcal{M}$ is a complete C^2_* Einstein orbifold with smooth boundary.

References

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