Brill–Noether theory over the Hurwitz space

A DISSERTATION SUBMITTED TO THE DEPARTMENT OF MATHEMATICS AND THE COMMITTEE ON GRADUATE STUDIES OF STANFORD UNIVERSITY IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

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Abstract

Historically, algebraic curves were defined as the solutions to a collection of polynomial equations inside of some ambient space. In the 19th century, however, mathematicians defined the notion of an abstract curve. With this perspective, the same abstract curve may sit in an ambient (projective) space in more than one way. The foundational *Brill–Noether theorem*, proved in the 1970s and 1980s, bridges these two perspectives by describing the maps of "most" abstract curves to projective spaces.

However, the theorem does not hold for all curves. In nature, we often encounter curves already in (or mapping to) a projective space, and the presence of such a map may force the curve to have unexpected maps to other projective spaces! The first case of this is a curve that already has a map to the projective line $\mathbb{P}^1$. From the 1990s through the late 2010s, several mathematicians investigated this first case. They found that the space of maps of such a curve to other projective spaces can have multiple components of varying dimensions and eventually determined the dimension of the largest component.

In this thesis, I develop analogues of all the main theorems of Brill–Noether theory for curves that already have a map to $\mathbb{P}^1$. The moduli space of curves together with a map to $\mathbb{P}^1$ is called the *Hurwitz space*, so we call this work *Brill–Noether theory over the Hurwitz space*. One of the key ideas is to introduce a new invariant called the *splitting type*. This comes from studying vector bundles on $\mathbb{P}^1$ that arise in the following way. Suppose we are given a curve C together with a degree k map $f : C \to \mathbb{P}^1$. A map $C \to \mathbb{P}^r$ corresponds to a line bundle L on C. The push forward f_*L is a rank k vector bundle on $\mathbb{P}^1$, so it splits as a direct sum of line bundles $f_*L \cong \mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_k)$. The tuple $\vec{e} = (e_1, \ldots, e_k)$ is called the splitting type.

In Chapter 2, we present some general results about families of vector bundles on $\mathbb{P}^1$, which was published in [51]. In general, given a family of vector bundles on $\mathbb{P}^1$ over a base B, it is natural to study the associated *splitting loci*, defined as the subvarieties of B where the restriction of our bundle to the fibers has a certain splitting type. We find that the classes of splitting loci are given by a universal formula in terms of naturally arising vector bundles on B.

In the remainder of the thesis, we describe the geometry of the splitting loci that arise in Brill–Noether theory over the Hurwitz space. Chapter 3 contains results about the local geometry of these splitting loci, determining their dimension and smoothness properties. This work was published in [50]. Chapter 4, which is joint work with Eric Larson and Isabel Vogt, contains results about their global geometry, showing for example that our splitting loci are irreducible when their dimension is positive.

Acknowledgements

I am very grateful for all of the people who have supported me and shared their mathematical insights with me throughout graduate school.

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A special thank you to Eric Larson and Isabel Vogt, my collaborators, mentors, and family, for their support in so many ways. All of the results in Chapter 4 are joint work with Eric and Isabel. Many of our best breakthrough moments happened while living together in their 800 square-foot apartment, after they invited me to live with them at the beginning of the 2020 lockdown. Our collaboration led to a much deeper understanding of Brill–Noether splitting loci, exceeding my previous dreams (and giving me new ones for the future!)

One of the key mathematical contributions in this thesis came from Dave Jensen and Kaelin Cook–Powell, who defined k-staircase tableaux, and explained their significance to me when they invited me to visit the University of Kentucky in Fall 2019. I thank them for generously sharing their work and thoughts on this problem with me.

I would also like to thank Geoffrey Smith, who knew of my interest in vector bundles on $\mathbb{P}^1$ and so introduced me to this topic by showing me Example 1.3.5. I am grateful for his insights and thoughtfulness in sharing them with me during his visit to Stanford in Fall 2018.

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Contents

Chapter 1

Overview

1.1 Background

Historically, algebraic curves were defined as living inside some projective space, as the solution set to a collection of polynomial equations. In 1851, however, Reimann defined the notion of an abstract curve. With this perspective, the same abstract curve may sit in projective space in more than one way. This naturally breaks the modern study of curves into two parts: understand all abstract curves, and understand all maps of a given abstract curve to projective space. The first part leads to the development and study of the *moduli space of curves*, a space in which each point corresponds to an abstract curve. The second part is the main goal of *Brill–Noether theory*, named after Alexander von Brill and Max Noether who first studied the following question in 1874.

Question 1.1.1. Given an algebraic curve C of genus g, what are all maps $C \to \mathbb{P}^r$ of given degree d?

The data of such a map is equivalent to a line bundle L of degree d on C, together with an $(r+1)$-dimensional space of global sections having no common zeros. Let $\mathrm{Pic}^d(C)$ be the space of degree d line bundles on C. This motivates the definition of the *Brill–Noether variety*

$$W^r_d(C) := \{L \in \mathrm{Pic}^d(C) : h^0(C, L) \geq r + 1\}.$$

Answering our initial Question 1.1.1 essentially boils down to understanding the following:

Question 1.1.2. Given an algebraic curve C of genus g, what is the geometry of $W^r_d(C)$?

I like to picture the varieties $W^r_d(C)$ inside $\mathrm{Pic}^d(C)$ as the result of a black and white picture obtained by shading each point $L \in \mathrm{Pic}^d(C)$ a darkness corresponding to the dimension of its space of global sections:

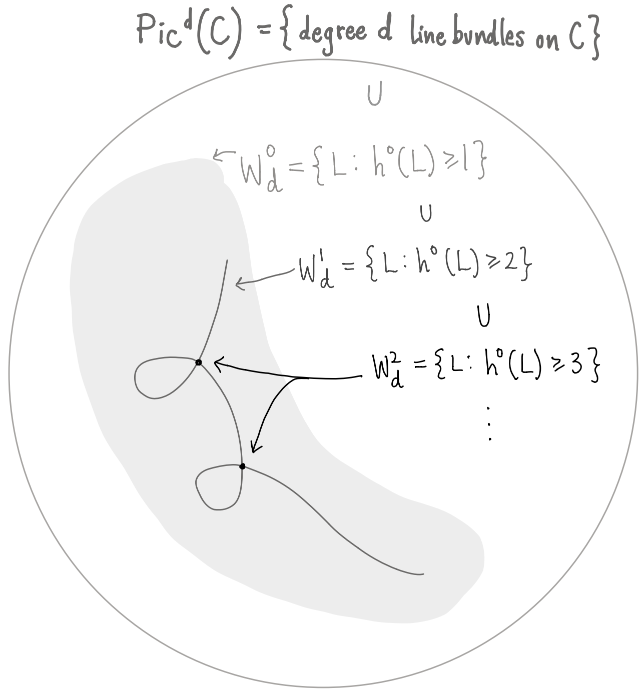


Figure 1.1: The black and white picture formed by Brill–Noether varieties $W^r_d(C)$

So far, we have just described $W^r_d(C)$ set theoretically. Let us now define $W^r_d(C)$ as a scheme. Let us label the projection maps

$$\begin{array}{ccc} C \times \mathrm{Pic}^d(C) & \xrightarrow{\alpha} & C \\ {\scriptstyle \nu}\downarrow & & \\ \mathrm{Pic}^d(C) & & \end{array}$$

Let $\mathcal{L}$ be a Poincaré line bundle on $C \times \mathrm{Pic}^d(C)$. This line bundle has the property that $\mathcal{L}|_{\nu^{-1}[L]} \cong L$. Now, let us fix a divisor D on C. On $C \times \mathrm{Pic}^d(C)$, we have an exact sequence of line bundles

$$0 \to \mathcal{L} \to \mathcal{L}(\alpha^{-1}D) \to \mathcal{L}(\alpha^{-1}D)|_{\alpha^{-1}D} \to 0.$$

Pushing forward by ν we obtain a sequence of sheaves on $\mathrm{Pic}^d(C)$:

$$0 \to \nu_*\mathcal{L} \to \nu_*\mathcal{L}(\alpha^{-1}D) \xrightarrow{\phi} \nu_*\mathcal{L}(\alpha^{-1}D)|_{\alpha^{-1}D} \to R^1\nu_*\mathcal{L} \to 0. \tag{1.1.1}$$

If we choose D to have suitably large degree N say, the second term in (1.1.1) will be a vector bundle of rank $d + N - g + 1$ by the theorem on cohomology and base change; the third term is also a vector bundle, of rank N. By cohomology and base change, we have

$$W^r_d(C) = \{[L] \in \mathrm{Pic}^d(C) : \dim\ker\phi|_{[L]} \geq r+1\}.$$

This endows $W^r_d(C)$ with a natural *scheme* structure, defined locally by the $(r+1)\times(r+1)$ minors of ϕ. (This scheme structure is equal to the Fitting support of the sheaf $R^1\nu_*\mathcal{L}$; the Fitting support of a sheaf is independent of choice of resolution, so our scheme is well-defined, independent of the choice of D.)

In the space of all $n \times m$ matrices, an elementary dimension count shows that the matrices whose kernel has dimension at least k have codimension $k(m-n+k)$. (I remember this with the phrase "(dim ker)(dim coker).") We call this the "expected codimension" for the determinantal locus. In particular, this gives rise to an "expected codimension" for $W^r_d(C)$:

$$(\dim\ker\phi)(\dim\operatorname{coker}\phi) = (r+1)(N-(N+d-g+1-(r+1))) = (r+1)(g-d+r).$$

Equivalently, the "expected dimension" of $W^r_d(C)$ is

$$\rho(g,r,d) := g-(r+1)(g-d+r).$$

This quantity is known as the *Brill–Noether number*. Brill and Noether conjectured in the 1870s that for a *general curve* C, we should have $\dim W^r_d(C) = \rho(g,r,d)$. (As written this is a bit sloppy, but standard phrasing. Unpacking a bit, if $\rho < 0$, we we mean that $W^r_d(C)$ should be empty; if $g \geq \rho \geq 0$, then it should measure $\dim W^r_d(C)$; and if $\rho \geq g$, then we mean $W^r_d(C) = \mathrm{Pic}^d(C)$.)

For a particular curve C, describing the varieties $W^r_d(C)$ pictured in Figure 1.1 may be very difficult. However, the varieties $W^r_d(C)$ are known to satisfy many nice properties when C is *general* in moduli (all "unusual" behavior will occur along a proper closed subvariety of the moduli space of curves $\mathcal{M}_g$). Describing $W^r_d(C)$ for a general curve was the combined efforts of several mathematicians from the 1970s and 1980s and represents one of the crowning achievements in the modern study of algebraic curves.

Theorem 1.1.3. *Let C be a general curve of genus g.*

1. $\dim W^r_d(C) = \rho(g,r,d)$. *(Griffith–Harris 1980 [36])*

2. *$W^r_d(C)$ is smooth away from $W^{r+1}_d(C)$. (Geiseker 1982 [35])*

3. *If $\rho = 0$, then*

$$\#W^r_d(C) = \#\left\{ \begin{array}{c} \text{fillings of an } (r+1)\times(g-d+r) \\ \text{diagram using } 1,\dots,g, \text{ with no} \\ \text{repeats, so it is increasing along} \\ \text{rows and columns} \end{array} \right\}.$$

More generally, if θ denotes the class of the theta divisor on $\mathrm{Pic}^d(C)$, then for $\rho \geq 0$, the class of $W^r_d(C)$ is given by

$$[W^r_d(C)] = \prod_{\alpha=0}^{r} \frac{\alpha!}{(g-d+r+\alpha)!} \cdot \theta^{(r+1)(g-d+r)}.$$

(Independently by Kempf 1971 [43], and Kleiman–Laksov 1972 [44])

4. *If $\rho > 0$, then $W^r_d(C)$ is irreducible. (Fulton–Lazarsfeld 1981 [33])*

5. *When $\rho \geq 0$, the* universal $\mathcal{W}^r_d$ *(defined below) has a* unique *irreducible component dominating the moduli space of curves. (Eisenbud–Harris 1987 [20])*

Each part of Theorem 1.1.3 says something about our picture in Figure 1.1. For example, part 4 tells us there is just one piece at each grayscale, unless that piece has dimension 0, in which case part 3 tells us exactly how many of those dots there are.

Example 1.1.4. For a general curve C of genus $g = 6$, we have $\dim W^1_4(C) = 0$ (by part 1) and $\#W^1_4(C) = 5$ (by part 3), corresponding to the following 5 fillings of a 2×3 grid:

1	3	5
2	4	6

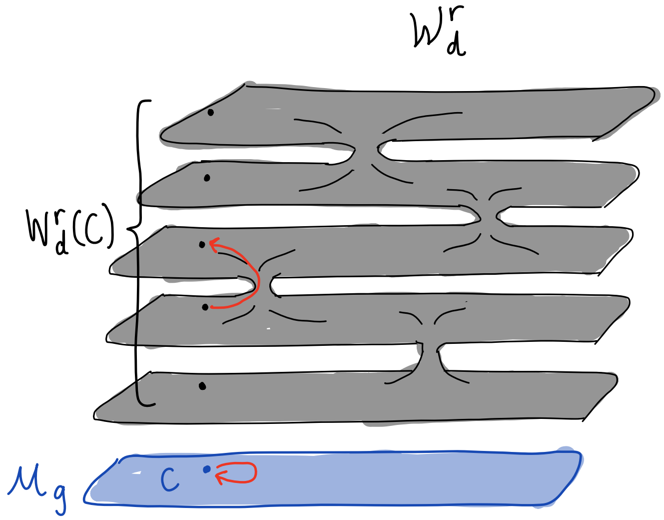
1	3	4
2	5	6

1	2	3
4	5	6

1	2	4
3	5	6

1	2	5
3	4	6

To picture part 5, we need to imagine the pictures of Figure 1.1 as we vary C, fitting into a large picture over the moduli space of curves $\mathcal{M}_g$. Together these form a space called the *universal* $\mathcal{W}^r_d$, which maps to $\mathcal{M}_g$ so that the fiber above each point $[C] \in \mathcal{M}_g$ is $W^r_d(C)$. Below is my cartoon of the universal $\mathcal{W}^r_d$ when $\rho = 0$.

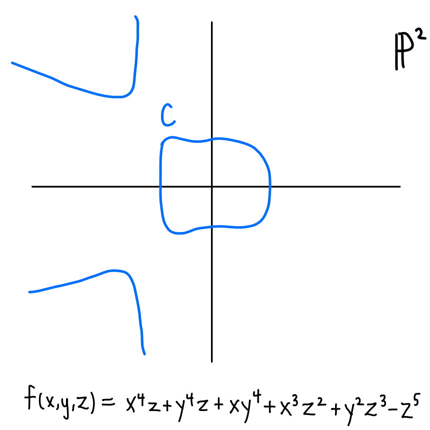


Part 5 says that no two points of $W^r_d(C)$ are distinguished from each other as we allow C to vary in moduli. That is, we can exchange any point with any other, as we move around some loop in $\mathcal{M}_g$ (pictured in red).

1.2 A next question

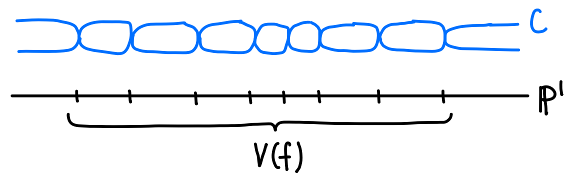
Theorem 1.1.3 is a wonderful theorem. However, there is a big difference between general curves, and the curves we meet in everyday life! Most numbers are transcendental, but we meet algebraic numbers all the time, because they can be described nicely as solutions to polynomial equations. Similarly, we often meet curves already living in (or mapping to) some projective space because that's how we can describe them nicely as solutions to polynomial equations. Such curves are often special, just by the virtue of the way we are able to describe them.

Example 1.2.1. Let $f(x, y, z)$ be a general homogeneous degree 5 polynomial. Then $C = V(f) \subset \mathbb{P}^2$ is a smooth curve of genus 6.



However, by Theorem 1.1.3 part 1, a general curve of genus 6 has no degree 5 map to $\mathbb{P}^2$ since $\rho(6,2,5) = -3 < 0$.

Example 1.2.2. Let $f(x,y)$ be a homogeneous polynomial of degree 8 with distinct roots in $\mathbb{P}^1$. Let C be the double cover of $\mathbb{P}^1$ branched along $V(f)$.



By the Riemann–Hurwitz formula, C has genus 3. However, a general curve of genus 3 has no degree 2 map to $\mathbb{P}^1$ since $\rho(3,1,2) = -1 < 0$.

More generally, if we meet C already with a map $C \to \mathbb{P}^s$ of degree k and $\rho(g,s,k) < 0$, then C must be special (by part 1 of Theorem 1.1.3). Hence, Theorem 1.1.3 no longer applies to tell us about the maps of C to other projective spaces $\mathbb{P}^r$. Thus, we must ask anew:

Question 1.2.3. Given a genus g curve together with a map $C \to \mathbb{P}^s$ of degree k, what is the geometry of $W^r_d(C)$?

In this thesis, we shall study the first case of this question: the case $s = 1$.

Question 1.2.4. Given a genus g, degree k cover $C \to \mathbb{P}^1$, what is the geometry of $W^r_d(C)$?

The moduli space of genus g curves together with a degree k map $C \to \mathbb{P}^1$ is called the *Hurwtiz space*, denoted $\mathcal{H}_{k,g}$. The space $\mathcal{H}_{k,g}$ is known to be irreducible when the characteristic of the ground field is 0 or greater than k (as proved by Fulton in [30]). All results in this thesis will assume the ground field has characteristic 0 or greater than k. It will therefore make sense to talk about *general covers* $C \to \mathbb{P}^1$. The main result of this thesis is an answer to Question 1.2.4 for general covers $C \to \mathbb{P}^1$.

Remark 1.2.5. In the case $s = 2$, the moduli space of curves C together with a degree k map $C \to \mathbb{P}^2$ is called the *Severi variety* and is also irreducible. However, for $s \geq 3$, the moduli space of curves together with a degree k map to $\mathbb{P}^s$ may have multiple components.

Before giving our main result, let us give a history of previous work on Question 1.2.4. Classical works give answers to Question 1.2.4 for $k = 2$ and $k = 3$. These first two results actually answer the question for *all* covers $C \to \mathbb{P}^1$.

1878 Clifford: complete answer $k = 2$ [7]

1946 Maroni: complete answer $k = 3$ [56]

Maroni saw that $W^r_d(C)$ can be reducible. In what follows, recall that the dimension of a reducible variety is defined to be the maximum of the dimensions of all of its components. Following the completion of the classical Brill–Noether theorem (Theorem 1.1.3) in the 1980s, there was a flurry of interest in Question 1.2.4 in the 1990s and early 2000s. The following results are for general degree k, genus g covers $C \to \mathbb{P}^1$:

1994 Coppens, Keem, Martens: determine dimensions of all components of $W^1_d(C)$ [10]

1996 Martens: Upper bounds on $\dim W^r_d(C)$ when k odd [57]

1996 Ballico, Keem: Upper bounds on $\dim W^r_d(C)$ [3]

1999 Coppens, Martens: $W^r_d(C)$ has a component of the expected dimension ρ when $d - g < r \leq k - 2$ [11]

2000 Coppens, Martens: partial progress $k = 4$ [12]

2002 Park: partial progress $k = 5$ [60]

2002 Coppens, Martens: $W^r_d(C)$ has components of the “wrong dimension" $\rho(g, \alpha - 1, d) - (r - \alpha + 1)k$ for α dividing r or $r + 1$ [13]

More recently, Pflueger proposed to study this problem via degeneration to chain of elliptic curves, attached so that the nodes differ by k-torsion. This approach was the beginning of the next wave of developments, and an important inspiration in my subsequent work. Again, these results are for general degree k, genus g covers $C \to \mathbb{P}^1$:

2016 Pflueger: an improved upper bound $\dim W^r_d(C)$: [61]

$$\dim W^r_d(C) \leq \rho_k(g, r, d) := \max_{\ell \in \{0, \ldots, r'\}} \rho(g, r - \ell, d) - \ell k. \tag{1.2.1}$$

where $r' = \min\{r, g - d + r - 1\}$. (The value where the above maximum is attained need not satisfy the divisibility conditions of Coppens–Martens [13].)

2017 Jensen, Ranganathan: $W^r_d(C)$ has a component of the maximum possible dimension $\rho_k(g, r, d)$ allowed by (1.2.1) [42]

Example 1.2.6. Let $f : C \to \mathbb{P}^1$ be a general genus 6, degree 3 cover. Such a curve lies on $\mathbb{P}^1 \times \mathbb{P}^1$ as a curve of bidegree $(3, 4)$. Let us write $\alpha : C \to \mathbb{P}^1$ for the projection onto the other $\mathbb{P}^1$ factor. Let $H = f^*\mathcal{O}_{\mathbb{P}^1}(1)$ and $D = \alpha^*\mathcal{O}_{\mathbb{P}^1}(1)$.

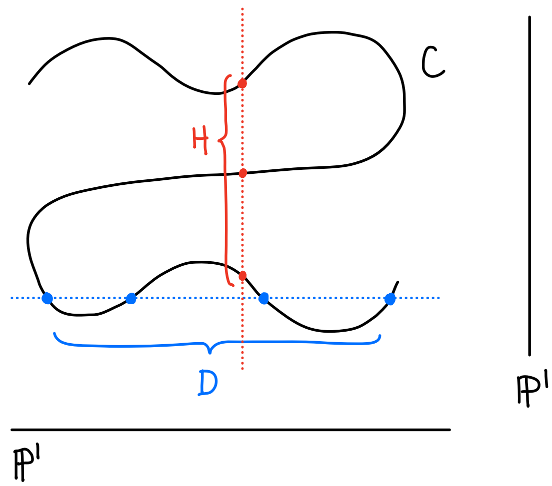


Figure 1.2: A curve of bidegree $(3, 4)$ on $\mathbb{P}^1 \times \mathbb{P}^1$

The Brill–Noether variety $W^1_4(C)$ has two components. One component is obtained from adding a base point to the degree three divisor H; this component is isomorphic to C. Having dimension 1, it is an example of the $\alpha = 1$ case of a component of the "wrong dimension" observed by Coppens–Martens in 2002. The other component is a single point, corresponding to D. This is an example of the $\alpha = 2 = r + 1$ case of [13], which yields a component of dimension $\rho(6, 1, 4) = 0$. (It is also then an example of the earlier 1999 result of Coppens–Martens.)

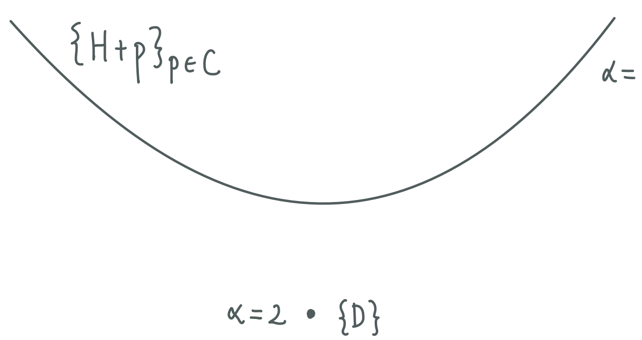


Figure 1.3: The components of $W^1_4(C)$.

Pflueger's formula (1.2.1) takes the form

$$\dim W^1_4(C) \leq \max\{0, 1\},$$

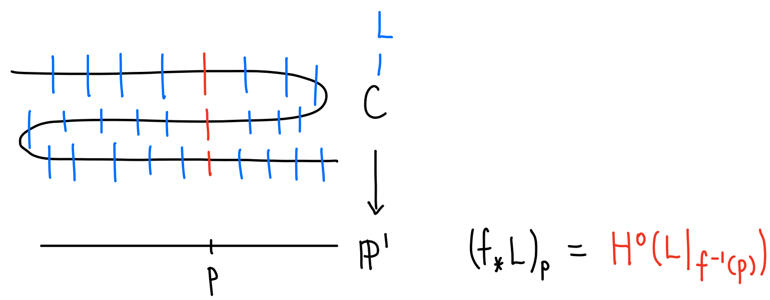
and Jensen–Ranganathan's result tells us that there does exist a component of dimension 1.

1.3 Results

Many questions about the geometry of $W^r_d(C)$ for general covers $C \to \mathbb{P}^1$ remain: What are the dimensions of all components? How many components are there? Where are they smooth? A key first step in answering these questions is to define a new invariant.

1.3.1 A new invariant: the splitting type

Given a line bundle L on such a curve C, the push forward f_*L is a rank k vector bundle on $\mathbb{P}^1$. The fiber of f_*L at a point $p \in \mathbb{P}^1$ is $H^0(L|_{f^{-1}(p)})$, which one can think of as the sum of the fibers of L along the preimage of p.



By Riemann-Roch, the degree of the push forward bundle f_*L is

$$\deg(f_*L) = \chi(C, L) - k = d - g + 1 - k. \tag{1.3.1}$$

Every vector bundle on $\mathbb{P}^1$ is isomorphic to $O(e_1) \oplus \cdots \oplus O(e_k)$ for some collection of integers $\vec{e} = (e_1, \ldots, e_k)$ with $e_1 \leq \cdots \leq e_k$. We call such a collection $\vec{e}$ the *splitting type* and abbreviate the corresponding sum of line bundles by $O(\vec{e})$. The specialization of splitting types follows certain rules. Given two splitting types $\vec{e}\,' = (e'_1, \ldots, e'_k)$ and $\vec{e} = (e_1, \ldots, e_k)$, we define a partial ordering by $\vec{e}\,' \leq \vec{e}$ if $O(\vec{e})$ can specialize to $O(\vec{e}\,')$, that is if $e'_1 + \ldots + e'_j \leq e_1 + \ldots + e_j$ for all j and $e'_1 + \ldots + e'_k = e_1 + \ldots + e_k$. For each rank and degree, there is a unique maximal splitting type, which we call *balanced*.

Definition 1.3.2. We define *Brill-Noether splitting loci* by

$$W^{\vec{e}}(C) = \{L \in \mathrm{Pic}^d(C) : f_*L \cong O(\vec{e}\,') \text{ for some } \vec{e}\,' \leq \vec{e}\}.$$

There is a natural scheme structure on $W^{\vec{e}}(C)$ as a splitting locus via Definition 2.2.1. We define the *expected codimension* of $W^{\vec{e}}(C)$ to be

$$u(\vec{e}) := h^1(\mathbb{P}^1, End(O(\vec{e}))) = \sum_{i<j} \max\{0, e_j - e_i - 1\}.$$

Note that the splitting type $\vec{e}$ is balanced if and only if $u(\vec{e}) = 0$, equivalently $|e_i - e_j| \leq 1$ for all i, j. This is where the name balanced comes from.

Remark 1.3.3. The splitting locus $W^{\vec{e}}(C)$ depends on the map $f : C \to \mathbb{P}^1$. However, we suppress f from the notation.

Let $H = f^*\mathcal{O}_{\mathbb{P}^1}(1)$ be the distinguished degree k line bundle that gives us our map $f : C \to \mathbb{P}^1$. One may readily check

$$W^{\vec{e}}(C) = \{L \in \mathrm{Pic}^d(C) : h^0(C, L(mH)) \geq h^0(\mathbb{P}^1, \mathcal{O}(\vec{e})(m)) \text{ for all } m\}. \tag{1.3.2}$$

Thus, splitting loci provide a refinement of the stratification of $\mathrm{Pic}^d(C)$ by Brill-Noether loci $W^r_d(C)$. In particular,

$$W^r_d(C) = \bigcup_{\substack{e_1+\ldots+e_k=d-g+1-k \\ h^0(\mathcal{O}(\vec{e}))\geq r+1}} W^{\vec{e}}(C) = \bigcup_{\substack{\vec{e} \text{ maximal:} \\ e_1+\ldots+e_k=d-g+1-k \\ h^0(\mathcal{O}(\vec{e}))\geq r+1}} W^{\vec{e}}(C), \tag{1.3.3}$$

The splitting types $\vec{w}$ which are maximal among those with $r+1$ global sections satisfy the following property: there exists some ℓ so that $|w_i - w_j| \leq 1$ whenever $i, j \leq \ell$ or $i, j \geq \ell$. We call a splitting type $\vec{w}$ with this property "balanced plus balanced" because it is the sum of two bundles of lower rank with balanced splitting types. Once the rank and degree are fixed, a "balanced plus balanced" splitting type is uniquely determined by the number of nonnegative summands. When the rank and degree are understood, we define $\vec{w}_{r,\ell}$ to be the splitting type with $r+1-\ell$ nonnegative parts that is maximal among those with $r+1$ global sections (see Lemma 3.5.1). In terms of these splitting loci, we can rewrite Pflueger's formula (1.2.1) suggestively to see

$$W^r_d(C) = \bigcup_{\ell} W^{\vec{w}_{r,\ell}}(C, f) \qquad \text{and} \qquad \rho_k(g, r, d) = \max_{\ell} \left(g - u(\vec{w}_{r,\ell})\right).$$

Example 1.3.4 (Example 1.2.6 continued)**.** Let $C \to \mathbb{P}^1$ be a general degree 3, genus 6 cover as in Example 1.2.6. By (1.3.1), the push forward of a degree 4 line bundle on C has degree -4. It turns out $\vec{w}_{2,1} = (-4, 0, 0)$ and $\vec{w}_{2,2} = (-3, -2, 1)$, so

$$W^1_4(C) = W^{(-4,0,0)}(C) \cup W^{(-3,-2,1)}(C).$$

We have $u(-4, 0, 0) = 6$ so $\dim W^{(-4,0,0)}(C) = 0$. On the other hand, $u(-3, -2, 1) = 5$ so

$\dim W^{(-3,-2,1)}(C) = 1$. Using (1.3.2), we can see that

$$W^{(-3,-2,1)}(C) = \{L \in \mathrm{Pic}^4(C) : h^0(C, L(-H)) \geq 1\} = \{H + p\}_{p \in C}.$$

Thus Figure 1.4 identifies each of the components from Figure 1.3 in terms of splitting loci. Their different dimensions are explained by the fact that these splitting types have different expected dimensions.

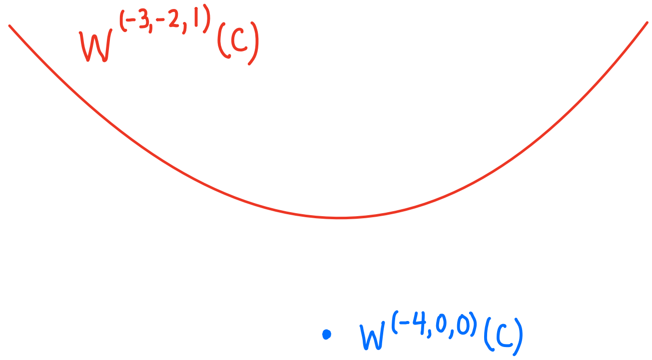
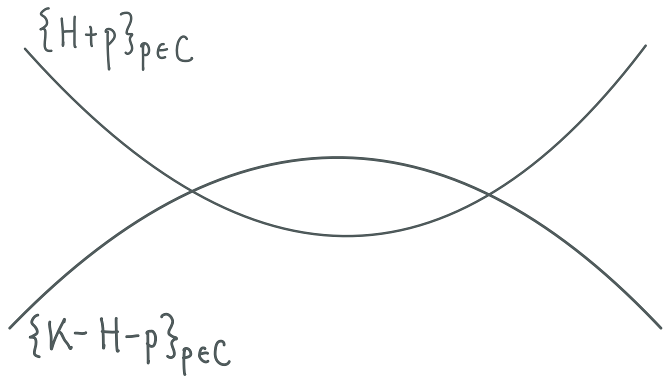


Figure 1.4: The components of $W^1_4(C)$ correspond to different splitting types.

Splitting loci capture more than just the components of $W^r_d(C)$, as the next example illustrates. At first glance, the next example may seem similar to the previous one, but it introduces an important new phenomenon. This example is what first interested me in this problem. It was shown to me by Geoffrey Smith when he visited Stanford in Fall 2018 and shared his insight that my work on families of vector bundles on $\mathbb{P}^1$ might have something interesting to say here.

Example 1.3.5. Let $f : C \to \mathbb{P}^1$ be a general degree 3, genus 5 cover. Let $H = f^*\mathcal{O}_{\mathbb{P}^1}(1)$. The Brill–Noether variety $W^1_4(C)$ has two components. One is obtained by adding a base point to H. Let K be the canonical line bundle, which is degree 8. Serre duality shows that $K - H - p$ is also in $W^1_4(C)$ for any point $p \in C$.



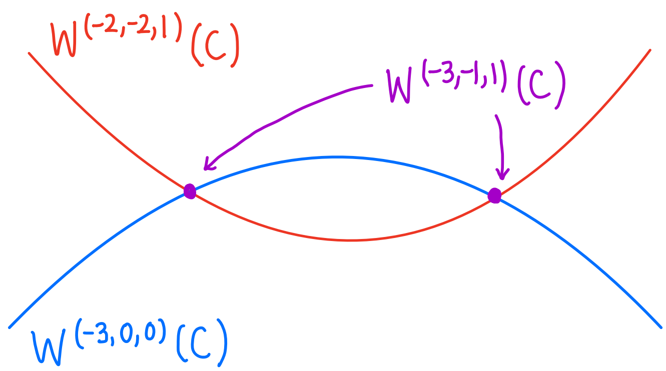
These two components are distinguished by splitting type of the push forward. Indeed, using (1.3.2) we see

$$W^{(-2,-2,1)}(C) = \{L \in \mathrm{Pic}^4(C) : h^0(C, L(-H)) \geq 1\} = \{H + p\}_{p \in C}$$
$$W^{(-3,0,0)}(C) = \{L \in \mathrm{Pic}^4(C) : h^0(C, L(H)) \geq 4\} = \{K - H - q\}_{q \in C}.$$

We recognize these splitting types as $\vec{w}_{1,1} = (-2, -2, 1)$ and $\vec{w}_{1,0} = (-3, 0, 0)$. Meanwhile, the intersection of the two components is the splitting locus

$$W^{(-3,-1,1)}(C) = \{L \in \mathrm{Pic}^4(C) : h^0(C, L(-H)) \geq 1 \text{ and } h^0(C, L(H)) \geq 4\}. \qquad (1.3.4)$$

If p_0 and q_0 denote the unique pair of points such that $p_0 + q_0 \sim K - 2H$, the intersection of the two components consists of the two points $H + p_0 \sim K - H - q_0$ and $H + q_0 \sim K - H - p_0$.



Notice that both conditions in (1.3.4) are needed to cut out $W^{(-3,-1,1)}(C)$. This example is the smallest example of a splitting locus where more than one condition is needed, and so is my favorite example. Note also that the intersection of the two components is not transverse (each component is codimension 4, and their intersection is codimension 5). This is part of what makes splitting loci more subtle and interesting.

As Examples 1.3.4 and 1.3.5 illustrate, I like to think of splitting loci as adding color to our earlier black and white picture in Figure 1.1. Figure 1.5 shows the full pictures for $\mathrm{Pic}^4(C)$ when $C \to \mathbb{P}^1$ is a general degree 3, genus 5 cover. The variety $W^1_4(C)$ we were

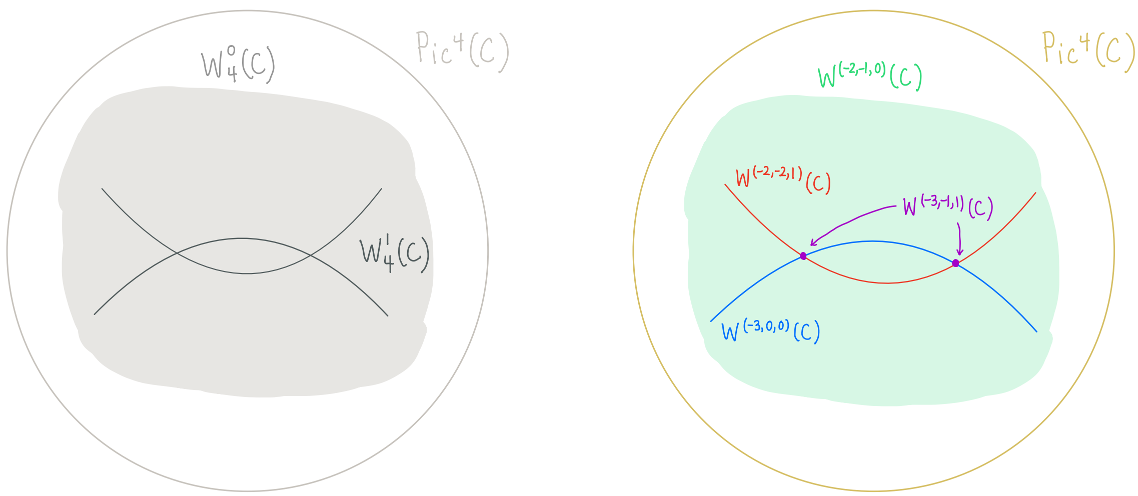


Figure 1.5: Keeping track of splitting loci is like adding color to a black and white picture.

studying earlier is of course contained in $W^0_4(C)$, which is just the locus of effective degree 4 divisors. Being the image of a generically finite map $\mathrm{Sym}^4 C \to \mathrm{Pic}^4(C)$, we see that $W^0_4(C)$ is irreducible of dimension $4 = \rho(5, 0, 4)$.

Part of the reason it was so difficult for people to make sense of Question 1.2.4 in the previous works listed in Section 1.2 is because they were still looking at the above picture in black and white. Once we have defined the splitting type, it is natural to conjecture that the different components should always correspond to different splitting types (different colors). When two colors intersect, they form new colors. Similarly, splitting loci don't just tell us about the components of $W^r_d(C)$, they also tell us how those components intersect. Finally, just as you can always make a gray-scale copy of a black and white picture, once you understand splitting loci, you can recover anything you wanted to know about $W^r_d(C)$.

1.3.6 Main theorems

Our main theorems describe the geometry of Brill–Noether splitting loci in a manner analogous to Theorem 1.1.3. We assume throughout that the characteristic of the ground field is 0 or greater than k.

Theorem 1.3.7. *Let $C \to \mathbb{P}^1$ be a general degree k, genus g cover.*

1. $\dim W^{\vec{e}}(C) = g - u(\vec{e})$.

2. *$W^{\vec{e}}(C)$ is smooth away from $W^{\vec{e}\,'}(C)$ for $\vec{e}\,' < \vec{e}$.*

3. *If θ denotes the class of the theta divisor on $\mathrm{Pic}^d(C)$, the class of $W^{\vec{e}}(C)$ is of the form*

$$[W^{\vec{e}}(C)] = \frac{N(\vec{e})}{u(\vec{e})!} \cdot \theta^{u(\vec{e})}$$

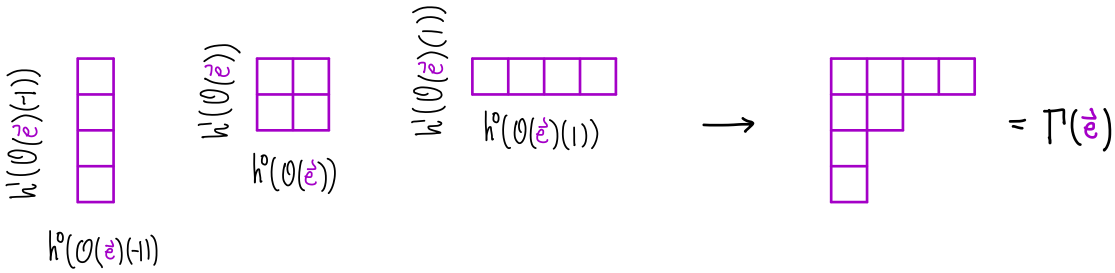
for a positive integer $N(\vec{e})$ which depends only on $\vec{e}$.

As a special case, Theorem 1.3.7 part 1 determines the dimensions of all components of $W^r_d(C)$ for general covers $C \to \mathbb{P}^1$, and part 2 shows the components are all generically smooth.

Corollary 1.3.8. *Let $C \to \mathbb{P}^1$ be a general degree k, genus g cover. Let $\vec{w}_{r,\ell}$ denote the rank k, degree $d - g + k - 1$ splitting type with $r + 1 - \ell$ nonnegative parts that is maximal among those with $r + 1$ global sections. Every component of $W^r_d(C)$ is generically smooth. If $k \leq g - d + 2r + 1$, then $W^r_d(C)$ is of pure dimension $\rho(g, r, d)$, or empty if this number is negative. Otherwise, every component has dimension $g - u(\vec{w}_{r,\ell}) = \rho(g, r - \ell, d) - \ell k$ for some $\max\{0, r + 2 - k\} \leq \ell \leq \min\{r, g - d + 2r + 1 - k\}$. Such a component exists for each ℓ with $g - u(\vec{w}_{r,\ell}) \geq 0$.*

In Theorem 1.3.7 part 3, since $N(\vec{e})$ depends only on $\vec{e}$ and not on g, setting $g = u(\vec{e})$, we have $N(\vec{e}) = \#W^{\vec{e}}(C)$. I proved part 3 using universal formulas for splitting loci that I will present in Chapter 2. Unfortunately, these formulas are difficult to compute explicitly, so although we can use them to determine the shape of the formula for $[W^{\vec{e}}(C)]$, computing the integer $N(\vec{e})$ in that way is infeasible (apart from a few special cases).

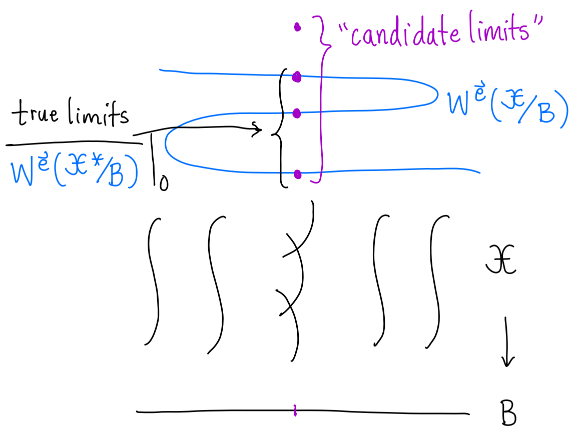
A key step towards determining $N(\vec{e})$ via degeneration was suggested to me by Dave Jensen and Kaelin Cook–Powell when I visited them at the University of Kentucky in Fall 2019. They introduced me to a combinatorial gadget they termed *k-staircase tableau* and explained its relevance in our degeneration. Given a splitting type $\vec{e}$, let us define a Young diagram $\Gamma(\vec{e})$ as the union of all rectangles that are $h^0(\mathbb{P}^1, \mathcal{O}(\vec{e})(m)) \times h^1(\mathbb{P}^1, \mathcal{O}(\vec{e}(m)))$ for integers m. (There are finitely many integers where both side lengths are non-zero.) The picture below explains the example $\vec{e} = (-3, -1, 1)$:

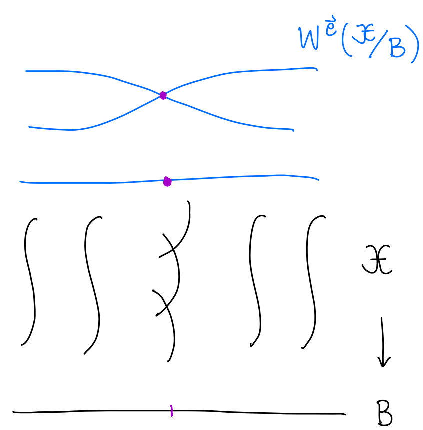


A filling of this diagram is called *k-regular* if it is increasing along rows and columns and any repeated symbols occur lattice distance a multiple of k apart. A filling is *efficient* if it uses the minimal number of symbols. Below are the two efficient 3-regular fillings for $\Gamma(-3,-1,1)$:

In [9], Jensen and Cook–Powell prove that the minimal number of symbols needed to fill $\Gamma(\vec{e})$ is always equal to $u(\vec{e})$ and use this to reprove my Theorem 1.3.7 part 1. Furthermore, they conjectured that $N(\vec{e})$ is equal to the number of efficient k-regular fillings of $\Gamma(\vec{e})$. For example, the two fillings above should be what tells us that there are two purple points in Example 1.3.5.

The major obstacle to proving this conjecture is to establish a *regeneration theorem*. Roughly speaking, the fillings of $\Gamma(\vec{e})$ correspond to "candidate limits" in our degeneration — but how do we know that these candidate limits truly are limits? And how do we know if the central fiber in our degeneration is reduced? In terms of a picture, we must show that the following things do not happen:





In joint work with Eric Larson and Isabel Vogt, we establish such a regeneration theorem. The proof relies on an intimate relationship between k-regular fillings and combinatorics of the affine symmetric group (to be discussed at length in Chapter 4). Our regeneration theorem is also the key to proving irreducibility and transitive monodromy.

Theorem 1.3.9. *Let $f : C \to \mathbb{P}^1$ be a general degree k cover of genus g.*

3. *(cont.) The integer $N(\vec{e})$ is equal to the number of efficient k-regular fillings of $\Gamma(\vec{e})$. Equivalently, it is equal to the number of reduced words for a certain element of the affine symmetric group (see Theorem 4.1.3 for a precise statement).*

4. *$W^{\vec{e}}(C)$ is irreducible when $g - u(\vec{e}) > 0$.*

5. *When $g - u(\vec{e}) \geq 0$, the universal $\mathcal{W}^{\vec{e}}$ has a* unique *component dominating the Hurwitz space $\mathcal{H}_{k,g}$ of degree k genus g covers of $\mathbb{P}^1$.*

Chapter 2 contains some general results about families of vector bundles on $\mathbb{P}^1$.
Chapter 3 contains the proof of Theorem 1.3.7.
Chapter 4 contains the proof of Theorem 1.3.9.

Chapter 2

Vector bundles on $\mathbb{P}^1$ bundles

This chapter contains results of mine on families of vector bundles on $\mathbb{P}^1$ more generally, which have been published in [51]. The main application of these results will be in the proof of Theorem 1.3.7 part 3. The reader primarily interested in Brill–Noether theory is welcome to skip forward to the next chapter and refer back to this one as necessary.

2.1 Introduction

Vector bundles on families of rational curves arise naturally in many geometric situations. The Grothendieck-Birkoff Theorem states that any vector bundle E on $\mathbb{P}^1$ splits as a direct sum of line bundles $E \cong \mathcal{O}_{\mathbb{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(e_r)$ for integers $e_1 \leq \cdots \leq e_r$. We call a non-decreasing collection of integers $\vec{e} = (e_1, \ldots, e_r)$ a *splitting type*, and abbreviate the corresponding sum of line bundles by $\mathcal{O}(\vec{e})$.

In this Chapter, we study this splitting phenomenon in families. The results hold over fields of any characteristic. Let W be a rank 2 vector bundle on a scheme B and form the $\mathbb{P}^1$ bundle $\pi : \mathbb{P}W \to B$. Given a vector bundle E on $\mathbb{P}W$, we define *strict splitting loci* of E by

$$\Sigma^{\circ}_{\vec{e}}(E) := \{b \in B : E|_{\pi^{-1}(b)} \cong \mathcal{O}(\vec{e})\} \subset B.$$

These are locally closed subvarieties of B. The *expected codimension* of $\Sigma^{\circ}_{\vec{e}}(E)$ is

$$u(\vec{e}) := h^1(\mathbb{P}^1, End(\mathcal{O}(\vec{e}))) = \sum_{i<j} \max\{0, e_j - e_i - 1\},$$

which is the dimension of the deformation space of the bundle $\mathcal{O}(\vec{e})$. Deformation theory shows (see e.g. [21, Ch. 14]) that if $\Sigma^\circ_{\vec{e}}(E)$ is non-empty,

$$\operatorname{codim} \Sigma^\circ_{\vec{e}}(E) \leq u(\vec{e}).$$

We write $\vec{e}\,' \leq \vec{e}$ if splitting type $\vec{e}$ can specialize to $\vec{e}\,'$, that is if $e'_1 + \ldots + e'_j \leq e_1 + \ldots + e_j$ for all j and $e'_1 + \ldots + e'_r = e_1 + \ldots + e_r$ (see Section 2.2). With this notion, we define *splitting loci* (set theoretically) by

$$\Sigma_{\vec{e}}(E) := \bigcup_{\vec{e}\,' \leq \vec{e}} \Sigma^\circ_{\vec{e}\,'}(E).$$

A priori, it is not clear how to construct the "right" scheme structure on $\Sigma_{\vec{e}}(E)$. This subtlety will be discussed in Section 2.4, where we confirm that the scheme structure we choose has the following minimality property: the tangent space at a point in the open stratum $\Sigma^\circ_{\vec{e}}(E) \subset \Sigma_{\vec{e}}(E)$ is precisely those maps $\operatorname{Spec} k[\epsilon]/(\epsilon^2) \to B$ so that the induced first order deformation of $E|_{\pi^{-1}(\operatorname{Spec} k)}$ is trivial.

Splitting loci often have geometric significance. The primary example in this thesis is the Brill–Noether splitting loci introduced in Definition 1.3.2. In the notation here, suppose $f : C \to \mathbb{P}^1$ is a degree k cover and $\mathcal{L}$ a Poincaré line bundle on $\operatorname{Pic}^d(C) \times C$. Then $\mathcal{E} = (id \times f)_* \mathcal{L}$ is a family of vector bundles on $\operatorname{Pic}^d(C) \times \mathbb{P}^1$ and we have

$$W^{\vec{e}}(C) = \Sigma_{\vec{e}}(\mathcal{E}).$$

Below are some other examples of naturally arising vector bundles on families of rational curves and their splitting loci.

1. Consider a projective variety $X \subset \mathbb{P}^n$ and suppose F parametrizes rational curves of a given degree on X. The family of curves C parametrized by F sits inside $X \times F$, and the normal bundle $\mathcal{N}_{C/X\times F}$ is a family of vector bundles on C whose splitting loci govern the local geometry of these curves inside X. The splitting of this bundle has been studied extensively for $X = \mathbb{P}^n$ (see for example [1, 15, 24, 25, 62, 64, 65]) and for other varieties (e.g. [17, 16, 34, 49, 45]). The present work answers the remark following Proposition 2.3 of [49], which asked for the classes of splitting loci of the normal bundle of the universal line on the universal hypersurface.

2. There is no analogous classification result for vector bundles on $\mathbb{P}^n$ for $n > 1$. One approach to studying vector bundles on $\mathbb{P}^n$ is by restricting to each line $\mathbb{P}^1 \subset \mathbb{P}^n$. For each vector bundle on $\mathbb{P}^n$, this gives rise to a family of vector bundles on lines whose splitting loci are called the loci of jumping lines and are important geometric invariants

of the bundle (see e.g. [26]). More generally, given a variety X with a vector bundle E, one may study E through the varieties of jumping curves in the moduli space of maps $\mathbb{P}^1 \to X$. When $X = \mathbb{G}(k, n)$ and E is the tautological bundle, these splitting loci describe the different types of rational scrolls in $\mathbb{P}^n$ (see e.g. [14, 54, 71]). As another example, [6] studies the case when X is the moduli space of stable vector bundles on a curve of genus at least 2.

The classes of splitting loci are natural invariants characterizing the degenerations of a vector bundle on a $\mathbb{P}^1$ bundle. We give a constructive proof that, when splitting loci occur in the correct codimension, these classes are given by a universal formula in terms of the Chern classes of naturally arising vector bundles on the base. In some cases, the formula is particularly simple.

Example 2.1.1. Suppose E is a rank 2, degree 0 vector bundle on $B \times \mathbb{P}^1$. On the open set $B \backslash \Sigma_{(-2,2)}$, the theorem on cohomology and base change shows that the pushforwards $\pi_* E$ and $\pi_* E(1)$ are locally free sheaves of ranks 2 and 4 respectively. There is a natural map between rank 4 vector bundles $\phi : \pi_* E \otimes H^0(\mathcal{O}(1)) \to \pi_* E(1)$, and

$$\Sigma_{(-1,1)} = \{b \in B : \operatorname{rank} \phi_b \leq 3\}.$$

If $\operatorname{codim} \Sigma_{(-1,1)} = 1$ then the class of $\Sigma_{(-1,1)}$ in $B \backslash \Sigma_{(-2,2)}$ is given by the Porteous formula (see e.g. [21, Thm. 12.4]):

$$[\Sigma_{(-1,1)}] = c_1(\pi_* E(1)) - 2c_1(\pi_* E).$$

If $\operatorname{codim} \Sigma_{(-2,2)} > 1$, then this formula also holds in the Chow ring of B by excision (see e.g. [21, Prop. 1.14]).

The above example worked because we could compute on the open subset of B where $\pi_* E(1)$ and $\pi_* E$ were locally free, and the splitting locus $(-1, 1)$ was determined by a single rank condition. The latter fails in general (see Example 2.2.3) and new ideas are needed to compute the classes of splitting loci in general.

We give a closed formula for degeneracy classes for certain splitting types, and an inductive algorithm that works for all splitting types.

Theorem 2.1.2. *Let E be a vector bundle on a $\mathbb{P}^1$ bundle $\mathbb{P}W \to B$. Assume that* $\operatorname{codim} \Sigma^\circ_{\vec{e}}(E) = u(\vec{e})$ *and* $\operatorname{codim} \Sigma^\circ_{\vec{e}\,'} > u(\vec{e})$ *for all $\vec{e}\,' < \vec{e}$. The class of $\Sigma_{\vec{e}}(E)$ in $A^*(B)$ is given by a universal formula, depending only on $\vec{e}$, in terms of Chern classes of $\pi_* \mathcal{O}_{\mathbb{P}W}(1), \pi_* E(m)$, and $\pi_* E(m-1)$ for m suitably large. This formula is computed by the procedure in Section 2.6.*

Remarks. (1) Even if the dimension of $\Sigma_{\vec{e}}(E)$ is larger than expected, the class resulting from this formula is still represented by a cycle supported on $\Sigma_{\vec{e}}(E)$. In particular, if the expected class for $\Sigma_{\vec{e}}(E)$ is non-zero, then $\Sigma_{\vec{e}}(E)$ must be non-empty.

(2) We carry out the inductive procedure in the first non-trivial case in Example 2.6.2. Although the implementation is computationally expensive, the fact that such universal formulas exist is useful. The existence of these formulas, together with closed formulas in special cases, are key ingredients in my original proof of non-emptiness of Brill-Noether splitting loci (given in Section 3.4).

(3) The push forwards $\pi_*E(m)$ and $\pi_*E(m-1)$ for m supplied by the algorithm in Section 2.6 will be locally free on a suitably large open subset, but need not be locally free on all of B. Using Lemma 2.3.2, one may express their Chern classes in terms of vector bundles $\pi_*E(i)$ and $\pi_*E(i-1)$ for any i large enough that $R^1\pi_*E(i-1) = 0$.

(4) One can deduce the classes of splitting loci on a general family of genus zero curves $C \to B$ up to 2-torsion by studying the fiber product

$$\begin{array}{ccc} C \times_B C & \xrightarrow{q} & C \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ C & \xrightarrow[\pi]{} & B \end{array}$$

The diagonal inside $C \times_B C$ is a degree 1 divisor on each fiber, making p into a $\mathbb{P}^1$ bundle. Given a vector bundle E on C, we can therefore compute the class of $\Sigma_{\vec{e}}(q^*E)$ on C, and this locus is $\pi^{-1}(\Sigma_{\vec{e}}(E))$. The relative tangent bundle T_π of $C \to B$ restricts to a degree 2 line bundle on each fiber. Thus, $\pi_*(c_1(T_\pi) \cdot [\Sigma_{\vec{e}}(q^*E)]) = 2[\Sigma_{\vec{e}}(E)]$.

The remainder of this Chapter is organized as follows. In Section 2.2, we review basic facts about splitting loci and describe an important example. In Section 2.3, we generalize results of Strømme in [70] to relative Quot schemes over $\mathbb{P}^1$ bundles. Section 2.4 describes the tangent spaces to splitting loci along open strata. In Section 2.5, we find classes of certain splitting loci where the techniques of Example 2.1.1 readily generalize. Finally, Section 2.6 proves Theorem 2.1.2 with an inductive procedure that computes the classes of all splitting loci.

2.2 Splitting loci

Given a vector bundle E on $\mathbb{P}^1$, knowing the splitting type of E is equivalent to knowing the list of integers $h^0(\mathbb{P}^1, E(m))$ for $m \in \mathbb{Z}$, or equivalently the list of integers $h^1(\mathbb{P}^1, E(m))$

for $m \in \mathbb{Z}$. The multiplicity of $\mathcal{O}(-j)$ as a summand of E is equal to the second difference function evaluated at j of the Hilbert function $m \mapsto h^0(\mathbb{P}^1, E(m))$ (see e.g. [22, Lemma 5.6]).

Let W be a rank 2 vector bundle on a scheme B and let $\mathbb{P}W := \mathrm{Proj}(\mathrm{Sym}^\bullet W^\vee)$ with projection map $\pi : \mathbb{P}W \to B$. On $\mathbb{P}W$ there is a natural surjection $\pi^* W^\vee \to \mathcal{O}_{\mathbb{P}W}(1)$, and on B an isomorphism $W^\vee \cong \pi_* \mathcal{O}_{\mathbb{P}W}(1)$. Note that if L is a line bundle on B, then there is a natural isomorphism $\mathbb{P}(W \otimes L) \cong \mathbb{P}W$ via which $\mathcal{O}_{\mathbb{P}(W\otimes L)}(1) = \mathcal{O}_{\mathbb{P}W}(1) \otimes \pi^* L^\vee$. By a $\mathbb{P}^1$ bundle $\mathbb{P}W$ on B, we will mean to remember the data of the rank 2 vector bundle W, or equivalently a choice of relative degree 1 line bundle $\mathcal{O}_{\mathbb{P}W}(1)$.

Given a vector bundle E on $\mathbb{P}W$, we write $E(m)$ for $E \otimes \mathcal{O}_{\mathbb{P}W}(1)^{\otimes m}$. Upper-semicontinuity of the ranks of cohomology of $E(m)$ on fibers determines which splitting loci can be in the closures of others. Given two splitting types $\vec{e} = (e_1, \ldots, e_r)$ with $e_1 \leq \cdots \leq e_r$ and $\vec{e}\,' = (e_1', \ldots, e_r')$ with $e_1' \leq \cdots \leq e_r'$, we define a partial ordering by $\vec{e}\,' \leq \vec{e}$ if all partial sums $e_1' + \ldots + e_j' \leq e_1 + \ldots + e_j$ and their total degree is equal $e_1' + \ldots + e_r' = e_1 + \ldots + e_r$. For each rank and degree, there is a unique maximal splitting type called the *balanced splitting type*, characterized by the condition that $|e_i - e_j| \leq 1$.

Set theoretically, *splitting loci* are defined by

$$\Sigma_{\vec{e}}(E) := \bigcup_{\vec{e}\,' \leq \vec{e}} \Sigma^\circ_{\vec{e}\,'}(E).$$

The cohomological conditions determining a splitting type and the theorem on cohomology and base change show that each splitting locus $\Sigma_{\vec{e}}(E)$ is a finite intersection of loci

$$\{b \in B : h^1(E(m)|_{\pi^{-1}(b)}) \geq n\} = \{b \in B : \dim(R^1\pi_* E(m))_b \geq n\}.$$

The latter has a natural scheme structure as defined by the $(n-1)$st Fitting ideal of $R^1\pi_* E(m)$.

Definition 2.2.1. We define $\Sigma_{\vec{e}}(E)$ as the intersection of determinantal schemes

$$\Sigma_{\vec{e}}(E) := \bigcap_{m \in \mathbb{Z}} \{b \in B : \dim(R^1\pi_* E(m))_b \geq h^1(\mathbb{P}^1, \mathcal{O}(\vec{e})(m))\}$$

defined by the Fitting supports of $R^1\pi_* E(m)$. There are only finitely many m for which these loci are not all of B.

In Section 2.4, we describe the tangent space to this intersection along the open stratum $\Sigma^\circ_{\vec{e}}(E) \subset \Sigma_{\vec{e}}(E)$. As discussed there, the geometry of $\Sigma_{\vec{e}}(E)$ along the more unbalanced loci is more subtle.

The locus $\Sigma_{\vec{e}}(E)$ is always closed, but in general, it may not be equal to the closure of $\Sigma^{\circ}_{\vec{e}}(E)$. If the splitting loci for $\vec{e}\,' < \vec{e}$ have the expected codimension then the support of $\Sigma_{\vec{e}}(E)$ is the closure of $\Sigma^{\circ}_{\vec{e}}(E)$.

Lemma 2.2.2. *Let E be a vector bundle on $\pi : \mathbb{P}^1 \times B \to B$ with B irreducible. If $\Sigma_{\vec{e}}(E)$ is non-empty, then every component of $\Sigma_{\vec{e}}(E)$ has at least the expected dimension. In particular, if all $\Sigma^{\circ}_{\vec{e}}(E)$ have the expected dimension, then the support of $\Sigma_{\vec{e}}(E)$ is the closure of $\Sigma^{\circ}_{\vec{e}}(E)$.*

Proof. Let $\mathcal{E}$ be the universal bundle over the moduli stack $\mathcal{B}$ of vector bundles on $\mathbb{P}^1$ bundles. Then $\Sigma_{\vec{e}}(\mathcal{E})$ has codimension $u(\vec{e})$ and $\Sigma_{\vec{e}}(E)$ is its preimage under the induced map $B \to \mathcal{B}$. Codimension can only decrease under pullback so $\operatorname{codim} \Sigma_{\vec{e}}(E) \leq u(\vec{e})$. This applies on any open set of B, so every component of $\Sigma_{\vec{e}}(E)$ has at least the expected dimension. If all splitting loci have the expected dimension, every component of $\Sigma_{\vec{e}}(E) \backslash \Sigma^{\circ}_{\vec{e}}(E)$ has dimension less than the expected dimension of $\Sigma_{\vec{e}}(E)$. Thus, all of the support of $\Sigma_{\vec{e}}(E) \backslash \Sigma^{\circ}_{\vec{e}}(E)$ must lie in the closure of $\Sigma^{\circ}_{\vec{e}}(E)$. □

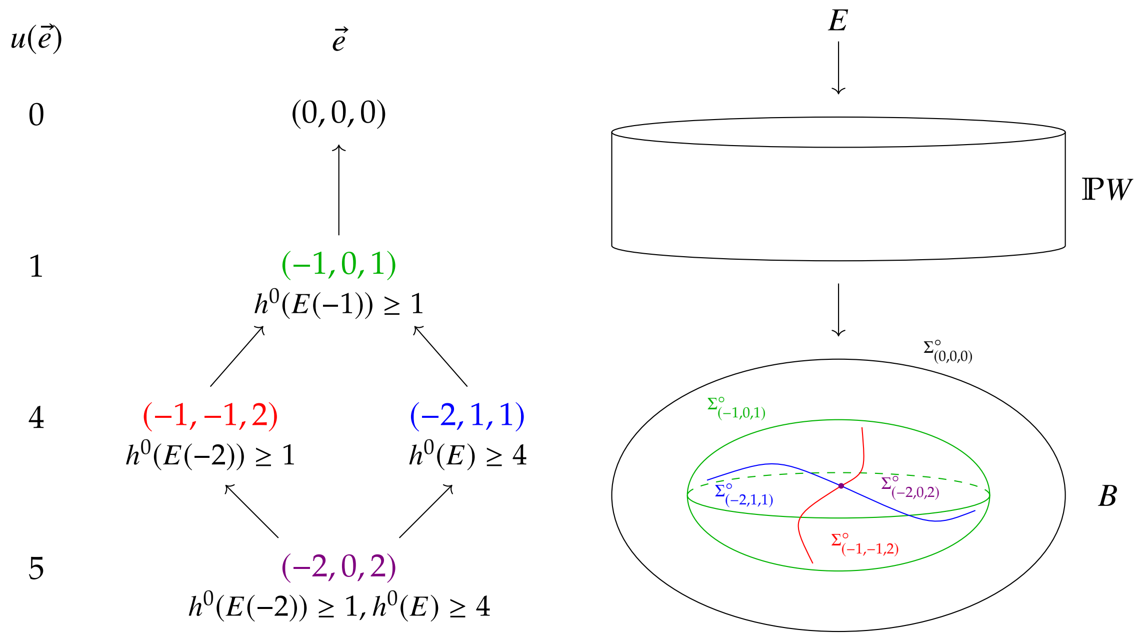
When E has rank 2, the possible splitting types are totally ordered with respect to $\leq$. However, in general they need not be.

Example 2.2.3 (Splitting type $(-2, 0, 2)$, to be revisited in Example 2.6.2)**.** The diagram below describes splitting loci for a rank 3 degree 0 vector bundle E on a $\mathbb{P}^1$ bundle. The cohomological conditions determining each splitting type are listed below it. An arrow between types indicates when one splitting type is below another in the partial ordering. Recall that the expected codimension for splitting type $\vec{e}$ is $u(\vec{e}) := h^1(\mathbb{P}^1, End(\mathcal{O}(\vec{e})))$.

It follows from the definitions that $\Sigma_{(-2,0,2)}$ is the intersection of $\Sigma_{(-1,-1,2)}$ and $\Sigma_{(-2,1,1)}$. In particular, the $(-2, 0, 2)$ splitting locus is not determined by a single rank condition. When splitting loci occur in the correct codimension, $\Sigma_{(-2,0,2)}$ has codimension 5, while have codimension 4. Hence, the intersection of $\Sigma_{(-1,-1,2)}$ and $\Sigma_{(-2,1,1)}$ is *not* transverse. This makes the task of computing splitting loci a delicate one in general (we cannot just intersect the classes of the corresponding determinantal loci in Definition 2.2.1.)

The diagram on the left shows indicates the partial ordering of splitting loci. When they occur in the expected codimension, this determines which splitting loci lie in the closure of another by Lemma 2.2.2.

Splitting loci for a rank 3, degree 0 vector bundle E



2.3 Relative Quot schemes of $\mathbb{P}^1$ bundles

Several of the results in this section generalize Strømme's work concerning vector bundles on trivial $\mathbb{P}^1$ bundles [70] to the case of non-trivial $\mathbb{P}^1$ bundles. Many of his proofs hold with appropriate modifications.

A key ingredient for explicit computation with splitting loci will be canonical resolutions of $R^1\pi_*E^\vee(-m)$ for certain m. To motivate these resolutions, we first explain the situation on a fixed $\mathbb{P}^1$. Suppose E is a globally generated vector bundle of rank r and degree k on $\mathbb{P}^1$, so there is a canonical surjection $H^0(\mathbb{P}^1, E) \otimes \mathcal{O}_{\mathbb{P}^1} \to E$ and $h^0(\mathbb{P}^1, E) = \chi(\mathbb{P}^1, E) = r + k$. It follows that the kernel of this surjection is rank k, degree $-k$ and therefore equal to $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus k}$ (as a subbundle of a trival bundle, all summands of the kernel are non-positive; moreover, any trivial summand would give a linear relation among the global sections). We also have $h^0(\mathbb{P}^1, E(-1)) = k$, so we can summarize the above observations with a sequence

$$0 \to H^0(\mathbb{P}^1, E(-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \to H^0(\mathbb{P}^1, E) \otimes \mathcal{O}_{\mathbb{P}^1} \to E \to 0.$$

The following lemma shows that this sequence globalizes suitably over $\mathbb{P}^1$ bundles. This generalizes [70, Prop. 1.1], which proves the case of trivial $\mathbb{P}^1$ bundles.

Lemma 2.3.1. *Let E be a vector bundle on a $\mathbb{P}^1$ bundle $\pi : \mathbb{P}W \to B$ and let $L = \det W^\vee$*

on B. *If* $R^1\pi_*E(-1) = 0$ *then there is a short exact sequence on* $\mathbb{P}W$

$$0 \to \pi^*(L \otimes \pi_*E(-1))(-1) \to \pi^*\pi_*E \to E \to 0.$$

Proof. Strømme's proof generalizes with suitable care. Let X be the fiber product of $\mathbb{P}W \to B$ with itself and consider the diagonal $\Delta \subset X$:

$$\begin{array}{ccc} \Delta \subset X & \xrightarrow{p} & \mathbb{P}W \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}W & \xrightarrow[\pi]{} & B. \end{array}$$

Suppose W is trivialized on some open by sections e_0 and e_1. Let $x_i = p^*e_i^\vee$ and $y_i = q^*e_i^\vee$ be the pullbacks of the corresponding dual sections of $\mathcal{O}_{\mathbb{P}W}(1)$. Then Δ is cut out locally by the vanishing of $(x_0e_0 + x_1e_1) \wedge (y_0e_0 + y_1e_1) = (x_0y_1 - x_1y_0)e_0 \wedge e_1$. This globalizes to realize Δ as the vanishing of a section of $q^*\pi^* \det W \otimes p^*\mathcal{O}_{\mathbb{P}W}(1) \otimes q^*\mathcal{O}_{\mathbb{P}W}(1)$. (In fact, this is the unique functorial construction of a line bundle of the correct degrees on fibers of p and q which is unaffected by twisting W by a line bundle on the base.) In particular, we have an exact sequence

$$0 \to q^*\pi^*L \otimes p^*\mathcal{O}_{\mathbb{P}W}(-1) \otimes q^*\mathcal{O}_{\mathbb{P}W}(-1) \to \mathcal{O}_X \to \mathcal{O}_\Delta \to 0.$$

Following Strømme, we tensor with p^*E, apply q_* and use the projection formula to obtain a long exact sequence on $\mathbb{P}W$:

$$\begin{aligned} 0 &\to \pi^*L \otimes q_*p^*E(-1) \otimes \mathcal{O}_{\mathbb{P}W}(-1) \to q_*p^*E \to E \\ &\to \pi^*L \otimes R^1q_*p^*E(-1) \otimes \mathcal{O}_{\mathbb{P}W}(-1) \to R^1q_*p^*E \to 0. \end{aligned} \tag{2.3.1}$$

By the theorem on cohomology and base change, $R^1q_*p^*E(-1) = \pi^*R^1\pi_*E(-1) = 0$, so the first row is exact. Similarly, $q_*p^*E(-1) = \pi_*\pi^*E(-1)$ and $q_*p^*E = \pi^*\pi_*E$, producing the desired sequence. □

We will apply Lemma 2.3.1 to suitable twists of vector bundles E. For an integer m, the condition $R^1\pi_*E(m-1) = 0$ is equivalent to the condition that the restriction of $E(m)$ to each fiber is globally generated, which in turn is equivalent to saying all summands of $E(m)$ restricted to any fiber are non-negative. In this case, taking the dual of the sequence in Lemma 2.3.1 expresses $E^\vee(-m)$ as the kernel of a map between twists of pullbacks of vector bundles from the base:

$$0 \to E^\vee(-m) \to \pi^*(\pi_*E(m))^\vee \xrightarrow{\psi} \pi^*(L \otimes \pi_*E(m-1))^\vee(1) \to 0. \tag{2.3.2}$$

Pushing forward, and recalling that $\pi_* \mathcal{O}_{\mathbb{P}W}(1) \cong W^\vee$, we obtain

$$0 \to \pi_* E^\vee(-m) \to (\pi_* E(m))^\vee \xrightarrow{\pi_*\psi} \pi^*(L \otimes \pi_* E(m-1))^\vee \otimes W^\vee \to R^1\pi_* E^\vee(-m) \to 0. \tag{2.3.3}$$

Sections 2.5 and 2.6 take advantage of these sequences to compute classes of splitting loci.

The proof of Lemma 2.3.1 also relates push forwards of various twists of E in the K-theory of B. Let $R\pi_* E$ denote the derived push forward $[\pi_* E] - [R^1\pi_* E]$ in $K(B)$. In addition, we define the following class in K theory depending only on W

$$\Theta(m) := \sum_{i=0}^{\lfloor m/2 \rfloor} (-1)^i \binom{m-i}{i} W^{\vee \otimes m-2i} \otimes L^{\vee \otimes m-i} \in K(B).$$

Note that $\operatorname{rank} \Theta(m) = m+1$, where rank is understood to extend linearly to K-theory.

Lemma 2.3.2. *If E is a vector bundle on $\pi : \mathbb{P}W \to B$, then*

$$R\pi_* E(-1) = R\pi_* E \otimes W^\vee \otimes L^\vee - R\pi_* E(1) \otimes L^\vee$$

in $K(B)$. More generally, by induction it follows that

$$R\pi_* E(-1) = (\Theta(m+1) - \Theta(m) \otimes W^\vee \otimes L^\vee) \otimes R\pi_* E(m) + \Theta(m) \otimes R\pi_* E(m-1).$$

Remark 2.3.3. The advantage of the second expression is that for suitably large m, $R\pi_* E(m) = \pi_* E(m)$ and $R\pi_* E(m-1) = \pi_* E(m-1)$ are vector bundles on B.

Proof. The first statement follows from tensoring (2.3.1) by $\mathcal{O}_{\mathbb{P}W}(1)$ and pushing forward to B. Setting $b_i = R\pi_* E(m-i)$, $x = W^\vee \otimes L^\vee$, and $y = -L^\vee$, we obtain a two-term recurrence relation of the form $b_{m+1} = x b_m + y b_{m-1}$ for all m. Packaging these in a generating function $f(t) = \sum_{i \geq 0} b_i t^i$, we see

$$f(t) = \frac{b_0 + b_1 t - x t b_0}{1 - xt - yt^2} \qquad \Rightarrow \qquad b_{m+1} = \theta(m) b_0 + \theta(m-1)(b_1 - x t b_0),$$

where $\theta(m)$ denotes the coefficient of degree m in the rational function $\frac{1}{1-xt-yt^2}$. Solving for this coefficient recovers our definition of $\Theta(m)$. □

In [70], Strømme describes an embedding of the Quot scheme of a trivial vector bundle on $\mathbb{P}^1$ into a product of Grassmannians. We require a generalization to relative Quot schemes over $\mathbb{P}^1$ bundles. Let $\mathcal{F}$ be a vector bundle on U and $\pi : \mathbb{P}W \to U$ a $\mathbb{P}^1$ bundle. Given some Hilbert polynomial, let $\mathrm{Quot}_{\pi^*\mathcal{F}}$ denote the relative Quot scheme of $\pi^*\mathcal{F}$ over $\mathbb{P}W \to U$.

The scheme $\mathrm{Quot}_{\pi^*\mathcal{F}}$ can be thought of as a fiber bundle over U where the fiber over $b \in U$ is Strømme's corresponding Quot scheme of the trivial bundle $\mathcal{F}_b \otimes \mathcal{O}_{\mathbb{P}W_b}$ on $\mathbb{P}W_b$. Let us label maps in the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Quot}_{\pi^*\mathcal{F}} \times_U \mathbb{P}W & \xrightarrow{q} & \mathbb{P}W \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathrm{Quot}_{\pi^*\mathcal{F}} & \xrightarrow[\gamma]{} & U. \end{array} \tag{2.3.4}$$

Then $\mathrm{Quot}_{\pi^*\mathcal{F}} \times_U \mathbb{P}W$ is equipped with a tautological sequence

$$0 \to \mathcal{S} \to q^*\pi^*\mathcal{F} \to \mathcal{Q} \to 0, \tag{2.3.5}$$

where $\mathcal{Q}$ is flat over $\mathrm{Quot}_{\pi^*\mathcal{F}}$. Let $-d$ be the degree of $\mathcal{S}$ restricted to a fiber of p. For each $m \geq d-1$, tensoring (2.3.5) with $q^*\mathcal{O}_{\mathbb{P}W}(m)$ and pushing forward to $\mathrm{Quot}_{\pi^*\mathcal{F}}$ gives rise to a natural injection

$$p_*\mathcal{S}(m) \hookrightarrow p_*(q^*(\pi^*\mathcal{F})(m)) = \gamma^*(\mathcal{F} \otimes \mathrm{Sym}^m W^\vee). \tag{2.3.6}$$

This induces a map of $\mathrm{Quot}_{\pi^*\mathcal{F}}$ to a corresponding Grassmann bundle over U. The following generalizes [70, Thm. 4.1].

Theorem 2.3.4. *Let* $\mathrm{Quot}_{\pi^*\mathcal{F}}$ *be the relative Quot scheme of* $\pi^*\mathcal{F}$ *over* $\pi : \mathbb{P}W \to U$ *for some Hilbert polynomial and let* p *be as in* (2.3.4). *Let* $-d$ *be the relative degree of the tautological subbundle* $\mathcal{S}$ *and let* $r_{d-1} = \mathrm{rank}\, p_*\mathcal{S}(d-1)$ *and* $r_d = \mathrm{rank}\, p_*\mathcal{S}(d)$. *There is an embedding*

$$\begin{array}{ccc} \mathrm{Quot}_{\pi^*\mathcal{F}} & \xrightarrow{\iota} & G(r_{d-1}, \mathcal{F} \otimes Sym^{d-1}W^\vee) \times G(r_d, \mathcal{F} \otimes Sym^d W^\vee) \\ & {\searrow}^{\gamma} & \downarrow{\scriptstyle \rho} \\ & & U \end{array}$$

such that the tautological subbundles S_{d-1} *and* S_d *on the Grassmann bundles on the right restrict to* $p_*\mathcal{S}(d-1)$ *and* $p_*\mathcal{S}(d)$. *Moreover, the image of* $\mathrm{Quot}_{\pi^*\mathcal{F}}$ *has class*

$$[\mathrm{Quot}_{\pi^*\mathcal{F}}] = c_{top}(S_{d-1}^\vee \otimes Q_d \otimes \rho^*(W^\vee \otimes L^\vee)),$$

where Q_d *denotes the tautological quotient bundle on the second factor Grassmann bundle.*

Proof. To see the map is an embedding, it suffices to check on fibers of $\mathrm{Quot}_{\pi^*\mathcal{F}} \to U$, which reduces us to Strømme's setting. To determine the image, Strømme uses a relationship between the maps in (2.3.6) for adjacent twists. Tensoring the sequence in Lemma 2.3.1 by $\mathcal{O}_{\mathbb{P}W}(1)$ and pushing forward gives rise to a natural map $\pi_* E(-1) \to \pi_* E \otimes W^\vee \otimes L^\vee$. In the case that $E = \pi^* \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}W}(m)$ for some vector bundle $\mathcal{F}$ on U and $m \geq 1$, this gives a map

$$\mathcal{F} \otimes \operatorname{Sym}^{m-1} W^\vee \to \mathcal{F} \otimes \operatorname{Sym}^m W^\vee \otimes W^\vee \otimes L^\vee.$$

The only modification needed in Strømme's proof is that his natural map j_m on page 262 should be replaced with the above. This results in replacing Strømme's 2-dimensional vector space H by the rank 2 vector bundle $\rho^*(W^\vee \otimes L^\vee)$ throughout the remainder of his Section 4. His proof then shows $\mathrm{Quot}_{\pi^*\mathcal{F}}$ is the zero locus of a natural map $S_{d-1} \to Q_d \otimes \rho^*(W^\vee \otimes L^\vee)$ on the product of Grassmann bundles, proving the formula for its class. □

2.4 The tangent space to splitting loci

In this section, we describe the tangent spaces to splitting degeneracy schemes and show they satisfy a certain minimality property. We also provide an alternative description of the tangent space that will appear in Section 2.6. Recall that, as a scheme, we have defined

$$\Sigma_{\vec{e}}(E) = \bigcap_m \{b \in B : \dim(R^1\pi_* E(m))_b \geq h^1(\mathcal{O}(\vec{e})(m))\},$$

where the schemes in the intersection on the right are defined by the appropriate Fitting ideals of $R^1\pi_* E(m)$.

Let $T = \operatorname{Spec} k[\epsilon]/(\epsilon^2)$. For $b \in B$, let $\mathrm{Mor}_b(T, B)$ denote the space of morphisms $T \to B$ sending the reduced point $0 = \operatorname{Spec} k \subset T$ to b. Given a vector bundle $\mathcal{E}$ on $\mathbb{P}^1 \times T$, we write $\mathcal{E}_0$ for the restriction to $\mathbb{P}^1 \times 0$. Given any $v : T \to B$, we have a fibered diagram

$$\begin{array}{ccc}
v'^* E & \longrightarrow & E \\
\big\downarrow & & \big\downarrow \\
T \times \mathbb{P}^1 & \xrightarrow{v'} & \mathbb{P}W \\
{\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
T & \xrightarrow[v]{} & B.
\end{array}$$

There is a natural map on tangent spaces

$$\delta_{E,b} : T_b B = \mathrm{Mor}_b(T, B) \to \mathrm{Def}(E|_{\pi^{-1}(b)}) = H^1(\mathcal{E}nd(E|_{\pi^{-1}(b)}))$$

that sends a map $v : T \to B$ to the induced first order deformation $v'^* E$. The tangent space to any scheme structure on a splitting locus of E contains $\ker(\delta_{E,b})$. We demonstrate that our schemes satisfy the following minimality property.

Lemma 2.4.1. *For $b \in \Sigma^\circ_{\vec{e}}(E) \subset \Sigma_{\vec{e}}(E)$, the tangent space is*

$$T_b\Sigma_{\vec{e}}(E) = \ker(\delta_{E,b}).$$

Remark 2.4.2. In [22, Conj. 5.1], Eisenbud-Shreyer conjecture that $\Sigma_{\vec{e}}(E)$ is reduced in the case where B is a versal deformation space of $\mathcal{O}^{\oplus r-1} \oplus \mathcal{O}(d)$. In this universal setting, $\delta_{E,b}$ is surjective for all $b \in \Sigma^\circ_{\vec{e}}(E)$, so Lemma 2.4.1 shows that $\Sigma_{\vec{e}}(E)$ is smooth along $\Sigma^\circ_{\vec{e}}(E)$. However, this does not rule out the possibility of embedded points along the more unbalanced locus $\Sigma_{\vec{e}}(E)\backslash\Sigma^\circ_{\vec{e}}(E)$. For the purposes of computing classes of degeneracy loci, our assumption that the more unbalanced locus $\Sigma_{\vec{e}}(E)\backslash\Sigma^\circ_{\vec{e}}(E)$ occurs in higher codimension means this subtlety does not affect the class.

Later, in Corollary 4.1.2, we shall prove [22, Conj. 5.1] by showing the stronger statement that, when B is a versal deformation space, $\Sigma_{\vec{e}}(E)$ is Cohen–Macaulay. A generically reduced Cohen–Macaulay scheme is always reduced. Since the above argument already shows the scheme is generically reduced, this will imply [22, Conj. 5.1].

First let us identify the tangent space to $\Sigma_{\vec{e}}(E)$.

Lemma 2.4.3. *Let $b \in \Sigma^\circ_{\vec{e}}(E) \subset \Sigma_{\vec{e}}(E)$. The tangent space to $\Sigma_{\vec{e}}(E)$ is*

$$T_b\Sigma_{\vec{e}}(E) = \{v \in \mathrm{Mor}_b(T,B) : R^1\pi'_*(v'^*E)(m) \text{ is free of rank } h^1(\mathcal{O}(\vec{e})(m)) \ \forall\ m\}.$$

Proof. Let $F \to G \to R^1\pi_*E(m)$ be a locally free resolution on B. If $v : T \to \Sigma_{\vec{e}}(E)$, then $v^*F \to v^*G \to v^*R^1\pi_*E(m)$ is a free resolution and the appropriate minors of $v^*F \to v^*G$ vanish on all of T. Some minor one size smaller is nonzero at the reduced point, hence a unit. Thus, the cokernel $v^*R^1\pi_*E(m)$ is free of the correct rank. Cohomology and base change shows that $v^*R^1\pi_*E(m) = R^1\pi'_*v'^*E(m)$. □

Lemma 2.4.1 is now implied by the following.

Lemma 2.4.4. *Let E be a vector bundle on $\mathbb{P}^1 \times T$ with $E_0 \cong \mathcal{O}(\vec{e})$. Label the projections*

$$\begin{array}{ccc} & \mathbb{P}^1 \times T & \\ {}^{\alpha}\swarrow & & \searrow^{\pi} \\ \mathbb{P}^1 & & T \end{array}$$

Suppose furthermore that $R^1\pi_*E(-m)$ is locally free of rank $h^1(\mathcal{O}(\vec{e})(-m))$ for all $m \geq \min\{e_i\}$. Then $E \cong \alpha^*\mathcal{O}(\vec{e})$ is the trivial deformation.

Proof. We induct on the rank of E. If E is balanced there is nothing to prove, as every deformation is trivial. After twisting, we may write $E_0 \cong \mathcal{O}(\vec{e}) = \mathcal{O}(-1)^{\oplus i} \oplus \mathcal{O}(\vec{a})$ where every summand of $\mathcal{O}(\vec{a})$ is nonnegative. As all summands of E_0 have degree > -2, cohomology and base change shows that $(R^1\pi_*E)_0 = 0$ and hence $R^1\pi_*E = 0$. By hypothesis, $R^1\pi_*E(-1)$ is free of rank i. By the proof of Lemma 2.3.1, there is a long exact sequence on T,

$$0 \to \pi^*(\pi_*E(-1))(-1) \to \pi^*\pi_*E \to E \to \pi^*(R^1\pi_*E(-1))(-1) \to \pi^*R^1\pi_*E \to 0.$$

In particular, we have a surjection $E \to \alpha^*\mathcal{O}(-1)^{\oplus i}$. Let F denote the kernel, which is locally free of lower rank with $F_0 \cong \mathcal{O}(\vec{a})$. For each $m \geq 0$, we have an exact sequence on $\mathbb{P}^1 \times T$

$$0 \to F(-m) \to E(-m) \to \alpha^*\mathcal{O}(-m-1) \to 0,$$

which pushes forward to give a sequence of vector bundles on T:

$$0 \to R^1\pi_*F(-m) \to R^1\pi_*E(-m) \to R^1\pi_*\alpha^*\mathcal{O}(-m-1)^{\oplus i} \to 0.$$

By hypothesis, $R^1\pi_*E(-m)$ is free of rank $h^1(\mathcal{O}(\vec{e})(-m))$. Meanwhile, the last term $R^1\pi_*\alpha^*\mathcal{O}(-m-1)^{\oplus i}$ is free of rank $h^1(\mathcal{O}(-1)^{\oplus i}(-m))$. It follows that $R^1\pi_*F(-m)$ is free of rank $h^1(\mathcal{O}(\vec{e})(-m)) - h^1(\mathcal{O}(-1)^{\oplus i}(-m)) = h^1(\mathcal{O}(\vec{a})(-m))$. By induction, $F \cong \alpha^*\mathcal{O}(\vec{a})$ is the trivial deformation. Now we see

$$0 \to \alpha^*\mathcal{O}(\vec{a}) \to E \to \alpha^*\mathcal{O}(-1)^{\oplus i} \to 0. \tag{2.4.1}$$

Finally, we have

$$H^1(T, Hom(\alpha^*\mathcal{O}(-1)^{\oplus i}, \alpha^*\mathcal{O}(\vec{a}))) = \bigoplus_{j=1}^{r-i} H^1(T, \alpha^*\mathcal{O}(a_j - 1))^{\oplus i} = 0,$$

so (2.4.1) must split. □

We also require another description of the tangent space.

Lemma 2.4.5. *Suppose $b \in \Sigma^\circ_{\vec{e}}(E)$. Write $\mathcal{O}(\vec{e}) = \mathcal{O}(-m_0)^{\oplus i} \oplus \alpha^*\mathcal{O}(\vec{a})$ where $a_i > -m_0$ for all i. Then*

$$\ker(\delta_{E,b}) = \{v \in \mathrm{Mor}_b(T, B) : \alpha^*\mathcal{O}(\vec{a}) \textit{ is a subsheaf of } v^*E\}$$

Proof. The left hand side is automatically contained in the right hand side. After an overall twist, we may assume that $m_0 = 0$. Now suppose $\alpha^*O(\vec{a})$ is a subsheaf of ν^*E. Let Q be the quotient, so we have a short exact sequence over $T \times \mathbb{P}^1$:

$$0 \to \alpha^*O(\vec{a}) \to E \to Q \to 0. \tag{2.4.2}$$

The $O(\vec{a})$ subsheaf of $E_0 = O(\vec{e})$ is unique and is a subbundle. It follows that Q is locally free and $Q_0 \cong O^{\oplus i}$. Hence $Q \cong \alpha^*O^{\oplus i}$ is trivial. Now, because all summands of $O(\vec{a})$ are positive, $H^1(T, Hom(\alpha^*O^{\oplus i}, \alpha^*O(\vec{a}))) = 0$, showing that (2.4.2) splits. □

2.5 Certain degeneracy classes

Suppose E is a degree ℓ, rank r vector bundle on $\mathbb{P}W \to B$. As before, let $L = \det W^\vee$. Finding the splitting loci of E is the same problem as finding splitting loci of twists $E(i)$, so from now on, we assume $0 \leq \ell < r$. We start by computing the classes of degeneracy loci of the form $\Sigma_{(-m,*,\ldots,*)}$, where $*$'s indicate a balanced remainder, i.e.

$$(-m, *, \ldots, *) = \left(-m, \left\lfloor \frac{\ell + m}{r - 1} \right\rfloor, \ldots, \left\lceil \frac{\ell + m}{r - 1} \right\rceil\right).$$

The expected codimension of $(-m, *, \ldots, *)$ is

$$h^1(\mathbb{P}^1, End(O(-m, *, \ldots, *))) = h^1(\mathbb{P}^1, Bal^\vee(-m)) = r(m - 1) + \ell + 1,$$

where Bal denotes the balanced bundle of rank $r - 1$ and degree $\ell + m$.

Assuming $\Sigma_{(-m-1,*,\ldots,*)}$ occurs in higher codimension than $\Sigma_{(-m,*,\ldots,*)}$, excision (see e.g. [21, Prop. 1.14]) allows us to calculate the class of $\Sigma_{(-m,*,\ldots,*)}$ on the open set $U_m = B \backslash \Sigma_{(-m-1,*,\ldots,*)}$. By the theorem on cohomology and base change, over U_m, the pushforwards $\pi_*E(m)$ and $\pi_*E(m - 1)$ are locally free of rank $(m + 1)r + \ell$ and $mr + \ell$ respectively. Let $\mathcal{F} := (\pi_*E(m))^\vee$ and $\mathcal{G} := (L \otimes \pi_*E(m - 1))^\vee$. Now equation (2.3.3) becomes

$$0 \to \pi_*E^\vee(-m) \to \mathcal{F} \xrightarrow{\pi_*\psi} \mathcal{G} \otimes W^\vee \to R^1\pi_*E^\vee(-m) \to 0.$$

We have that $\Sigma_{(-m,*,\ldots,*)}$ is precisely the locus where $\pi_*\psi : \mathcal{F} \to \mathcal{G} \otimes W^\vee$ fails to be injective on fibers. The expected codimension of this locus as a degeneracy locus of a map of vector bundles is

$$\operatorname{rank}(\mathcal{G} \otimes W^\vee) - \operatorname{rank} \mathcal{F} + 1 = r(m - 1) + \ell + 1.$$

Therefore, applying Porteous' formula (see e.g. [21, Thm. 12.4]) proves the following.

Lemma 2.5.1. *If* $\operatorname{codim}\Sigma_{(-m,*,\dots,*)} = r(m-1)+\ell+1$ *and* $\operatorname{codim}\Sigma_{(-m-1,*,\dots,*)} > r(m-1)+\ell+1$*, then*

$$[\Sigma_{(-m,*,\dots,*)}] = \left[\frac{c(\mathcal{G}\otimes W^\vee)}{c(\mathcal{F})}\right]_{r(m-1)+\ell+1} \tag{2.5.1}$$

where we formally invert the denominator and the subscript indicates that we take the component of the resulting class in that degree.

Remark 2.5.2. One might hope more generally to compute the class of $\Sigma_{(-m^i,*,\dots,*)}$ as the locus where $\dim\ker\psi \geq i$. The codimension is correct to apply Porteous' formula, and this does indeed yield the class of $\Sigma_{(-m^i,*,\dots,*)}$ *restricted to* U_m. However, in general, $\operatorname{codim} U_m^c$ may be smaller than $\operatorname{codim}\Sigma_{(-m^i,*,\dots,*)}$, so the class can contain contributions from U_m^c. Nevertheless, this is approach is helpful if one has a family that is bounded in some way (see for example [49, Section 4]).

2.6 The inductive algorithm

In general, splitting loci are determined by a sequence of cohomological conditions on the fibers, but the conditions are not transverse (see Example 2.2.3). This indicates that we need something more refined than Porteous calculations on the base.

The Porteous formula finds the class of where a map of vector bundles drops rank by pulling back to a Grassmann bundle, computing the class of where the universal subbundle includes into the kernel and pushing forward the result. To get an algorithm for arbitrary splitting types, instead of tracking degeneracy of the map $\pi_*\psi$ in (2.3.3), we need to track high degree subsheaves of the kernel of the map ψ in (2.3.2). We do this by pulling back to an appropriate relative Quot scheme. Utilizing Theorem 2.3.4, our answer also winds up being a pushforward of natural classes on a product of Grassmann bundles (and reduces to the Porteous formula in the special case).

Algorithm and Proof of Theorem 2.1.2

Fix some splitting type $\vec{e}$. We assume that $\Sigma_{\vec{e}}^\circ$ is codimension $u(\vec{e})$ and $Y = \Sigma_{\vec{e}}\backslash\Sigma_{\vec{e}}^\circ$ has codimension greater than $u(\vec{e})$. Inductively, we can assume we know a formula for the expected classes of splitting loci for lower rank bundles, in terms of Chern classes of pushforwards of twists of the vector bundle. As before, let $U_m = B\backslash\Sigma_{(-m-1,*,\dots,*)}$. Fix m large enough that $(\mathcal{O}(\vec{e}))(m)$ is globally generated and $\operatorname{codim} U_m^c > h^1(End(\mathcal{O}(\vec{e}))$. We will carry out our calculation of the class of $\Sigma_{\vec{e}}$ on $U = U_m\backslash Y$, allowing us to assume $\Sigma_{\vec{e}} = \Sigma_{\vec{e}}^\circ$. The result will hold on all of B by excision.

Let $\mathcal{F} := (\pi_* E(m))^\vee$ and $\mathcal{G} := (L \otimes \pi_* E(m-1))^\vee$ be the vector bundles on U as in the previous section. Then (2.3.2) becomes the exact sequence

$$0 \to E^\vee(-m) \xrightarrow{\psi} \pi^*\mathcal{F} \to (\pi^*\mathcal{G})(1) \to 0 \tag{2.6.1}$$

on $\mathbb{P}W$. Finding where E has splitting type $O(\vec{e})$ is the same finding where $E^\vee(-m)$ has splitting type $O(\vec{e})^\vee(-m)$. Let us write $O(\vec{e})^\vee(-m) = O(-m_0)^{\oplus i} \oplus O(\vec{a})$ where each $a_j > -m_0$. Let $d = -\deg O(\vec{a})$ and $s = \operatorname{rank} O(\vec{a}) = r - i$. Any vector bundle admitting a subsheaf of splitting type $\vec{a}$ is at least as unbalanced as $O(\vec{e})^\vee(-m)$. Therefore, our splitting locus is also described as

$$\Sigma_{\vec{e}} = \{b : \text{ there exists } O(\vec{a}') \hookrightarrow \ker \psi_b \text{ for } \vec{a}' \leq \vec{a}\}.$$

To describe the latter, let $\mathrm{Quot}_{\pi^*\mathcal{F}}$ be the relative Quot scheme of $\pi^*\mathcal{F}$ over $\mathbb{P}W \to U$ parametrizing quotients with Hilbert polynomial $h(n) = (\operatorname{rank}\mathcal{F} - s)(n+1) + d$. On a curve, a subsheaf of a locally free sheaf is locally free, so this is equivalent to parametrizing locally free subsheaves of rank s and degree $-d$. Thus, we think of $\mathrm{Quot}_{\pi^*\mathcal{F}}$ as a fiber bundle over U where the fiber over $b \in U$ is the Quot scheme parametrizing all quotients of $\mathcal{F}_b \otimes O_{\mathbb{P}W_b}$ where the subsheaf has rank and degree equal to $O(\vec{a})$. Let us label maps of the fiber product as in (2.3.4) and the tautological bundles as in (2.3.5).

Consider the composition

$$\phi : \mathcal{S} \to q^*\pi^*\mathcal{F} \xrightarrow{q^*\psi} q^*(\pi^*\mathcal{G})(1).$$

We have $\mathcal{S}_b \hookrightarrow (\ker\psi)_b$ when ϕ_b is the zero map. In other words, when ϕ vanishes, considered as a section of the vector bundle

$$p_* Hom(\mathcal{S}, q^*(\pi^*\mathcal{G})(1)) = p_*(\mathcal{S}^\vee \otimes O(1) \otimes p^*\gamma^*\mathcal{G}) = p_*(\mathcal{S}^\vee(1)) \otimes \gamma^*\mathcal{G},$$

which is locally free by the theorem on cohomology and base change. Let σ denote the top Chern class of this vector bundle and let $Z_{\vec{a}}$ be the closure of the splitting locus in $\mathrm{Quot}_{\pi^*\mathcal{F}}$ over which $\mathcal{S}$ splits as $O(\vec{a})$. The splitting loci of $\mathcal{S}$ on each fiber of $\mathrm{Quot}_{\pi^*\mathcal{F}} \to U$ are the splitting loci on Strømme's Quot scheme. These are all described as quotients of open subsets of $Hom(O(\vec{a}'), \mathcal{F}_b \otimes O_{\mathbb{P}^1})$ by $Aut(O(\vec{a}'))$ and hence occur in the expected codimension.

Lemma 2.6.1. *We have* $[\Sigma_{\vec{e}}] = \gamma_*(\sigma \cdot [Z_{\vec{a}}])$.

Proof. With our assumption $\Sigma_{\vec{e}} = \Sigma_{\vec{e}}^\circ$, any subsheaf of $\ker\psi_b$ of splitting type $\vec{a}' \leq \vec{a}$ is unique (and actually $\vec{a}' = \vec{a}$). Thus, by construction, γ sends $V(\phi) \cap Z_{\vec{a}}$ one-to-one onto $\Sigma_{\vec{e}}$.

Lemmas 2.4.1 and 2.4.5 show that γ is an isomorphism on tangent spaces. In particular, $[\Sigma_{\vec{e}}] = \gamma_*([V(\phi) \cap Z_{\vec{a}}])$. It also follows that

$$\begin{aligned}\operatorname{codim} V(\phi) \cap Z_{\vec{a}} &= u(\vec{e}) + \operatorname{fiberdim}(\gamma) \\ &= \operatorname{codim} Z_{\vec{a}} + h^1(\mathbb{P}^1, End(\mathcal{O}(-m_0)^i \oplus Bal)) + \operatorname{fiberdim}(\gamma),\end{aligned}$$

where Bal denotes the balanced bundle of rank s and degree $-d$. The fibers of γ have dimension $h^0(\mathbb{P}^1, Hom(Bal, \mathcal{O}^{\oplus \operatorname{rank} \mathcal{F}}) - h^0(\mathbb{P}^1, End(Bal))$. Therefore, the codimension of $V(\phi)$ inside $Z_{\vec{a}}$ is

$$h^1(\mathbb{P}^1, Hom(Bal, \mathcal{O}(-m_0)^{\oplus i})) + h^0(\mathbb{P}^1, Hom(Bal, \mathcal{O}^{\oplus \operatorname{rank} \mathcal{F}})) - h^0(\mathbb{P}^1, End(Bal)). \tag{2.6.2}$$

Now apply $Hom(Bal, -)$ to the exact sequence

$$0 \to \mathcal{O}(-m_0)^{\oplus i} \oplus Bal \to \mathcal{O}^{\oplus \operatorname{rank} \mathcal{F}} \to \mathcal{O}(1)^{\oplus \operatorname{rank} \mathcal{G}} \to 0$$

on $\mathbb{P}^1$ and use the long exact sequence in cohomology to see that (2.6.2) is equal to $h^0(\mathbb{P}^1, Hom(Bal, \mathcal{O}(1)^{\oplus \operatorname{rank} \mathcal{G}})) = \operatorname{rank}(p_*(\mathcal{S}^\vee(1)) \otimes \gamma^* \mathcal{G})$. This shows that $Z_{\vec{a}}$ and $V(\phi)$ meet in the expected codimension. Let us write $V(\phi) = V_0 \cup V_1$ where V_0 is the expected codimension and every component of V_1 has strictly larger dimension. Then σ differs from $[V_0]$ by a class supported in V_1, and $V_1 \cap Z_{\vec{a}} = \varnothing$. Therefore,

$$\sigma \cdot [Z_{\vec{a}}] = [V_0] \cdot [Z_{\vec{a}}] = [V_0 \cap Z_{\vec{a}}] = [V(\phi) \cap Z_{\vec{a}}]$$

and the result follows. □

To compute the pushforward in Lemma 3.2.5, we use Theorem 2.3.4 to embed $\operatorname{Quot}_{\pi^*\mathcal{F}}$ into a product of Grassmann bundles and adopt the notation of that diagram. Our goal is to express σ and $[Z_{\vec{a}}]$ as pullbacks of natural classes under ι^*. First, consider the Chern classes of $(p_*\mathcal{S}^\vee(1))^\vee$. Using Serre duality and Lemma 2.3.2, we have the following equality in K-theory:

$$\begin{aligned}(p_*\mathcal{S}^\vee(1))^\vee &= R^1 p_*(\mathcal{S}(-1) \otimes q^* \mathcal{O}_{\mathbb{P}W}(-2) \otimes p^* \gamma^* L) = -Rp_*\mathcal{S}(-3) \otimes \gamma^* L \\ &= (\Theta(d+2) \otimes W^\vee - \Theta(d+3) \otimes L) \otimes p_*\mathcal{S}(d) - \Theta(d+2) \otimes p_*\mathcal{S}(d-1) \otimes L.\end{aligned} \tag{2.6.3}$$

In the second line, we have used $Rp_*\mathcal{S}(d-1) = p_*\mathcal{S}(d-1)$ and the pullbacks by γ are implicit. This determines a polynomial β_d in the Chern classes of W, S_{d-1}, S_d such that

$c((p_*\mathcal{S}^\vee(1))^\vee) = \iota^*\beta_d$. For example, in the case where W is trivial, (2.6.3) simplifies to

$$(p_*\mathcal{S}^\vee(1))^\vee = (d+2)p_*\mathcal{S}(d) - (d+3)p_*\mathcal{S}(d-1) \qquad \text{from which} \qquad \beta_d = \frac{c(S_d)^{d+2}}{c(S_{d-1})^{d+3}},$$

where we formally invert the denominator. To obtain an expression for σ, we use a formula for the top Chern class of a tensor product of vector bundles (see e.g. [21, Cor. 12.3]). Noting that $\operatorname{rank}(p_*\mathcal{S}^\vee(1))^\vee = d + 2s$, this gives

$$\sigma = c_{top}(p_*(\mathcal{S}^\vee(1)) \otimes \gamma^*\mathcal{G}) = \Delta^{d+2s}_{\operatorname{rank}\mathcal{G}}\left(\frac{c(\gamma^*\mathcal{G})}{c((p_*(\mathcal{S}^\vee(1)))^\vee)}\right) = \iota^*\Delta^{d+2s}_{\operatorname{rank}\mathcal{G}}\left(\frac{c(\rho^*\mathcal{G})}{\beta_d}\right). \quad (2.6.4)$$

Above, Δ^a_b denotes the standard determinantal class: given an input class x in Chow, let x_i be the component in degree i and define

$$\Delta^a_b(x) = \Delta_{b,\dots,b}(x) = \det\begin{pmatrix} x_b & x_{b+1} & \cdots & x_{b+a-1} \\ x_{b-1} & x_b & \cdots & x_{b+a-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{b-a+1} & x_{b-a+2} & \cdots & x_b \end{pmatrix}.$$

Since $\operatorname{rank}\mathcal{S} < r$, by our inductive hypothesis, we can assume we know a formula for $[Z_{\vec{a}}]$ in terms of Chern classes of bundles $p_*\mathcal{S}(i)$. Using Lemma 2.3.2, we can write $[Z_{\vec{a}}] = \iota^*\alpha$ where α is some polynomial in the Chern classes of S_{d-1}, S_d and W. Finally, using push-pull and the class of $\operatorname{Quot}_{\pi^*\mathcal{F}}$ given in Theorem 2.3.4, we have

$$\begin{aligned}
[\Sigma_{\vec{e}}] &= \gamma_*(\sigma \cdot [Z_{\vec{a}}]) = \rho_*\iota_*(\sigma \cdot [Z_{\vec{a}}]) \\
&= \rho_*\iota_*\left(\iota^*\Delta^{d+2s}_{\operatorname{rank}\mathcal{G}}\left(\frac{c(\rho^*\mathcal{G})}{\beta_d}\right) \cdot \iota^*\alpha\right) \\
&= \rho_*\left([\operatorname{Quot}_{\pi^*\mathcal{F}}] \cdot \Delta^{d+2s}_{\operatorname{rank}\mathcal{G}}\left(\frac{c(g^*\mathcal{G})}{\beta_d}\right) \cdot \alpha\right) \\
&= \rho_*\left(c_{top}(S^\vee_{d-1} \otimes Q_d \otimes \rho^*(W^\vee \otimes L^\vee)) \cdot \Delta^{d+2s}_{\operatorname{rank}\mathcal{G}}\left(\frac{c(\rho^*\mathcal{G})}{\beta_d}\right) \cdot \alpha\right). \qquad (2.6.5)
\end{aligned}$$

The expression we need to push forward in (2.6.5) can be solved for explicitly in terms of the pullbacks of Chern classes of $W, \mathcal{F}$, and $\mathcal{G}$ and the Chern classes of the tautological bundles. The push forwards of all polynomials in the Chern classes of S_d and S_{d-1} are polynomials in the Chern classes of $\mathcal{F}$ and W, determined by [40, Cor. 2.6]. Thus, we have all the necessary ingredients to compute the classes of splitting loci in terms of the Chern classes of W, $\mathcal{F} = (\pi_*E(m))^\vee$ and $\mathcal{G} = (L \otimes \pi_*E(m-1))^\vee$.

Example 2.6.2 (Splitting type $(-2,0,2)$, Example 2.2.3 revisited)**.** We explain how to find the class of $\Sigma_{(-2,0,2)}$ using the general algorithm, supposing W is trivial for simplicity. The stratum $\Sigma_{(-2,0,2)}$ is codimension 5 so we may take $m = 2$. We have

$$\mathcal{O}((-2,0,2))^\vee(-2) = \mathcal{O}(-4) \oplus \mathcal{O}(-2) \oplus \mathcal{O},$$

so $\vec{a} = (-2,0)$. On U, the bundle $\mathcal{F} = (\pi_* E(2))^\vee$ has rank 9 and $\mathcal{G} = (\pi_* E(1))^\vee$ has rank 6. We form the relative Quot scheme $\mathrm{Quot}_{\pi^*\mathcal{F}}$, parametrizing rank 2, degree -2 subsheaves of $\mathcal{F}$ on the fibers of $\mathbb{P}^1 \times U \to U$. The bundle $p_*\mathcal{S}(1)$ has rank 2 and $p_*\mathcal{S}(2)$ has rank 4, so Theorem 2.3.4 embeds $\mathrm{Quot}_{\pi^*\mathcal{F}}$ into the product of Grasmmann bundles

$$\iota : \mathrm{Quot}_{\pi^*\mathcal{F}} \hookrightarrow G(2, \mathcal{F}^{\oplus 2}) \times_U G(4, \mathcal{F}^{\oplus 3}),$$

where the universal bundle S_1 (resp. S_2) restricts to $p_*\mathcal{S}(1)$ (resp. $p_*\mathcal{S}(2)$). Moreover,

$$[\mathrm{Quot}_{\pi^*\mathcal{F}}] = c_{top}(S_1^\vee \otimes Q_2)^2 = \left(\Delta^2_{23}\left[\frac{c(Q_2)}{c(S_1)}\right]\right)^2 = \left(\Delta^2_{23}\left[\frac{c(\rho^*\mathcal{F})^3}{c(S_1)c(S_2)}\right]\right)^2.$$

On $\mathrm{Quot}_{\pi^*\mathcal{F}}$, the locus where $\mathcal{S}$ has splitting type $(-2,0)$ is the same as where $\mathcal{S}(1)$ has splitting type $(-1,1)$. This is given by the universal rank 2 formula:

$$[Z_{(-2,0)}] = \left[\frac{c((p_*\mathcal{S}(1))^\vee)^2}{c((p_*\mathcal{S}(2))^\vee)}\right]_1 = c_1(p_*\mathcal{S}(2)) - 2c_1(p_*\mathcal{S}(1)) = \iota^*(c_1(S_2) - 2c_1(S_1)).$$

Then (2.6.5) says

$$[\Sigma_{(-2,0,2)}] = \rho_*\left(\left(\Delta^2_{23}\left[\frac{c(\rho^*\mathcal{F})^3}{c(S_1)c(S_2)}\right]\right)^2 \cdot \Delta^6_6\left[c(\rho^*\mathcal{G})\frac{c(S_1)^5}{c(S_2)^4}\right] \cdot (c_1(S_2) - 2c_1(S_1))\right).$$

The class inside the outer parenthesis is codimension 129 and the relative fiber dimension of ρ is 124, so the pushforward is a codimension 5 class on the base.

Explicit computation

Let $f_i = c_i(\mathcal{F}) = c_i((\pi_* E(m))^\vee)$ and $g_i = c_i(\mathcal{G}) = c_i((\pi_* E(m-1))^\vee)$. To implement the algorithm, we first stored all push forwards of monomials in the Chern classes of S_1 and S_2, as determined by [40, Cor. 2.6]. This precomputation took 5 days on 6 cores. Then, we expanded the above class as a polynomial in $\rho^* f_i, \rho^* g_i$ and the Chern classes of S_1 and S_2 and computed the push forward. The second step took 2 days on 6 cores and produced the following formula:

$$
\begin{aligned}
[\Sigma_{(-2,0,2)}] = {} & 4f_1^4g_1 - 8f_1^3g_1^2 + 4f_1^2g_1^3 - 3f_1^3f_2 - 6f_1^2f_2g_1 \\
& + 13f_1f_2g_1^2 - 4f_2g_1^3 + 8f_1^2g_1g_2 - 8f_1g_1^2g_2 + 6f_1f_2^2 + 3f_1^2f_3 - 2f_2^2g_1 \\
& + 2f_1f_3g_1 - 5f_3g_1^2 - 6f_1f_2g_2 - 2f_2g_1g_2 + 4g_1g_2^2 - 8f_1g_1g_3 + 8g_1^2g_3 \\
& - 6f_2f_3 - 3f_1f_4 + 2f_4g_1 + 6f_3g_2 + 6f_2g_3 - 6g_2g_3 + 2g_1g_4 + 3f_5 - 6g_5.
\end{aligned}
\tag{2.6.6}
$$

Sage code for these processes may be found at `http://web.stanford.edu/~hlarson/`. Factoring the push forward through the product of Grassmann bundles greatly increases the codimension of the class we need to push forward. If one could push forward from $\mathrm{Quot}_{\pi^*\mathcal{F}}$ directly, this should yield substantial improvements in run time.

Chapter 3

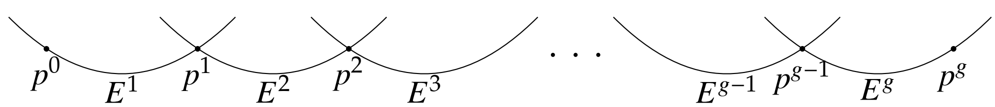
Brill–Noether splitting loci: Part I

We return now to the study of Brill–Noether splitting loci and the proof of Theorem 1.3.7, which determines their dimension, smoothness, and shape of the formula for their classes. The contents of this chapter was published in [50]

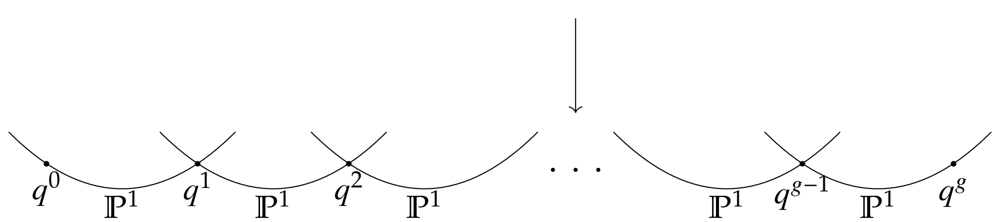
3.1 Our degeneration

Degeneration to chains of elliptic curves have been used previously in Brill-Noether theory, by Osserman [59], and earlier by Welters [72]. Here, to study curves which are degree k covers of $\mathbb{P}^1$, we degenerate to a chain of elliptic curves *whose attachment points differ by k torsion* (as first suggested by Pflueger [61, Remark 1.14]). This same degeneration will be used both in this chapter and the next.

Let $X = E^1 \cup_{p^1} E^2 \cup_{p^2} \cdots \cup_{p^{g-1}} E^g$ be a chain of g elliptic curves:



Let $f^i \colon E^i \to \mathbb{P}^1$ be degree k maps. Pasting these maps together, we get a map $f\colon X \to P$, where P denotes a chain of g rational curves, attached at points $q^i = f(p^i)$:



If all the f^i *are totally ramified at* p^{i-1} *and* p^i, then the theory of admissible covers implies that f is a limit of smooth k-gonal curves. (The theory of admissible covers was developed by Harris and Mumford in characteristic zero [39]; see also Section 5 of [55] for a characteristic-independent proof of this fact.) In other words, there is a map $\mathfrak{f}\colon \mathcal{X} \to \mathcal{P}$ between families of curves of genus g and 0 respectively over the base $B = \operatorname{Spec} K[[t]]$, such that the general fiber of $\mathfrak{f}$ is a smooth k-gonal curve and the special fiber of $\mathfrak{f}$ is f. Moreover, we may suppose that the total space $\mathcal{X}$ is smooth, that $\mathcal{P} \to B$ is the base-change of a family $\mathcal{P}_0 \to B_0$ with smooth total space via a map $\beta\colon B \to B_0$, and that $\mathfrak{f}$ is totally ramified along sections $\mathfrak{p}^0$ and $\mathfrak{p}^g$ of $C \to B$ whose special fibers are p^0 and p^g respectively.

A map $f^i\colon E^i \to \mathbb{P}^1$, of degree k totally ramified at p^{i-1} and p^i, exists if and only if $p^i - p^{i-1} \in \operatorname{Pic} E^i$ is k-torsion. To keep things as generic as possible, we therefore suppose for the remainder of the document that $p^i - p^{i-1}$ has order *exactly* k in $\operatorname{Pic} E^i$.

Remark 3.1.1 (A note on "general" degree k covers)**.** When the characteristic of the ground field is 0 or greater than k, then $\mathcal{H}_{k,g}$ is irreducible [30], so it makes sense to talk about a general cover $C \to \mathbb{P}^1$. However, many results later on in the thesis will still make sense in characteristic less than k if we take a *general* degree k cover to mean one in a component of $\mathcal{H}_{k,g}$ containing the above deformation of X.

3.2 Dimension

A notable difference in our approach in this chapter compared to previous work is that we do not use the framework of limit linear series: Instead of tracking vanishing sequences of different limit line bundles, we describe the sections that smooth from a fixed limit. The key technical difficulty will be to understand how the compatibility conditions from two nodes on the same component interact.

Remark 3.2.1. Later, in Chapter 4, we shall use the theory of limit linear series to prove Theorem 1.3.9. While the work of Chapter 4 will reprove the dimension statement, I have not seen a way to use those techniques to prove smoothness. Thus, I have also included my original argument for the dimension, which will be built upon in the next Section 3.3 to prove smoothness.

A consequence of our analysis will be that, for a general degree k cover $C \to \mathbb{P}^1$,

$$\dim\{L \in \operatorname{Pic}^d(C) : h^0(\mathbb{P}^1, End(f_*L)) \geq \delta + k^2\} \leq g - \delta. \tag{3.2.1}$$

for all $\delta \geq 0$. Since $End(f_*L)$ is rank k^2 and degree 0 on $\mathbb{P}^1$,

$$u(f_*L) = h^1(\mathbb{P}^1, End(f_*L)) = h^0(\mathbb{P}^1, End(f_*L)) - k^2,$$

so (3.2.1) implies $\dim W^{\vec{e}}(C) \leq g - u(\vec{e})$ for all $\vec{e}$. (Notice that (3.2.1) does not refer to a particular splitting type!)

Basic cohomological observations determine all push forwards of line bundles from elliptic curves.

Lemma 3.2.2. *Let E be an elliptic curve and $f\colon E \to \mathbb{P}^1$ a degree k map. Let L be a line bundle of degree $d = a + nk$ on X with $0 \leq a < k$. We have*

$$f_*L = \begin{cases} \mathcal{O}_{\mathbb{P}^1}(n-2) \oplus \mathcal{O}_{\mathbb{P}^1}(n-1)^{\oplus k-2} \oplus \mathcal{O}_{\mathbb{P}^1}(n) & \text{if } L = f^*\mathcal{O}_{\mathbb{P}^1}(n) \\ \mathcal{O}_{\mathbb{P}^1}(n-1)^{\oplus k-a} \oplus \mathcal{O}_{\mathbb{P}^1}(n)^{\oplus a} & \text{otherwise.} \end{cases}$$

Proof. Let $H = f^*\mathcal{O}_{\mathbb{P}^1}(1)$. By the projection formula, $f_*L = \mathcal{O}_{\mathbb{P}^1}(n) \otimes f_*L(-nH)$, so it suffices to consider the case $n = 0$. First observe that $h^0(E, \mathcal{O}_E) = h^1(E, \mathcal{O}_E) = 1$. The only rank k vector bundle on $\mathbb{P}^1$ with this cohomology is $\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus k-2} \oplus \mathcal{O}_{\mathbb{P}^1}$ so this must be $f_*\mathcal{O}_E$, completing the first case. Now suppose L is non-trivial of degree $0 \leq a < k$. By Serre duality, $h^1(E, L) = h^0(E, L^\vee) = 0$, implying all summands of f_*L are degree at least -1. Riemann-Roch shows $h^0(E, L) = a$ and moreover, $h^0(E, L(-H)) = 0$, because in this case $\deg L(-H) < 0$. It follows that $f_*L = \mathcal{O}(-1)^{\oplus k-a} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus a}$, completing the second case. □

Let $\mathcal{X} \to \mathcal{P} \to B$ be our degeneration from Section 3.1. We write $\widetilde{\mathcal{X}} \to \widetilde{\mathcal{P}}$ for the family over the general fiber. The central fiber $X = \mathcal{X}_0$ of our degeneration is compact type. In particular, given a degree d line bundle $\widetilde{\mathcal{L}}$ on $\widetilde{\mathcal{X}}$ and partition $d = d^1 + \ldots + d^g$, there is a unique extension $\mathcal{L}$ to $\mathcal{X}$ so that the limit $\mathcal{L}_0 = \mathcal{L}|_{\mathcal{X}_0}$ restricts to a degree d^i line bundle L^i on E^i. We wish to bound $h^0(\mathcal{P}_t, End(\mathfrak{f}_*\mathcal{L}_t))$ for general $t \in B$. To do so, we study the subspace $V_{\mathcal{L}} \subset H^0(\mathcal{P}_0, End(\mathfrak{f}_*\mathcal{L}_0))$ of sections which arise as limits of sections in $H^0(\mathcal{P}_t, End(\mathfrak{f}_*\mathcal{L}_t))$ as $t \to 0$.

For simplicity, let us fix a partition $d^1 + \ldots + d^g = d$ and write

$$\operatorname{Pic}^d(X) := \operatorname{Pic}^{d^1}(E^1) \times \operatorname{Pic}^{d^2}(E^2) \times \cdots \times \operatorname{Pic}^{d^g}(E^g).$$

(In Chapter 4, we will need to work with *all* possible partitions of d to understand more about the limits on our central fiber, but interestingly, in this Chapter, one fixed partition will provide sufficient information.) Let $\alpha^i : E^i \to X$ denote the inclusion. Given $\mathcal{L}_0 = (L^1, \ldots, L^g) \in \operatorname{Pic}^d(X)$, we have a short exact sequence on X

$$0 \to \mathcal{L}_0 \to \bigoplus_{i=1}^{g} (\alpha^i)_* L^i \to \bigoplus_{i=1}^{g-1} \mathcal{O}_{p^i} \to 0.$$

Let $\beta^i : P^i \to \mathcal{P}_0$ be the inclusion. For each i, let $F^i = (f^i)_* L^i$. Applying $\mathfrak{f}_*$ to the above, we obtain an exact sequence on $\mathcal{P}_0$

$$0 \to \mathfrak{f}_*\mathcal{L}_0 \to \bigoplus_{i=1}^{g} (\beta^i)_* F^i \to \bigoplus_{i=1}^{g} \mathcal{O}_{f(p^i)} \to 0. \tag{3.2.2}$$

The restriction of $\mathfrak{f}_*\mathcal{L}_0$ to each component P^i has an isomorphism

$$(\mathfrak{f}_*\mathcal{L}_0)|_{P^i} \cong F^i \oplus \mathcal{O}_{\mathfrak{f}(p^{i-1})}^{\oplus k-1} \oplus \mathcal{O}_{\mathfrak{f}(p^i)}^{\oplus k-1}. \tag{3.2.3}$$

In what follows, we will write $L^i|_{kp}$ for $L^i|_{(f^i)^{-1}(f^i(p))}$. Above, $\mathcal{O}_{\mathfrak{f}(p^{i-1})}^{\oplus k-1}$ is identified with the subspace of $H^0(L^{i-1}|_{kp^{i-1}})$ of functions vanishing at p^{i-1} and $\mathcal{O}_{\mathfrak{f}(p^i)}^{\oplus k-1}$ is identified with the subspace of $H^0(L^{i+1}|_{kp^i})$ of functions vanishing at p^i. The splitting of the middle term is defined by the map sending a section σ to $\sigma|_{kp^{i-1}} - \sigma(p^{i-1}) \in H^0(L^{i-1}|_{kp^{i-1}})$. We think of the $k-1$ factors $\mathcal{O}_{f(p^{i-1})}^{\oplus k-1}$ as remembering the values of the first $k-1$ derivatives of σ at p^{i-1} along E^{i-1}, and similarly the $k-1$ factors $\mathcal{O}_{f(p^i)}$ as remembering the values of the first $k-1$ derivatives of σ at p^i along E^{i+1}.

Applying $\mathcal{H}om(\mathfrak{f}_*\mathcal{L}_0, -)$ to (3.2.2) and using (3.2.3), we obtain an injection of sheaves

$$\mathcal{E}nd(\mathfrak{f}_*\mathcal{L}_0) \hookrightarrow \bigoplus_{i=1}^{g} \mathcal{H}om(\mathfrak{f}_*\mathcal{L}_0, (\beta^i)_* F^i) \cong \bigoplus_{i=1}^{g} \mathcal{H}om((\mathfrak{f}_*\mathcal{L}_0)|_{P^i}, F^i) \cong \bigoplus_{i=1}^{g} \mathcal{E}nd(F^i).$$

The last isomorphism follows because there are no non-zero maps from the torsion summands to a locally free sheaf. Taking global sections yields an inclusion

$$\iota : H^0(\mathcal{P}_0, \mathcal{E}nd(\mathfrak{f}_*\mathcal{L}_0)) \hookrightarrow \bigoplus_{i=1}^{g} H^0(P^i, \mathcal{E}nd(F^i)).$$

We want to describe the image under ι of the subspace $V_{\mathcal{L}} \subset H^0(\mathcal{P}_0, \mathcal{E}nd(\mathfrak{f}_*\mathcal{L}_0))$ of sections that arise as limits from smooth curves. One necessary condition on each factor is described in the following definition. In what follows $\mathbb{F}$, denotes the ground field, which is algebraically closed of any characteristic.

Definition 3.2.3. Let L be a line bundle on an elliptic curve with a degree k map $f : X \to \mathbb{P}^1$ and let $F = f_* L$. Given a point p of total ramification of f, we say $\phi \in H^0(\mathbb{P}^1, \mathcal{E}nd(F))$ is *order preserving at* p if $\mathrm{ord}_p(\phi(\sigma)) \geq \mathrm{ord}_p \sigma$ for all $\sigma \in H^0(U, L)$ for any $U \ni p$. Equivalently, the restriction $\mathrm{res}\,\phi \in \mathrm{End}(H^0(L|_{kp})) \cong \mathrm{End}(\mathbb{F}[x]/(x^k))$ is lower triangular with respect to the basis $1, x, x^2, \dots, x^{k-1}$. Note that the diagonal entries

$$d_p^{(j)}(\phi) := (\operatorname{res}\phi)(x^j)/x^j|_{x=0}$$

are independent of choice of local coordinate x.

We now describe agreement conditions near the nodes that are satisfied by every element of $\iota(V_{\mathcal{L}})$.

Lemma 3.2.4. *Given $\mathcal{L}_0 = (L^1, \dots, L^g) \in \operatorname{Pic}^d(X)$, let $\mathcal{L}$ be a line bundle on $\mathcal{X}$ such that $\mathcal{L}|_X = \mathcal{L}_0$. Let $V_{\mathcal{L}} \subset H^0(\mathcal{P}_0, End(\mathfrak{f}_*\mathcal{L}_0))$ be the subspace of sections that can be extended to $H^0(\mathcal{P}, End(\mathfrak{f}_*\mathcal{L}))$. If $(\phi_1, \dots, \phi_g) \in \iota(V_{\mathcal{L}})$ then the following conditions hold for each $i = 1, \dots, g-1$:*

1. *ϕ_i is order preserving at p^i*
2. *ϕ_{i+1} is order preserving at p^i*
3. *We have $d_{p^i}^{(0)}(\phi_i) = d_{p^i}^{(0)}(\phi_{i+1})$ and $d_{p^i}^{(j)}(\phi_i) = d_{p^i}^{(k-j)}(\phi_{i+1})$ for $j = 1, \dots, k-1$.*

Proof. It suffices to work locally around p^i. Let $\mathbb{F}$ be the ground field, which is algebraically closed of any characteristic. We may choose formal local coordinates x, y, t near p^i so that $\widehat{\mathcal{O}}_{X,p^i} = \mathbb{F}[\![x,y]\!]/(xy-t)$ and $\widehat{\mathcal{O}}_{\mathcal{P},\mathfrak{f}(p^i)} = \mathbb{F}[\![a,b]\!]/(ab-t^k)$ and the map $\mathfrak{f}$ is described locally by a map $\mathbb{F}[\![a,b,t]\!]/(ab-t^k) \to \mathbb{F}[\![x,y,t]\!]/(xy-t)$ such that $a \mapsto x^k u^{-1}$ and $b \mapsto y^k u$ for u a power series in x, y with constant coefficient 1 (see [55, Section 5.3]. (If the characteristic of $\mathbb{F}$ does not divide k, we can extract a kth root of $u|_{x=0}$ and absorb it into y and extract a kth root of $u^{-1}|_{y=0}$ and absorb it into x, and thereby assume $u = 1$ as in [55, p. 57, Section 4].)

Since $\mathcal{L}$ is locally free, a section of $End(\mathfrak{f}_*\mathcal{L})$ is given locally near $\mathfrak{f}(p^i)$ by an an endomorphism ψ of $\mathbb{F}[\![x,y,t]\!]/(xy-t)$ viewed as an $\mathbb{F}[\![a,b,t]\!]/(ab-t^k)$ module. On the central fiber, the monomials $1, x, x^2, \dots, x^{k-1}, y, y^2, \dots, y^{k-1}$ generate $\mathbb{F}[\![x,y]\!]/(xy)$ as a module over $\mathbb{F}[\![a,b]\!]/(ab)$. By Nakayama's lemma, these monomials generate $\mathbb{F}[\![x,y,t]\!]/(xy-t)$ as an $\mathbb{F}[\![a,b,t]\!]/(ab-t^k)$ module. Because ψ is a module homomorphism, we have

$$\begin{aligned}(y^k u)\cdot\psi(x^j) = b\cdot\psi(x^j) = \psi(b\cdot x^j) &= \psi(y^{k-j}\cdot u\cdot(y^j x^j)) \\ &= \psi(y^{k-j}\cdot u\cdot t^j) = \psi(y^{k-j}\cdot u)\cdot t^j = \psi(y^{k-j}\cdot u)\cdot x^j y^j. \qquad (3.2.4)\end{aligned}$$

Hence, x^j divides $\psi(x^j)$. A similar argument shows that y^j divides $\psi(y^j)$. Thus, ψ is order-preserving, so conditions (1) and (2) are satisfied. Moreover, since $xy = t$, we see that

$x^i y^j$ divides

$$\psi(x^i y^j) = \begin{cases} t^i \psi(y^{j-i}) = x^i y^i \psi(y^{j-i}) & \text{if } i \leq j \\ t^j \psi(x^{i-j}) = x^j y^j \psi(x^{i-j}) & \text{if } j \leq i \end{cases}$$

for all i, j. Since $u = 1 + (x, y)$, it follows that $\psi(y^{k-j} \cdot u) = \psi(y^{k-j}) + y^{k-j} \cdot (x, y)$. Dividing both sides of (3.2.4) by $x^j y^k$, we see that

$$u \cdot \frac{\psi(x^j)}{x^j} = \frac{\psi(y^{k-j} \cdot u)}{y^{k-j}} = \frac{\psi(y^{k-j})}{y^{k-j}} + (x, y).$$

We have $d^{(j)}_{p^i}(\phi_i) = (\psi(x^j)/x_j)|_{x=0,y=0}$, so setting $x = y = 0$ in the equation above establishes part (3) for $j = 1, \ldots, k-1$. The case $d^{(0)}_{p^i}(\phi_i) = d^{(0)}_{p^i}(\phi_{i+1})$ follows from the fact that both are equal to the constant term of $\psi(1)$. It follows that any collection $(\phi_1, \ldots, \phi_g)$ which arises as a limit of a section defined on smooth curves must satisfy these local compatibility properties near the nodes. □

Notice that conditions (1) and (2) of Lemma 3.2.4 each represent $k(k-1)/2$ linear conditions on ϕ_i and ϕ_{i+1}. Condition (3) represents another k linear conditions on ϕ_i and ϕ_{i+1}, for a total of k^2 possible linear conditions near each node. Our next task is to show that these conditions are all independent for general $(L^1, \ldots, L^g)$, and bound the dimension of the subvarieties inside $\mathrm{Pic}^d(\mathcal{X}_0)$ where they fail to be independent by a certain amount. The key technical lemma is to establish when the constraints on $\phi_i \in H^0(P^i, End(F^i))$ coming from the two different nodes p^i and p^{i+1} are independent.

Lemma 3.2.5. *Suppose we have an elliptic curve with a degree k map $f\colon E \to \mathbb{P}^1$ which is totally ramified over two distinct points $p, q \in E$. Let L be a line bundle on E which is not isomorphic to $f^*\mathcal{O}_{\mathbb{P}^1}(n)$ for any n, and set $F = f_*L$. Let $W_p \subset H^0(\mathbb{P}^1, End(F))$ (respectively $W_q \subset H^0(\mathbb{P}^1, End(F))$) denote the subspace of sections which are order preserving at p (respectively q). Then,*

1. *We have*

$$\dim W_q = \begin{cases} \frac{k(k+1)}{2} + 1 & \text{if } L \cong \mathcal{O}_E(mq) \text{ for some } m \\ \frac{k(k+1)}{2} & \text{otherwise} \end{cases}$$

and

$$\dim W_p \cap W_q = \begin{cases} k+1 & \text{if } L \cong \mathcal{O}_E(np + mq) \text{ for some } m, n \\ k & \text{otherwise.} \end{cases}$$

2. *The map* $\mathfrak{d}_p : W_p \cap W_q \to \mathbb{F}^{\oplus k}$ *defined by*

$$\phi \mapsto \left(d_p^{(0)}(\phi), \ldots, d_p^{(k-1)}(\phi)\right)$$

is surjective.

3. *We have* $\ker(\mathfrak{d}_p) \cap W_p \cap W_q = \ker(\mathfrak{d}_q) \cap W_p \cap W_q$.

4. *If* $\phi \in \ker(\mathfrak{d}_p) \cap W_p \cap W_q$, *then* ϕ *can be represented by a matrix with at most one non-zero entry.*

Proof. The rough idea is to choose a decomposition of F so that the condition of being order preserving at p is that a matrix for an endomorphism is lower triangular, while the condition of being order preserving at q is that the matrix is upper triangular. We shall see that if $L \not\cong \mathcal{O}_E(np+mq)$, then the conditions to be order preserving and p and at q are independent, while if $L \cong \mathcal{O}_E(np+mq)$ we obtain one less condition. Twisting L by $f^*\mathcal{O}_{\mathbb{P}^1}(1)$ does not change $End(F)$, so we assume $k \leq \deg L < 2k$.

We first prove the case when $L \not\cong \mathcal{O}_E(np+mq)$. By Lemma 3.2.2,

$$F \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus k-a} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus a}$$

where $a = \deg L - k$. We now describe a specific isomorphism $\mathcal{O}_{\mathbb{P}^1}^{\oplus k-a} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus a} \to F$. It will suffice to give the map on global sections. Let $s, t \in H^0(E, f^*\mathcal{O}_{\mathbb{P}^1}(1))$ denote sections defining the map f with $V(s) = kq$ and $V(t) = kp$. For each $0 \leq j \leq a-1$, and $\alpha, \beta \in \mathbb{F}$, there is a section $\tau_j(\alpha, \beta) = (\alpha s + \beta t) \cdot u_j \in H^0(E, L)$ where $V(u_j) = jp + (a-1-j)q + r_j$. Note that $r_j \neq p, q$ by assumption. For each j, the $\tau_j(\alpha, \beta)$ span a copy of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ inside $H^0(\mathbb{P}^1, F) = H^0(E, L)$. For $a \leq j \leq k-1$, we choose $\sigma_j \in H^0(E, L)$ so that $V(\sigma_j) = jp + (k+a-1-j)q + r_j$, where again $r_j \neq p, q$. These σ_j are non-vanishing on fibers of f, so each corresponds to an $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ factor inside $H^0(\mathbb{P}^1, F) = H^0(E, L)$. With respect to this decomposition of F, an element of $H^0(\mathbb{P}^1, End(F))$ is represented by a block upper triangular matrix where the two diagonal blocks consist of elements of $\mathbb{F}$ and the upper block consists of linear forms.

$$\phi = \begin{pmatrix} c_{0,0} & \cdots & c_{0,a-1} & \alpha_{0,a}s + \beta_{0,a}t & \cdots & \alpha_{0,k-1}s + \beta_{0,k-1}t \\ c_{1,0} & \cdots & c_{1,a-1} & \alpha_{1,a}s + \beta_{1,a}t & \cdots & \alpha_{1,k-1}s + \beta_{1,k-1}t \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{d-1,0} & \cdots & c_{a-1,a-1} & \alpha_{a-1,a}s + \beta_{a-1,a}t & \cdots & \alpha_{a-1,k-1}s + \beta_{a-1,a-1}t \\ 0 & \cdots & 0 & c_{a,a} & \cdots & c_{a,k-1} \\ 0 & \cdots & 0 & c_{a+1,a} & \cdots & c_{a+1,k-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & c_{k-1,a} & \cdots & c_{k-1,k-1} \end{pmatrix} \tag{3.2.5}$$

For $\ell \geq a$ and $j \geq a$, the coefficients $\alpha_{\ell,j}$ and $\beta_{\ell,j}$ specify which $\tau_\ell(\alpha_{\ell,j}, \beta_{\ell,j})$ appears in the image of σ_j with respect to our chosen decomposition of $H^0(\mathbb{P}^1, F)$. The condition for ϕ to be order preserving at p is that $\alpha_{\ell,j} = 0$ for all ℓ, j and $c_{\ell,j} = 0$ for all $\ell < j$. Hence, $\dim W_p = k(k+1)/2$. The condition for ϕ to be order preserving at q is that $\beta_{\ell,j} = 0$ for all ℓ, j and $c_{\ell,j} = 0$ for all $\ell > j$. It follows that $\dim W_p \cap W_q = k$, proving part (1). Moreover, with this decomposition, we have that $d_p^{(n)}(\phi) = c_{n,n}$ and $d_q^{(n)}(\phi)$ is also given by an appropriate diagonal entry. That is, $\mathfrak{d}_p$ and $\mathfrak{d}_q$ are related by a permutation of coordinates on $\mathbb{F}^{\oplus k}$, so (3) follows. As the diagonal entries of any $\phi \in W_p \cap W_q$ are unconstrained, the map $\mathfrak{d}_p$ is a surjection, proving (2), and $\ker(\mathfrak{d}_p) \cap W_p \cap W_q = \{0\}$ proving (4).

Now suppose $L \cong \mathcal{O}_E(np + mq) \not\cong \mathcal{O}_E(kp)$. Without loss of generality, we may assume that $n \geq m > 0$. Since $n + m = a + k$ with $a < k$, we also have $n > a$. Again, we have $F \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus k-a} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus a}$, but the argument in the previous paragraph must be modified because $r_{n-1} = p$ and $r_n = q$ (or when $n = k$, we have $r_{k-1} = p$ and $r_0 = q$). Instead, for $a \leq j \leq k-1$, we choose $\sigma_j \in H^0(\mathbb{P}^1, F)$ so that $\mathrm{ord}_p(\sigma_j) = j$ and

$$\mathrm{ord}_q(\sigma_j) = \begin{cases} k + a - n & \text{if } j = n \\ k + a - n - 1 & \text{if } j = n - 1 \\ k + a - j - 1 & \text{otherwise.} \end{cases}$$

If $n = k$, then the first case above never occurs, but we must take $\tau_0(\alpha, \beta) = (\alpha s + \beta t) \cdot u_0$ where $V(u_0) = aq$. Otherwise, the vanishing orders of τ_j and u_j may be taken as before. Again, this decomposition has the property that $\mathfrak{d}_p$ and $\mathfrak{d}_q$ are given by taking diagonal entries, establishing (3).

If $n \neq k$, then the condition for ϕ in (3.2.5) to be order preserving at p is that $\alpha_{\ell,j} = 0$ for all ℓ, j and $c_{\ell,j} = 0$ for all $\ell < j$. The condition for ϕ to be order preserving at q is that all

$\beta_{\ell,j} = 0$; and $c_{\ell,j} = 0$ for all $\ell > j$ with $(\ell, j) \neq (n, n-1)$; and $c_{n-1,n} = 0$. Note that $c_{n,n-1}$ need not vanish because $\mathrm{ord}_q(\sigma_n) > \mathrm{ord}_q(\sigma_{n-1})$.

In the case $n = k$, the condition for ϕ to be order preserving at q is that $c_{\ell,j} = 0$ for all $\ell > j$ and $\beta_{\ell,j} = 0$ for all $(\ell, j) \neq (0, k-1)$. Note that $\beta_{0,k-1}$ is not required to vanish because $\mathrm{ord}_q(\sigma_{k-1}) = a - 1 < a = \mathrm{ord}_q(u_0)$ so $\beta_{0,k-1}$ is not required to vanish. Thus, $\dim W_q = \frac{k(k+1)}{2} + 1$. Note that $n = k$ corresponds to the case when $L \cong \mathcal{O}((n+m)q)$. Our explicit description shows that $\dim W_p \cap W_q = k + 1$ (part (1)), and the intersection consists of matrices with arbitrary diagonal entries (part (2)) and at most one non-zero off-diagonal entry (part (4)). ◻

Having characterized necessary compatibility conditions at the nodes and when they are independent, we now prove (3.2.1). This will be subsumed by the results of the next section, but we include it here as the proof indicates subvarieties of $\mathrm{Pic}^d(\mathcal{X}_0)$ where the limits of line bundles with a certain splitting type must live.

Lemma 3.2.6. *Let $f\colon C \to \mathbb{P}^1$ be a general genus g, degree k cover. Then*

$$\dim W^{\vec{e}}(C) \leq g - u(\vec{e}).$$

Proof. The case $g = 1$ was proved in Lemma 3.2.2, so we assume $g > 1$. Since we are also assuming $k > 2$, we can choose a degree distribution $d = d^1 + \ldots + d^g$ so that no d^i is a multiple of k. In particular, given $\mathcal{L}_0 = (L^1, \ldots, L^g) \in \mathrm{Pic}^d(\mathcal{X}_0)$, we may assume that $L^i \not\cong f^*\mathcal{O}(n)$. Define

$$\epsilon_i = \begin{cases} 1 & \text{if } i \neq 1, g \text{ and } L^i \cong \mathcal{O}_{E^i}(np^{i-1} + mp^i) \\ 1 & \text{if } i = 1 \text{ and } L^1 \cong \mathcal{O}_{E^1}(mp^1) \\ 1 & \text{if } i = g \text{ and } L^g \cong \mathcal{O}_{E^g}(np^{g-1}) \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 3.2.4 and Lemma 3.2.5 (1) and (2),

$$\begin{aligned} \dim V_{\mathcal{L}} &\leq \sum_{i=1}^{g} \dim\{\phi \in H^0(P^i, End(F^i)) : \phi \text{ order preserving at nodes on } E^i\} - k(g-1) \\ &\leq \frac{k(k+1)}{2} + \epsilon_1 + \frac{k(k+1)}{2} + \epsilon_g + \sum_{i=2}^{g-1}(k + \epsilon_i) - k(g-1) \\ &\leq k^2 + \delta \end{aligned}$$

where δ is the number of i for which $L^i \cong \mathcal{O}_{E^i}(np^{i-1} + mp^i)$ for some m, n (with $m = 0$ if $i = 1$ and $n = 0$ if $i = g$). In particular, the codimension of the subvariety of line bundles $\mathcal{L}_0$ in $\mathrm{Pic}^d(X)$ for which $V_{\mathcal{L}} \geq k^2 + \delta$ is at least δ. This implies that for general $\mathfrak{f}_t : \mathcal{X}_t \to \mathcal{P}_t$ in the family $\mathfrak{f} \colon \mathcal{X} \to \mathcal{P}$,

$$\dim\{L \in \mathrm{Pic}^d(\mathcal{X}_t) : h^0(\mathcal{P}_t, End((\mathfrak{f}_t)_* L)) \geq \delta + k^2\} \leq g - \delta.$$

To finish, note that $h^0(\mathcal{P}_t, End((\mathfrak{f}_t)_* L)) \geq k^2 + \delta$ implies $u((\mathfrak{f}_t)_* L) \geq \delta$, and so for each $\vec{e}$, we have $\dim W^{\vec{e}}(\mathcal{X}_t) \leq g - u(\vec{e})$ for general t. By upper semi-continuity, this upper bound on $\dim W^{\vec{e}}(C)$ holds for general degree k covers $f \colon C \to \mathbb{P}^1$. □

Remark 3.2.7. Our proof has established that the limit of any line bundle with splitting type $\vec{e}$ is of the form $(L^1, \ldots, L^g)$ where $L^i \cong \mathcal{O}_{E^i}(np^{i-1} + mp^i)$ for at least $u(\vec{e})$ values of i. In Chapter 4, we shall see some further constraints on these limits that allow only certain collections of values for i, m, n, as dictated by fillings of k-staircase tableaux.

3.3 Smoothness

Given a degree k cover $C \to \mathbb{P}^1$, let

$$W^{\vec{e}}(C)^\circ := \{L \in \mathrm{Pic}^d(C) : f_* L \cong \mathcal{O}(\vec{e})\}$$

denote the strict splitting locus. In this section, we prove that $W^{\vec{e}}(C)^\circ$ is smooth for general $f \colon C \to \mathbb{P}^1$. This should be thought of as an analogue of Theorem 1.1.3 part 2 which was first proved by Gieseker [35], and later by Eisenbud-Harris [18] using a degeneration with elliptic tails.

For every $L \in \mathrm{Pic}^d(C)$, there is a natural map

$$H^1(\mathbb{P}^1, f_* \mathcal{O}_C) = H^1(C, \mathcal{O}_C) = \mathrm{Def}^1(L) \to \mathrm{Def}^1(f_* L) = H^1(\mathbb{P}^1, End(f_* L)), \tag{3.3.1}$$

sending a first order deformation of L to the induced deformation of the push forward. Above, the first equality follows from the fact that f is finite, hence affine. The map (3.3.1) is realized by taking cohomology of the map of sheaves $\eta : f_* \mathcal{O}_C \to End(f_* L)$ that locally sends a function z on C to the endomorphism “multiplication by z" on L, viewed as an $\mathcal{O}_{\mathbb{P}^1}$ module. The kernel of (3.3.1) is the tangent space to $W^{\vec{e}}(C)$ at a point $L \in W^{\vec{e}}(C)^\circ$. Thus, our goal is to show that (3.3.1) is surjective for all $L \in \mathrm{Pic}^d(C)$. Indeed, this implies that for every $L \in W^{\vec{e}}(C)^\circ$, we have

$$\dim T_L W^{\vec{e}}(C)^\circ \leq g - \dim \mathrm{Def}^1(f_* L) = g - u(\vec{e}) \leq \dim W^{\vec{e}}(C)^\circ,$$

so it is smooth.

We proceed by showing that the Serre dual of (3.3.1),

$$\mu : H^0(\mathbb{P}^1, End(f_*L) \otimes \omega_{\mathbb{P}^1}) \to H^0(\mathbb{P}^1, (f_*O_C)^\vee \otimes \omega_{\mathbb{P}^1}), \tag{3.3.2}$$

is injective. The kernel of this map is the "obstruction to smoothness." We think of $H^0(\mathbb{P}^1, End(f_*L) \otimes \omega_{\mathbb{P}^1})$ as the subspace of $H^0(\mathbb{P}^1, End(f_*L))$ vanishing at two prescribed points. The map μ is thus a restriction of the map on global sections induced by

$$\widetilde{\mu} : End(f_*L) \cong End(f_*L)^\vee \to (f_*O_C)^\vee,$$

which is the composition of the canonical isomorphism $End(f_*L) \cong End(f^*L)^\vee$ with the map dual to η. For any vector bundle F, this isomorphism $End(F) \cong End(F)^\vee$ is induced by the perfect pairing $End(F) \times End(F) \to O$ given by $(\phi, \psi) \mapsto \mathrm{Tr}(\phi \cdot \psi)$. Therefore, $\widetilde{\mu}$ sends an endomorphism $\phi \in End(f_*L)(U)$ to the linear functional on $(f_*O_C)(U)$ given by $z \mapsto \mathrm{Tr}(\phi \cdot \eta(z))$. We will need to know that this map is non-zero on certain elements over components of our degeneration.

Lemma 3.3.1. *Let $f\colon E \to \mathbb{P}^1$ be a degree k map of an elliptic curve to $\mathbb{P}^1$ and let $L \in \mathrm{Pic}^d(C)$. If $\phi \in H^0(\mathbb{P}^1, End(f_*L))$ is represented by a matrix with a single nonzero entry, then $\widetilde{\mu}(\phi) \neq 0$.*

Proof. For each open subset $U \subset \mathbb{P}^1$, we have a commutative diagram

$$\begin{array}{ccc} H^0(\mathbb{P}^1, End(f_*L)) & \xrightarrow{\widetilde{\mu}} & H^0(\mathbb{P}^1, (f_*O_E)^\vee) \\ \downarrow & & \downarrow \\ H^0(U, End(f_*L)) & \xrightarrow[(\widetilde{\mu})|_U]{} & H^0(U, (f_*O_E)^\vee). \end{array}$$

It suffices to show that the image of ϕ in the lower right is nonzero. Choose U small enough that L is trivialized on $f^{-1}(U)$ and f_*O_E is trivialized on U, so we have isomorphisms $(f_*L)|_U \cong (f_*O_C)|_U \cong O_U^{\oplus k}$. By hypothesis, there exists a basis so that, $\phi|_U : O_U^{\oplus k} \to O_U^{\oplus k}$ is represented by a matrix with one non-zero entry, say in the (i, j) slot. These basis vectors of $O_U^{\oplus k}$ correspond to non-zero functions in $O_C(f^{-1}(U))$. Shrinking U further if necessary, we may assume they are non-vanishing. The ratio of the jth basis element over the ith basis element therefore now defines a function $z \in O_C(f^{-1}(U))$ such that the (j,i) entry of $\eta(z)$ is non-zero. Hence, $\mathrm{Tr}(\phi|_U \cdot \eta(z)) \neq 0$, showing $\widetilde{\mu}(\phi|_U)$ is non-zero. □

We now deduce the desired injectivity by studying limits on the central fiber of our degeneration from the previous section, continuing all notation developed there. Recall that for each $\mathcal{L}_0 \in \mathrm{Pic}^d(\mathcal{X}_0)$, we understood $V_{\mathcal{L}} \subset H^0(End(\mathfrak{f}_*\mathcal{L}_0))$ through its image under the inclusion ι as compatible tuples $(\phi_1, \dots, \phi_g)$ in $\bigoplus_{i=1}^g H^0(P^i, End(F^i))$.

Lemma 3.3.2. *For general $\mathcal{X}_t \to \mathcal{P}_t$ in our degeneration,*

$$\mu_t : H^0(\mathcal{P}_t, End(f_*\mathcal{L}_t) \otimes \omega_{\mathcal{P}_t}) \to H^0(\mathcal{P}_t, (f_*O_{\mathcal{X}_t})^\vee \otimes \omega_{\mathcal{P}_t}),$$

is injective for all $\mathcal{L}_t \in \mathrm{Pic}^d(\mathcal{X}_t)$. Hence, if it is non-empty, $W^{\vec{e}}(C)^\circ$ is smooth for general degree k covers $f : C \to \mathbb{P}^1$.

Proof. Let ω be the relative dualizing sheaf of $\mathcal{P} \to B$. Recall that p^0 and p^g are points of total ramification of $\mathfrak{f}|_{\mathcal{X}_0}$ that are distinct from the nodes. We set $\zeta_1 = f(p^0) \in P^1$ and $\zeta_g = f(p^g) \in P^g$, so we have an isomorphism $\omega|_{\mathcal{P}_0} \cong O_{\mathcal{P}_0}(-\zeta_1 - \zeta_g)$.

Let $\mathcal{L}$ be given and define $V_{\mathcal{L}}(-\zeta_1 - \zeta_g) \subset V_{\mathcal{L}}$ to be the subspace of sections vanishing at ζ_1 and ζ_g. We have a commutative diagram

$$\begin{array}{ccccc} V_{\mathcal{L}}(-\zeta_1-\zeta_g) & \longrightarrow & H^0(\mathcal{P}_0, End(\mathfrak{f}_*\mathcal{L}_0)\otimes\omega|_{\mathcal{P}_0}) & \xrightarrow{\mu} & H^0(\mathcal{P}_0, (\mathfrak{f}_*O_{\mathcal{X}_0})^\vee\otimes\omega|_{\mathcal{P}_0}) \\ & \searrow^{\iota} & \downarrow & & \downarrow \\ & & H^0(\mathcal{P}_0, End(\mathfrak{f}_*\mathcal{L}_0)) & \xrightarrow{\widetilde{\mu}} & H^0(\mathcal{P}_0, (\mathfrak{f}_*O_{\mathcal{X}_0})^\vee) \\ & & \downarrow & & \downarrow \\ & & \bigoplus_{i=1}^g H^0(P^i, End(F^i)) & \xrightarrow[\oplus\widetilde{\mu}_i]{} & \bigoplus_{i=1}^g H^0(P^i, ((f^i)_*O_{E^i})^\vee). \end{array}$$

By upper semi-continuity, injectivity of μ_t for general t follows from showing the composition along the top row is injective. We will show that the composition from the upper left to the lower right along the bottom is injective.

For each i, let $W_p^i \subset H^0(P^i, End(F^i))$ denote the subspace of endomorphisms on component i that are order preserving at p. In addition, let $\mathfrak{d}_p^i : W_p^i \to \mathbb{F}^{\oplus k}$ be defined by $\phi_i \mapsto (d_p^{(0)}(\phi_i), \dots, d_p^{(k-1)}(\phi_i))$. Let $(\phi_1, \dots, \phi_g)$ be a compatible tuple. By Lemma 3.2.4, each ϕ_i is order preserving at the nodes on component i. Note that, taking a matrix representation for ϕ_1 as in (3.2.5), the condition $\phi_1(\zeta_1) = 0$ is that $c_{j,\ell} = 0$ and $\alpha_{\ell,j} = 0$ for *all* ℓ, j. Thus, if $\phi_1(\zeta_1) = 0$, then $\phi_1 \in W_{p_0}^1 \cap \ker(\mathfrak{d}_{p_0}^1)$. Similarly, if $\phi_g(\zeta_g) = 0$ then $\phi_g \in W_{p^g} \cap \ker(\mathfrak{d}_{p^g}^g)$. Lemma 3.2.4 (3), implies that if $\phi_{i-1} \in \ker(\mathfrak{d}_{p_i}^{i-1})$ then $\phi_i \in \ker(\mathfrak{d}_{p^i}^i)$. Meanwhile, Lemma 3.2.5 (3) implies that if $\phi_i \in \ker(\mathfrak{d}_{p^{i-1}}^i)$ then $\phi_i \in \ker(\mathfrak{d}_{p^i}^i)$. Thus, the vanishing of diagonal entries “propagates" down the chain. In summary, Lemma 3.2.4 and Lemma 3.2.5 (3) give

$$\iota(V_{\mathcal{L}}(-\zeta_1 - \zeta_g)) \subseteq \left\{(\phi_1, \dots, \phi_g) : \ \phi_i \in W^i_{p^{i-1}} \cap W^i_{p^i} \cap \ker(\mathfrak{d}^i_{p^i}) \text{ for } 1 \leq i \leq g\right\}.$$

If $g > 1$, then we can choose a degree distribution so that $\deg(L^i)$ is never a multiple of k, and hence L^i is never $f^*\mathcal{O}(n)$. Lemma 3.2.5 (4) then ensures that each ϕ_i is represented by a matrix with at most one-nonzero entry. In the case $g = 1$, we need an additional argument if $L^1 = f^*\mathcal{O}(n)$. In this case, by Lemma 3.2.2 choosing any splitting of f_*L^1 induces a splitting $End(f_*L^1) \cong \mathcal{O}(-2) \oplus \mathcal{O}(-1)^{\oplus 2k-4} \oplus \mathcal{O}^{\oplus(k-2)^2+2} \oplus \mathcal{O}(1)^{\oplus 2k-4} \oplus \mathcal{O}(2)$, so all but one of the matrix entries of a global section consist of a constant or linear form, and one entry is quadratic. After imposing vanishing at ζ_1 and ζ_g, only the quadratic entry can be non-zero. In either case, all ϕ_i have at most one non-zero entry, so Lemma 3.3.1 now shows that the composition of the inclusion ι with $\oplus\widetilde{\mu}_i$ is injective. □

3.4 Existence

In this section, we exploit the combinatorial structure of splitting loci stratifications to deduce existence from a simple calculation. This relies on universal enumerative formulas for splitting loci presented in Chapter 2. To make use of Theorem 2.1.2, we need the Chern classes of the relevant vector bundles involved.

Let $f : C \to \mathbb{P}^1$ be a degree k, genus g cover and consider the following commuting triangle

$$\begin{array}{ccc} C \times \mathrm{Pic}^d(C) & \xrightarrow{f \times \mathrm{id}} & \mathbb{P}^1 \times \mathrm{Pic}^d(C) \\ & \searrow^{\nu} & \downarrow \pi \\ & & \mathrm{Pic}^d(C). \end{array}$$

Let $\mathcal{L}$ be a Poincaré line bundle on $C \times \mathrm{Pic}^d(C)$, that is, a line bundle with the property that $\mathcal{L}|_{\nu^{-1}[L]} \cong L$ (see e.g. [2, IV.2]). The push forward $\mathcal{E} := (f \times \mathrm{id})_*\mathcal{L}$ is a vector bundle on $\mathbb{P}^1 \times \mathrm{Pic}^d(C)$ with the property that $\mathcal{E}|_{\pi^{-1}[L]} \cong f_*L$.

Lemma 3.4.1. *Let θ denote the class of the theta divisor on* $\mathrm{Pic}^d(C) = \mathrm{Jac}(C)$. *Then we have $c_i(\pi_*\mathcal{E}(m)) = (-1)^i\theta^i/i!$ modulo classes supported on* $\mathrm{Supp}(R^1\pi_*\mathcal{E}(m))$. *The total Chern class is $c(\pi_*\mathcal{E}(m)) = e^{-\theta}$ away from* $\mathrm{Supp}(R^1\pi_*\mathcal{E}(m))$.

Proof. We have a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\ f\ } & \mathbb{P}^1 \\ \alpha\uparrow & & \uparrow\beta \\ C \times \mathrm{Pic}^d(C) & \xrightarrow{f\times\mathrm{id}} & \mathbb{P}^1 \times \mathrm{Pic}^d(C) \\ & \searrow^{\nu} & \downarrow\pi \\ & & \mathrm{Pic}^d(C) \end{array}$$

Let $H = f^*(\mathcal{O}_{\mathbb{P}^1}(1))$. By the projection formula,

$$\mathcal{E}(m) = ((f \times \mathrm{id})_*\mathcal{L}) \otimes \beta^*\mathcal{O}_{\mathbb{P}^1}(m) = (f \times \mathrm{id})_*(\mathcal{L} \otimes \alpha^* H^{\otimes m}),$$

and so $\pi_*\mathcal{E}(m) = \nu_*(\mathcal{L} \otimes \alpha^* H^{\otimes m})$. We have that $\mathcal{L} \otimes \alpha^* H^{\otimes m}$ is the pullback of a Poincaré line bundle on $C \times \mathrm{Pic}^{d+mk}(C)$ via the identification $\mathrm{Pic}^d(C) \to \mathrm{Pic}^{d+mk}(C)$ given by tensoring with $H^{\otimes m}$. Thus, it suffices to treat the case $m = 0$, where we wish to determine the Chern class of $\pi_*\mathcal{E} = \nu_*\mathcal{L}$.

If $d \geq 2g - 1$, then the argument in [2, p. 317-319] shows that $c(\nu_*\mathcal{L}) = e^{-\theta}$. We claim that for arbitrary d, this formula holds modulo classes supported on $R^1\nu_*\mathcal{L}$. Indeed, let Γ be a collection of n points so that $d + n \geq 2g - 1$. We have an exact sequence on $C \times \mathrm{Pic}^d(C)$

$$0 \to \mathcal{L} \to \mathcal{L}(\Gamma) \to \mathcal{L}(\Gamma)|_\Gamma \to 0.$$

Moreover, $R^1\nu_*\mathcal{L}(\Gamma) = R^1\nu_*\mathcal{L}(\Gamma)|_\Gamma = 0$ and $\nu_*(\mathcal{L}(\Gamma)|_\Gamma)$ is numerically trivial (see [2, p. 309]). It follows that $c(\nu_*\mathcal{L}) = c(\nu_*\mathcal{L}(\Gamma))$ modulo classes supported on $R^1\pi_*\mathcal{L}$. □

Given the Chern classes of $\pi_*\mathcal{E}(m)$, the classes of splitting degeneracy loci are (in theory) computable by the techniques of Chapter 2.

Example 3.4.2. Continuing Example 1.3.5, the classes of the Brill-Noether splitting degeneracy loci on $\mathrm{Pic}^4(C)$ for C a general trigonal curve of genus 5 are

$$[W^{(-2,-1,0)}(\mathcal{E})] = \theta, \qquad [W^{(-2,-2,1)}(\mathcal{E})] = [W^{(-3,0,0)}(\mathcal{E})] = \frac{\theta^4}{24}, \qquad [W^{(-3,-1,1)}(\mathcal{E})] = \frac{\theta^5}{60}.$$

The first three classes are computed using Lemma 2.5.1. The last class comes from twisting and substituting the Chern classes from Lemma 3.4.1 into the universal formula found in Equation (2.6.6). The class of a point in $\mathrm{Pic}^4(C)$ is $\frac{1}{5!}\theta^5$. In particular, $[W^{(-3,-1,1)}(\mathcal{E})]$ is twice the class of a point. This two is our two purple points in Example 1.3.5! Also, $[W^1_4(C)] = [W^{(-2,-2,1)}(\mathcal{E})] + [W^{(-3,0,0)}(\mathcal{E})] = \frac{\theta^4}{12}$ is the class computed via Theorem 1.1.3 part 2, due to Kempf/Klieman–Laksov.

The universal formulas guaranteed by Theorem 2.1.2 are difficult to compute in general, but Lemma 3.4.1 implies the following remarkable fact. Given a splitting type $\vec{e}$, let $|\vec{e}| = e_1 + \ldots + e_k$.

Lemma 3.4.3. *Fix k and $\vec{e} = (e_1, \ldots, e_k)$. Given $f : C \to \mathbb{P}^1$ a genus g curve with degree k map to $\mathbb{P}^1$, let $d = g + k + |\vec{e}| - 1$. The expected class of $W^{\vec{e}}(C)$ in $\mathrm{Pic}^d(C)$ is $a_{\vec{e}} \cdot \theta^{u(\vec{e})}$ for some constant $a_{\vec{e}} \in \mathbb{Q}$ depending only on $\vec{e}$ (independent of g).*

Proof. The loci $W^{\vec{e}}(C)$ are splitting loci of the rank k, degree $|\vec{e}|$ vector bundle $\mathcal{E} = (f \times \mathrm{id})_* \mathcal{L}$ on $\mathbb{P}^1 \times \mathrm{Pic}^d(C)$. By Theorem 2.1.2, the expected class of $W^{\vec{e}}(C)$ is given by a universal formula, depending only on $\vec{e}$, in terms of the Chern classes of $\pi_* \mathcal{E}(m)$ for suitably large m. The ith Chern class of this vector bundle is a multiple of θ^i that does not depend on g by Lemma 3.4.1. □

Remark 3.4.4. For a fixed k, a choice of $|\vec{e}|$ determines an allowed difference

$$d - g = k + |\vec{e}| - 1.$$

(See (1.3.1).) Lemma 3.4.3 is therefore akin to the observation that the formula for the class of $W^r_d(C)$ in Theorem 1.1.3 part 3 depends only on the difference $d - g$.

Remark 3.4.5. In the notation of Theorem 1.3.7 part 3, we will have $N(\vec{e}) = u(\vec{e})! \cdot a_{\vec{e}}$.

Lemma 3.4.3 allows us to leverage the combinatorics of the partial ordering of splitting types to deduce existence from calculations for certain special splitting types. Following the notation in Lemma 2.5.1, let us write $(-n, *, \ldots, *)$ to denote the splitting type of $\mathcal{O}_{\mathbb{P}^1}(-n) \oplus B(k-1, |\vec{e}| + n)$.

Lemma 3.4.6. *For every $\vec{e}$, there exists n such that $(-n, *, \ldots, *) \leq \vec{e}$. We have*

$$a_{(-n,*,\ldots,*)} = \frac{1}{u(-n, *, \ldots, *)!}.$$

Proof. We may take $n = -(|\vec{e}| + e_1 k)$. Notice that $\mathrm{Supp}(R^1 \pi_* \mathcal{E}(n-1)) = \overline{\Sigma}_{(-n-1,*,\ldots,*)}(\mathcal{E})$, which has codimension larger than $u(-n, *, \ldots, *)$ by Lemma 3.2.6. Therefore, we may calculate the class of $\overline{\Sigma}_{(-n,*,\ldots,*)}(\mathcal{E})$ on the complement of $\mathrm{Supp}(R^1 \pi_* \mathcal{E}(n-1))$. On the complement, Lemma 3.4.1 says that $c((\pi_* \mathcal{E}(n-1))^\vee) = c((\pi_* \mathcal{E}(n))^\vee) = e^\theta$. By Lemma 2.5.1, we have

$$[\overline{\Sigma}_{(-n,*,\ldots,*)}(\mathcal{E})] = \left[\frac{c((\pi_* \mathcal{E}(n-1))^\vee)^2}{c((\pi_* \mathcal{E}(n))^\vee)} \right]_{u(-n,*,\ldots,*)} = \left[\frac{(e^\theta)^2}{e^\theta} \right]_{u(-n,*,\ldots,*)} = \frac{\theta^{u(-n,*,\ldots,*)}}{u(-n, *, \ldots, *)!}$$

as desired. □

Proof of Theorem 1.3.7. We will show that $a_{\vec{e}}$ is non-zero for all $\vec{e}$. Since its expected class is non-zero, this will imply $W^{\vec{e}}(C)$ is non-empty whenever $u(\vec{e}) \leq g$. Then, Lemmas 2.2.2 and 3.2.6 show that $W^{\vec{e}}(C)$ has dimension $g - u(\vec{e})$ (part 1). Lemma 3.3.2 shows $W^{\vec{e}}(C)^{\circ}$ is smooth (part 2).

Fix $\vec{e}$ and choose n such that $(-n, *, \ldots, *) \leq \vec{e}$. Choose any $g' \geq u(-n, *, \ldots, *)$ and let $f' : C' \to \mathbb{P}^1$ be a general point of $\mathcal{H}_{k,g'}$. Let $d' = g' + k + |\vec{e}| - 1$. By Lemma 3.4.6, $W^{(-n,*,\ldots,*)}(C') \subset \mathrm{Pic}^{d'}(C')$ is non-empty. Thus, $W^{\vec{e}}(C') \subset \mathrm{Pic}^{d'}(C')$ is non-empty too. By Lemmas 2.2.2 and 3.2.6, $\operatorname{codim} W^{\vec{e}}(C') = u(\vec{e})$. Being non-empty of the expected codimension on a projective variety, $[W^{\vec{e}}(C')] = a_{\vec{e}} \cdot \theta^{u(\vec{e})} \neq 0$ on $\mathrm{Pic}^{d'}(C')$. Hence $a_{\vec{e}} \neq 0$, as desired. □

3.5 Components of $W^r_d(C)$

To characterize contributions of splitting loci to the components of $W^r_d(C)$ and prove Corollary 1.3.8, we are interested in splitting types that are maximal with respect to the partial ordering among those satisfying $h^0(\mathcal{O}(\vec{e})) \geq r + 1$. (Recall Equation 1.3.3.) Continuing our colored pictures analogy as in Figure 1.5, one might think of these splitting types as the “primary colors" of our palette.

Lemma 3.5.1. *Let $d' = d - g + 1 - k$ and suppose $r > d - g$. The maximal splitting types of rank k, degree d' among those satisfying $h^0(\mathcal{O}(\vec{e})) \geq r + 1$ are*

$$\vec{w}_{r,\ell} := B(k - r - 1 + \ell, d' - \ell) \oplus B(r + 1 - \ell, \ell)$$

for $\max\{0, r + 2 - k\} \leq \ell \leq r$ *such that* $\ell = 0$ *or* $\ell \leq g - d + 2r + 1 - k$. *Moreover,*

$$u(\vec{w}_{r,\ell}) = g - \rho(g, r - \ell, d) + \ell k.$$

Remark 3.5.2. If $r \leq d - g$ we automatically have $W^r_d(C) = \mathrm{Pic}^d(C)$. As Pflueger points out in [61, Remarks 1.6 and 3.2], the codimension $g - \rho(g, r - \ell, d) + \ell k$ is quadratic in ℓ, achieving its minimum at $\ell_0 = \frac{1}{2}(g - d + 2r + 1 - k)$. Our lower bound $r + 2 - k$ is the same distance from the minimum ℓ_0 as Pflueger's upper bound $g - d + r - 1$. From this, it is not hard to see that the minimum over ℓ in our range is the same as Pflueger's minimum.

Proof. The assumption $r > d - g$ implies $k - r - 1 + \ell < \ell - d'$ so $B(k - r - 1 + \ell, d' - \ell)$ consists of entirely negative summands. Requiring that the rank of this vector bundle is positive gives our lower bound $\ell \geq r + 2 - k$.

First we show every $\vec{e}$ with $h^0(\mathcal{O}(\vec{e})) \geq r+1$ is less than $\vec{w}_{r,\ell}$ for some ℓ. We may write $\mathcal{O}(\vec{e}) = N \oplus P$ where N consists of negative summands, and P consists of nonnegative summands. If $h^0(\mathbb{P}^1, P) > r+1$, then the splitting type obtained from $\vec{e}$ by decreasing the largest summand by one and increasing the lowest summand by one is more balanced than $\vec{e}$ and still has at least $r+1$ sections. Hence, it suffices to consider the case $h^0(\mathbb{P}^1, P) = r+1$. Then, $\vec{e} \leq \vec{w}_{r,\ell}$ for $\ell = \deg P$.

Suppose $\vec{e}$ is minimal among splitting types with $\vec{e} > \vec{w}_{r,\ell}$. Then by construction, $\vec{e}$ is obtained from $\vec{w}_{r,\ell}$ by lowering a summand in $B(r+1-\ell, \ell)$ and raising a summand in $B(k-r-1+\ell, d'-\ell)$. Hence, $\mathcal{O}(\vec{e})$ has less than $r+1$ global sections unless $\ell > 0$ and $B(k-r-1+\ell, d'-\ell)$ has a summand of degree -1. In that case, we see $\vec{w}_{r,\ell} < \vec{w}_{r,\ell-1}$. Thus, $\vec{w}_{r,\ell}$ is maximal precisely when $\ell = 0$ or all summands of $B(k-r-1+\ell, d'-\ell)$ are degree at most -2. The latter means $2(k-r-1+\ell) \leq \ell - d'$, which is equivalent to $\ell \leq g-d+2r+1-k$.

Finally, the expected codimension of $\vec{w}_{r,\ell}$ is

$$\begin{aligned} u(\vec{w}_{r,\ell}) = h^1(\mathbb{P}^1, End(\vec{w}_{r,\ell})) &= h^1(\mathbb{P}^1, Hom(B(r+1-\ell,\ell), B(k-r-1+\ell, d'-\ell)) \\ &= -\chi(\mathbb{P}^1, Hom(B(r+1-\ell,\ell), B(k-r-1+\ell, d'-\ell))) \\ &= \ell(k-r-1+\ell) - (d'-\ell)(r+1-\ell) - (r+1-\ell)(k-r-1+\ell) \\ &= \ell k - (r+1-\ell)(d-g-r+\ell). \end{aligned}$$

□

Example 3.5.3. The following table lists the “balanced plus balanced" splitting types of rank 5 and degree −4 with at least 4 global sections. The first three are maximal.

ℓ	0	1	2	3
$\vec{w}_{3,\ell}$	$(-4,0,0,0,0)$	$(-3,-2,0,0,1)$	$(-2,-2,-2,1,1)$	$(-2,-2,-2,-1,3)$
$u(\vec{w}_{3,\ell})$	12	11	12	15

Notice that $w_{3,3} < w_{3,2}$ in the partial ordering, showing necessity of the condition $\ell \leq g-d+2r+1-k$ in Lemma 3.5.1. Corollary 1.3.8 says that for a general pentagonal curve, every component of $W^3_g(C)$ has dimension $g-11$ or $g-12$. Moreover, there is at least one component of dimension $g-11$ and at least two components of dimension $g-12$ when these quantities are nonnegative.

Proof of Corollary 1.3.8. Equation (1.3.3) and Lemma 3.5.1 show that $W^r_d(C)$ is the union of $W^{\vec{w}_{r,\ell}}(C)$ for $\max\{0, r+2-k\} \leq \ell \leq r$ such that $\ell = 0$ or $\ell \leq g-d+2r+1-k$.

Theorem 1.3.7 parts 1 and 2 assert that $W^{\vec{w}_{r,\ell}}(C)^\circ$ is smooth of pure dimension $g - u(\vec{w}_{r,\ell})$ whenever this quantity is nonnegative. $\square$

Chapter 4

Brill–Noether splitting loci: Part II

All work in this chapter is joint with Eric Larson and Isabel Vogt. In this collaboration, we learned more about Brill–Noether splitting loci by identifying the limits of line bundles with a given splitting type in our degeneration. In particular, this allows for results about the global geometry of Brill–Noether splitting loci, which were not accessible with the techniques of the previous chapter.

4.1 The results in more detail

In this chapter we will prove the following results. We shall write $\rho' := \rho'(g, \vec{e}) = g - u(\vec{e})$. This continues Theorem 1.3.7. (The numbering parallels Theorem 1.1.3.)

Theorem 4.1.1. *Suppose that the characteristic of the ground field is zero, or greater than k. Let $f\colon C \to \mathbb{P}^1$ be a general degree k cover of genus g, and let $\vec{e}$ be any splitting type.*

2. *(cont.) $W^{\vec{e}}(C)$ is normal, Cohen–Macaulay, and smooth away from the union of the $W^{\vec{e}'}(C) \subset W^{\vec{e}}(C)$ having codimension 2 or more.*

3. *(cont.) The integer $N(\vec{e})$ is equal to the number of efficient k-regular fillings of $\Gamma(\vec{e})$. Equivalently, it is equal to the number of reduced words for a certain element of the affine symmetric group (see Theorem 4.1.3 for a precise statement).*

4. *$W^{\vec{e}}(C)$ is irreducible when $\rho' > 0$.*

5. *When $\rho' \geq 0$, the universal $\mathcal{W}^{\vec{e}}$ has a* unique *component dominating the Hurwitz space $\mathcal{H}_{k,g}$ of degree k genus g covers of $\mathbb{P}^1$.*

See Remark 4.1.5 for more details on the characteristic assumptions.

It turns out that Theorem 4.1.1 part 2 together with Theorem 1.3.7 part 2 implies a conjecture of Eisenbud and Schreyer regarding the equations of splitting loci on versal deformation spaces. Suppose $\vec{e}' \leq \vec{e}$; let $\mathcal{F}$ on $\mathbb{P}^1 \times \mathrm{Def}(\mathcal{O}(\vec{e}'))$ be the versal deformation of $\mathcal{O}(\vec{e}')$. Consider the subscheme $\overline{\Sigma}_{\vec{e}}(\mathcal{F}) \subseteq \mathrm{Def}(\mathcal{O}(\vec{e}'))$, defined by the Fitting support for where $\mathrm{rk}\, R^1\pi_*\mathcal{F}(m) \geq h^1(\mathbb{P}^1, \mathcal{O}(\vec{e})(m))$. Eisenbud and Schreyer conjecture that $\overline{\Sigma}_{\vec{e}}(\mathcal{F})$ is reduced (Conjecture 5.1 [22]). We saw earlier Remark 2.4.2 that $\overline{\Sigma}_{\vec{e}}(\mathcal{F})$ is generically reduced, but did not rule out the possibility of embedded components along the locus $\overline{\Sigma}_{\vec{e}}(\mathcal{F}) \smallsetminus \Sigma_{\vec{e}}(\mathcal{F})$ of "worse" splitting types.

Corollary 4.1.2. *Suppose that the characteristic of the ground field does not divide k (c.f. Remark 4.1.5). Then $\overline{\Sigma}_{\vec{e}}(\mathcal{F})$ is normal and Cohen–Macaulay (and hence reduced).*

Proof. Let $f\colon C \to \mathbb{P}^1$ be a general cover of genus $g \geq u(\vec{e}')$ and let $L \in W^{\vec{e}'}(C)$. By Theorem 1.3.7 part 2, the induced map from $\mathrm{Pic}^d(C)$ near L to $\mathrm{Def}(f_*L) = \mathrm{Def}(\mathcal{O}(\vec{e}'))$ is smooth when the characteristic of the ground field does not divide k. Thus, the fact that $W^{\vec{e}}(C)$ is normal and Cohen–Macaulay implies $\overline{\Sigma}_{\vec{e}}(\mathcal{F})$ is normal and Cohen–Macaulay. □

To further explain Theorem 4.1.1 part 3, let W be a Coxeter group with generating set S, and let $w \in W$ be an element. Define

$$R(w) := \text{number of reduced words for } (W, S) \text{ equal to } w.$$

Determination of the integers $R(w)$ is a well-studied problem in combinatorics, starting with Stanley's computation of $R(w)$ for Coxeter groups of type A (i.e. the symmetric groups), and his proposal for a systematic study of $R(w)$ for other Coxeter groups, in 1984 [66]. This problem has since been solved completely for other finite Coxeter groups — including of type B by Haiman in 1992 [38], and of type D by Billey and Haiman in 1995 [4] — and partial progress has been made for some infinite Coxeter groups by Eriksson, Fan, and Stembridge in a series of papers from the late 1990s [27, 28, 29, 67, 68, 69].

Of particular relevance to us are the Coxeter systems of type $\widetilde{A}$, known as *affine symmetric groups*. Explicitly, these are groups generated by elements s_j with $j \in \mathbb{Z}/k\mathbb{Z}$, subject to relations

$$s_j^2 = 1, \quad s_j s_{j'} = s_{j'} s_j \text{ if } j - j' \neq \pm 1, \quad \text{and} \quad (s_j s_{j+1})^3 = 1.$$

Alternatively, elements of the affine symmetric group can be realized as permutations $f\colon \mathbb{Z} \to \mathbb{Z}$ such that

$$f(x+k) = f(x) + k \quad \text{and} \quad \sum_{x=1}^{k} f(x) = \sum_{x=1}^{k} x = \frac{k(k+1)}{2};$$

here s_j corresponds to the simple transposition defined by

$$f(x) = \begin{cases} x+1 & \text{if } x \equiv j \mod k; \\ x-1 & \text{if } x \equiv j+1 \mod k; \\ x & \text{otherwise.} \end{cases}$$

For the affine symmetric group, Eriksson [27] gave recursive formulas for $R(w)$, and showed that for fixed k the generating function for $R(w)$ is rational.

It turns out that the efficient k-regular fillings of the diagram $\Gamma(\vec{e})$ mentioned in Section 1.3.6 are in natural correspondence with reduced words for a certain element in the affine symmetric group. Our regeneration theorem relates components of the Brill–Noether splitting locus on the central fiber to these fillings, or equivalently reduced words in the affine symmetric group. As a consequence, the count of points (when $\rho' = 0$) on the general fiber is equal to the count on the central fiber. Therefore we obtain:

Theorem 4.1.3. *Given a splitting type $\vec{e}$, define $w(\vec{e})$ to be the affine symmetric group element that sends (for $1 \leq \ell \leq k$):*

$$\ell \mapsto \chi(\mathcal{O}(\vec{e})(-e_{k+1-\ell})) - \#\{\ell' : e_{\ell'} \geq e_{k+1-\ell}\} + \#\{\ell' : \ell' \geq k+1-\ell \text{ and } e_{\ell'} = e_{k+1-\ell}\}.$$

Then

$$N(\vec{e}) = R(w(\vec{e})).$$

In particular, the integers $N(\vec{e})$ grow rapidly, and may be easily computed in any desired case using Eriksson's recursions mentioned above. For example, $N(2,7,18,18,28,28)$ is the integer

$$25867977167969459670048709047628541850991022718608668059259099938720 \approx 2.6 \cdot 10^{67}.$$

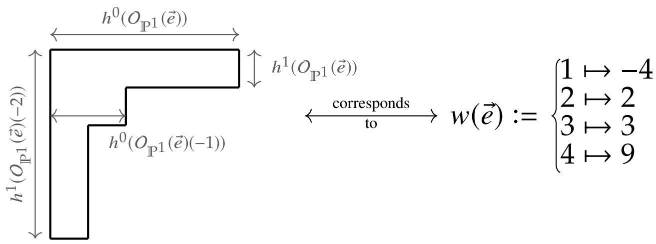
One can also check that the description of $N(\vec{e})$ in Theorem 4.1.3 agrees with the conjectural value of $N(\vec{e})$ proposed by Cook-Powel–Jensen, and hence proves Conjecture 1.6 of [9].

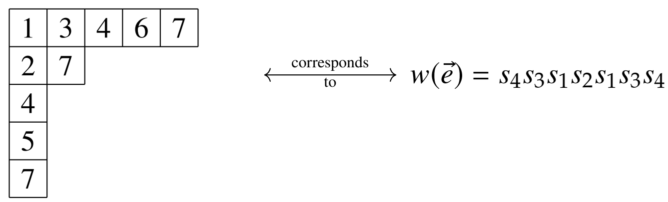
4.1.4 Overview of Techniques

The degeneration we use is to a chain of elliptic curves, the same as in the previous chapter (see Section 3.1). In Section 4.2, we identify the sorts of objects that look like they might be a limit of line bundles in $W^{\vec{e}}(C)$; we call these *$\vec{e}$-positive limit line bundles*.

This locus of $\vec{e}$-positive limit line bundles has an intricate combinatorial structure: In Section 4.3 we show that its components are in bijection with certain fillings of a certain

Young diagram $\Gamma(\vec{e})$. In Section 4.4, we relate these fillings to the reduced word problem for the affine symmetric group. As a preview, for example, the splitting type $\vec{e} = (-2, 0, 0, 2)$ corresponds to the Young diagram to the right. When $g = u(\vec{e}) = 7$, there are six $\vec{e}$-positive limit line bundles on the central fiber, corresponding to six fillings, one of which is shown below:





We then prove our regeneration theorem, which is the heart of this chapter since it provides the bridge between the combinatorics of the central fiber and the geometry of the general fiber. Because the components of W^r_d have the "wrong" dimension, naively applying the techniques used by Eisenbud and Harris to prove their regeneration theorem in [19] necessarily produces too many equations. Our key insight is that the combinatorial structure coming from the affine symmetric group forces the limit linear series associated to a *general* $\vec{e}$-positive limit line bundle to "break up" into minimally-interacting pieces that can be regenerated almost independently. This allows us to avoid overcounting equations, and prove a regeneration theorem in Section 4.5. However, this "breaking up" happens a priori only set-theoretically. We then upgrade this to a scheme-theoretic regeneration theorem in Section 4.6 by showing that the locus of $\vec{e}$-positive limit line bundles on the central fiber is reduced.

Having established the regeneration theorem, we then deduce the fundamental global geometric properties of Brill–Noether splitting loci in Sections 4.7–4.9.

Remark 4.1.5 (A note on our ground field)**.** Since the conclusion of Theorem 4.1.1 is geometric, we suppose for the remainder of the document that our ground field K is algebraically closed.

The assumption that the characteristic of K is zero or greater than k is used only to guarantee the irreducibility of $\mathcal{H}_{k,g}$ (as proved by Fulton in [30]), and hence to be able to state Theorem 4.1.1 in terms of a “general" degree k cover. However, in any characteristic not dividing k, the conclusion of Theorem 4.1.1 holds for *some* component of $\mathcal{H}_{k,g}$. In particular, Corollary 4.1.2 actually only requires that the characteristic does not divide k. Certain parts of Theorem 4.1.1 require even weaker assumptions. In any characteristic, we have Theorem 4.1.1 part 3 and weaker versions of part 2 and 4:

2.’ $W^{\vec{e}}(C)$ is reduced and Cohen–Macaulay.

4.’ $W^{\vec{e}}(C)$ is connected if $\rho' > 0$.

This chapter is organized so that characteristic assumptions are made as late as possible. All of Sections 4.2 – 4.7 make no assumptions on the characteristic of the ground field. Sections 4.8 and 4.9 assume that the ground field has characteristic not dividing k.

Remark 4.1.6 (A note on Hurwitz spaces)**.** Our arguments show the a priori stronger statement that there exists a smooth degree k cover $f : C \to \mathbb{P}^1$ *with two points of total ramification* satisfying Theorem 4.1.1 part 2 – 4. Moreover, in Theorem 4.1.1 part 5, the Hurwitz space can be replaced with a component of the stack $\mathcal{H}_{k,g,2}$ parameterizing degree k genus g covers of $\mathbb{P}^1$ with two marked points of total ramification (see Definition 4.9.3).

4.2 Limits of Line Bundles

In this section, let $\mathfrak{f} : \mathcal{C} \to \mathcal{P} \to B$ be a family of degree k genus g covers, over a smooth irreducible base B, which is smooth over the generic point B^*, and has smooth total space $\mathcal{C}$. (Prior to Section 4.9, the only case of interest will be when B is the spectrum of a DVR.) We suppose that *all* fibers (including over non-closed points) of $\mathcal{C} \to B$ are *chain curves*, i.e. of the form $C^1 \cup_{p^1} \cup \cdots \cup_{p^{n-1}} C^n$, with all C^i smooth. (The integer n will depend on which fiber we consider.) Equivalently, all *geometric* fibers of $\mathcal{C} \to B$ are chain curves, and these chain curves can be oriented (i.e. the two ends can be distinguished) in a way which is consistent over B. This second condition holds, in particular, if $\mathcal{C} \to B$ has a section whose value at any geometric point $C^1 \cup_{p^1} \cup \cdots \cup_{p^{n-1}} C^n$ is supported in $C^1 \smallsetminus \{p^1\}$ (which allows us to consistently pick which end of the chain is “left” and “right”).

Similarly, we suppose that all fibers of $\mathcal{P} \to B$ are chain curves with all components $P^i \simeq \mathbb{P}^1$, and that the map $\mathfrak{f} : \mathcal{C} \to \mathcal{P}$ respects this structure. Finally, we suppose that for each fiber the maps $f^i : C^i \to P^i$ are totally ramified at the nodes p^{i-1} and p^i (note that this condition is vacuous if $\mathcal{C}$ is smooth).

Note that such covers include our degeneration $\mathcal{X} \to \mathcal{P} \to B$ from the previous section as the special case where B is the spectrum of a DVR and all C^i have genus 1. Similarly, this includes $\mathcal{P}_0 \xrightarrow{\sim} \mathcal{P}_0 \to B_0$ as the special case where all C^i have genus 0.

In this section, we address the following two fundamental questions:

1. Suppose $\mathcal{L}^*$ is a line bundle of degree d on the generic fiber $C^* = C \times_B B^*$. What data do we obtain on a special fiber over $b \in B$?

2. If $\mathfrak{f}_* \mathcal{L}^*$ has splitting type $\vec{e}$, what conditions must this data on a special fiber satisfy?

These questions are local on B. Shrinking B if necessary, we may suppose that every component of the singular locus Δ of $\mathfrak{f}$ meets the fiber over $b \in B$. In other words, writing

$$C = C \times_B b = C^1 \cup_{p^1} \cup \cdots \cup_{p^{n-1}} C^n,$$

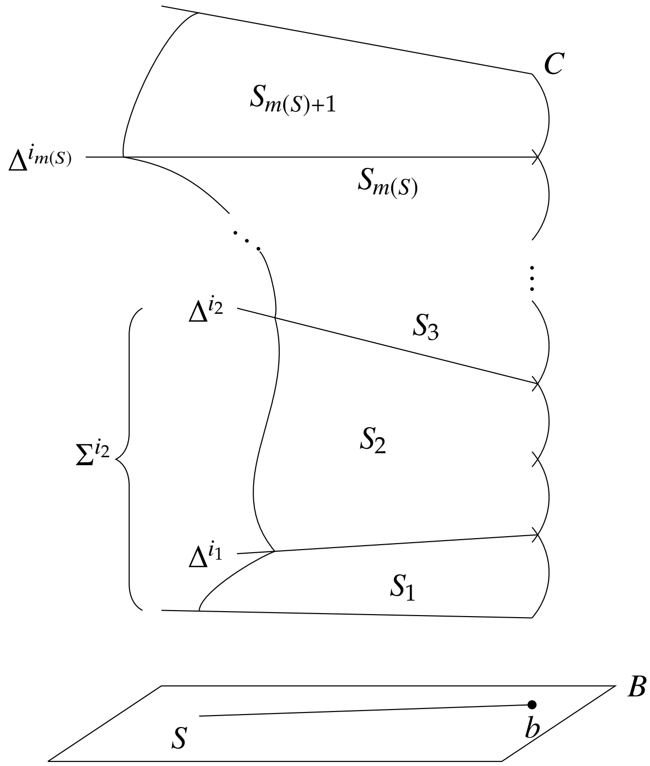
every component of Δ contains some p^i.

We now turn to Question (1) above. Since C is smooth, we may extend $\mathcal{L}^*$ to a line bundle $\mathcal{L}$ on C. However, this extension is only unique up to twisting by divisors on C that do not meet the generic fiber, i.e. which do not dominate B. We now describe a basis for such divisors.

Since $C \to B$ is a family of chain curves, each component of Δ contains at most one p^i. Because $C \to B$ is a family of nodal curves, $\mathfrak{f} \colon \Delta \to B$ is unramified. Moreover, because the versal deformation space of a node is $\operatorname{Spec} K[[x, y, t]]/(xy - t) \to \operatorname{Spec} K[[t]]$, and the total space C is smooth, the image under $\mathfrak{f}$ of any component of Δ is a smooth divisor in B. Consequently, Δ is smooth of codimension 2 in C. Thus, each p^i is contained in a unique component of Δ^i.

Putting this together, there are exactly $n - 1$ components of Δ, one containing each node of C. Label these components $\Delta^1, \Delta^2, \ldots, \Delta^{n-1}$, so that Δ^i contains p^i.

Consider any component S of $\mathfrak{f}(\Delta)$, and let $\{i_1, i_2, \ldots, i_{m(S)}\}$ denote the set of i such that $\mathfrak{f}(\Delta^i) = S$. (As we range through all components of $\mathfrak{f}(\Delta)$, these sets form a partition of $\{1, 2, \ldots, n\}$.) Then, because $C \to B$ is a family of chain curves, $\mathfrak{f}^{-1}(S) = S_1 \cup S_2 \cup \cdots \cup S_{m(S)+1}$ has exactly $m(S) + 1$ components, meeting pairwise along the Δ^{i_j}:



As shown in the above diagram, these components are indexed so that:

$$S_j \cap C = \begin{cases} C^1 \cup \cdots \cup C^{i_1} & \text{if } j = 1; \\ C^{i_{m(S)}+1} \cup \cdots \cup C^n & \text{if } j = m(S)+1; \\ C^{i_{j-1}+1} \cup \cdots \cup C^{i_j} & \text{otherwise.} \end{cases} \quad \text{and} \quad S_j \cap S_{j'} = \begin{cases} \Delta^{i_j} & \text{if } j' = j+1; \\ \emptyset & \text{if } j' > j+1. \end{cases}$$

For $1 \leq j \leq m(S)$, we define

$$\Sigma^{i_j} = S_1 + S_2 + \cdots + S_j \quad \text{which satisfies} \quad \Sigma^{i_j} \cap C = C^1 + C^2 + \cdots + C^{i_j}.$$

By construction, every divisor on C supported on $\mathfrak{f}^{-1}(S)$ is a unique linear combination of the Σ^{i_j} and $\mathfrak{f}^{-1}(S)$. Repeating this construction for every component S of $\mathfrak{f}(\Delta)$, we will have defined divisors Σ^i for all $1 \leq i \leq n-1$.

Example 4.2.1. When B is the spectrum of a DVR, and b is the special fiber, then we have $\Sigma^i = C^1 + C^2 + \cdots + C^i$.

Now suppose that D is any irreducible divisor such that $\mathfrak{f}(D)$ is a divisor on B not contained in $\mathfrak{f}(\Delta)$. Then the generic fiber of C over $\mathfrak{f}(D)$ is irreducible, so D is a multiple of $\mathfrak{f}^{-1}(\mathfrak{f}(D))$. Putting this together, we learn that any divisor on C that does not dominate B can be written uniquely as a linear combination of the Σ^i and the pullback of a divisor on B.

Note that twisting by the pullback of a divisor on B does not change $\mathcal{L}|_C$, and that twisting by the Σ^i changes the $\mathcal{L}|_{C^j}$ as follows:

$$\mathcal{L}(\Sigma^i)|_{C^j} \simeq \begin{cases} \mathcal{L}|_{C^j}(p^i) & \text{if } j = i; \\ \mathcal{L}|_{C^j}(-p^i) & \text{if } j = i+1; \\ \mathcal{L}|_{C^j} & \text{otherwise.} \end{cases} \tag{4.2.1}$$

In particular, for any *degree distribution* $\vec{d} = (d^1, d^2, \ldots, d^n)$ with $d = \sum d^i$, there is an extension $\mathcal{L}_{\vec{d}}$ of $\mathcal{L}^*$ to C so that $\mathcal{L}_{\vec{d}}|_C$ has degree $\vec{d}$ (i.e. has degree d^i on C^i), which is unique up to twisting by the pullback of a divisor on B. Moreover, any one extension $\mathcal{L}_{\vec{d}}$ determines all other extensions (up to pullbacks of divisors on B) via the above relation.

Restricting to the fiber C over b, we conclude that for each such degree distribution $\vec{d}$, there is a unique limit $L_{\vec{d}} := \mathcal{L}_{\vec{d}}|_C$ of degree $\vec{d}$. Moreover, any one limit $L_{\vec{d}}$ determines all other limits via repeatedly applying the relation:

$$L_{(d^1,d^2,\ldots,d^i+1,d^{i+1}-1,\ldots,d^g)}|_{C^j} \simeq \begin{cases} L_{(d^1,d^2,\ldots,d^g)}|_{C^j}(p^i) & \text{if } j = i; \\ L_{(d^1,d^2,\ldots,d^g)}|_{C^j}(-p^i) & \text{if } j = i+1; \\ L_{(d^1,d^2,\ldots,d^g)}|_{C^j} & \text{otherwise.} \end{cases} \tag{4.2.2}$$

The following definition thus encapsulates the data we obtain on any fiber:

Definition 4.2.2. Let

$$\operatorname{Pic}^d C := \frac{\bigsqcup_{\vec{d}:\sum d^i = d} \operatorname{Pic}^{\vec{d}} C}{\sim},$$

where $\sim$ denotes the equivalence relation generated by (4.2.2). We call elements L of $\operatorname{Pic}^d C$ *limit line bundles* of degree d, and write $L_{\vec{d}}$ for the corresponding line bundle on C of degree $\vec{d}$.

If $D = C^i \cup C^{i+1} \cup \cdots \cup C^j \subset C$ is any connected curve, we write L^D for the "restriction of L to D as a limit line bundle of degree d". More formally, for any degree distribution $(d^i, d^{i+1}, \ldots, d^j)$ on D with $d^i + d^{i+1} + \cdots + d^j = d$, we have

$$(L^D)|_{(d^i,d^{i+1},\ldots,d^j)} = L_{(0,\ldots,0,d^i,d^{i+1},\ldots,d^j,0,\ldots,0)}|_D.$$

For ease of notation when $C = X$ (respectively $C = P$) is our chain of g elliptic (respectively rational) curves, we set $L^i = L^{E^i}$ (respectively $L^i = L^{P^i}$). These are limit line bundles on smooth curves, which are just ordinary line bundles.

In other words, if we fix a degree distribution $\vec{d}$ with $\sum d^i = d$, then we have a natural isomorphism $\operatorname{Pic}^d C \simeq \operatorname{Pic}^{\vec{d}} C$; but $\operatorname{Pic}^d C$ exists without fixing a degree distribution (although its elements do not then yet correspond naturally to line bundles on C). Note that $\operatorname{Pic}^d C$ is a torsor for $\operatorname{Pic}^\circ C \simeq \prod \operatorname{Pic}^0 C^i$, and that there are natural tensor product maps $\operatorname{Pic}^{d_1} C \times \operatorname{Pic}^{d_2} C \to \operatorname{Pic}^{d_1+d_2} C$.

Example 4.2.3. Consider the family appearing in Section 3.1. When $\mathcal{L}^* = \mathcal{O}_{C^*}(m) := \mathfrak{f}^*\mathcal{O}_{\mathcal{P}^*}(m)$, we obtain limit line bundles $\mathcal{O}_C(m)$. These can be described in terms of the geometry of the central fiber alone: For instance, if we fix the degree distribution $(mk, 0, \ldots, 0)$, we have

$$\mathcal{O}_C(m)_{(mk,0,\ldots,0)}|_{C^i} = \begin{cases} \mathcal{O}_{C^1}(m) := (f^1)^*\mathcal{O}_{\mathbb{P}^1}(m) & \text{if } i = 1; \\ \mathcal{O}_{C^i} & \text{otherwise.} \end{cases}$$

By slight abuse of notation, we write $\mathcal{O}_{\mathcal{P}}(m)^i := \beta^*\mathcal{O}_{\mathcal{P}_0}(m)^i$, where $\beta\colon \mathcal{P} \to \mathcal{P}_0$ is the base-change of $\beta\colon B \to B_0$ appearing in Section 3.1.

This then provides an answer to the first question posed at the beginning of the section: To a line bundle $\mathcal{L}^*$ on C^* on the generic fiber, we can associate a limit line bundle L of degree d on C.

We now turn to the second question: Suppose that $\mathfrak{f}_*\mathcal{L}^*$ has splitting type $\vec{e}$. What can we say about the associated limit line bundle L? First of all,

$$\chi(L) = \chi(\mathcal{L}^*) = \chi(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\vec{e})),$$

and so

$$d = g - 1 + \chi(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\vec{e})). \tag{4.2.3}$$

Moreover, since $\mathcal{L}^*$ has splitting type $\vec{e}$, we have

$$h^0(C^*, \mathcal{L}^*(m)) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\vec{e})(m)) = \sum_{\ell=1}^{k} \max(0, e_\ell + m + 1) \quad \text{for any } m. \tag{4.2.4}$$

By semicontinuity, the limit line bundle L therefore satisfies

$$h^0(C, L(m)_{\vec{d}}) \geq \sum_{\ell=1}^{k} \max(0, e_\ell + m + 1) \quad \text{for any degree distribution } \vec{d} \text{ with } \sum_{i=1}^{n} d^i = d + mk. \tag{4.2.5}$$

The following definition thus encapsulates the conditions our data on the central fiber must satisfy:

Definition 4.2.4. We say that a limit line bundle $L \in \mathrm{Pic}^d(C)$ is *$\vec{e}$-positive* if it satisfies (4.2.3) and (4.2.5).

This then provides an answer to the second question posed at the beginning of the section: If $\mathfrak{f}_*\mathcal{L}^*$ has splitting type $\vec{e}$, then the associated limit line bundle L must be $\vec{e}$-positive.

In fact, there is a proper *scheme* $W^{\vec{e}}(\mathcal{C})$ over B whose fibers over every point parameterize $\vec{e}$-positive line bundles on the corresponding fiber of $\mathcal{C} \to B$. This scheme will be an intersection of determinantal loci (over all degree distributions). To construct this scheme, work locally on the base near $b \in B$ as above, and write $\pi\colon \mathrm{Pic}^d(\mathcal{C}/B) \times_B \mathcal{C} \to \mathrm{Pic}^d(\mathcal{C}/B)$ for the projection map. For any degree distribution $\vec{d}$ on $C := \mathcal{C} \times_B b$ of $d + mk$, we obtain a universal bundle $\mathcal{L}(m)_{\vec{d}}$. For each m and $\vec{d}$, there is a natural scheme structure on

$$\begin{aligned}\{L \in \mathrm{Pic}^{\vec{d}}(\mathcal{C}/B) : h^0(\pi^{-1}(L), L(m)) \geq h^0(\mathbb{P}^1, \mathcal{O}(\vec{e})(m))\} \\ = \{L \in \mathrm{Pic}^d(\mathcal{C}/B) : \mathrm{rk}(R^1\pi_*\mathcal{L}(m)_{\vec{d}})|_L \geq h^1(\mathbb{P}^1, \mathcal{O}(\vec{e})(m))\},\end{aligned}$$

defined by the Fitting support for where $\mathrm{rk}\, R^1\pi_*\mathcal{L}(m)_{\vec{d}} \geq h^1(\mathbb{P}^1, \mathcal{O}(\vec{e})(m))$, as we now recall. The Fitting supports of a coherent sheaf are defined by the appropriately sized determinantal loci of a resolution by vector bundles and are independent of the resolution (see for example Section 20.2 of [23]).

An often-used resolution of $R^1\pi_*\mathcal{L}(m)_{\vec{d}}$ is constructed as follows. Let $D_{\vec{d}} \subset \mathcal{C}$ be a sufficiently relatively ample divisor (relative to $\vec{d}$), so that $\pi_*[\mathcal{L}(m)_{\vec{d}}(D_{\vec{d}})]$ and $\pi_*[\mathcal{L}(m)_{\vec{d}}(D_{\vec{d}})|_{D_{\vec{d}}}]$ are vector bundles on $\mathrm{Pic}^d(\mathcal{C}/B)$. Pushing forward the exact sequence

$$0 \to \mathcal{L}(m)_{\vec{d}} \to \mathcal{L}(m)_{\vec{d}}(D_{\vec{d}}) \to \mathcal{L}(m)_{\vec{d}}(D_{\vec{d}})|_{D_{\vec{d}}} \to 0$$

by π we see that the restriction map

$$\pi_*[\mathcal{L}(m)_{\vec{d}}(D_{\vec{d}})] \to \pi_*[\mathcal{L}(m)_{\vec{d}}(D_{\vec{d}})|_{D_{\vec{d}}}]$$

provides a resolution of $R^1\pi_*\mathcal{L}(m)_{\vec{d}}$. Using the scheme structure defined by the appropriate minors, we define

$$W^{\vec{e}}(\mathcal{C}) := \bigcap_{m,\vec{d}} \left\{L \in \mathrm{Pic}^{\vec{d}}(\mathcal{C}/B) : \mathrm{rk}(R^1\pi_*\mathcal{L}(m)_{\vec{d}})|_L \geq h^1(\mathbb{P}^1, \mathcal{O}(\vec{e})(m))\right\}.$$

Since $h^1(\mathbb{P}^1, \mathcal{O}(\vec{e})(m)) = 0$ for m large, only finitely many terms in the intersection are proper subschemes of $\mathrm{Pic}^{\vec{d}}(\mathcal{C}/B)$.

4.3 Classification of $\vec{e}$-Positive Limit Line Bundles

Returning to notation of Section 3.1, in this section we classify $\vec{e}$-positive line bundles on the central fiber X. The following description in terms of k-staircase tableaux is an observation due to Cook-Powell–Jensen in the tropical setting [9]. Here, we provide a self-contained proof in the classical setting.

For any $0 \leq i \leq g$, and any degree distribution $\vec{d}$, write

$$X^{\leq i} = E^1 \cup E^2 \cup \cdots \cup E^i \quad \text{and} \quad d^{\leq i} = d^1 + d^2 + \cdots + d^i.$$

Definition 4.3.1. For a limit line bundle L, and $1 \leq i \leq g-1$, and $n \geq 1$, define

$$a_n^i(L) = \min\{\alpha : \text{we have } h^0(X^{\leq i}, L_{\vec{d}}|_{X^{\leq i}}) \geq n \text{ for any degree distribution } \vec{d} \text{ with } d^{\leq i} = \alpha\}.$$

We extend this to $i = g$ via

$$a_n^g(L) = \min \Big\{\alpha : \text{for some } m \text{ and } \epsilon \text{ with } d + mk = \alpha + \epsilon \text{ and } \epsilon \geq 0, \text{ we have}$$
$$h^0(X, L(m)_{\vec{d}(m)}) \geq n + \epsilon \text{ for any degree distribution } \vec{d}(m) \text{ with } d(m)^{\leq g} = d + mk\Big\},$$

and to $i = 0$ via

$$a_n^0(L) = n - 1.$$

For $1 \leq i \leq g-1$, unwinding the definition of a_n^i, there exists a degree distribution $\vec{d}$ with $d^{\leq i} = a_n^i - 1$ satisfying $h^0(X^{\leq i}, L_{\vec{d}}|_{X^{\leq i}}) \leq n-1$. Furthermore, since vanishing at a single point imposes at most one condition on global sections, there exists a degree distribution $\vec{d}$ with $d^{\leq i} = a_n^i$ witnessing $h^0(X^{\leq i}, L_{\vec{d}}|_{X^{\leq i}}) = n$, such that not every section of $L_{\vec{d}}|_{X^{\leq i}}$ vanishes at p^i.

Proposition 4.3.2. *We have $a_n^i > a_{n-1}^i$.*

Proof. The case $i = 0$ is clear by definition.

When $1 \leq i \leq g-1$, let $\vec{d}$ be a degree distribution with $d^{\leq i} = a_{n-1}^i$ witnessing $h^0(X^{\leq i}, L_{\vec{d}}|_{X^{\leq i}}) = n-1$. This implies $a_n^i > a_{n-1}^i$ as desired.

Finally, when $i = g$, we claim that for any m and ϵ with $d + mk = a_{n-1}^g + \epsilon$, there is some degree distribution $\vec{d}(m)$ with $d(m)^{\leq g} = d + mk$ such that $h^0(X, L(m)_{\vec{d}(m)}) \leq (n-1) + \epsilon$. Indeed, if not, then $h^0(X, L(m)_{\vec{d}(m)}) \geq (n-1) + (\epsilon + 1)$ for every such degree distribution, which would contradict the definition of a_{n-1}^g because $d + mk = (a_{n-1}^g - 1) + (\epsilon + 1)$. This implies $a_n^g > a_{n-1}^g$ as desired. ☐

Proposition 4.3.3. *We have $a_n^i \geq a_n^{i-1}$. If equality holds, then $L^i \simeq \mathcal{O}_{E^i}(a_n^{i-1}p^{i-1} + (d - a_n^{i-1})p^i)$.*

Remark 4.3.4. Our proof will show that if equality holds when $i = 1$ (respectively $i = g$) then $a_n^0 = 0$ (respectively $a_n^{g-1} \equiv d \bmod k$). Thus the formula given for L^i is independent of choice of p^0 and p^g.

Proof. We separately consider the following cases:

The Case $i = 1$: For any degree distribution $\vec{d}$, the line bundle $L_{\vec{d}}|_{E^1} \simeq L^1(-(d-d^1)p^1)$ is of degree d^1 on a genus 1 curve and hence by Reimann–Roch has a $\max(0, d^1)$-dimensional space of global sections unless $d^1 = 0$ and $L^1(-dp^1) \simeq \mathcal{O}_{E^1}$. Hence, there is no degree distribution $\vec{d}$ such that $d^1 \leq a_n^0 - 1 = n - 2$ and $h^0(E^1, L_{\vec{d}}|_{E^1}) \geq n$. Furthermore, there is no such degree distribution with $d^1 = a_n^0 = n - 1$ and $h^0(L_{\vec{d}}|_{E^1}) \geq n$ unless $a_n^0 = 0$ and $L^1 = \mathcal{O}_{E^1}(dp^1) = \mathcal{O}_{E^1}(a_n^0 p^0 + (d - a_n^0)p^1)$.

The Case $2 \leq i \leq g - 1$: Let $\vec{d}$ be a degree distribution such that $d^{\leq i-1} = a_n^{i-1} - 1$ and

$$h^0(X^{\leq i-1}, L_{\vec{d}}|_{X^{\leq i-1}}) < n.$$

We may further assume that $d^i = 0$. Then

$$h^0(X^{\leq i}, L_{\vec{d}}|_{X^{\leq i}}) \leq h^0(X^{\leq i-1}, L_{\vec{d}}|_{X^{\leq i-1}}) + h^0(E^i, L_{\vec{d}}|_{E^i}(-p^{i-1})) < n.$$

Therefore $a_n^i \geq a_n^{i-1}$.

Furthermore, there exists a degree distribution $\vec{d}$ with $d^{\leq i-1} = a_n^{i-1}$ and $d^i = 0$ witnessing $h^0(X^{\leq i-1}, L_{\vec{d}}|_{X^{\leq i-1}}) = n$, and $h^0(X^{\leq i-1}, L_{\vec{d}}|_{X^{\leq i-1}}(-p^{i-1})) = n - 1$. Thus

$$h^0(X^{\leq i}, L_{\vec{d}}|_{X^{\leq i}}) = h^0(E^i, L_{\vec{d}}|_{E^i}) + n - 1.$$

If $a_n^i = a_n^{i-1}$, then to ensure this degree distribution has enough sections, $h^0(E^i, L_{\vec{d}}|_{E^i}) > 0$. Since $L_{\vec{d}}|_{E^i}$ has degree zero, this implies $L_{\vec{d}}|_{E^i} \simeq \mathcal{O}_{E^i}$. Applying (4.2.2),

$$L_{\vec{d}}|_{E^i} \simeq L^i(-a_n^{i-1}p^{i-1} - (d - a_n^{i-1})p^i),$$

so this implies the desired condition.

The Case $i = g$: Let $\vec{d}$ be a degree distribution such that $d^{\leq g-1} = a_n^{g-1} - 1$ and

$$h^0(X^{\leq g-1}, L_{\vec{d}}|_{X^{\leq g-1}}) < n.$$

Let m and $\epsilon \geq 0$ be any integers such that $d + mk = a_n^{g-1} - 1 + \epsilon$. Define

$$\vec{d}(m) := (d^1, d^2, \dots, d^{g-1}, d^g + mk).$$

Then $d^g + mk = \epsilon$. Thus

$$h^0(X, L(m)_{\vec{d}(m)}) \leq h^0(X^{\leq g-1}, L(m)_{\vec{d}(m)}|_{X^{\leq g-1}}) + h^0(E^g, L(m)_{\vec{d}(m)}|_{E^g}(-p^{g-1})) < n + \epsilon.$$

Therefore $a_n^g \geq a_n^{g-1}$.

Furthermore, there exists a degree distribution $\vec{d}$ with $d^{\leq g-1} = a_n^{g-1}$, witnessing

$$h^0(X^{\leq g-1}, L_{\vec{d}}|_{X^{\leq g-1}}) = n \quad \text{and} \quad h^0(X^{\leq g-1}, L_{\vec{d}}|_{X^{\leq g-1}}(-p^{g-1})) = n - 1.$$

Let m and $\epsilon \geq 0$ be any integers such that $d + mk = a_n^{g-1} + \epsilon$. Define $\vec{d}(m)$ as above; as before, $d^g + mk = \epsilon$. We have

$$h^0(X, L(m)_{\vec{d}(m)}) = h^0(E^g, L(m)_{\vec{d}(m)}|_{E^g}) + n - 1.$$

If $a_n^g = a_n^{g-1}$, then for some such choice of m and ϵ, we must have $h^0(E^g, L(m)_{\vec{d}(m)}|_{E^g}) > \epsilon$. Since $\deg L(m)_{\vec{d}(m)}|_{E^g} = d^g + mk = \epsilon$, this implies $\epsilon = 0$ and $L(m)_{\vec{d}(m)}|_{E^g} \simeq \mathcal{O}_{E^g}$. Applying (4.2.2),

$$L(m)_{\vec{d}(m)}|_{E^g} \simeq L^g(m)(-a_n^{g-1}p^{g-1}) \simeq L^g((mk - a_n^{g-1})p^{g-1}) \simeq L^g(-a_n^{g-1}p^{g-1} - (d - a_n^{g-1})p^g),$$

so this is exactly the desired condition. □

We now repackage this information as follows:

Definition 4.3.5. For $n \geq 1$, write

$$f_n(i) = i + n - 1 - a_n^i,$$

and define

$$h(n) := h_{\vec{e}}(n) = \max\left\{h^1(\mathbb{P}^1, \mathcal{O}(\vec{e})(m)) : m \text{ satisfies } h^0(\mathbb{P}^1, \mathcal{O}(\vec{e})(m)) \geq n\right\}$$
$$= \max\left\{\sum_{\ell=1}^{k} \max(0, -e_\ell - m - 1) : m \text{ satisfies } \sum_{\ell=1}^{k} \max(0, e_\ell + m + 1) \geq n\right\}.$$

Note that $h(n)$ is nonincreasing and is zero for n large.

Proposition 4.3.6. *If L is $\vec{e}$-positive, then $f_n(g) \geq h(n)$.*

Proof. Suppose that m satisfies $h^0(\mathbb{P}^1, \mathcal{O}(\vec{e})(m)) \geq n$; let $\epsilon = h^0(\mathbb{P}^1, \mathcal{O}(\vec{e})(m)) - n \geq 0$. By (4.2.5), we have

$$h^0(X, L(m)_{\vec{d}(m)}) \geq h^0(\mathbb{P}^1, \mathcal{O}(\vec{e})(m)) = n + \epsilon,$$

for any degree distribution $\vec{d}(m)$ with $d(m)^{\leq g} = d + mk$. Therefore by Definition 4.3.1, we have

$$a_n^g \leq d + mk - \epsilon = d + mk - h^0(\mathbb{P}^1, \mathcal{O}(\vec{e})(m)) + n.$$

Thus,

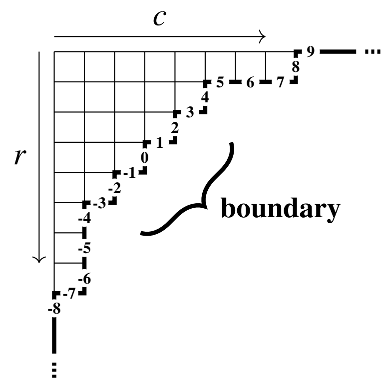
$$f_n(g) \geq g + n - 1 - [d + mk - h^0(\mathbb{P}^1, \mathcal{O}(\vec{e})(m)) + n] = h^1(\mathbb{P}^1, \mathcal{O}(\vec{e})(m)).$$

Therefore $f_n(g) \geq h(n)$. □

The inequality of Proposition 4.3.6 forces equality to hold in Proposition 4.3.3 for many values of i and n. To keep track of when equality holds, we use a combinatorial object that we will term a *k-staircase tableau*.

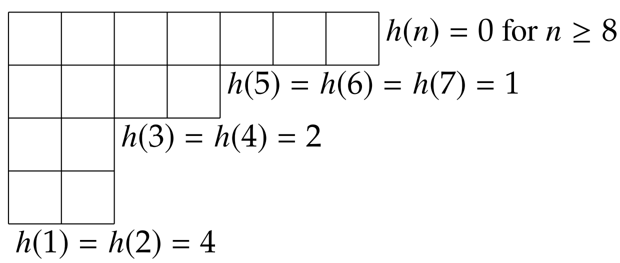
Definition 4.3.7. A *Young diagram* is a finite collection of boxes arranged in left-justified rows, such that the number of boxes in each row is nonincreasing. We index the boxes by their row and column (r, c), beginning with $(1, 1)$, and we define the *diagonal index* of a box to be $c - r$.

The *boundary* of a Young diagram is the sequence of line segments formed by the right-most edges of the last box in every row and the bottom-most edge of the last box in every column. For convenience, we extend this to infinity below and to the right of the diagram. We index the boundary segments by the diagonal index of the box above (if the segment is horizontal), or to the right (if the segment is vertical).



Because $h(n)$ is nonincreasing and zero for n large, the data of the function $h(n)$ (which is defined for positive integers n) is thus the same as the data of a Young diagram, where we put $h(n)$ boxes in the nth column.

Definition 4.3.8. For a splitting type $\vec{e}$, we write $\Gamma(\vec{e})$ for the Young diagram determined by $h_{\vec{e}}(n)$ in the above manner. We call a Young diagram of the form $\Gamma(\vec{e})$ for some $\vec{e}$ a *k-staircase*.



Example: $\Gamma(\vec{e})$ for $\vec{e} = (-4, -2, 0, 0)$

For each $\vec{e}$-positive line bundle L, we will use the functions $f_n(i)$ to build a filling T of $\Gamma(\vec{e})$. Namely, we have $f_n(0) = 0$ and $f_n(g) \geq h(n)$, and by Proposition 4.3.3, $f_n(i) \leq f_n(i-1) + 1$. Therefore, f_n assumes every value between 0 and $h(n)$ inclusive. Our filling T of $\Gamma(\vec{e})$ is obtained by placing $\min\{i : f_c(i) = r\}$ in the rth row of the cth column.

Proposition 4.3.9. *If i is in the rth row of the cth column of T, then*

$$L^i \simeq \mathcal{O}_{E^i}((c - r + i - 1)p^{i-1} + (d - (c - r + i - 1))p^i).$$

In particular, if i appears in multiple boxes of T, then it follows that all such boxes have the same value of $c - r$ modulo k. Moreover, this filling is increasing along rows and columns.

Proof. Given r and c, suppose i is the first time for which $f_c(i) = r$. Because this is a new maximum, we must have $f_c(i-1) = r - 1$, which implies $a_c^{i-1} = a_c^i = c - r + i - 1$. By

Proposition 4.3.3,

$$L^i = O_{E^i}(a_n^{i-1}p^{i-1} + (d - a_n^{i-1})p^i) = O_{E^i}((c - r + i - 1)p^{i-1} + (d - (c - r + i - 1))p^i),$$

as desired. In particular, if i appears in multiple boxes of T, then since $p^{i-1} - p^i$ is exactly k-torsion in $\mathrm{Pic}^0 E^i$, all such boxes have the same value of $c - r$ modulo k.

We now show that the filling is increasing along rows and columns. Since $f_c(i) \leq f_c(i-1) + 1$, the function f_c must attain the value r before it attains $r + 1$. This shows the filling is increasing down column c. Meanwhile, by Proposition 4.3.2, we have $a^i_{c-1} < a^i_c$ and so $f_{c-1}(i) \geq f_c(i)$. It follows that $\min\{i : f_{c-1}(i) = r\} \leq \min\{i : f_c(i) = r\}$ (the larger function must attain r at an earlier or same time). However, if equality holds, the first part of this proposition says that $c - r \equiv (c-1) - r \bmod k$, which is impossible. Thus, $\min\{i : f_{c-1}(i) = r\} < \min\{i : f_c(i) = r\}$, which shows the filling is increasing along row r. □

Definition 4.3.10. A filling T of a Young diagram is called *k-regular* if it is increasing along rows and columns, and all boxes containing the same symbol i have the same value of $c - r$ modulo k. We write $T[i] \in \mathbb{Z}/k\mathbb{Z} \cup \{*\} = \{1, 2, \dots, k, *\}$ for this common value of $c - r$ modulo k if i appears in T; if i does not appear in T then we set $T[i] = *$. We call a k-regularly filled k-staircase a *k-staircase tableau*.

For the remainder of the chapter, all fillings of any Young diagram will be assumed to be k-regular.

Definition 4.3.11. Given a tableau T, we define a corresponding reduced subscheme of $\mathrm{Pic}^d(X)$ by

$$W^T(X) := \left\{L \in \mathrm{Pic}^d(X) : L^i \simeq O_{E^i}((T[i] + i - 1)p^{i-1} + (d - (T[i] + i - 1))p^i) \text{ if } T[i] \neq *\right\}.$$

Similarly, given a diagram Γ, we define

$$W^\Gamma(X) := \bigcup_{\substack{T \text{ filling} \\ \text{of } \Gamma}} W^T(X).$$

In this language, Proposition 4.3.9 states that $W^{\vec{e}}(X)_{\mathrm{red}} \subseteq W^{\Gamma(\vec{e})}(X)$. In fact, we will see later that $W^{\vec{e}}(X) = W^{\Gamma(\vec{e})}(X)$.

4.4 Combinatorics

In the previous section, we classified limit $\vec{e}$-positive line bundles in terms of k-staircase tableaux. Such tableaux are special cases of a more general class of tableaux known as *k-core tableaux*, which are well-studied due to their relationship with the affine symmetric group (see [47] and [53], or for an overview see Section 1.2 of [46]). We recall the basic facts about this relationship here (without proof) in the next two subsections, and use them to deduce the structure results for k-staircase tableaux that are needed for the proof of the regeneration theorem. This explicit description of $W^\Gamma(X)$ will also be used directly in the proofs of all of our main theorems.

4.4.1 k-cores and the affine symmetric group.

Recall that the *affine symmetric group* $\widetilde{S}_k$ is the group of permutations $f\colon \mathbb{Z} \to \mathbb{Z}$ such that

$$f(x+k) = f(x) + k \quad \text{and} \quad \sum_{x=1}^{k} f(x) = \sum_{x=1}^{k} x = \frac{k(k+1)}{2}.$$

Such permutations automatically satisfy

$$f(x) \not\equiv f(y) \pmod{k} \quad \text{for} \quad x \not\equiv y \pmod{k}. \tag{4.4.1}$$

The affine symmetric group is generated by transpositions s_j (for $j \in \mathbb{Z}/k\mathbb{Z}$) satisfying

$$s_j(x) = \begin{cases} x+1 & \text{if } x \equiv j \mod k; \\ x-1 & \text{if } x \equiv j+1 \mod k; \\ x & \text{otherwise,} \end{cases}$$

with relations

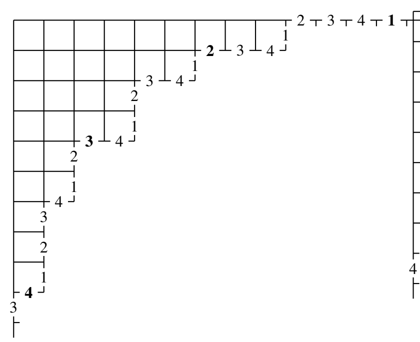
$$s_j^2 = 1, \quad s_j s_{j'} = s_{j'} s_j \text{ if } j - j' \neq \pm 1, \quad \text{and} \quad (s_j s_{j+1})^3 = 1.$$

For ease of notation, we include the identity $e = s_*$ as a generator (so generators are indexed by $\mathbb{Z}/k\mathbb{Z} \cup \{*\}$).

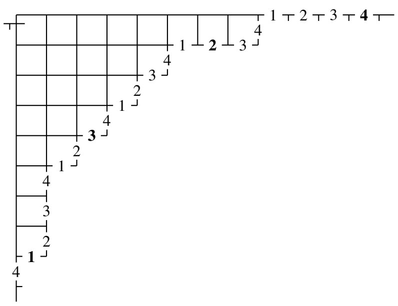
Each line segment making up the boundary of a Young diagram is either vertical or horizontal. The following key definition generalizes the notion of a k-staircase.

Definition 4.4.2. A sequence $\{\gamma_j\}$ of vertical and horizonal line segments is called *k-convex* if γ_j is vertical only if γ_{j-k} is also vertical. A Young diagram is called a *k-core* if its boundary is k-convex. A (k-regular) filling of a k-core will be called a *k-core tableau*.

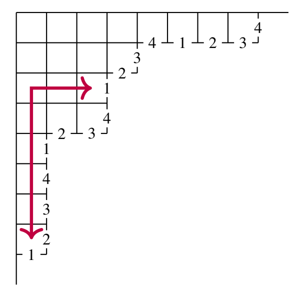
In the literature, k-cores are also frequently defined in terms of their *hook lengths*, which are the number of boxes to the right or bottom of a given box (including the given box). Namely, a Young diagram is a k-core if and only if no hook lengths are divisible by k, or equivalently if and only if no hook lengths are equal to k.



A 4-staircase is 4-core.



Another 4-core that is not a 4-staircase.



A diagram that is not a 4-core.

A sequence $\{\gamma_j\}$ is k-convex if each residue class of segments is composed of an infinite sequence of vertical segments followed by an infinite sequence of horizontal segments. Thus, to specify a k-core, it suffices to give a collection $\{t_1, \dots, t_k\}$ of integers (distinct mod k), representing the first horizontal segment in each residue class. Such data is a priori determined up to addition of an overall constant (i.e. $\{t_j\} \mapsto \{t_j + \delta\}$); the indexing of boundary segments in Definition 4.3.7 corresponds to the unique normalization so that

$$\sum_{j=1}^{k} t_j = \sum_{j=1}^{k} j = \frac{k(k+1)}{2}.$$

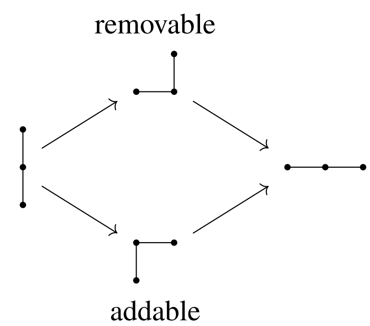
Therefore, k-cores are in bijection with elements of $\widetilde{S_k}/S_k$ — by sending $\{t_j\}$ to the coset of permutations sending $\{1, 2, \dots, k\}$ to $\{t_1, t_2, \dots, t_k\}$. There is a distinguished coset representative f satisfying $f(1) < f(2) < \cdots < f(k)$.

Definition 4.4.3. If Γ is a k-core and $x \in \mathbb{Z}$, we define $\Gamma(x)$ to be the value of this distinguished permutation applied to x; if T is a k-core tableau of shape Γ, we define $T(x) = \Gamma(x)$.

In the definition of a k-convex sequence $\{\gamma_j\}$, we could equivalently have considered pairs of adjacent line segments (γ_j, γ_{j+1}). Then, the mod k residue class of pairs of boundary segments

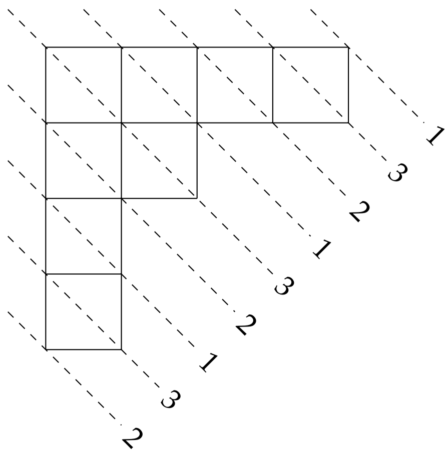
$$\{\dots, (\gamma_j, \gamma_{j+1}), (\gamma_{j+k}, \gamma_{j+k+1}), \dots\}$$

is composed of a sequence of (vertical, vertical) segments, followed by a (possibly empty) sequence of *either* (vertical, horizontal) *or* (horizontal, vertical) corners, followed by a sequence of (horizontal, horizontal) segments. In other words, the mod k residue classes of pair of boundary segments in a k-core always progress along *one* of the following trajectories:



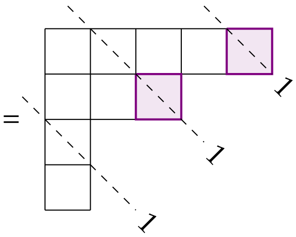
The configuration of a vertical and then horizontal segment is called an *addable corner*, and the configuration of a horizontal and then vertical segment is called a *removable corner*.

This gives a natural (left) action of the affine symmetric group $\widetilde{S}_k$ on the set of k-cores. Namely, $s_j \cdot \Gamma$ is the k-core obtained from Γ by adding a box in all addable corners whose diagonal index has residue class j (if such addable corners exist), or removing a box from all removable corners whose diagonal index has residue class j (if such removable corners exist), or doing nothing (if no such addable or removable corners exist).

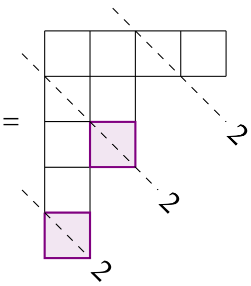


A k-core diagram Γ.

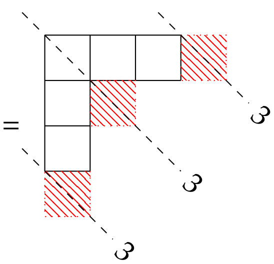
$s_1 \cdot \Gamma =$



$s_2 \cdot \Gamma =$



$s_3 \cdot \Gamma =$



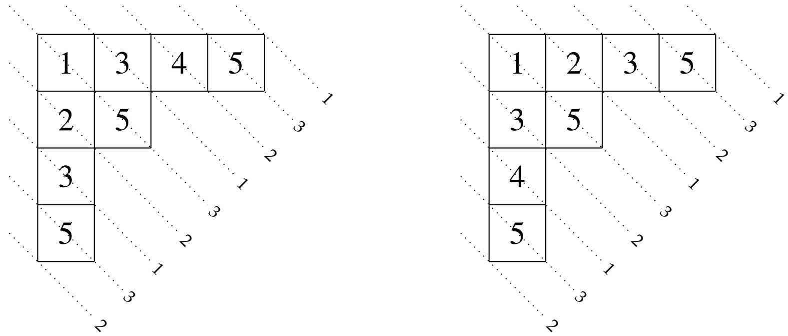
One easily checks that this respects the relations for the affine symmetric group, and that under this action,

$$\{k\text{-core diagrams}\} \leftrightarrow \widetilde{S}_k / S_k$$

is an $\widetilde{S}_k$-equivariant bijection of sets.

4.4.4 k-core tableaux and the word problem

Given a k-core Γ, let $w \in \widetilde{S}_k$ be the representative of the corresponding coset with $w(1) < w(2) < \cdots < w(k)$. If $w = s_{j_g} s_{j_{g-1}} \cdots s_{j_1}$ is a word for w in $\widetilde{S}_k$, then we obtain a filling of Γ: Indeed, we build Γ from the empty k-core by consecutively applying the s_{j_i}; this determines a k-core tableau of shape Γ where any box added in the ith step contains the symbol i.



$w = s_3 s_2 s_1 s_2 s_3$ $w = s_3 s_1 s_2 s_1 s_3$

The two efficient fillings of this 3-staircase diagram.

Lapointe and Morse showed in [47] that this completely describes efficiently filled k-core tableaux. Namely:

1. Any efficient filling (i.e. with the fewest possible symbols) arises in this way from a unique *reduced word* for w (i.e. a word with the fewest possible non-identity generators). Conversely, any reduced word gives an efficient filling (and if the word is reduced then no boxes are ever removed). See Section 8 of [47].

2. The minimal number of symbols needed to fill Γ, which we will denote $u(\Gamma)$, is exactly the number of boxes in Γ whose hook length is less than k. See Lemma 31 of [47].

3. Efficient fillings can be constructed inductively: Suppose Γ has a removable corner whose diagonal index has residue class j, so that $s_j \cdot \Gamma$ is strictly contained in Γ. Then we have $u(s_j \cdot \Gamma) = u(\Gamma) - 1$. See Proposition 22 of [47]. In particular:

   (a) An efficient filling of Γ whose largest symbol appears in a box with diagonal index of residue j restricts to an efficient filling of $s_j \cdot \Gamma$.

   (b) An efficient filling of $s_j \cdot \Gamma$ can be completed to an efficient filling of Γ whose largest symbol appears in a box with diagonal index of residue j.

4.4.5 Reduction to efficient tableaux

One consequence of this final property (3b) is that we need only consider efficient tableaux for our geometric problem.

Proposition 4.4.6. *Let T be a k-core tableau of shape Γ. Then there is an efficiently filled k-core tableau T' of shape Γ with $W^T(X) \subseteq W^{T'}(X)$. In particular,*

$$W^\Gamma(X) = \bigcup_{\substack{T \text{ efficient} \\ \text{filling of } \Gamma}} W^T(X).$$

Proof. We argue by induction on $u(\Gamma)$; the base case $u(\Gamma) = 0$ is tautological. For the inductive step, let t be the largest symbol appearing in T, and $j = T[t]$ (c.f. Definition 4.3.10). Let $T_\circ$ be the restriction of T to $s_j \cdot \Gamma$. By our inductive hypothesis, there is an efficient filling $T'_\circ$ of $s_j \cdot \Gamma$ with $W^{T_\circ}(X) \subseteq W^{T'_\circ}(X)$. If T' is the completion of $T'_\circ$ to a filling of Γ using the additional symbol t, then $W^T(X) \subseteq W^{T'}(X)$ as desired. □

4.4.7 Truncations

The following will be a convenient way of packaging the data of an efficient filling as necessary for our regeneration theorem.

Definition 4.4.8. Let T be an efficiently filled k-core tableau, corresponding to a reduced word $s_{j_g} \cdots s_{j_2} s_{j_1}$. Define $T^{\leq t}$ to be the tableau formed by the boxes of T with symbols up to t, i.e. corresponding to the reduced word $s_{j_t} \cdots s_{j_2} s_{j_1}$.

In particular, for each t and ℓ, we obtain an integer which we refer to as the ℓth *truncation* at time t:

$$T^{\leq t}(\ell) = (s_{j_t} \cdots s_{j_2} s_{j_1})(\ell).$$

We now summarize several properties of the $T^{\leq t}(\ell)$. First of all, by construction, we have:

$$T^{\leq 0}(\ell) = \ell. \tag{4.4.2}$$

Moreover, by (4.4.1),

$$T^{\leq t}(\ell_1) \not\equiv T^{\leq t}(\ell_2) \pmod{k} \quad \text{for} \quad \ell_1 \not\equiv \ell_2 \pmod{k}. \tag{4.4.3}$$

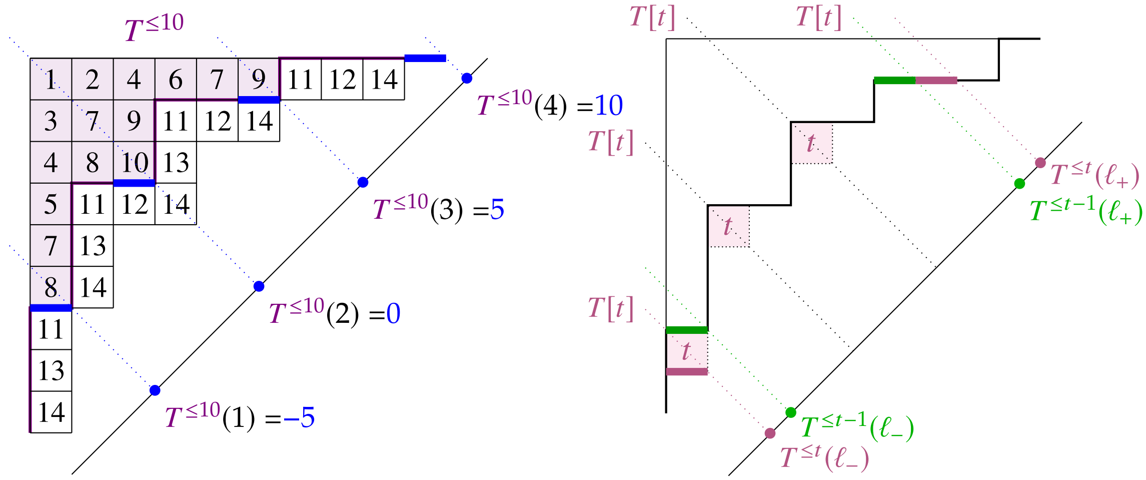
If s_{j_t} is the identity (equivalently if $T[t] = *$), then $T^{\leq t}(\ell) = T^{\leq t-1}(\ell)$ for all ℓ. Otherwise, s_{j_t} is a simple transposition, and there are exactly two values of $\ell \in \{1, 2, \ldots, k\}$, say ℓ_- and ℓ_+, for which $T^{\leq t}(\ell)$ changes:

$$T^{\leq t}(\ell) = \begin{cases} T^{\leq t-1}(\ell) & \text{if } \ell \neq \ell_\pm; \\ T^{\leq t-1}(\ell_\pm) \pm 1 & \text{if } \ell = \ell_\pm. \end{cases} \tag{4.4.4}$$

In this case, we say that t is *increasing* for ℓ_+ and *decreasing* for ℓ_-. Combining (4.4.3) and (4.4.4), we have

$$T^{\leq t}(\ell_+) - T^{\leq t}(\ell_-) \equiv T^{\leq t-1}(\ell_-) - T^{\leq t-1}(\ell_+) \equiv 1 \pmod{k}. \tag{4.4.5}$$

The following pictures illustrate the behavior of the $T^{\leq t}(\ell)$ at fixed time t, and the relation between times $t-1$ and t:



As can be seen in the diagram above, we have the relation

$$T[t] \equiv T^{\leq t}(\ell_-) \pmod{k}. \tag{4.4.6}$$

As can be seen in the right diagram, $T^{\leq t-1}(\ell_-)$ is an edge of the leftmost addable corner whose diagonal index has residue class $T[t]$, while $T^{\leq t-1}(\ell_+)$ lies to the right of the rightmost addable corner whose diagonal index has residue class $T[t]$. Thus,

$$T^{t-1}(\ell_-) \leq T^{t-1}(\ell_+) - (k-1) < T^{t-1}(\ell_+) \quad \text{and} \quad T^{t}(\ell_-) \leq T^{t}(\ell_+) - (k+1) < T^{t}(\ell_+). \tag{4.4.7}$$

Proposition 4.4.9. *If $\ell_1 > \ell_2$ then*

$$\left\lfloor \frac{T^{\leq t}(\ell_1) - T^{\leq t}(\ell_2)}{k} \right\rfloor \tag{4.4.8}$$

is a non-decreasing function of t. In particular, the truncations are "sorted," i.e.

$$T^{\leq t}(\ell_1) > T^{\leq t}(\ell_2) \quad \textit{if} \quad \ell_1 > \ell_2.$$

Proof. We will prove this by induction on t. From (4.4.3) and (4.4.4), we have

$$\left\lfloor \frac{T^{\leq t}(\ell_1) - T^{\leq t}(\ell_2)}{k} \right\rfloor = \left\lfloor \frac{T^{\leq t-1}(\ell_1) - T^{\leq t-1}(\ell_2)}{k} \right\rfloor + \begin{cases} 1 & \text{if } (\ell_1,\ell_2) = (\ell_+,\ell_-); \\ -1 & \text{if } (\ell_1,\ell_2) = (\ell_-,\ell_+); \\ 0 & \text{otherwise.} \end{cases}$$

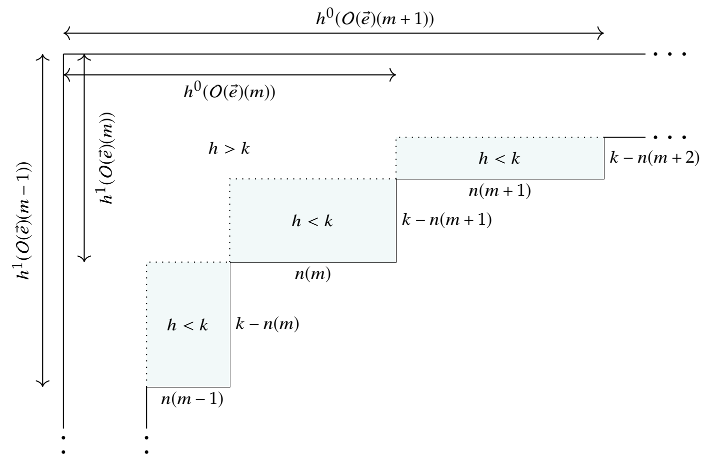
It thus remains to see that $\ell_- < \ell_+$. But this follows from (4.4.7), given our inductive hypothesis that the truncations are sorted. □

4.4.10 k-staircases

Let $\vec{e}$ be a splitting type; write $d_1 > d_2 > \cdots > d_s$ for the distinct parts of $\vec{e}$, and $m_1, m_2, \ldots, m_s$ for the corresponding multiplicities. (Note that $e_1 \leq e_2 \leq \cdots \leq e_k$ but we have $d_1 > d_2 > \cdots > d_S^*$!) The integers $1, 2, \ldots, k$ are then naturally in bijection with pairs (j, n) with $1 \leq j \leq s$ and $1 \leq n \leq m_j$ (via lexicographic order).

Proposition 4.4.11. *Every k-staircase is a k-core, and $u(\Gamma(\vec{e})) = u(\vec{e})$.*

Proof. Write $n(m) := \#\{\ell : e_\ell \geq -m\}$. The k-staircase $\Gamma(\vec{e})$ has the following form:



Since $n(m)$ is a nondecreasing function of m, the boxes in the shaded regions have hook length $h < k$, and the remaining boxes have $h > k$. In particular, no box has $h = k$, so $\Gamma(\vec{e})$ is a k-core. Counting up the number of boxes with $h < k$, we obtain

$$\begin{aligned}
u(\Gamma(\vec{e})) &= \sum_m n(m) \cdot (k - n(m+1)) \\
&= \sum_m \sum_{e_{\ell_1} \geq -m} \sum_{e_{\ell_2} < -(m+1)} 1 \\
&= \sum_{\ell_1, \ell_2} \#\{m : e_{\ell_2} + 1 < -m \leq e_{\ell_1}\} \\
&= \sum_{\ell_1, \ell_2} h^1(\mathbb{P}^1, O_{\mathbb{P}^1}(e_{\ell_2} - e_{\ell_1})) \\
&= h^1(\mathbb{P}^1, \mathrm{End}(O_{\mathbb{P}^1}(\vec{e}))) \\
&= u(\vec{e}).
\end{aligned}$$

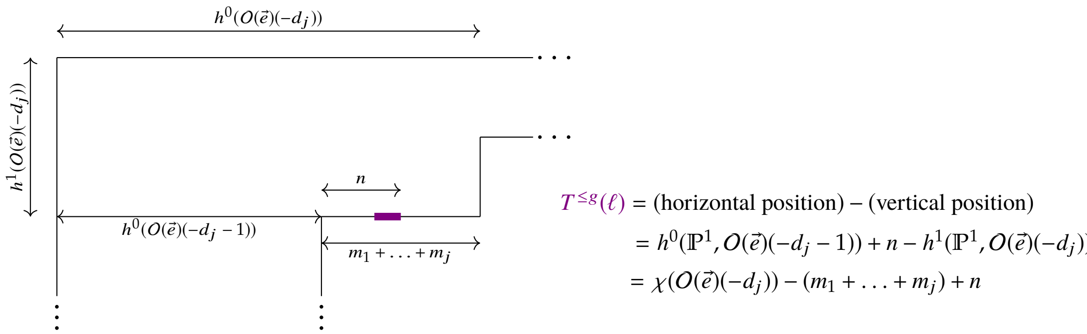
□

Remark 4.4.12. The fact that $u(\Gamma(\vec{e})) = u(\vec{e})$ already establishes that for $C \to \mathbb{P}^1$ a general degree k genus g cover, $\dim W^{\vec{e}}(C) \leq \dim W^{\Gamma(\vec{e})}(X) = g - u(\vec{e})$.

Proposition 4.4.13. *We have*

$$T^{\leq g}(j,n) = \chi(\mathcal{O}(\vec{e})(-d_j)) - (m_1 + \cdots + m_j) + n.$$

Proof. We use the lengths labeled in the diagram below to calculate the diagonal index of the first horizontal segment in every residue class along the boundary of $\Gamma(\vec{e})$:



as desired. □

We conclude the section with two results on the relationship between the truncations with same value of j, respectively distinct values of j.

Corollary 4.4.14. *If $n_1 \geq n_2$, then $T^{\leq t}(j,n_1) - T^{\leq t}(j,n_2) \leq k-1$.*

Proof. By Propositions 4.4.9 and 4.4.13,

$$\left\lfloor \frac{T^{\leq t}(j,n_1) - T^{\leq t}(j,n_2)}{k} \right\rfloor \leq \left\lfloor \frac{T^{\leq g}(j,n_1) - T^{\leq g}(j,n_2)}{k} \right\rfloor = \left\lfloor \frac{n_1 - n_2}{k} \right\rfloor = 0.$$ □

Corollary 4.4.15. *If t is decreasing for (j_-, n_-) and increasing for (j_+, n_+), then $j_- < j_+$.*

Proof. By (4.4.7), we have $T^{\leq t}(j_-, n_-) \leq T^{\leq t}(j_+, n_+) - (k+1)$. Therefore by Proposition 4.4.9, we have $j_- \leq j_+$. Moreover, by Corollary 4.4.14, we have $j_- \neq j_+$. □

Corollary 4.4.16. *If $j' > j$, then $T^{\leq g}(j',n') - T^{\leq t}(j',n') \geq T^{\leq g}(j,n) - T^{\leq t}(j,n)$.*

Proof. It suffices to consider the case that $j' = j+1$. Because the truncations remains sorted (Proposition 4.4.9), we have

$$T^{\leq t}(j+1, m_{j+1}) - T^{\leq t}(j+1, n') \geq m_{j+1} - n' = T^{\leq g}(j+1, m_{j+1}) - T^{\leq g}(j+1, n') \quad (4.4.9)$$

$$T^{\leq t}(j,n) - T^{\leq t}(j,1) \geq n - 1 = T^{\leq g}(j,n) - T^{\leq g}(j,1). \quad (4.4.10)$$

Moreover, by Proposition 4.4.9, we also have

$$\left\lfloor \frac{T^{\leq g}(j+1, m_{j+1}) - T^{\leq g}(j,1)}{k} \right\rfloor \geq \left\lfloor \frac{T^{\leq t}(j+1, m_{j+1}) - T^{\leq t}(j,1)}{k} \right\rfloor.$$

By Proposition 4.4.13, we have $T^{\leq g}(j+1, m_{j+1}) - T^{\leq g}(j,1) \equiv -1 \bmod k$, so this implies

$$T^{\leq g}(j+1, m_{j+1}) - T^{\leq g}(j,1) \geq T^{\leq t}(j+1, m_{j+1}) - T^{\leq t}(j,1). \tag{4.4.11}$$

Adding (4.4.9), (4.4.10), and (4.4.11), we obtain

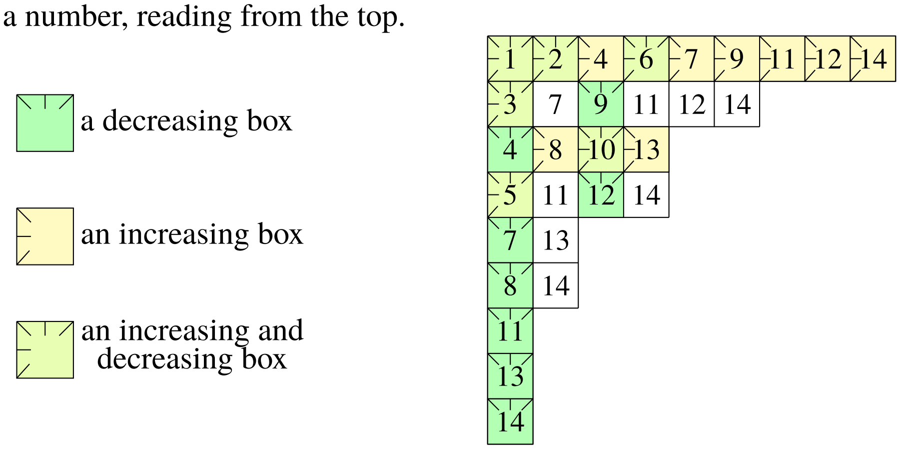
$$T^{\leq t}(j,n) - T^{\leq t}(j+1,n') \geq T^{\leq g}(j,n) - T^{\leq g}(j+1,n'),$$

as desired. □

4.4.17 Alternative construction of truncations: Paths on the tableau

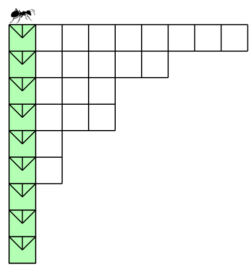
The truncations can alternatively be described directly in terms of the tableau, without going through the affine symmetric group. In this section, we give a brief exposition of this. We omit proofs since the construction of the truncations given above is logically sufficient for the remainder.

We call a box in a tableau *decreasing* if it is filled with the first instance of a number, reading from the left. Dually, we call a box *increasing* if it is filled with the first instance of a of a number, reading from the top.

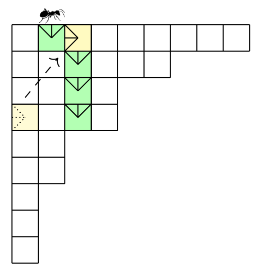


For an efficient filling, one can show that the number of decreasing boxes in a row is a nonincreasing function of the row, and is always at most $k-1$. For $0 \leq i \leq k$, we call the collection of boxes which are the ith decreasing box in each row the *ith decreasing cascade* of the tableau. Dually, we define the *increasing cascades*. (The kth cascade is always empty.) One then shows that the translation of the $(k+1-i)$th increasing cascade of a tableau up

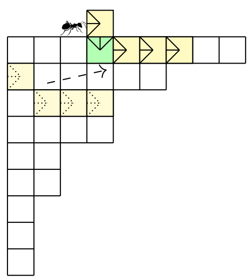
$k+1-i$ and over i "meshes" with the ith decreasing cascade to form a "walking path" and consists of symbols in increasing order.



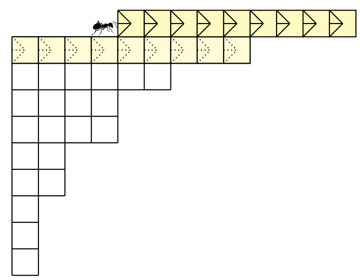
The 1st walking path.



The 2nd walking path.



The 3rd walking path.



The 4th walking path.

We think of these walking paths as giving instructions to a collection of k ants starting at $(0,\ell)$ for $1 \leq \ell \leq k$. At time t, if the symbol t appears as the next box in the ℓth walking path, the ℓth ant steps onto the corresponding box. The diagonal index of the ℓth ant at time t recovers $T^{\leq t}(\ell)$; the symbol t is decreasing (respectively increasing) for ℓ if t is a decreasing (respectively increasing) box in the ℓth walking path.

4.5 Set-Theoretic Regeneration

In this section, we show that every point of $W^{\Gamma(\vec{e})}(X)$ arises as a limit of line bundles with splitting type $\vec{e}$. In particular, the fiber over 0 of the closure of $W^{\vec{e}}(\mathcal{X}^*/B^*)$ coincides *set-theoretically* with $W^{\Gamma(\vec{e})}(X)$, and therefore also with $W^{\vec{e}}(X)_{\text{red}}$. For this task, we will use the language of limit linear series, as developed by Eisenbud and Harris [19].

Then, in the following section, we will show that $W^{\vec{e}}(X)$ is reduced. Since the closure of $W^{\vec{e}}(\mathcal{X}^*/B^*)$ is a priori contained scheme-theoretically in $W^{\vec{e}}(X)$, this will upgrade our set-theoretic regeneration theorem to a scheme-theoretic regeneration theorem. In other words, this will show that the fiber over 0 of the closure of $W^{\vec{e}}(\mathcal{X}^*/B^*)$ is equal to $W^{\Gamma(\vec{e})}(X)$ as schemes.

Recall that, for a splitting type $\vec{e}$, we write $d_1 > \cdots > d_{\mathcal{S}}^*$ for the distinct entries of $\vec{e}$, and $m_1, \ldots, m_{\mathcal{S}}^*$ for the corresponding multiplicities. (Note that $e_1 \leq e_2 \leq \cdots \leq e_k$ but $d_1 > d_2 > \cdots > d_{\mathcal{S}}^*$!) If $f \colon C \to \mathbb{P}^1$ is a smooth degree k cover, the locus $W^{\vec{e}}(C)$ can be described (set-theoretically) as follows. A line bundle L is in $W^{\vec{e}}(C)$ if and only if L possesses a collection for $j = 1, \ldots, \mathcal{S}^* - 1$ of linear series $V_j \subset H^0(C, L(-d_j))$ with $\dim V_j = h^0(\mathbb{P}^1, \mathcal{O}(\vec{e})(-d_j))$. Moreover, the V_j may be chosen so that the image of the natural map

$$V_{j-1} \otimes H^0(\mathbb{P}^1, \mathcal{O}(d_{j-1} - d_j)) \to H^0(C, L(-d_{j-1})) \otimes H^0(\mathbb{P}^1, \mathcal{O}(d_{j-1} - d_j)) \to H^0(C, L(-d_j)) \tag{4.5.1}$$

is contained in V_j. We call such a collection $\{V_1, \ldots, V_{\mathcal{S}}^*\}$ satisfying (4.5.1) an *$\vec{e}$-nested linear series* and V_j the linear series at *layer* j. By convention, we set $V_0 := \{0\}$.

Now suppose that $p \in C$ is a point of total ramification for f, and let $\mathfrak{I}_j(V_{j-1})$ denote the image of (4.5.1). Call the ramification indices of V_j at p that are *not* ramification indices of $\mathfrak{I}_j(V_{j-1})$ at p the *new ramification indices for layer j at p*. An $\vec{e}$-nested linear series will be called *non-colliding at p* if the new ramification indices at p are distinct from each other, and from all ramification indices at lower layers, modulo k. For $j' < j$, if $V_{j'}$ has a section vanishing to order a at p, then there are sections vanishing to orders $a, a + k, \ldots, a + (d_{j'} - d_j)k$ in $\mathfrak{I}_j(V_{j-1})$.

Lemma 4.5.1. *If $\{V_1, \ldots, V_s\}$ is a $\vec{e}$-nested linear series that is non-colliding at some $p \in C$, then*

$$\dim \mathfrak{I}_j(V_{j-1}) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\vec{e})(-d_j)) - m_j.$$

Hence, the number of new ramification indices at layer j is exactly m_j.

Proof. We induct on j. When $j = 1$, we have $\dim \mathfrak{I}_1(V_0) = 0$ by definition. Suppose that for all $j' < j$, there are $m_{j'}$ new sections at layer j', say $\sigma_{j',1}, \ldots, \sigma_{j',m_{j'}}$, of distinct vanishing orders mod k. Then $\mathfrak{I}_j(V_{j-1})$ is spanned by the image of

$$\bigoplus_{j'<j} \langle \sigma_{j',1}, \ldots, \sigma_{j',m_{j'}} \rangle \otimes H^0(\mathbb{P}^1, \mathcal{O}(d_{j'} - d_j)) \to H^0(C, L(-d_j)).$$

Considering orders of vanishing at p, we see that the map above is injective, so

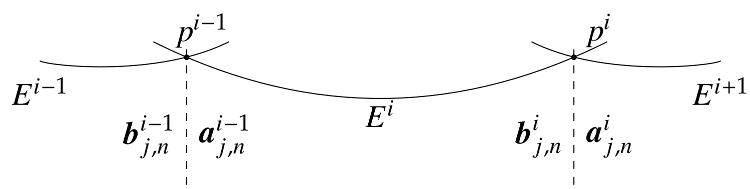
$$\dim \mathfrak{I}_j(V_{j-1}) = \sum_{j'<j} m_{j'} h^0(\mathbb{P}^1, \mathcal{O}(d_{j'} - d_j)) = h^0(\mathbb{P}^1, \mathcal{O}(\vec{e})(-d_j)) - m_j. \qquad \square$$

The above notions extend readily to limit linear series on our chain curve X: we call a collection of limit linear series $\{\mathcal{V}_1, \ldots, \mathcal{V}_{\mathcal{S}}^*\}$ an *$\vec{e}$-nested limit linear series* if for each component $E^i \to \mathbb{P}^1$, the collection of aspects $\{\mathcal{V}_1(E^i), \dots, \mathcal{V}_S^*(E^i)\}$ form an $\vec{e}$-nested linear series; we say this limit linear series is *non-colliding* if $\{\mathcal{V}_1(E^i), \dots, \mathcal{V}_S^*(E^i)\}$ is non-colliding at p^{i-1} and p^i.

In what follows, new ramification indices will be denoted in bold: $\boldsymbol{a}^i_{j,n}$ for $n = 1, \dots, m_j$ will be the new ramification indices in layer j at p^i on the component E^{i+1}. In terms of the $\boldsymbol{a}^i_{j,n}$, the ramification indices of $V_j(E^{i+1})$ at p^i are

$$\{\boldsymbol{a}^i_{j',n} + \delta k : \delta = 0, \dots, d_j - d_{j'}, n = 1, \dots, m_{j'}, j' = 1, \dots, j\}. \tag{4.5.2}$$

We define $\boldsymbol{b}^i_{j,n} := d - d_j k - \boldsymbol{a}^i_{j,n}$; these represent the new ramification indices in layer j at p^i on the component E^i (if our limit linear series is refined).



For ease of notation, we will sometimes replace (j, n) by its corresponding lexigraphical order

$$\ell := \ell(j, n) = m_1 + \ldots + m_{j-1} + n,$$

and so write $\boldsymbol{a}^i_{j,n} = \boldsymbol{a}^i_\ell$ and $\boldsymbol{b}^i_{j,n} = \boldsymbol{b}^i_\ell$.

Let T be any efficient tableau of shape $\Gamma(\vec{e})$. Our argument for the regeneration theorem will proceed in two basic steps. First, we show that a general $\vec{e}$-positive line bundle in $W^T(X)$ arises from a refined, non-colliding $\vec{e}$-nested limit linear series. Then, we prove a regeneration theorem for refined, non-colliding $\vec{e}$-nested limit linear series.

4.5.2 From tableaux to limit linear series

We explain how to construct nested limit linear series from tableaux. We will need to know our proposed new ramification indices increase across layers, as established in the following lemma.

Lemma 4.5.3. *Let T be an efficient tableau of shape $\Gamma(\vec{e})$. Define*

$$\boldsymbol{a}^i_{j,n} := T^{\leq i}(j, n) + i - 1 \quad (so\ \boldsymbol{b}^i_{j,n} = d - d_j k - \boldsymbol{a}^i_{j,n} = d - d_j k - i + 1 - T^{\leq i}(j, n)). \tag{4.5.3}$$

If $j' < j$, then $\boldsymbol{a}^i_{j',n'} < \boldsymbol{a}^i_{j,n}$ and $\boldsymbol{b}^i_{j',n'} < \boldsymbol{b}^i_{j,n}$.

Proof. For each i, Lemma 4.4.9 says $\boldsymbol{a}^i_1 < \boldsymbol{a}^i_2 < \boldsymbol{a}^i_3 < \cdots$. To obtain the statement for the $\boldsymbol{b}^i_{j,n}$, we first apply Proposition 4.4.13 to obtain

$$\boldsymbol{b}^g_{j,n} = d - d_j k - g + 1 - [\chi(\mathcal{O}(\vec{e})(-d_j)) - (m_1 + \cdots + m_j) + n] = m_1 + \ldots + m_j - n.$$

In particular,

$$\boldsymbol{b}^g_{j',n'} = m_1 + \ldots + m_{j'} - n' < m_1 + \ldots + m_j - n = \boldsymbol{b}^g_{j,n}. \tag{4.5.4}$$

We then rewrite Lemma 4.4.16 in terms of the $\boldsymbol{b}$'s to obtain

$$\boldsymbol{b}^i_{j',n'} - \boldsymbol{b}^g_{j',n'} \leq \boldsymbol{b}^i_{j,n} - \boldsymbol{b}^g_{j,n}. \tag{4.5.5}$$

The claim now follows from adding (4.5.4) and (4.5.5). □

Remark 4.5.4. Although $\boldsymbol{b}^i_{j',n'} < \boldsymbol{b}^i_{j,n}$ for $j' < j$ (increasing across layers), we have $\boldsymbol{b}^i_{j,1} > \boldsymbol{b}^i_{j,2} > \cdots > \boldsymbol{b}^i_{j,m_j}$ (decreasing within a layer).

Given a tableau T of shape $\Gamma(\vec{e})$, we now show that a general line bundle in $W^T(X)$ posseses a unique $\vec{e}$-nested limit linear series with the proposed ramification.

Lemma 4.5.5. *Let T be a tableau of shape $\Gamma(\vec{e})$, and let $\boldsymbol{a}^i_\ell$ and $\boldsymbol{b}^i_\ell$ be as defined in the previous lemma. A general line bundle L in $W^T(X)$ possesses a unique $\vec{e}$-nested limit linear series whose new ramification indices at p^i are are (exactly) $\boldsymbol{a}^i_\ell$ for the E^{i+1}-aspects and $\boldsymbol{b}^i_\ell$ for the E^i-aspects. This $\vec{e}$-nested limit linear series is refined and non-colliding.*

Proof. Equation (4.4.3) ensures that the proposed ramification indices are non-colliding. Equation (4.5.3) ensures the limit linear series is refined (c.f. (4.5.2)).

Fix i; we will build an $\vec{e}$-nested linear series on E^i, which will be the E^i-aspect of our desired $\vec{e}$-nested limit linear series. If $T[i] = *$, then L^i is a general degree d line bundle. Moreover, for any (j,n), we have $T^{\leq i-1}(j,n) = T^{\leq i}(j,n)$, so

$$\boldsymbol{a}^{i-1}_{j,n} + \boldsymbol{b}^i_{j,n} = \boldsymbol{a}^{i-1}_{j,n} + d - d_j k - \boldsymbol{a}^i_{j,n} = (T^{\leq i-1}(j,n) + i - 2) + d - d_j k - (T^{\leq i}(j,n) + i - 1) = d - d_j k - 1.$$

Thus, $L(-d_j)^i$ has a unique (up to rescaling) section $\sigma_{j,n}$ vanishing to orders exactly $\boldsymbol{a}^{i-1}_{j,n}$ at p^{i-1} and $\boldsymbol{b}^i_{j,n}$ at p^i. The unique linear series on E^i at layer j having the prescribed ramification is therefore

$$V_j(E^i) = \bigoplus_{(j',n'):j'\leq j} H^0(\mathcal{O}_{\mathbb{P}^1}(d_{j'} - d_j)) \cdot \sigma_{j',n'}.$$

We now suppose that i appears in T. Let $\ell_\pm$ be the indices such that i is decreasing for ℓ_- and increasing for ℓ_+, and write $\ell_\pm = (j_\pm, n_\pm)$. By Corollary 4.4.15, we have $j_- < j_+$. The

$\boldsymbol{a}_\ell^{i-1}$ and $\boldsymbol{a}_\ell^i$ are related via:

$$\boldsymbol{a}_{\ell_\pm}^{i-1} = T^{\leq i-1}(\ell_\pm) + i - 2 = T^{\leq i}(\ell_\pm) + i - 2 \mp 1 = \boldsymbol{a}_{\ell_\pm}^i - 1 \mp 1 \tag{4.5.6}$$

$$\boldsymbol{a}_\ell^{i-1} = T^{\leq i-1}(\ell) + i - 2 = T^{\leq i}(\ell) + i - 2 = \boldsymbol{a}_\ell^i - 1 \text{ for } \ell \neq \ell_\pm. \tag{4.5.7}$$

Equivalently,

$$\boldsymbol{a}_\ell^{i-1} + \boldsymbol{b}_\ell^i = \begin{cases} d - d_j k - 1 \mp 1 & \text{if } \ell = \ell_\pm \\ d - d_j k - 1 & \text{if } \ell \neq \ell_\pm \end{cases} \tag{4.5.8}$$

Futhermore, by Equation (4.4.5),

$$\boldsymbol{a}_{\ell_-}^{i-1} \equiv \boldsymbol{a}_{\ell_+}^{i-1} + 1 \quad \text{and} \quad \boldsymbol{a}_{\ell_+}^{i} \equiv \boldsymbol{a}_{\ell_-}^{i} + 1 \qquad (\text{mod } k). \tag{4.5.9}$$

By definition of $W^T(X)$, we have

$$L^i \simeq O_{E^i}(d_{j_-})(\boldsymbol{a}_{\ell_-}^{i-1} p^{i-1} + \boldsymbol{b}_{\ell_-}^i p^i) \simeq O_{E^i}(d_{j_+})((\boldsymbol{a}_{\ell_+}^{i-1} + 1) p^{i-1} + (\boldsymbol{b}_{\ell_+}^i + 1) p^i). \tag{4.5.10}$$

We now build the layers inductively (starting with the tautological case $V_0(E^i) = \{0\}$). After we have built layer $V_{j-1}(E^i)$, write $\ell = (j, n)$, and let

$$S_\ell := H^0(E^i, L^i(-d_j)(-\boldsymbol{a}_\ell^{i-1} p^{i-1} - \boldsymbol{b}_\ell^i p^i)) \subset H^0(E^i, L^i(-d_j))$$

be the subspace of sections having vanishing order at least $\boldsymbol{a}_\ell^{i-1}$ at p^{i-1} and $\boldsymbol{b}_\ell^i$ at p^i. We must show that S_ℓ posseses a section vanishing to order exactly $\boldsymbol{a}_\ell^{i-1}$ at p^{i-1} and $\boldsymbol{b}_\ell^i$ at p^i, and that this section is essentially unique, in the sense that, along with $\mathfrak{I}_j(V_{j-1}(E^i)) \cap S_\ell$, it generates S_ℓ.

For $\ell \neq \ell_\pm$, the line bundle $L^i(-d_j)(-\boldsymbol{a}_\ell^{i-1} p^{i-1} - \boldsymbol{b}_\ell^i p^i)$ has degree 1. Equation (4.4.3) implies $\boldsymbol{a}_\ell^{i-1} \not\equiv \boldsymbol{a}_{\ell_-}^{i-1}$ mod k, so this line bundle is not equal to $O_{E^i}(p^i)$. Similarly, since $\boldsymbol{a}_\ell^{i-1} \not\equiv \boldsymbol{a}_{\ell_+}^{i-1}$ mod k, this line bundle is not equal to $O_{E^i}(p^{i-1})$. Thus $\dim S_\ell = 1$ and S_ℓ consists of sections of the exact vanishing order desired.

When $\ell = \ell_-$, we have $L^i(-d_{j_-})(-\boldsymbol{a}_{\ell_-}^{i-1} p^{i-1} - \boldsymbol{b}_{\ell_-}^i p^i) = O_{E^i}$. Thus $\dim S_{\ell_-} = 1$, and the section with the required vanishing orders corresponds to the constant section of $L^i(-d_{j_-})$.

Finally, when $\ell = \ell_+$, we have $L^i(-d_{j_+})(-\boldsymbol{a}_{\ell_+}^{i-1} p^{i-1} - \boldsymbol{b}_{\ell_+}^i p^i) = O_{E^i}(p^{i-1} + p^i)$ and $\dim S_{\ell_+} = 2$. However, we shall show that $\mathfrak{I}_j(V_{(j_+)-1}(E^i))$ contains the 1-dimensional subspace of sections

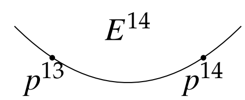
$$H^0(O_{E^i}) \subset H^0(O_{E^i}(p^{i-1} + p^i)) = S_{\ell_+} \subset H^0(E^i, L^i(-d_{j_+}))$$

that vanish to order $\boldsymbol{a}^{i-1}_{\ell_+} + 1$ at p^{i-1} and $\boldsymbol{b}^{i}_{\ell_+} + 1$ at p^i. By (4.5.2), this follows in turn from

$$\begin{aligned} \boldsymbol{a}^{i-1}_{\ell_+} + 1 &\equiv \boldsymbol{a}^{i-1}_{\ell_-} \quad \text{and} \quad \boldsymbol{b}^{i}_{\ell_+} + 1 \equiv \boldsymbol{b}^{i}_{\ell_-} \qquad \pmod k \\ \boldsymbol{a}^{i-1}_{\ell_+} + 1 &\geq \boldsymbol{a}^{i-1}_{\ell_-} \quad \text{and} \quad \boldsymbol{b}^{i}_{\ell_+} + 1 \geq \boldsymbol{b}^{i}_{\ell_-}. \end{aligned}$$

The first line follows from (4.5.9). The second line follows from Lemma 4.5.3 because $j_- < j_+$. □

Example 4.5.6 ($g = 20, \vec{e} = (-6,-4,-2,-2,0)$)**.** Let T be the tableau on the next page. The locus $W^T(X)$ is 1-dimensional, corresponding to the fact that 13 does not appear in T, and so L^{13} may be any degree 10 line bundle on E^{13}. We assume L^{13} is not a linear combination of the nodes. The table on the page following the tableau lists the new ramification indices $\boldsymbol{a}^i_\ell$ and $\boldsymbol{b}^i_\ell$. The table to the right lists all ramification indices for $V_j(E^{14})$ at each layer j. The ramification indices at p^{13} are on the left and those for p^{14} are on the right. The new ramification indices are bold, and images of a section in higher layers are given the same color.



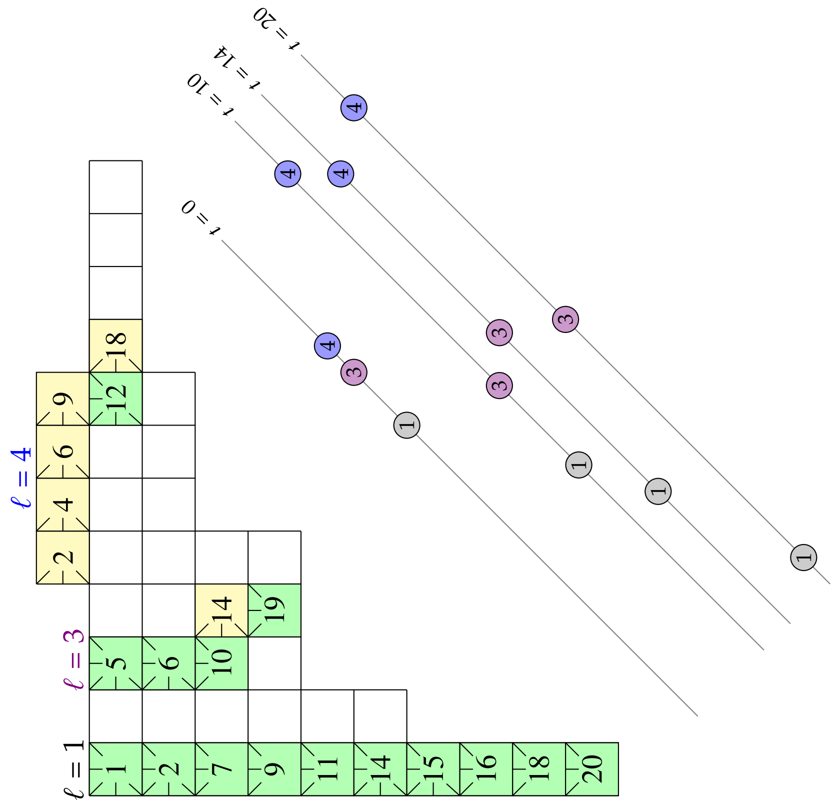
Layer $j = 1$	**8**	**2**
Layer $j = 2$	8	2
	11	**6**
	12	(7)
	(13)	**8**
	18	12
Layer $j = 3$	8	2
	11	6
	12	7
	13	8
	16	**10**
	17	11
	18	12
	19	13
	21	16
	22	17
	23	18
	28	22

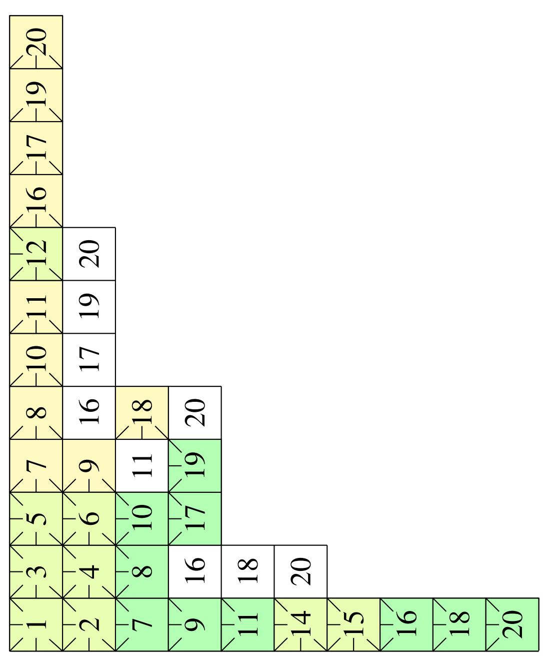
The number 14 is decreasing for the first truncation ($\ell_- = 1 = (1,1)$), so we have $L^{14} = \mathcal{O}_{E^{14}}(\mathbf{8}p^{13} + \mathbf{2}p^{14})$.

There are two new sections in layer 2, drawn in red ($\ell = 2 = (2,1)$) and purple ($\ell = 3 = (2,2)$). The number 14 is increasing for the third truncation ($\ell_+ = 3 = (2,2)$), and the old ramification indices that are one more than the new ramification indices are as follows:

$$\begin{aligned} \boldsymbol{a}^{13}_{2,2} + 1 = \mathbf{12} + 1 = (13) = \mathbf{8} + 5 = \boldsymbol{a}^{13}_{1,1} + k \\ \boldsymbol{b}^{14}_{2,2} + 1 = \mathbf{6} + 1 = (7) = \mathbf{2} + 5 = \boldsymbol{b}^{14}_{1,1} + k \end{aligned}$$

There is one new section in layer 3 ($\ell = 4 = (3,1)$), colored blue.


ℓ = 4
ℓ = 3
ℓ = 1
t = 0
t = 10
t = 14
t = 20



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Layer 1																				
$\ell = 1$	0 10	0 10	0 9	1 8	2 7	3 6	4 6	4 5	5 5	5 4	6 4	6 3	7 2	8 2	8 2	8 2	8 1	9 1	9 0	10 0
Layer 2																				
$\ell = 2$	1 18	2 17	3 17	3 17	3 16	4 15	5 14	6 14	6 13	7 12	8 11	9 10	10 9	11 8	12 6	14 5	15 5	15 4	16 3	17 2
$\ell = 3$	2 17	3 16	4 15	5 14	6 14	6 14	6 13	7 12	8 11	9 11	9 10	10 9	11 8	12 6	14 5	15 4	16 3	17 2	18 2	18 1
Layer 3																				
$\ell = 4$	3 26	4 24	6 23	7 21	9 20	10 18	12 17	13 16	14 14	16 13	17 12	18 12	18 11	19 10	20 9	21 8	22 7	23 5	25 4	26 3

4.5.7 Regeneration for refined, non-colliding nested limit linear series

Theorem 4.5.8 (Regeneration). *Suppose $\boldsymbol{a}^i_\ell$ and $\boldsymbol{b}^i_\ell$ are the new ramification indices at p^i for a refined, non-colliding $\vec{e}$-nested limit linear series. Let $\mathcal{X} \to \mathcal{P} \to B$ be the family of curves constructed in Section 3.1. There is a quasi-projective scheme $\widetilde{W}$ over B whose general fiber is contained in the space of $\vec{e}$-nested linear series on $\mathcal{X}^*$ and whose special fiber is the space of $\vec{e}$-nested limit linear series on X with the (strictly) specified ramification. Every component of $\widetilde{W}$ has dimension at least* $\dim \operatorname{Pic}^d(\mathcal{X}/B) - u(\vec{e})$.

Proof. We will construct a variety $\widetilde{W}^{\mathrm{fr}}$ which parametrizes compatible framings of nested linear series and surjects onto the desired $\widetilde{W}$. Set $\chi := \deg(\mathcal{O}(\vec{e})) + k = d - g + 1$. We retain notation as above so $d_1 > \cdots > d^*_S$ are the distinct degrees appearing in $\vec{e}$ and $m_1, \ldots, m^*_S$ are the corresponding multiplicities. Let $\operatorname{Pic} := \operatorname{Pic}^d(\mathcal{X}/B)$ and label maps as in the diagram below.

$$\begin{array}{ccc} \mathcal{X} \times_B \operatorname{Pic} & \longrightarrow & \mathcal{X} \\ & & \downarrow \mathfrak{f} \\ \downarrow \pi & & \mathcal{P} \\ & & \downarrow \varphi \\ \operatorname{Pic} & \xrightarrow[\eta]{} & B \end{array}$$

Let $\mathcal{L}$ be the universal limit line bundle on $\mathcal{X} \times_B \operatorname{Pic} \xrightarrow{\pi} \operatorname{Pic}$. Recall that $\mathcal{L}^i = \mathcal{L}_{(0,\ldots,0,d,0,\ldots0)}$ has degree d on component E^i and degree 0 on all other components. In addition, recall that $\mathcal{O}_{\mathcal{P}}(n)^i = \mathcal{O}_{\mathcal{P}}(n)_{(0,\ldots,0,n,0,\ldots,0)}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(n)$ on the smooth fibers of $\mathcal{P} \to B$; on the central fiber, it has degree n on P^i and degree 0 on all other components. (See Example 4.2.3.)

Let D be an effective divisor of relative degree N on $\mathcal{X} \to B$ so that D meets each component of the central fiber with sufficiently large degree. By slight abuse of notation we denote by D the pullback of this divisor to $\mathcal{X} \times_B \operatorname{Pic}$. By cohomology and base change, $\pi_*\mathcal{L}(-d_j)^i(D)$ is a vector bundle on $\operatorname{Pic}^{d-d_jk+N}(\mathcal{X}/B)$, which we identify with Pic via tensoring with $\mathcal{O}_{\mathcal{X}}(-d_j)(D)$; its rank is $N + \chi - d_jk$.

For each component E^i of the central fiber, we are going to build a tower

$$G^i_{S^*-1} \xrightarrow{\psi^i_{S^*-1}} G^i_{S^*-2} \xrightarrow{\psi^i_{S^*-2}} \cdots \to G^i_1 \xrightarrow{\psi^i_1} G^i_0 := \operatorname{Pic},$$

where each G^i_j is a Grassmann bundle $\operatorname{Gr}(m_j, Q^i_j)$ for Q^i_j a vector bundle over an open $U^i_{j-1} \subset G^i_{j-1}$. We will write $\mathcal{S}^i_j$ for the tautological subbundle on G^i_j. This tower of Grassmann bundles will parametrize $\vec{e}$-nested linear series on E^i. The bundle $\mathcal{S}^i_j$ will correspond to the space of "new sections at layer j" (i.e. sections at layer j modulo those coming from lower layers).

We build the tower by constructing the Q^i_j inductively. By convention, set $G^i_0 := \mathrm{Pic}$ and $U^i_0 := \mathrm{Pic}$. We begin by defining Q^i_1 as the push forward $Q^i_1 := \pi_*\mathcal{L}(-d_1)^i(D)$, which we have seen above is a vector bundle.

Now suppose, by induction, that we have defined Q^i_{j-1} as a quotient

$$(\psi^i_1 \circ \cdots \circ \psi^i_{j-2})^* \pi_* \mathcal{L}(-d_{j-1})^i(D) \to Q^i_{j-1},$$

defined on some open $U^i_{j-2} \subset G^i_{j-2}$. Let $(\mathcal{S}^i_{j-1})'$ be the pullback of the tautological bundle, i.e. the bundle so that the diagram below is a fiber square:

$$\begin{array}{ccc} (\psi^i_1 \circ \cdots \circ \psi^i_{j-1})^* \pi_* \mathcal{L}(-d_{j-1})^i(D) & \longrightarrow & (\psi^i_{j-1})^* Q^i_{j-1} \\ \uparrow & & \uparrow \\ (\mathcal{S}^i_{j-1})' & \longrightarrow & \mathcal{S}^i_{j-1}. \end{array}$$

The bundle $(\mathcal{S}^i_{j-1})'$ will correspond to the full linear series at layer $j-1$ (not just the new sections).

The layer j comparison maps of (4.5.1) fit together in our family as the map

$$\begin{array}{c} (\mathcal{S}^i_{j-1})' \otimes (\eta \circ \psi^i_1 \circ \cdots \circ \psi^i_{j-1})^* \varphi_* \mathcal{O}_{\mathcal{P}}(d_{j-1} - d_j)^i \\ \downarrow \\ (\psi^i_1 \circ \cdots \circ \psi^i_{j-1})^* \pi_* \mathcal{L}(-d_{j-1})^i(D) \otimes (\eta \circ \psi^i_1 \circ \cdots \circ \psi^i_{j-1})^* \varphi_* \mathcal{O}_{\mathcal{P}}(d_{j-1} - d_j)^i \\ \downarrow \\ (\psi^i_1 \circ \cdots \circ \psi^i_{j-1})^* \pi_* \mathcal{L}(-d_j)^i(D). \end{array}$$

As in the proof of Lemma 4.5.1, the above composition has rank at most $h^0(\mathcal{O}(\vec{e})(-d_j)) - m_j$. Define $U^i_{j-1} \subset G^i_{j-1}$ to be the open where the composition has exactly this rank. On U^i_{j-1}, define Q^i_j to be the cokernel of the compositon. Its rank $q_j := \mathrm{rk}\, Q^i_j$ is

$$\begin{aligned} q_j &= \mathrm{rk}(\pi_* \mathcal{L}(-d_j)^i(D)) - (h^0(\mathcal{O}(\vec{e})(-d_j)) - m_j) \\ &= N + \chi(\mathcal{O}(\vec{e})(-d_j)) - (h^0(\mathcal{O}(\vec{e})(-d_j)) - m_j) \\ &= N + m_j - h^1(\mathcal{O}(\vec{e})(-d_j)). \end{aligned} \tag{4.5.11}$$

Now we introduce a space parametrizing "lifted projective frames" over our tower of Grassmann bundles. Recalling that Q^i_j is a quotient of $(\psi^i_1 \circ \cdots \circ \psi^i_{j-1})^* \pi_* \mathcal{L}(-d_j)^i(D)$, let $\widetilde{G}^i_j \to G^i_j$ be the space of lifts of m_j-dimensional subspaces of Q^i_j to m_j-dimensional subspaces of $(\psi^i_1 \circ \cdots \circ \psi^i_{j-1})^* \pi_* \mathcal{L}(-d_j)^i(D)$ (so $\widetilde{G}^i_j$ is an open inside

$$\mathrm{Gr}(m_j, (\psi^i_1 \circ \cdots \circ \psi^i_{j-1})^* \pi_* \mathcal{L}(-d_j)^i(D))).$$

Let $\widetilde{\mathcal{S}}^i_j$ denote the tautological bundle on $\widetilde{G}^i_j$, and let $\mathrm{Fr}(\widetilde{\mathcal{S}}^i_j) \to \widetilde{G}^i_j$ be the space of projective frames of $\widetilde{\mathcal{S}}^i_j$.

For p a node on a component E^i, we inductively define

$$F^{i,p}_1 := \mathrm{Fr}(\widetilde{\mathcal{S}}^i_1) \quad \text{and} \quad F^{i,p}_j := F^{i,p}_{j-1} \times_{G^i_{j-1}} \mathrm{Fr}(\widetilde{\mathcal{S}}^i_j),$$

which encodes framing information for component i of all sections up to layer j. Note that $F^{i,p}_j$ does not depend on p — however, we include the p to highlight that we will be imposing conditions on these frames at the node p. We define a "master frame space"

$$F := F^{1,p^1}_{s-1} \times F^{2,p^1}_{s-1} \times F^{2,p^2}_{s-1} \times F^{3,p^2}_{s-1} \times F^{3,p^3}_{s-1} \times \cdots \times F^{g-1,p^{g-2}}_{s-1} \times F^{g-1,p^{g-1}}_{s-1} \times F^{g,p^{g-1}}_{s-1}$$

where all products are over Pic. This maps to

$$G := G^1_{s-1} \times G^2_{s-1} \times G^2_{s-1} \times G^3_{s-1} \times G^3_{s-1} \times \cdots \times G^{g-1}_{s-1} \times G^{g-1}_{s-1} \times G^g_{s-1}.$$

Next we compute $\dim F$. The relative dimension of $\mathrm{Fr}(\widetilde{\mathcal{S}}^i_j)$ over G^i_j is

$$m_j \cdot (\mathrm{rk}\, \pi_* \mathcal{L}(-d_j)^i(D) - 1) = m_j \cdot [h^0(\mathcal{O}(\vec{e})(-d_j)) - 1]. \tag{4.5.12}$$

Since each ψ^i_j is relative dimension $m_j(q_j - m_j)$, we have

$$\dim F^{i,p}_{s-1} = \dim \mathrm{Pic} + \sum_{j=1}^{\mathcal{S}^*-1} m_j \cdot [h^0(\mathcal{O}(\vec{e})(-d_j)) - 1] + m_j(q_j - m_j).$$

All together, we find that

$$\begin{aligned}\dim F &= \dim \operatorname{Pic} + (2g-2)\left(\sum_{j=1}^{S^*-1} m_j \cdot [h^0(\mathcal{O}(\vec{e})(-d_j)) - 1] + m_j(q_j - m_j)\right)\\ &= \dim \operatorname{Pic} + (2g-2)\left(\sum_{j=1}^{S^*-1} m_j \cdot [h^0(\mathcal{O}(\vec{e})(-d_j)) - 1] + m_j(N - h^1(\mathcal{O}(\vec{e})(-d_j)))\right)\\ &= \dim \operatorname{Pic} + (2g-2)\left(\sum_{j=1}^{S^*-1} m_j(N + \chi(\mathcal{O}(\vec{e})(-d_j)) - 1)\right). \qquad (4.5.13)\end{aligned}$$

We now construct a subvariety $\widetilde{W}^{\mathrm{fr}} \subset F$ that parametrizes compatible projective frames of $\vec{e}$-nested limit linear series. The image of $\widetilde{W}^{\mathrm{fr}}$ in G will be the desired variety $\widetilde{W}$.

We will impose three types of conditions on frames; the first two will be pull-backs of conditions on G.

1. First, we require that the spaces of sections corresponding to the same component are equal.

2. Second we require that the space of sections vanish along D.

3. Finally, we impose compatibility conditions for the two frames labeled with the same node.

4.5.9 Compatibility along components

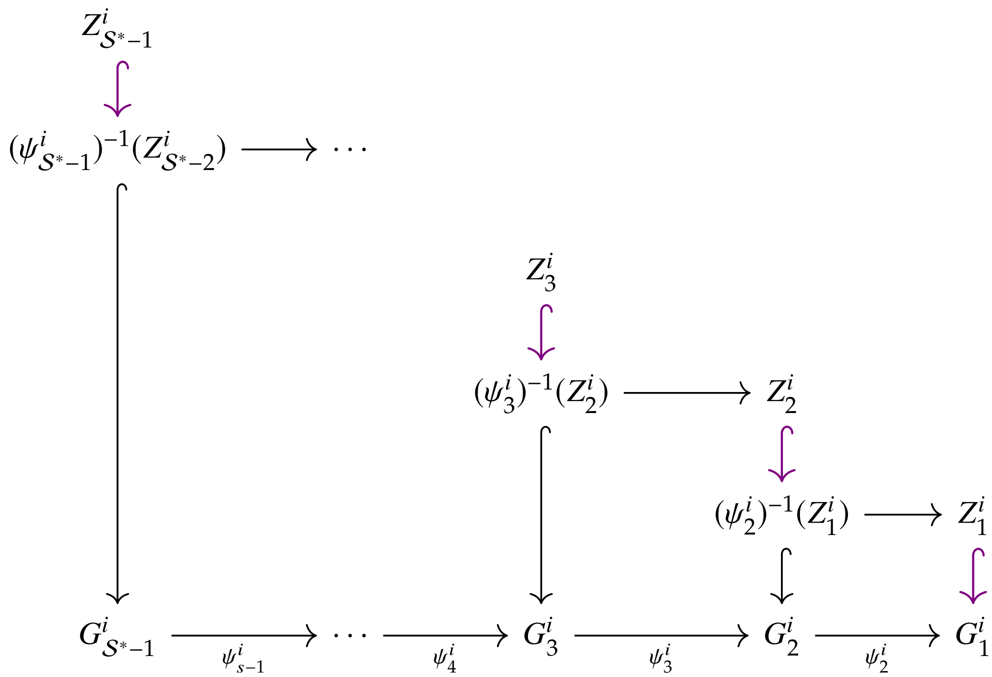
The first of these conditions is represented by restricting to a diagonal. This imposes

$$\begin{aligned}\operatorname{codim}\left(G^1_{s-1} \times G^2_{s-1} \times \cdots \times G^g_{s-1} \hookrightarrow G\right) &= \dim G^2_{s-1} + \ldots + \dim G^{g-1}_{s-1}\\ &= (g-2)\sum_{j=1}^{s-1} m_j(q_j - m_j)\\ &= (g-2)\sum_{j=1}^{s-1} m_j(N - h^1(\mathcal{O}(\vec{e})(-d_j))) \qquad (4.5.14)\end{aligned}$$

conditions.

4.5.10 Vanishing along D

After étale base change, we may assume that each component of D meets only one E^i. Let D^i be the union of components of D meeting E^i, and let $Z^i_j \subset G^i_j$ be the locus where the nested linear series vanishes on D^i (up to layer j). To determine an upper bound on the codimension of $Z^i_{s-1} \subset G^i_{\mathcal{S}^*-1}$, we count the number of equations needed to describe $Z^i_j \subset (\psi^i_j)^{-1}(Z^i_{j-1})$ at each layer.



We start with the locus $Z^i_1 \subset G^i_1$, which is defined by the vanishing of the composition

$$\mathcal{S}^i_1 \to (\psi^i_1)^* Q^i_1 = (\psi^i_1)^* \pi_* \mathcal{L}(-d_1)^i(D) \to (\psi^i_1)^* \pi_* (\mathcal{L}(-d_1)^i(D) \otimes \mathcal{O}_{D^i}).$$

This represents $m_1 \deg(D^i)$ equations.

On $(\psi^i_j)^{-1} Z^i_{j-1}$, evaluation

$$(\psi^i_1 \circ \cdots \circ \psi^i_{j-1})^* \pi_* (\mathcal{L}(-d_j)^i(D)) \to (\psi^i_1 \circ \cdots \circ \psi^i_{j-1})^* \pi_* (\mathcal{L}(-d_j)^i(D) \otimes \mathcal{O}_{D^i})$$

factors through Q^i_j. Therefore, $Z^i_j \subset (\psi^i_{j-1})^{-1}(Z^i_{j-1})$ is the locus where the composition

$$\mathcal{S}^i_j \to (\psi^i_j)^* Q^i_j \to (\psi^i_1 \circ \cdots \circ \psi^i_j)^* \pi_* (\mathcal{L}(-d_j)^i(D) \otimes \mathcal{O}_{D^i})$$

vanishes. This represents $m_j \deg(D^i)$ equations.

Totaling over the layers, we see that every component of $Z^i_{\mathcal{S}^*-1}$ has codimension at most

$$\operatorname{codim}(Z^i_{\mathcal{S}^*-1} \subset G^i_{\mathcal{S}^*-1}) \leq (m_1 + \ldots + m_{s-1}) \deg(D^i).$$

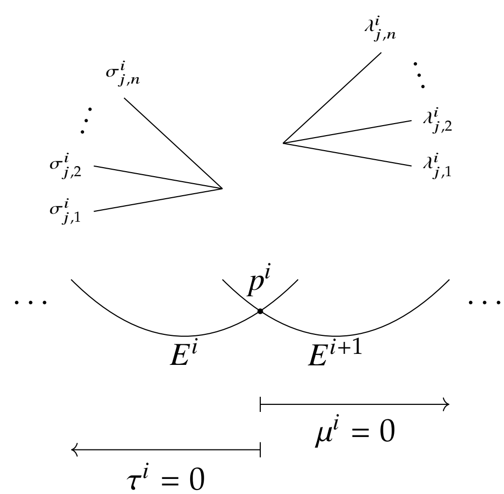
Taking the product over all components, every component of $Z^1_{\mathcal{S}^*-1} \times_{\mathrm{Pic}} \cdots \times_{\mathrm{Pic}} Z^g_{\mathcal{S}^*-1}$ has codimension at most

$$\operatorname{codim}(Z^1_{\mathcal{S}^*-1} \times_{\mathrm{Pic}} \cdots \times_{\mathrm{Pic}} Z^g_{\mathcal{S}^*-1} \subset G^1_{s-1} \times G^2_{s-1} \times \cdots \times G^g_{s-1}) \leq (m_1 + \ldots + m_{s-1})N. \tag{4.5.15}$$

4.5.11 Compatibility at nodes

For each node p^i we describe equations on $F^{i,p^i}_{s-1} \times F^{i+1,p^i}_{s-1}$ that impose ramification conditions on both frames at p^i in the central fiber, and say the two frames are equal up to translation by old sections on the general fiber.

Let $\sigma^i_{j,1}, \ldots, \sigma^i_{j,m_j}$ be coordinates on the $\mathrm{Fr}(\widetilde{\mathcal{S}}^i_j)$ component of F^{i,p^i}_{s-1} (the universal framing of $\widetilde{\mathcal{S}}^i_j$ associated to p^i). Similarly, let $\lambda^i_{j,1}, \ldots, \lambda^i_{j,m_j}$ be coordinates on the $\mathrm{Fr}(\widetilde{\mathcal{S}}^{i+1}_j)$ component of F^{i+1,p^i}_{s-1} (the universal framing of $\widetilde{\mathcal{S}}^{i+1}_j$ associated to p^i). Let τ^i be the constant section of $\mathcal{O}_{\mathcal{X}}(X^{\leq i})$ (that vanishes to the left of p^i), and let μ^i be the constant section of $\mathcal{O}_{\mathcal{X}}(X^{>i})$ (that vanishes to the right of p^i).



Suppose that for layer j, the specified new ramification indices at p^i on E^{i+1} are

$$\boldsymbol{a}^i_{j,1}, \boldsymbol{a}^i_{j,2}, \ldots, \boldsymbol{a}^i_{j,m_j} \text{ (desired ramification of the } \lambda^i_{j,n}\text{'s)},$$

and that the new ramification indices at p^i on E^i are the

$$\boldsymbol{b}^i_{j,n} = d - d_j k - \boldsymbol{a}^i_{j,n} \text{ (desired ramification of the } \sigma^i_{j,n}\text{'s).}$$

We define a closed subvariety $Y^i \subset F^{i,p^i}_{s-1} \times F^{i+1,p^i}_{s-1}$ by the conditions (for $j = 1, \dots, s-1$, and $n = 1, \dots, m_j$):

$$\sigma^i_{j,n} \otimes (\tau^i)^{\boldsymbol{a}^i_{j,n}} = \lambda^i_{j,n} \otimes (\mu^i)^{\boldsymbol{b}^i_{j,n}}, \tag{4.5.16}$$

viewed as elements of the projectivization of

$$\pi_*\mathcal{L}(-d_j)^i(D + \boldsymbol{a}^i_{j,n} X^{\le i}) \cong \pi_*\mathcal{L}(-d_j)^{i+1}(D + \boldsymbol{b}^i_{j,n} X^{>i}).$$

The isomorphism above comes from the fact that

$$\mathcal{L}(-d_j)^i \simeq \mathcal{L}(-d_j)^{i+1}(-(d - d_j k)X^{\le i})$$

and $\mathcal{O}(X^{\le i}) \cong \mathcal{O}(-X^{>i})$ and $\boldsymbol{a}^i_{j,n} + \boldsymbol{b}^i_{j,n} = d - d_j k$.

Away from the central fiber, τ^i and μ^i are non-zero, so (4.5.16) says the new sections $\sigma^i_{j,n}$ and $\lambda^i_{j,n}$ are equal (up to scaling).

In the central fiber, $\tau^i|_{X^{>i}}$ is not a zero divisor and vanishes only at p^i, while $\mu^i|_{X^{\le i}}$ is not a zero a divisor and vanishes only at p^i. Thus condition (4.5.16) says that $\sigma^i_{j,n}$ and $\lambda^i_{j,n}$ are determined by the restrictions $\sigma^i_{j,n}|_{X^{\le i}}$ and $\lambda^i_{j,n}|_{X^{>i}}$, which can be anything of the desired vanishing order at p^i:

$$\sigma^i_{j,n}|_{X^{\le i}} \text{ vanishes to order at least } \boldsymbol{b}^i_{j,n} \text{ at } p^i \tag{4.5.17}$$

$$\lambda^i_{j,n}|_{X^{>i}} \text{ vanishes to order at least } \boldsymbol{a}^i_{j,n} \text{ at } p^i \tag{4.5.18}$$

$$\lambda^i_{j,n}|_{X^{\le i}} = \sigma^i_{j,n}|_{X^{\le i}} \cdot \frac{\tau^i|_{X^{\le i}}^{\otimes \boldsymbol{a}^i_{j,n}}}{\mu^i|_{X^{\le i}}^{\otimes \boldsymbol{b}^i_{j,n}}} \quad \text{and} \quad \sigma^i_{j,n}|_{X^{>i}} = \lambda^i_{j,n}|_{X^{>i}} \cdot \frac{\mu^i|_{X^{>i}}^{\otimes \boldsymbol{b}^i_{j,n}}}{\tau^i|_{X^{>i}}^{\otimes \boldsymbol{a}^i_{j,n}}}. \tag{4.5.19}$$

Since τ^i vanishes on $X^{\le i}$, and μ^i vanishes on $X^{>i}$, in most cases (4.5.19) simplifies to: $\lambda^i_{j,n}|_{X^{\le i}} = 0$ (unless $\boldsymbol{a}^i_{j,n} = 0$), and $\sigma^i_{j,n}|_{X^{>i}} = 0$ (unless $\boldsymbol{b}^i_{j,n} = 0$). In particular, when $j > 1$, both $\boldsymbol{a}^i_{j,n}$ and $\boldsymbol{b}^i_{j,n}$ are positive, so (4.5.19) simplifies to $\lambda^i_{j,n}|_{X^{\le i}} = 0$ and $\sigma^i_{j,n}|_{X^{>i}} = 0$.

For each (j, n), equation (4.5.16) represents

$$\operatorname{rk} \pi_*\mathcal{L}(-d_j)^i(D + \boldsymbol{a}^i_{j,n} X^{\le i}) - 1 = \operatorname{rk} \pi_*\mathcal{L}(-d_j)^i(D) - 1 = N + \chi(\mathbb{P}^1, \mathcal{O}(\vec{e})(-d_j)) - 1$$

conditions. Thus, every component of Y^i has codimension at most

$$\operatorname{codim}(Y^i \subset F^{i,p^i}_{s-1} \times F^{i+1,p^i}_{s-1}) \le \sum_{j=1}^{s-1} m_j(N + \chi(\mathbb{P}^1, \mathcal{O}(\vec{e})(-d_j)) - 1),$$

and so every component of $Y^1 \times Y^2 \times \cdots \times Y^{g-1}$ has codimension at most

$$\operatorname{codim}(Y^1 \times Y^2 \times \cdots \times Y^{g-1} \subset F) \le (g-1) \sum_{j=1}^{s-1} m_j(N + \chi(\mathbb{P}^1, \mathcal{O}(\vec{e})(-d_j)) - 1). \quad (4.5.20)$$

Imposing all the conditions of Sections 4.5.9 – 4.5.11 defines a closed subvariety of F. Let us additionally remove the locus over the central fiber where any ramification index of the frame is larger than the specified ramification index. We call the resulting quasiprojective variety $\widetilde{W}^{\mathrm{fr}}$. The image of $\widetilde{W}^{\mathrm{fr}} \to G$ is the desired $\widetilde{W}$.

4.5.12 Fiber dimensions

Finally, let us count the dimension of fibers $\widetilde{W}^{\mathrm{fr}} \to G$. At each layer j, the master frame space F parameterizes $2g-2$ lifted projective frames of dimension m_j.

On the general fiber, our equations specify that these frames are equal in $g-1$ pairs. The fiber dimension $\widetilde{W}^{\mathrm{fr}} \to G$ is thus equal to (c.f. (4.5.12)):

$$(g-1) \sum_{j=1}^{s-1} m_j(h^0(\mathcal{O}(\vec{e})(-d_j)) - 1). \quad (4.5.21)$$

On the special fiber, let $\{\sigma^i_{j,n}, \lambda^i_{j,n}\}$ denote a point of $\widetilde{W}^{\mathrm{fr}}$. The other points in the same fiber are then obtained by applying linear transformations Σ^i and Λ^i to the $\sigma^i_{j,n}$ and $\lambda^i_{j,n}$, of a particular form we will now explain. Let x^i and y^i be sections of $\mathcal{O}_P(1)^i$ that vanish at $q^{i-1} = f(p^{i-1})$ and $q^i = f(p^i)$ respectively; via pullback, we think of them as sections of $\mathcal{O}_X(1)^i$ on X vanishing at p^{i-1} and p^i respectively. Then to $\sigma^i_{j,n}$, the linear transformation Σ^i may add any section from a previous layer whose image in layer j has higher vanishing order; explicitly, we may add $\sigma^i_{j',n'}$ times any monomial $(x^i)^\delta \cdot (y^i)^{d_{j'}-d_j-\delta}$ (with $0 \le \delta \le d_{j'} - d_j$) such that

$$\boldsymbol{b}^i_{j',n'} + (d_{j'} - d_j - \delta)k > \boldsymbol{b}^i_{j,n}. \quad (4.5.22)$$

Similarly, to $\lambda^i_{j,n}$, the linear transformation Λ^i may add $\lambda^i_{j',n'}$ times any monomial $(x^i)^\delta \cdot$

$(y^i)^{d_{j'}-d_j-\delta}$ (with $0 \le \delta \le d_{j'} - d_j$) such that

$$\boldsymbol{a}^i_{j',n'} + \delta k > \boldsymbol{a}^i_{j,n}. \tag{4.5.23}$$

The fiber dimension is therefore the total number of monomials satisfying (4.5.22), plus the number satisfying (4.5.23). Recall that $\boldsymbol{b}^i_{j',n'} + \boldsymbol{a}^i_{j',n'} = d - d_{j'}k$ and $\boldsymbol{b}^i_{j,n} + \boldsymbol{a}^i_{j,n} = d - d_j k$, and $\boldsymbol{a}^i_{j',n'} \not\equiv \boldsymbol{a}^i_{j,n} \pmod{k}$ unless $(j', n') = (j, n)$. Therefore, every monomial satisfies exactly one of (4.5.22) or (4.5.23), except when $(j', n') = (j, n)$. The fiber dimension is therefore

$$(g-1)\sum_{j=1}^{s-1} m_j \left(-1 + \sum_{j' \le j} m_{j'}(d_{j'} - d_j + 1)\right) = (g-1)\sum_{j=1}^{s-1} m_j (h^0(\mathcal{O}(\vec{e})(-d_j)) - 1). \tag{4.5.24}$$

Note that this is the same as the fiber dimension on the general fiber.

4.5.13 Final dimension estimate

Recalling the dimension of F and totaling the equations imposed in Sections 4.5.9 – 4.5.11, we find that every component $\widetilde{W}'$ of $\widetilde{W}$ has dimension

$$\begin{aligned}
\dim \widetilde{W}' &\geq \dim F - (\text{number of defining equations}) - (\text{fiber dimension}) \\
&= \dim \mathrm{Pic} + (2g-2)\left(\sum_{j=1}^{\mathcal{S}^*-1} m_j(N + \chi(\mathcal{O}(\vec{e})(-d_j)) - 1)\right) \\
&\qquad \qquad (\dim F, \text{ c.f. } (4.5.13)) \\
&\quad - (g-2)\sum_{j=1}^{s-1} m_j(N - h^1(\mathcal{O}(\vec{e})(-d_j))) \\
&\qquad \qquad (\text{diagonal condition, c.f. } (4.5.14)) \\
&\quad - N\sum_{j=1}^{\mathcal{S}^*-1} m_j \qquad (\text{vanishing along } D, \text{ c.f. } (4.5.15)) \\
&\quad - (g-1)\sum_{j=1}^{s-1} m_j(N + \chi(\mathcal{O}(\vec{e})(-d_j)) - 1) \\
&\qquad \qquad (\text{compatibility at nodes, c.f. } (4.5.20)) \\
&\quad - (g-1)\sum_{j=1}^{s-1} m_j(h^0(\mathcal{O}(\vec{e})(-d_j)) - 1) \\
&\qquad \qquad (\text{fiber dimension, c.f.} (4.5.21) \text{ and } (4.5.24)) \\
&= \dim \mathrm{Pic} - \sum_{j=1}^{\mathcal{S}^*-1} m_j h^1(\mathcal{O}(\vec{e})(-d_j)) \\
&= \dim \mathrm{Pic} - u(\vec{e}).
\end{aligned}$$

□

The regeneration theorem allows us to show that the $\vec{e}$-nested limit linear series built from tableaux as in Lemma 4.5.5 arise as limits from smooth curves, implying that $W^{\Gamma(\vec{e})}(X)$ is contained in the closure of $W^{\vec{e}}(\mathcal{X}^*/B^*)$.

Corollary 4.5.14. *Let T be an efficiently filled tableau of shape $\Gamma(\vec{e})$. Then $W^T(X)$ is contained in the closure of $W^{\vec{e}}(\mathcal{X}^*/B^*)$.*

Proof. Let $\widetilde{W}$ be the quasiprojective scheme with ramification corresponding to T as constructed in Theorem 4.5.8. By Lemma 4.5.5, a generic line bundle in $W^T(X)$ is in the image of $\widetilde{W}$ in $\mathrm{Pic}^d(X)$. Moreover, the uniqueness statement in Lemma 4.5.5 together with the fact that $\dim W^T(X) = g - u(\vec{e})$ shows that the restriction of $\widetilde{W}$ to the central fiber has an irreducible component Y of dimension $g - u(\vec{e})$ that dominates $W^T(X)$. By Theorem 4.5.8, the dimension of Y is less than the dimension of any component of $\widetilde{W}$. Let Y' be an irreducible component of $\widetilde{W}$ containing Y, which necessarily dominates the base. The closure of the image of Y' in $\mathrm{Pic}^d(\mathcal{X}/B)$ contains $W^T(X)$ and is contained in the closure of $W^{\vec{e}}(\mathcal{X}^*/B^*)$. □

4.6 Reducedness and Cohen–Macaulayness

4.6.1 Reducedness

At the central fiber $X = E^1 \cup_{p^1} \cdots \cup_{p^{g-1}} E^g$, the schemes $W^{\vec{e}}(X)$ are defined as intersections of determinantal loci of the form

$$W^r_{\vec{d}}(X) = \{L \in \mathrm{Pic}^{\vec{d}}(X) : h^0(X, L) \geq r + 1\},$$

for various degree distributions $\vec{d} = (d^1, \ldots, d^g)$ and integers r.

We will first show that any such determinental locus is a union of preimages of reduced points under various projection maps $\mathrm{Pic}^{\vec{d}}(X) \to \prod_{i \in S} \mathrm{Pic}^{d^i}(E^i)$ (for $S \subseteq \{1, 2, \ldots, g\}$). Then, we will show that the class of such varieties is closed under intersection, thus establishing that $W^{\vec{e}}(X)$ is reduced. Combined with Corollary 4.5.14, this will show that

$$W^{\vec{e}}(X) = W^{\Gamma}(X).$$

Let $\pi \colon X \times \mathrm{Pic}^{\vec{d}}(X) \to \mathrm{Pic}^{\vec{d}}(X)$ be the projection and let $\mathcal{L} := \mathcal{L}_{\vec{d}}$ be a universal line bundle on $X \times \mathrm{Pic}^{\vec{d}}(X)$. Additionally, let D be a divisor on X, contained in the smooth locus, and such that $D^i := D \cap E^i$ is of sufficiently high degree. Recall that, in terms of the natural map

$$\phi \colon \pi_* \mathcal{L}(D) \to \pi_*(\mathcal{L}(D)|_D),$$

the loci of interest are

$$W^r_{\vec{d}}(X) = \{L \in \mathrm{Pic}^{\vec{d}}(X) : \dim \ker \phi|_L \geq r + 1\}.$$

The scheme structure is given by the $(n+1) \times (n+1)$ minors of ϕ, where

$$n = \mathrm{rk}\, \pi_* \mathcal{L}(D) - (r + 1) = d - g - r + \deg(D).$$

We will describe the rank loci of ϕ in terms of evaluation maps on the normalization of

X. Let $\nu\colon E^1 \sqcup \cdots \sqcup E^g \to X$ be the normalization and let $\pi^i\colon E^i \times \operatorname{Pic}^{d^i}(E^i) \to \operatorname{Pic}^{d^i}(E^i)$ be the projection. In addition, let $\operatorname{pr}^i\colon \operatorname{Pic}^{\vec{d}}(X) \to \operatorname{Pic}^{d^i}(E^i)$ be the projection and let $\mathcal{L}_i$ be a universal line bundle on $E^i \times \operatorname{Pic}^{d^i}(E^i)$ such that $\mathcal{L}|_{E^i\times \operatorname{Pic}^{\vec{d}}(X)} = (\operatorname{Id}\times \operatorname{pr}^i)^*\mathcal{L}_i$. These maps fit into the diagram:

$$
\begin{array}{ccccc}
 & & \bigsqcup_{i=1}^g E^i \times \operatorname{Pic}^{\vec{d}}(X) & & \\
 & \nearrow & & \searrow^{\nu\times\operatorname{Id}} & \\
E^i \times \operatorname{Pic}^{\vec{d}}(X) & & \xrightarrow{\quad\subset\quad} & & X \times \operatorname{Pic}^{\vec{d}}(X) \\
\downarrow{\scriptstyle \operatorname{Id}\times\operatorname{pr}^i} & & & & \downarrow{\scriptstyle \pi} \\
E^i \times \operatorname{Pic}^{d^i}(E^i) & & & & \operatorname{Pic}^{\vec{d}}(X) \\
 & \searrow_{\pi^i} & & \swarrow_{\operatorname{pr}^i} & \\
 & & \operatorname{Pic}^{d^i}(E^i) & &
\end{array}
$$

There is a commuting diagram of vector bundles on $\operatorname{Pic}^{\vec{d}}(X)$:

$$
\begin{array}{ccccccccc}
 & & & & \bigoplus_{i=1}^g (\operatorname{pr}^i)^*(\pi^i)_*\mathcal{L}_i(D^i) & & & & \\
 & & & & \| & & & & \\
0 & \longrightarrow & \pi_*\mathcal{L}(D) & \longrightarrow & \pi_*(\nu\times\operatorname{Id})_*(\nu\times\operatorname{Id})^*\mathcal{L}(D) & \xrightarrow{\eta} & \bigoplus_{i=1}^{g-1} \pi_*(\mathcal{L}(D)|_{p^i}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & & & \\
 & & \pi_*(\mathcal{L}(D)|_D) & \xrightarrow{\sim} & \pi_*(\nu\times\operatorname{Id})_*(\nu\times\operatorname{Id})^*\mathcal{L}(D)|_{\nu^{-1}(D)} & & & & \\
 & & & & \| & & & & \\
 & & & & \bigoplus_{i=1}^g (\operatorname{pr}^i)^*(\pi^i)_*\mathcal{L}_i(D^i)|_{D^i} & & & &
\end{array}
$$

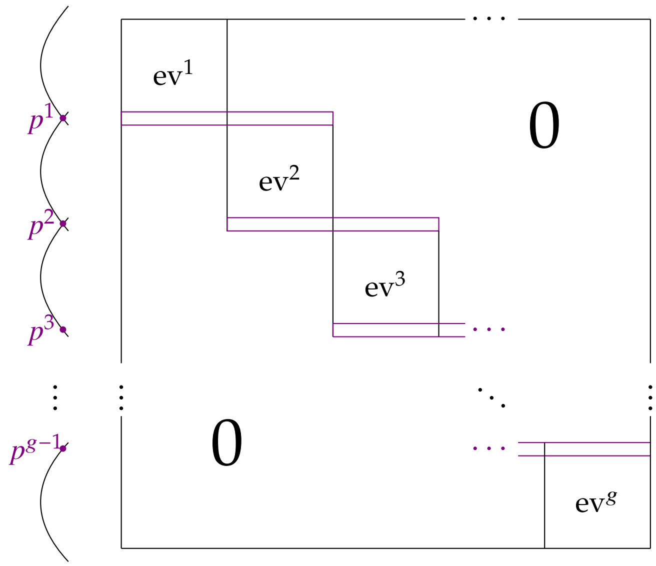
The top row is exact by our assumption that D is sufficiently positive. Since sequences of vector bundles split locally, the rank loci of ϕ are corresponding rank loci of $\psi \oplus \eta$:

$$
\{L \in \operatorname{Pic}^{\vec{d}}(X) : \operatorname{rk}\phi \le n\} = \{L \in \operatorname{Pic}^{\vec{d}}(X) : \operatorname{rk}(\psi\oplus\eta) \le n + (g-1)\}. \tag{4.6.1}
$$

The restriction of $\psi \oplus \eta$ to each summand $(\operatorname{pr}^i)^*(\pi^i)_*\mathcal{L}_i(D^i)$ is $(\operatorname{pr}^i)^*\operatorname{ev}^i$, where

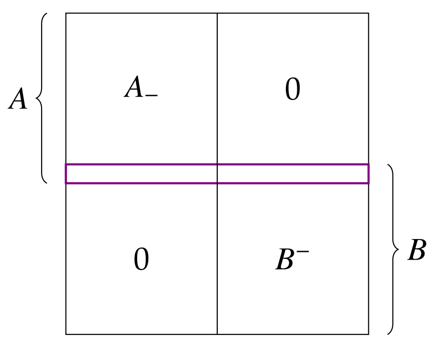
$$
\operatorname{ev}^i\colon (\pi^i)_*\mathcal{L}_i(D^i) \to (\pi^i)_*(\mathcal{L}_i(D^i)|_{p^{i-1}\cup p^i \cup D^i})
$$

is the map that evaluates a section on E^i at the points of D^i and the nodes p^{i-1} and p^i (or just p^1 on E^1 or just p^{g-1} on E^g). The matrix for $\psi \oplus \eta$ is almost block diagonal.



The following lemma helps us describe the rank loci of such almost block diagonal matrices scheme-theoretically.

Lemma 4.6.2. *Let* $\mathrm{Mat}(x, y)$ *denote the affine space of* $x \times y$ *matrices. Suppose that* S *and* T *are given schemes, and* $A\colon S \to \mathrm{Mat}(x_1, y_1)$ *and* $B\colon T \to \mathrm{Mat}(x_2, y_2)$ *are two families of matrices. Let* $M\colon S \times T \to \mathrm{Mat}(x_1 + x_2, y_1 + y_2 - 1)$ *be the family of matrices where the upper left entries are given by* A*, the lower right entries are given by* B*, and the rest of the entries are* 0.



Let $A_-\colon S \to \mathrm{Mat}(x_1 - 1, y_1)$ *be the composition of* A *with the map that removes the bottom row. Similarly, let* $B^-\colon T \to \mathrm{Mat}(x_2, y_2 - 1)$ *be the composition of* B *with the map that removes the top row. Let*

$$S_a = \{s \in S : \mathrm{rk}\, A(s) \le a\} \qquad T_b = \{t \in T : \mathrm{rk}\, B(t) \le b\}$$
$$S_a^- = \{s \in S : \mathrm{rk}\, A_-(s) \le a\} \qquad T_b^- = \{t \in T : \mathrm{rk}\, B^-(t) \le b\},$$

be defined scheme-theoretically by the vanishing of appropriately sized minors. Then

$$\{(s,t) \in S \times T : \operatorname{rk} M(s,t) \leq n\} = \bigcup_{a+b=n} (S_a \times T_b) \cup \bigcup_{a+b=n-1} (S_a^- \times T_b^-) \tag{4.6.2}$$

as schemes. In particular, if all rank loci of A, A_-, B, and B^- are reduced, then all rank loci of M are reduced.

Proof. The scheme structure on the left hand side of (4.6.2) is defined by the $(n+1) \times (n+1)$ minors of M. Any such minor contains a columns meeting A and b columns meeting B for some $a + b = n + 1$.

First consider a square submatrix of M that uses the common row (colored violet in the diagram). If its determinant is nonzero, then the submatrix contains at least $a - 1$ rows meeting A_- and at least $b - 1$ rows meeting B^-. Either there are a rows meeting A_-, and its determinant is an $a \times a$ minor of A_- times a $b \times b$ minor of B; or there are b rows meeting B^-, and its determinant is an $a \times a$ minor of A times a $b \times b$ minor of B^-. Now consider a square submatrix of M that does not use the common row. If its determinant is nonzero, then it is a product of an $a \times a$ minor of A_- and a $b \times b$ minor of B^-. Ranging over all minors with this distribution of columns we obtain elements that generate

$$I(S_{a-1}^-) \cdot I(T_{b-1}) + I(S_{a-1}) \cdot I(T_{b-1}^-) = I((S_{a-1}^- \times T \cup S \times T_{b-1}) \cap (S_{a-1} \times T \cup S \times T_{b-1}^-)),$$

where $S_{-1} = S_{-1}^- = T_{-1} = T_{-1}^- = \varnothing$ by convention.

We therefore find that

$$\{(s,t) \in S \times T : \operatorname{rk} M(s,t) \leq n\} = \bigcap_{a+b=n+1} (S_{a-1}^- \times T \cup S \times T_{b-1}) \cap (S_{a-1} \times T \cup S \times T_{b-1}^-).$$

Because A_- is obtained from A by removing a single row, the rank loci of A and A_- are nested

$$\varnothing = S_{-1}^- \subseteq S_0 \subseteq S_0^- \subseteq S_1 \subseteq S_1^- \subseteq S_2 \subseteq S_2^- \subseteq \cdots \subseteq S_{n-1} \subseteq S_{n-1}^- \subseteq S_n \subseteq S,$$

and similarly for B and B^-

$$\varnothing = T_{-1}^- \subseteq T_0 \subseteq T_0^- \subseteq T_1 \subseteq T_1^- \subseteq T_2 \subseteq T_2^- \subseteq \cdots \subseteq T_{n-1} \subseteq T_{n-1}^- \subseteq T_n \subseteq T.$$

The claim now follows from the following general fact regarding such intersections. □

Lemma 4.6.3. *Given nested sequences of schemes*

$$\emptyset = Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_m \subseteq Y \qquad \text{and} \qquad \emptyset = Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_m \subseteq Z,$$

we have

$$\bigcap_{i+j=m} (Y_i \times Z) \cup (Y \times Z_j) = \bigcup_{i+j=m+1} Y_i \times Z_j$$

as schemes.

Proof. In general, intersection does not distribute across unions scheme-theoretically. However, we first show that if $A_1 \subset A_2$ and B are subschemes of any scheme, then

$$A_2 \cap (A_1 \cup B) = A_1 \cup (A_2 \cap B).$$

as schemes. That is, intersection distributes across union if appropriate containments are satisfied. It suffices to prove the statement in the affine case, where this becomes a statement about ideals. Suppose $I_1 := I(A_1) \supset I_2 := I(A_2)$ and $J := I(B)$. Then we must show

$$I_2 + (I_1 \cap J) = I_1 \cap (I_2 + J).$$

If $a \in I_2$ and $b \in I_1 \cap J$, then it's clear that $a + b \in I_1$ and $a + b \in I_2 + J$. Now suppose we have $a \in I_2$ and $b \in J$ so that $a + b \in I_1$. Since $I_2 \subset I_1$, it follows that $b \in I_1$. Hence $a + b \in I_2 + I_1 \cap J$.

To prove the lemma, we induct on m. The case $m = 0$ is immediate ($\emptyset = \emptyset$). Suppose we know the result for chains of length one less. We want to study

$$\begin{aligned}\bigcap_{i+j=m} (Y_i \times Z) \cup (Y \times Z_j) &= (Y \times Z_m) \cap [(Y_1 \times Z) \cup (Y \times Z_{m-1})] \cap \bigcap_{\substack{i+j=m\\ i\geq 2}} (Y_i \times Z) \cup (Y \times Z_j) \\ &= [(Y \times Z_{m-1}) \cup (Y_1 \times Z_m)] \cap \bigcap_{\substack{i+j=m\\ i\geq 2}} (Y_i \times Z) \cup (Y \times Z_j) \\ &= (Y_1 \times Z_m) \cup \left[(Y \times Z_{m-1}) \cap \bigcap_{\substack{i+j=m\\ i\geq 2}} (Y_i \times Z) \cup (Y \times Z_j)\right].\end{aligned}$$

Now the term in large square brackets is the intersection of complementary unions of the chains

$$\emptyset = Y_0 \subseteq Y_2 \subseteq Y_3 \subseteq \cdots \subseteq Y_m \subseteq Y \qquad \text{and} \qquad \emptyset = Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_{m-1} \subseteq Z,$$

which have length one less. The result now follows by induction. □

We now study the rank loci of evaluation maps on elliptic curves.

Lemma 4.6.4. *Let E be an elliptic curve and $\mathcal{L}$ a universal line bundle on $\pi\colon E \times \mathrm{Pic}^d(E) \to \mathrm{Pic}^d(E)$ with $d \geq 1$. Suppose D is an effective divisor and*

$$\mathrm{ev}\colon \pi_*\mathcal{L} \to \pi_*(\mathcal{L}|_{D\times\mathrm{Pic}^d(E)})$$

is the evaluation map. For any n, the scheme

$$S_n = \{L \in \mathrm{Pic}^d(E) : \mathrm{rk}\,\mathrm{ev}|_L \leq n\}$$

is either empty, all of $\mathrm{Pic}^d(E)$, or a single reduced point. In particular, it is reduced.

Proof. We have that S_n is empty or all of $\mathrm{Pic}^d(E)$ unless $\deg D = d$ and $n = d-1$. When $\deg D = d$, the evaluation map is between vector bundles that both have rank d. The locus where the map drops rank is cut out by the determinant, which is a section of $\det(\pi_*\mathcal{L})^\vee \otimes \det(\pi_*(\mathcal{L}|_{D\times\mathrm{Pic}^d(E)}))$. Set theoretically, the vanishing of this section is supported on $L = \mathcal{O}(D) \in \mathrm{Pic}^d(E)$. To see it is reduced, we compute its degree:

$$\deg\left(\det(\pi_*\mathcal{L})^\vee \otimes \det(\pi_*(\mathcal{L}|_{D\times\mathrm{Pic}^d(E)}))\right) = -\deg c_1(\pi_*\mathcal{L}) + \deg c_1(\pi_*(\mathcal{L}|_{D\times\mathrm{Pic}^d(E)}))$$

Using Grothendieck–Riemann–Roch (noting that the relative Todd class is trivial since E is an elliptic curve):

$$= -\deg \mathrm{ch}_2(\mathcal{L}) + \deg \mathrm{ch}_2(\mathcal{L}|_{D\times\mathrm{Pic}^d(E)})$$

Using the additivity of Chern characters in exact sequences:

$$= -\deg \mathrm{ch}_2(\mathcal{L}(-D\times\mathrm{Pic}^d(E)))$$
$$= -\frac{1}{2}\deg c_1(\mathcal{L}(-D\times\mathrm{Pic}^d(E)))^2$$

Given an identification $E\times\mathrm{Pic}^d(E) \cong E\times E$, the line bundle $\mathcal{L}(-D\times\mathrm{Pic}^d(E))$ can be represented by the diagonal Δ minus a fiber f. Since $\Delta^2 = 0$ (by adjunction) and $f^2 = 0$, we obtain $(\Delta - f)^2 = -2$. Thus:

$$= 1.$$ □

Lemma 4.6.5. *Let $\mathcal{S}$ be the collection of subvarieties of* $\mathrm{Pic}^{\vec{d}}(X)$ *that are unions of reduced preimages of points via projections* $(\prod_{i\in S}\mathrm{pr}^i)\colon \mathrm{Pic}^{\vec{d}}(X) \to \prod_{i\in S}\mathrm{Pic}^{d^i}(E^i)$ *for some* $S \subseteq \{1,\dots,g\}$. *Then $\mathcal{S}$ is closed under union and intersection.*

Proof. It is clear that $\mathcal{S}$ is closed under union. The intersection statement is clear set-theoretically, so it suffices to show that the intersection of two elements of $\mathcal{S}$ is reduced.

Suppose $A, B \in \mathcal{S}$ and $p \in A \cap B$. Choose étale coordinates x_i on $\mathrm{Pic}^{d^i}(E^i)$ near $\mathrm{pr}^i(p)$. Then étale-locally, A and B are reduced unions of coordinate linear spaces. Equivalently, $I(A)$ and $I(B)$ are reduced monomial ideals (i.e. generated by monomials in the x_i whose exponents are 0 or 1). The class of such ideals is closed under addition. □

Putting together Lemmas 4.6.2, 4.6.4, and 4.6.5, we deduce the desired reducedness property.

Theorem 4.6.6. *For any $r \geq 0$ and degree distribution $\vec{d}$, the scheme $W^r_{\vec{d}}(X)$ is a union of preimages of reduced points via projections* $\mathrm{Pic}^{\vec{d}}(X) \to \prod_{i\in S}\mathrm{Pic}^{d_i}(E_i)$ *for subsets* $S \subseteq \{1,\dots,g\}$. *It follows that $W^{\vec{e}}(X)$ is reduced.*

Proof. By (4.6.1),

$$W^r_{\vec{d}}(X) = \{L \in \mathrm{Pic}^{\vec{d}}(X) : \mathrm{rk}(\eta \oplus \psi)|_L \leq d + \deg(D) - (r+1)\}.$$

Applying Lemma 4.6.4 with $D = D^i \cup p^i \cup p^{i-1}$ (respectively $D = D^i \cup p^i$ or $D^i \cup p^{i-1}$ or D^i), we see that the rank loci of the maps ev^i (respectively ev^i with top row or bottom row or top and bottom row removed) are all in $\mathcal{S}$. Repeated application of Lemma 4.6.2 then shows that the rank loci of $\eta \oplus \psi$ are all in $\mathcal{S}$. Thus, $W^{\vec{e}}(X)$ is an intersection of subschemes in $\mathcal{S}$, so $W^{\vec{e}}(X)$ is in $\mathcal{S}$, and in particular is reduced. □

Corollary 4.6.7. *We have $W^{\vec{e}}(X) = W^{\Gamma(\vec{e})}(X)$ is reduced, and equal scheme-theoretically to the closure of $W^{\vec{e}}(\mathcal{X}^*/B^*)$ in the central fiber.*

Proof. We have shown the following containments. (In order: by construction c.f. Definition 4.2.4 and following discussion, by Theorem 4.6.6, by Proposition 4.3.9, and by Corollary 4.5.14 respectively.)

$$\overline{W^{\vec{e}}(\mathcal{X}^*/B^*)}|_0 \subseteq W^{\vec{e}}(X) = W^{\vec{e}}(X)_{\mathrm{red}} \subseteq W^{\Gamma(\vec{e})}(X) \subseteq \overline{W^{\vec{e}}(\mathcal{X}^*/B^*)}|_0.$$

Therefore all containments are equalities. □

4.6.8 Cohen–Macaulayness

For any k-convex diagram Γ, we will prove that $W^\Gamma(X)$ is Cohen–Macaulay by inducting on g. The key to running our induction is the following standard fact:

Lemma 4.6.9 (See, for example, Proposition 4.1 of [41])**.** *Suppose A and B are Cohen–Macaulay and $A \cap B$ is codimension 1 in both A and B. Then $A \cup B$ is Cohen–Macaulay if and only if $A \cap B$ is Cohen–Macaulay.*

Theorem 4.6.10. *Given any k-convex shape Γ, the scheme $W^\Gamma(X)$ is Cohen–Macaulay. In particular, $W^{\vec{e}}(X)$ is Cohen–Macaulay, and therefore $W^{\vec{e}}(C)$ is Cohen–Macaulay for a general degree k genus g cover $f : C \to \mathbb{P}^1$.*

Proof. We will induct on g. For the base case $g = 1$, we know $W^\Gamma(X)$ is either all of $\mathrm{Pic}^d(E^1)$, or a single reduced point. For the inductive step, suppose we are given $L \in W^\Gamma(X)$. Let

$$\pi\colon\ \mathrm{Pic}^d(X) \cong \prod_{i=1}^{g} \mathrm{Pic}^d(E^i) \to \mathrm{Pic}^d(X^{\leq g-1}) \cong \prod_{i<g} \mathrm{Pic}^d(E^i)$$

be the projection map, i.e. $(M^1, \dots, M^g) \mapsto (M^1, \dots, M^{g-1})$. Let $\iota_L\colon \mathrm{Pic}^d(X^{\leq g-1}) \to \mathrm{Pic}^d(X)$ be the section which sends $(M^1, \dots, M^{g-1}) \mapsto (M^1, \dots, M^{g-1}, L^g)$. Define

$$A := \bigcup_{T : g \in T} W^T(X) \qquad \text{and} \qquad B := \bigcup_{T : g \notin T} W^T(X).$$

If $L \in A$, then L^g is a linear combination of p^{g-1} and p^g, and Γ has a removable corner in the corresponding residue j. Recall that $u(s_j \cdot \Gamma) = u(\Gamma) - 1$ (c.f. Section 4.4.4 item (3)).

Then $A = \iota_L(W^{s_j \cdot \Gamma}(X^{\leq g-1}))$ in a neighborhood of L, so A is Cohen–Macaulay by induction. Meanwhile, we have $B = \pi^{-1}(W^\Gamma(X^{\leq g-1}))$, so B is Cohen–Macaulay by induction. Moreover, $A \cap B = \iota_L(W^\Gamma(X^{\leq g-1}))$ in a neighborhood of L, so is Cohen–Macaulay by induction. Since $A \cap B$ is codimension 1 in both A and B, Lemma 4.6.9 implies that $W^\Gamma(X) = A \cup B$ is Cohen–Macaulay.

Since Cohen–Macaulayness is an open condition, $W^{\vec{e}}(C)$ is Cohen–Macaulay for a general degree k cover $f : C \to \mathbb{P}^1$. □

Remark 4.6.11. As explained in the introduction, this also establishes Cohen-Macaulayness, and hence reducedness of universal splitting loci (Corollary 4.1.2). The authors suspect that a more direct argument may be given by just degenerating the $\mathbb{P}^1$ (without considering another curve). Such an argument would likely remove the hypothesis that the characteristic does not divide k. Reducedness confirms that the scheme structure on splitting loci obtained from Fitting supports in Definition 2.2.1 — i.e. non-transverse intersections of determinantal loci — is the “correct" scheme structure. Cohen-Macaulyness supports the perspective that, despite this failure of transversality, splitting loci ought to behave like determinantal loci (see e.g. Chapter 1 where analogues of the Porteous formula were given).

4.7 Connectedness

In this section, we show $W^{\vec{e}}(C)$ is connected when $g > u(\vec{e})$, where $f: C \to \mathbb{P}^1$ is a general degree k cover. Since “geometrically connected and geometrically reduced” is an open condition in flat proper families [37, Theorem 12.2.4(vi)], and we have already shown that $W^{\vec{e}}(X)$ is reduced (c.f. Theorem 4.6.6), it suffices to see that $W^{\vec{e}}(X) = W^{\Gamma(\vec{e})}(X)$ is connected. In other words, we want to show the transitivity of the following equivalence relation:

Definition 4.7.1. We say two k-core tableau T and T' with the same shape Γ *meet* if $W^T(X)$ and $W^{T'}(X)$ intersect. We say that T and T' are *connected* if $W^T(X)$ and $W^{T'}(X)$ are in the same connected component of $W^\Gamma(X)$.

By definition, “connected” is the equivalence relation generated by “meet.” Unwinding Definition 4.3.11, T and T' meet if and only if for every t, either:

$$T[t] = *, \quad T'[t] = *, \quad \text{or} \quad T[t] \equiv T'[t] \pmod{k}.$$

The simplest example of two tableaux that are connected is the following.

Example 4.7.2. Suppose that T is a tableau filled with a subset of the symbols $\{1, \dots, g\}$ that does not include the symbol N. Let N' be the smallest symbol greater than (respectively largest symbol less than) N appearing in T. Then T meets the tableau obtained from T by replacing the symbol N' with N. Applying this repeatedly, every tabuleau is connected to the tableau obtained by relabeling symbols via an order preserving map from the subset of symbols used to any other subset of u symbols.

By Example 4.7.2, it suffices to show that all efficiently-filled tableau filled with symbols $\{1, \dots, u\}$ are connected when $g > u$. To do this, we use the relations in the affine symmetric group

$$s_j s_{j'} = s_{j'} s_j \text{ (for } j - j' \neq \pm 1) \quad \text{and} \quad s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1}.$$

These relations give rise to two basic moves, known as the *braid moves* for the affine symmetric group, between reduced words:

$$s_{j_u} \cdots s_{j_1} \leftrightarrow s_{j_u} \cdots s_{j_{i+2}} s_{j_i} s_{j_{i+1}} s_{j_{i-1}} \cdots s_{j_1} \quad (\text{for } j_i - j_{i+1} \neq \pm 1) \qquad F^i$$
$$s_{j_u} \cdots s_{j_1} \leftrightarrow s_{j_u} \cdots s_{j_{i+3}} s_{j_{i+1}} s_{j_i} s_{j_{i+1}} s_{j_{i-1}} \cdots s_{j_1} \quad (\text{for } j_i = j_{i+1} \pm 1 = j_{i+2}) \qquad S^i$$

Given our identification of reduced words with efficient k-core tableaux, this is equivalent to the following moves on tableaux:

Definition 4.7.3. Let T be an efficiently filled k-core tableau. We define the two *braid moves* as follows:

F If $T[i] - T[i+1] \not\equiv \pm 1 \bmod k$, define the *flip* F^iT by

$$(F^iT)[t] = \begin{cases} T[i+1] & \text{if } t = i; \\ T[i] & \text{if } t = i+1; \\ T[t] & \text{otherwise.} \end{cases}$$

S Similarly, if $T[i] \equiv T[i+2]$ and $T[i+1] \equiv T[i] \pm 1 \bmod k$, define the *shuffle* S^iT by

$$(S^iT)[t] = \begin{cases} T[i] & \text{if } t = i+1; \\ T[i+1] & \text{if } t \in \{i, i+2\}; \\ T[t] & \text{otherwise.} \end{cases}$$

Note that both braid moves are involutions, i.e. $F^iF^iT = T$ and $S^iS^iT = T$. It is known that we can get from any reduced word to any other reduced word — or equivalently from any efficient tableau to any other efficient tableau — by applying a sequence of braid moves (c.f. Theorem 3.3.1(ii) of [5]). Therefore, it suffices to prove that T and F^iT (when defined), respectively T and S^iT (when defined), are connected.

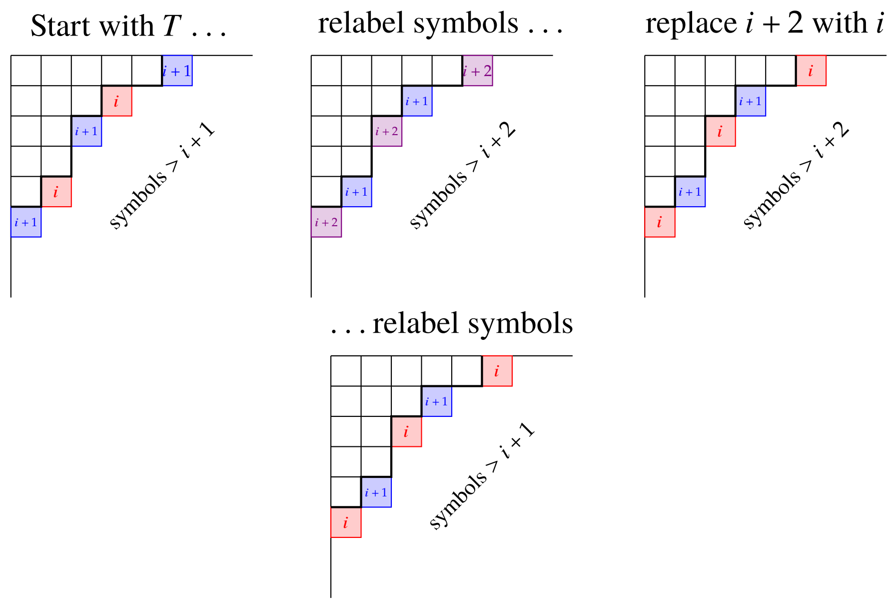
Lemma 4.7.4. *Suppose that T is an efficient filling of a k-core Γ with symbols $\{1, \ldots, u\}$, and that F^iT is defined. If $g > u(\Gamma)$, then T and F^iT are connected.*

Proof. Start with T and relabel symbols according to

$$\{1, \ldots, u\} \mapsto \{1, \ldots, i-1, i+1, \ldots, u+1\}$$

(see Example 4.7.2). In this filling, all $(i+2)$'s may be replaced with i's because no $i+1$

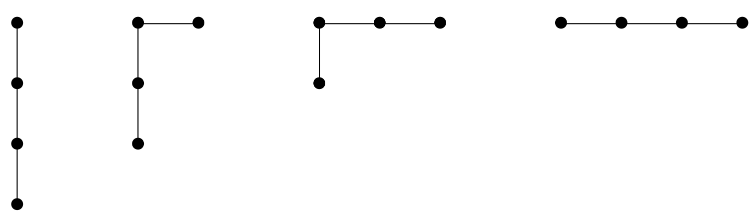
appears to the upper-left of an $i+2$. After this replacement, we further relabel symbols according to $\{1,\dots,i+1,i+3,\dots,u+1\} \mapsto \{1,\dots,u\}$.



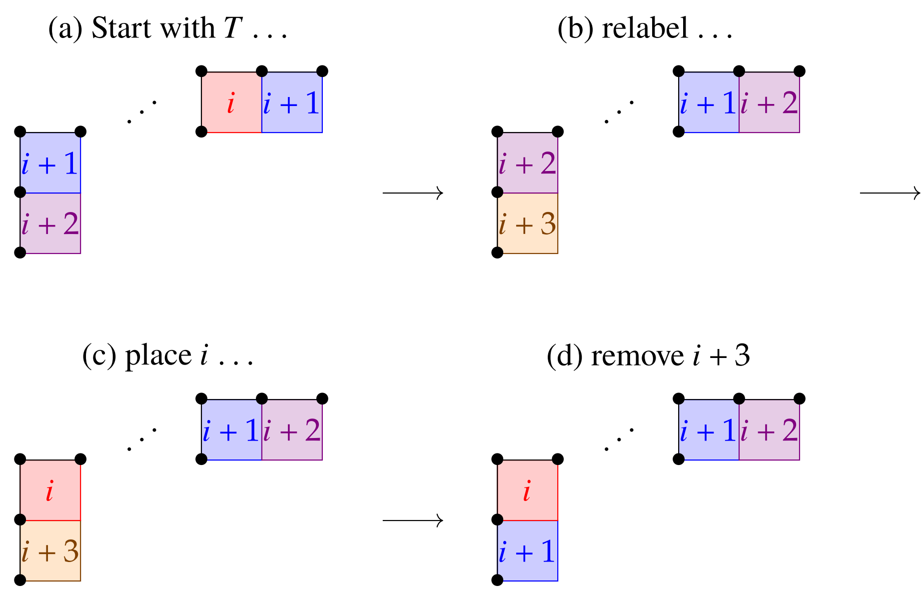
The resulting tableau is F^iT. Each tableau in this sequence is connected to the previous, so T and F^iT are connected. ◻

Lemma 4.7.5. *Suppose that T is an efficient filling of a k-core Γ with symbols $\{1,\dots,u\}$, and that S^iT is defined. If $g > u(\Gamma)$, then T and S^iT are connected.*

Proof. Because S^i is an involution, after possibly replacing T with S^iT, it suffices to treat the case $T[i+1]-T[i] \equiv 1 \pmod{k}$. Let $j = T[i] = T[i+2]$. Note that $T^{\leq i-1}$ has addable corners with diagonal indices of residues j and $j+1$ (because S^iT is defined). In other words, the boundary segments of $T^{\leq i}$ neighboring boxes with diagonal index of residue class j or $j+1$ must proceed through:



To see that T and S^iT are connected when $g > u$, consider the following sequence of tableaux. First relabel T according to $\{1,\dots,u\} \mapsto \{1,\dots,i-1,i+1,\dots,u+1\}$ to get filling (b). Then place i in the positions of $T^{\leq i-1}$'s addable corner with diagonal index of residue $j+1$ to get filling (c). Now all instances of $i+3$ may be replaced with $i+1$ to give filling (d). Tableau (d) is a relabeling of S^iT by $\{1,\dots,u\} \mapsto \{1,\dots,i+2,i+4,\dots,u+1\}$.



(a) Start with T . . . (b) relabel . . . (c) place i . . . (d) remove $i+3$

Each tableau above is connected to the previous, so this shows that T and S^iT are connected.

□

This establishes that $W^{\vec{e}}(X)$ is connected. Since it is also reduced (Theorem 4.6.6) and "geometrically connected and geometrically reduced" is an open condition in flat proper families, we have therefore proven:

Theorem 4.7.6. *If $f\colon C \to \mathbb{P}^1$ is a general degree k cover, and $g > u(\vec{e})$, then $W^{\vec{e}}(C)$ is connected.*

4.8 Normality and Irreducibility

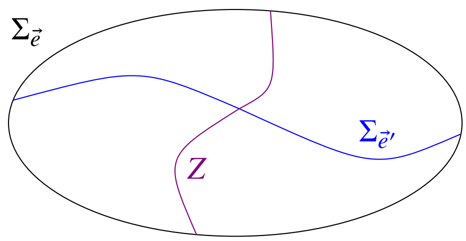
For the remainder of the chapter, we suppose that the characteristic of our ground field does not divide k. Let $f\colon C \to \mathbb{P}^1$ be a general degree k cover. In this section, we show that $W^{\vec{e}}(C)$ is smooth away from more unbalanced splitting loci of codimension 2 or more. Since we have already established that $W^{\vec{e}}(C)$ is Cohen–Macaulay (Theorem 4.6.10), this implies that $W^{\vec{e}}(C)$ is normal by Serre's criterion ($R_1 + S_2$). Combining this with connectedness (Theorem 4.7.6), this establishes that $W^{\vec{e}}(C)$ is irreducible when $g > u(\vec{e})$.

To do this, we will prove the following general result regarding splitting stratifications. Suppose $\mathcal{E}$ is a family of vector bundles on $\pi\colon B \times \mathbb{P}^1 \to B$. For any splitting type $\vec{e}$, recall that the scheme structure on the closed splitting locus $\Sigma_{\vec{e}} \subset B$ is defined as an intersection of determinantal loci. More precisely, the locus $\Sigma_{\vec{e}}$ is the intersection over all m of the Fitting support for $\operatorname{rk} R^1\pi_*\mathcal{E}(m) \geq h^1(\mathbb{P}^1, \mathcal{O}(\vec{e})(m))$ (Definition 2.2.1). We use $\Sigma_{\vec{e}}^\circ \subset \Sigma_{\vec{e}}$ to denote the open where the splitting type is exactly $\vec{e}$.

Proposition 4.8.1. *Suppose $\mathcal{E}$ is a family of vector bundles on $\pi\colon B \times \mathbb{P}^1 \to B$ such that all open splitting loci $\Sigma^\circ_{\vec{e}}$ are smooth of the expected codimension $u(\vec{e})$. Then $\Sigma_{\vec{e}}$ is smooth away from all splitting loci of codimension ≥ 2 inside $\Sigma_{\vec{e}}$.*

Proof. Since the statement is local on B, we may assume that B is affine.

Suppose $\Sigma_{\vec{e}'} \subset \Sigma_{\vec{e}}$ has codimension 1 for some $\vec{e}' \leq \vec{e}$. Let $Z \subset \Sigma_{\vec{e}}$ be the union of all *other* splitting loci properly contained in $\Sigma_{\vec{e}}$. We will show that $\Sigma^\circ_{\vec{e}'} \subset \Sigma_{\vec{e}} \smallsetminus Z$ is a Cartier divisor. It will follow that $\Sigma_{\vec{e}}$ is smooth along $\Sigma^\circ_{\vec{e}'}$, as we now explain. If $\Sigma_{\vec{e}}$ were singular at some $b \in \Sigma^\circ_{\vec{e}'}$, then $\dim T_b\Sigma_{\vec{e}} > \dim \Sigma_{\vec{e}} = \dim \Sigma^\circ_{\vec{e}'} + 1$. Assuming $\Sigma^\circ_{\vec{e}'} \subset \Sigma_{\vec{e}} \smallsetminus Z$ is Cartier, we would find $\dim T_b\Sigma^\circ_{\vec{e}'} \geq \dim T_b\Sigma_{\vec{e}} - 1 > \dim \Sigma^\circ_{\vec{e}'}$, forcing $\Sigma^\circ_{\vec{e}'}$ to be singular at b, which contradicts the assumption that all open splitting loci are smooth. Repeating the argument for each divisorial $\Sigma_{\vec{e}'} \subset \Sigma_{\vec{e}}$ shows that $\Sigma_{\vec{e}}$ is smooth away from all codimension 2 subsplitting loci.



First, we show that to have $\vec{e}' < \vec{e}$ with $u(\vec{e}') = u(\vec{e}) + 1$, the splitting types must have a special shape. Let r be the smallest index such that $e'_r < e_r$ and let s be the largest index such that $e'_s > e_s$. (The fact that $\vec{e}' < \vec{e}$ means $r < s$.) Then

$$\vec{e}' \leq \vec{e}_{rs} := (e_1, \ldots, e_{r-1}, e_r - 1, e_{r+1}, \ldots, e_{s-1}, e_s + 1, e_{s+1}, \ldots, e_k) < \vec{e}, \tag{4.8.1}$$

from which we see

$$\begin{aligned}
1 &= u(\vec{e}') - u(\vec{e}) \\
&\geq u(\vec{e}_{rs}) - u(\vec{e}) \\
&= \sum_{\ell \neq r,s} \Big([h^1(\mathcal{O}(e_\ell - e_r + 1)) + h^1(\mathcal{O}(e_\ell - e_s - 1)) + h^1(\mathcal{O}(e_r - 1 - e_\ell)) + h^1(\mathcal{O}(e_s + 1 - e_\ell))] \\
&\qquad - [h^1(\mathcal{O}(e_\ell - e_r)) + h^1(\mathcal{O}(e_\ell - e_s)) + h^1(\mathcal{O}(e_r - e_\ell)) + h^1(\mathcal{O}(e_s - e_\ell))] \Big) \\
&\quad + [h^1(\mathcal{O}(e_r - e_s - 2)) + h^1(\mathcal{O}(e_s - e_r + 2))] - [h^1(\mathcal{O}(e_r - e_s)) + h^1(\mathcal{O}(e_s - e_r))] \\
&= \#\{\ell : e_r - 1 \leq e_\ell \leq e_s - 1\} + \#\{\ell : e_r + 1 \leq e_\ell \leq e_s + 1\} + \begin{cases} 1 & \text{if } e_r = e_s; \\ 2 & \text{otherwise.} \end{cases}
\end{aligned}$$

It follows that the non-negative quantities $\#\{\ell : e_r - 1 \leq e_\ell \leq e_s - 1\} = \#\{\ell : e_r + 1 \leq e_\ell \leq$

$e_s + 1\} = 0$, and $e_r = e_s$, and $\vec{e}' = \vec{e}_{rs}$. In other words, after twisting (down by $e_r = e_s$), we may assume $\mathcal{O}(\vec{e}) = N \oplus \mathcal{O}^{\oplus m} \oplus P$, where all parts of N have degree at most -2, all parts of P have degree at least 2, and $\mathcal{O}(\vec{e}') = N \oplus \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus(m-2)} \oplus \mathcal{O}(1) \oplus P$.

Away from Z, we will show that the vector bundle $\mathcal{E}$ on $\pi \colon \mathbb{P}^1 \times (\Sigma_{\vec{e}} \setminus Z) \to (\Sigma_{\vec{e}} \setminus Z)$ splits as $\mathcal{E} = \mathcal{N} \oplus \mathcal{T} \oplus \mathcal{P}$ where for any $b \in \Sigma_{\vec{e}} \setminus Z$,

$$\mathcal{N}|_b \simeq N, \quad \mathcal{P}|_b \simeq P, \quad \text{and} \quad \mathcal{T}|_b \simeq \begin{cases} \mathcal{O}^{\oplus m} & \text{if } b \notin \Sigma_{\vec{e}'}; \\ \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus(m-2)} \oplus \mathcal{O}(1) & \text{if } b \in \Sigma_{\vec{e}'}. \end{cases} \tag{4.8.2}$$

To construct $\mathcal{N}$ and $\mathcal{P}$, let $\mathcal{Q}(-2)$ be the cokernel of $\pi^*\pi_*\mathcal{E}(-2) \to \mathcal{E}(-2)$, which is locally free. Define $\mathcal{P}$ by the exact sequence

$$0 \to \mathcal{P}(-2) \to \mathcal{E}(-2) \to \mathcal{Q}(-2) \to 0.$$

This sequence splits if the induced map $H^0(\mathcal{H}om(\mathcal{Q}(-2), \mathcal{E}(-2))) \to H^0(\mathcal{H}om(\mathcal{Q}(-2), \mathcal{Q}(-2)))$ is surjective, which in turn follows if we show $H^1(\mathcal{H}om(\mathcal{Q}(-2), \mathcal{P}(-2))) = 0$. Now, $\mathcal{P}(-2)$ is globally generated on each fiber, and $\mathcal{Q}(-2)$ has negative summands on each fiber, so $\mathcal{H}om(\mathcal{Q}(-2), \mathcal{P}(-2))$ has positive summands on every fiber. Thus, by the theorem on cohomology and base change, $R^1\pi_*\mathcal{H}om(\mathcal{Q}(-2), \mathcal{P}(-2)) = 0$ and so $H^1(\mathcal{H}om(\mathcal{Q}(-2), \mathcal{P}(-2))) = 0$ because B is affine.

Next, define $\mathcal{N}$ so that $\mathcal{N}(1)$ is the cokernel of $\pi^*\pi_*\mathcal{Q}(1) \to \mathcal{Q}(1)$, and define $\mathcal{T}$ by the sequence

$$0 \to \mathcal{T} \to \mathcal{Q} \to \mathcal{N} \to 0.$$

The same argument as before shows that this sequence splits too. Thus, $\mathcal{E} = \mathcal{Q} \oplus \mathcal{P} = \mathcal{N} \oplus \mathcal{T} \oplus \mathcal{P}$. By construction, this splitting satisfies (4.8.2).

On $\Sigma_{\vec{e}} \setminus Z$, the fibers of $R^1\pi_*\mathcal{T}(-1)$ have rank at most 1. We also have that $\pi_*\mathcal{T}$ and $\pi_*\mathcal{T}(1)$ are locally free on $\Sigma_{\vec{e}} \setminus Z$. The equations that cut out $\Sigma_{\vec{e}'} \subset B$ are the same as the equations that cut out $\Sigma_{\vec{e}} \subset B$ except at one twist: namely, when we ask for the rank of $R^1\pi_*\mathcal{E}(-1)$. To cut out $\Sigma_{\vec{e}}$, we ask that $\operatorname{rk} R^1\pi_*\mathcal{E}(-1) \geq h^1(\mathbb{P}^1, \mathcal{O}(\vec{e})(-1)) := n$, whereas to cut out $\Sigma_{\vec{e}'}$, we ask that $\operatorname{rk} R^1\pi_*\mathcal{E}(-1) \geq h^1(\mathcal{O}(\vec{e}')(-1)) = n + 1$.

To study these equations, we must build a resolution of $R^1\pi_*\mathcal{E}(-1)$ by vector bundles. On $\Sigma_{\vec{e}} \setminus Z$, the theorem on cohomology and base change shows $R^1\pi_*\mathcal{N}(-1)$ is a vector bundle, and $R^1\pi_*\mathcal{P}(-1) = 0$. We also have a resolution by vector bundles

$$U_1 := \pi_*\mathcal{T} \otimes H^0(\mathbb{P}^1, \mathcal{O}(1)) \xrightarrow{\psi} U_2 := \pi_*\mathcal{T}(1) \to R^1\pi_*\mathcal{T}(-1) \to 0,$$

where U_1 and U_2 have the same rank $2m$. Therefore,

$$0 \oplus U_1 \xrightarrow{0\oplus\psi} R^1\pi_*\mathcal{N}(-1) \oplus U_2 \to R^1\pi_*\mathcal{N}(-1) \oplus R^1\pi_*\mathcal{T}(-1) = R^1\pi_*\mathcal{E}(-1)$$

is a resolution of $R^1\pi_*\mathcal{E}(-1)$ by vector bundles. Now, $\Sigma^\circ_{\vec{e}}$, is cut out inside $\Sigma_{\vec{e}} \setminus Z$ by the condition that $\dim \operatorname{coker}(0 \oplus \psi) \geq n+1$, or equivalently that $\dim \operatorname{coker} \psi \geq 1$. This locus is cut by the vanishing of its determinant, which we can view as a section of the line bundle

$$\bigwedge^{2m}(\pi_*\mathcal{T} \otimes H^0(\mathbb{P}^1, \mathcal{O}(1)))^\vee \otimes \bigwedge^{2m}(\pi_*\mathcal{T}(1)).$$

Hence, $\Sigma^\circ_{\vec{e}} \subset \Sigma_{\vec{e}} \setminus Z$ is Cartier, as desired. □

In Section 3.3, we showed that for $f\colon C \to \mathbb{P}^1$ a general degree k cover, the strict Brill-Noether splitting loci $W^{\vec{e}}(C)^\circ$ are smooth of the expected dimension when the characteristic of the ground field does not divide k. The above proposition thus implies $W^{\vec{e}}(C)$ is smooth away from all splitting loci of codimension 2 or more. Together with Theorem 4.6.10, we have thus shown:

Theorem 4.8.2. *Let $f\colon C \to \mathbb{P}^1$ be a general genus g, degree k cover. Then $W^{\vec{e}}(C)$ is smooth away from all codimension 2 splitting loci. Thus $W^{\vec{e}}(C)$ is normal.*

Combined with 4.7.6, we have therefore proven:

Theorem 4.8.3. *If $f\colon C \to \mathbb{P}^1$ is a general degree k cover, and $g > u(\vec{e})$, then $W^{\vec{e}}(C)$ is irreducible.*

4.9 Monodromy

Our final task is to show that the universal $\mathcal{W}^{\vec{e}}$ has a unique irreducible component dominating a component of the unparameterized Hurwitz stack $\mathcal{H}_{k,g}$ when $g \geq u(\vec{e})$. When $g > u(\vec{e})$ — or when $k = 2$ in which case $N(\vec{e}) = 1$ for all $\vec{e}$ — we have that $W^{\vec{e}}(C)$ is irreducible for C general. So for the remainder of this section, we suppose $g = u(\vec{e})$ and $k > 2$.

When $g = u(\vec{e})$, we have shown that $W^{\vec{e}}(X)$ is a reduced finite set of line bundles. Using this, we obtain:

Lemma 4.9.1. *Let $\mathfrak{f}^*\colon \mathcal{X}^* \to \mathcal{P}^*$ be a deformation of $f\colon X \to P$ to a smooth cover, with smooth total space (c.f. Section 3.1). If $\mathcal{C} \to \mathcal{P}' \to B$ is a family of smooth covers containing $\mathfrak{f}^*$, over a reduced base B, then $W^{\vec{e}}(\mathcal{C}/B) \to B$ is étale near $\mathfrak{f}^*$.*

Proof. Because $u(\vec{e}) = g$, every component of $W^{\vec{e}}(C/B)$ has dimension at least $\dim B$. Moreover, $W^{\vec{e}}(C/B) \to B$ is proper, and the fiber over $\mathfrak{f}^*$ is a finite set of reduced points. Since B is reduced, we conclude that the map is étale near $\mathfrak{f}^*$. □

Since $f\colon X \to P$ is separable, $\mathfrak{f}^*$ is also separable. In particular, its cotangent complex is punctual, so $\mathcal{H}_{k,g}$ is smooth at $\mathfrak{f}^*$. We can therefore apply this lemma to the universal family over a component B of $\mathcal{H}_{k,g}$.

In greater generality, suppose that B is any irreducible base, and $\pi\colon W \to B$ is étale near $b \in B$. Then, any irreducible component of W dominating B meets $\pi^{-1}(b)$, and every point of $\pi^{-1}(b)$ is contained in a unique irreducible component of W, which dominates B. To show W has a unique irreducible component dominating B, it thus suffices to show that any two points of $\pi^{-1}(b)$ are contained in the same irreducible component of W. In particular, suppose that B' is irreducible and the image of $B' \to B$ meets b. If $W \times_B B' \to B'$ has a unique irreducible component dominating B', then W has a unique irreducible component dominating B.

Therefore, if there is *some* family C/B' over a reduced irreducible base B', containing $\mathfrak{f}^*$, so that $W^{\vec{e}}(C/B') \to B'$ has a unique component dominating B', then the universal $\mathcal{W}^{\vec{e}}$ has a unique component dominating our component of $\mathcal{H}_{k,g}$. The argument will proceed in the following steps:

1. In Section 4.9.2, we define the stack $\mathcal{H}_{k,g,2}$ of degree k genus g covers with total ramification at 2 points, and partial and total compactifications thereof:

$$\mathcal{H}_{k,g,2} \subseteq \bar{\mathcal{H}}^{\text{sm-ch}}_{k,g,2} \subseteq \bar{\mathcal{H}}^{\text{ch}}_{k,g,2} \subseteq \bar{\mathcal{H}}_{k,g,2}.$$

   The universal curve $C^{\text{sm-ch}}$ over $\bar{\mathcal{H}}^{\text{sm-ch}}_{k,g,2}$ will be a family of chain curves with smooth total space. We may therefore construct the universal $\vec{e}$-positive locus $W^{\vec{e}}(C^{\text{sm-ch}})$ as in Section 4.2. We will also observe that:

   (a) X lies in $\bar{\mathcal{H}}^{\text{sm-ch}}_{k,g,2}$, and $\mathcal{H}_{k,g,2}$ contains a deformation of X as in Lemma 4.9.1,

   (b) $\bar{\mathcal{H}}^{\text{ch}}_{k,g,2}$ is smooth, so X lies in a *unique* component $\bar{\mathcal{H}}^{\text{sm-ch},\circ}_{k,g,2}$ of $\bar{\mathcal{H}}^{\text{sm-ch}}_{k,g,2}$,

   (c) $W^{\vec{e}}(C^{\text{sm-ch}}) \to \bar{\mathcal{H}}^{\text{sm-ch}}_{k,g,2}$ is étale near X.

We may therefore reduce to studying $W^{\vec{e}}(C^{\text{sm-ch}})$. (The reason for this first reduction is to sidestep questions related to the existence of a nice compactification of the Hurwitz space in positive characteristic, given that we will need these compactifications of $\mathcal{H}_{k,g,2}$ anyways later in our argument.) In light of (1)(c), our problem is now to show that every two points of $W^{\vec{e}}(X)$ lie in the same irreducible component of $W^{\vec{e}}(C^{\text{sm-ch}})$.

Recall that $W^{\vec{e}}(X)$ is the reduced finite set of line bundles L_T indexed by the efficient fillings T of $\Gamma(\vec{e})$ (c.f. Definition 4.3.11). Therefore, it suffices to see that for any two tableaux T and T' of shape $\Gamma(\vec{e})$, the irreducible components of $W^{\vec{e}}(C^{\text{sm-ch}})$ containing T and T' coincide. Because any two tableaux can be connected via a sequence of braid moves (c.f. Section 4.7), it suffices to show that L_T and L_{F^iT} (respectively L_T and L_{S^iT}), when defined, lie in the same irreducible component.

2. In Section 4.9.5 (respectively Section 4.9.7), we restrict $W^{\vec{e}}(C^{\text{sm-ch}})$ to certain families in $\bar{\mathcal{H}}^{\text{sm-ch},\circ}_{k,g,2}$ whose closures contain X; these families arise by smoothing nodes of X and are themselves parameterized by $\mathcal{H}^{\circ}_{k,2,2}$ (respectively $\mathcal{H}^{\circ}_{k,3,2}$).

   The restriction of $W^{\vec{e}}(C^{\text{sm-ch}})$ to these families is not irreducible. Nonetheless, for each T such that F^iT (respectively S^iT) is defined, we describe substacks $Y_F(i,T)$ (respectively $Y_S(i,T)$) of the restriction of $W^{\vec{e}}(C^{\text{sm-ch}})$ to these families; the fiber over X of the closure of $Y_F(i,T)$ (respectively $Y_S(i,T)$) in $W^{\vec{e}}(C^{\text{sm-ch}})$ consists of L_T and L_{F^iT} (respectively L_T and L_{S^iT}).

3. Finally, in Section 4.9.9, we prove that $Y_F(i,T)$ and $Y_S(i,T)$ are irreducible, thereby establishing that L_T and L_{F^iT} (respectively L_T and L_{S^iT}) lie in the same irreducible component of $W^{\vec{e}}(C^{\text{sm-ch}})$, as desired.

   We do this by observing that $Y_F(i,T)$ (respectively $Y_S(i,T)$) extends naturally over the entirety of $\bar{\mathcal{H}}_{k,2,2}$ (respectively $\bar{\mathcal{H}}_{k,3,2}$). Moreover, they remain generically étale over a certain boundary stratum R_2 (respectively R_3), where we can write down explicit equations and check irreducibility.

4.9.2 Covers with 2 points of total ramification

Definition 4.9.3. Let $\mathcal{M}_{g,2}$ denote the moduli stack of curves of genus g with 2 marked points, and $\bar{\mathcal{M}}_{g,2}$ denote its Deligne–Mumford compactification by stable curves. Write $\mathcal{H}_{k,g,2} \subseteq \mathcal{M}_{g,2}$ for the substack of $(C,p,q) \in \mathcal{M}_{g,2}$ with $\mathcal{O}_C(kp) \simeq \mathcal{O}_C(kq)$. (In other words, for a scheme B, the B-points of $\mathcal{H}_{k,g,2}$ parameterize relative smooth curves $C \to B$ equipped with a pair of sections $\{p,q\}$, such that B can be covered by opens U with $\mathcal{O}_{C|_U}(kp) \simeq \mathcal{O}_{C|_U}(kq)$.)

Write $\bar{\mathcal{H}}_{k,g,2} \subseteq \bar{\mathcal{M}}_{g,2}$ for the closure of $\mathcal{H}_{k,g,2}$. Let $\bar{\mathcal{H}}^{\text{sm-ch}}_{k,g,2} \subseteq \bar{\mathcal{H}}_{k,g,2}$ (respectively $\bar{\mathcal{H}}^{\text{ch}}_{k,g,2} \subseteq \bar{\mathcal{H}}_{k,g,2}$) denote the open substack parameterizing chains of smooth (respectively irreducible) curves where the marked points are on opposite ends. Let $C^{\text{sm-ch}}$ denote the universal curve over $\bar{\mathcal{H}}^{\text{sm-ch}}_{k,g,2}$.

Any $(C, p, q) \in \mathcal{M}_{g,2}$ lies in $\mathcal{H}_{k,g,2}$ if and only if C admits a map $C \to \mathbb{P}^1$ of degree k, totally ramified at p and q. Such a map is unique up to $\operatorname{Aut}\mathbb{P}^1$, so there is a natural map $\mathcal{H}_{k,g,2} \to \mathcal{H}_{k,g}$. The boundary can be understood explicitly in a similar fashion. Suppose $C = C^1 \cup_{p^1} \cup_{p^2} \cdots \cup_{p^{n-1}} C^n$ with $p = p^0 \in C^1$ and $q = p^n \in C^n$ is a chain of irreducible curves where the marked points are on opposite ends. By the theory of admissible covers (c.f. Section 5 of [55]), $(C, p, q) \in \bar{\mathcal{H}}^{\text{ch}}_{k,g,2}$ if and only if C admits a map of degree k to a chain of n copies of $\mathbb{P}^1$ totally ramified over the p^i. Such a map exists if and only if $p^i - p^{i-1}$ is k-torsion in $\operatorname{Pic}^0(C^i)$ for $i = 1, \ldots, n$, in which case it is unique.

Note that by construction, (X, p^0, p^g) lies in $\bar{\mathcal{H}}^{\text{sm-ch}}_{k,g,2}$, and the deformation constructed in Section 3.1 lies in $\mathcal{H}_{k,g,2}$.

Lemma 4.9.4. *$\bar{\mathcal{H}}^{\text{ch}}_{k,g,2}$ is smooth, as is the total space $C^{\text{sm-ch}}$ of the universal curve over $\bar{\mathcal{H}}^{\text{sm-ch}}_{k,g,2}$.*

Proof. Let $C = C^1 \cup_{p^1} C^2 \cup_{p^2} \cdots \cup_{p^{n-1}} C^n$ be a point of $\bar{\mathcal{H}}^{\text{ch}}_{k,g,2}$, with marked points $p = p^0 \in C^1$ and $q = p^n \in C^n$. Let $f' \colon C \to P$ be the unique map of degree k to a chain of n copies of $\mathbb{P}^1$ totally ramified over the p^i. Because the characteristic does not divide k by assumption, f' is separable.

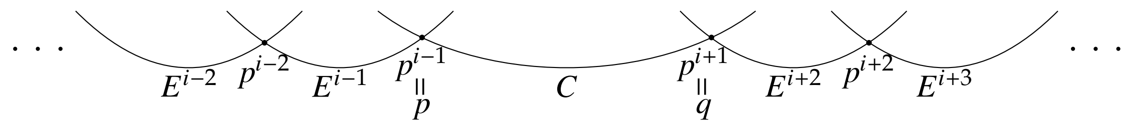
By formal patching (c.f. Lemma 5.6 of [55]), a deformation of f' is uniquely determined by deformations in formal neighborhoods of every branch point b of f', and every node q^i of P. Near a branch point b, the deformation space is smooth since the relative cotangent complex is punctual. Near a node q^i, write x and y for local coordinates on C^i and C^{i+1} at p^i. Since the map is totally ramified and k is not a multiple of the characteristic, $a = x^k$ and $b = y^k$ give local coordinates on the two copies of $\mathbb{P}^1$ meeting at q^i. A local versal deformation space is then smooth of dimension 1: a versal deformation with coordinate t is $\operatorname{Spec} K[[x, y, t]]/(xy - t) \to \operatorname{Spec} K[[a, b, t]]/(ab - t^k)$. Thus $\bar{\mathcal{H}}^{\text{ch}}_{k,g,2}$ is smooth.

Finally, along the fiber C, the map $C^{\text{sm-ch}} \to \bar{\mathcal{H}}^{\text{sm-ch}}_{k,g,2}$ is smooth away from the p^i. Therefore the only possible singularities of the total space $C^{\text{sm-ch}}$ along C occur at the p^i. In a formal neighborhood of p^i, the total space $C^{\text{sm-ch}}$ is a pullback under a smooth map of the total space $\operatorname{Spec} K[[x, y, t]]/(xy - t)$ of the universal source over the versal deformation space appearing above, which is smooth. □

In particular, X lies in a unique irreducible component $\bar{\mathcal{H}}^{\text{sm-ch},\circ}_{k,g,2}$ of $\bar{\mathcal{H}}^{\text{sm-ch}}_{k,g,2}$. The work in Section 4.2 defines a universal $\vec{e}$-positive locus $W^{\vec{e}}(C^{\text{sm-ch}})$, which is proper over $\bar{\mathcal{H}}^{\text{sm-ch}}_{k,g,2}$, and whose fiber over X is $W^{\vec{e}}(X)$, which we have shown is a reduced finite set. By regeneration (Theorem 4.5.14), every point of $W^{\vec{e}}(X)$ lies in a component of $W^{\vec{e}}(C^{\text{sm-ch}})$ dominating $\bar{\mathcal{H}}^{\text{sm-ch},\circ}_{k,g,2}$. Since $\bar{\mathcal{H}}^{\text{sm-ch},\circ}_{k,g,2}$ is reduced by Lemma 4.9.4, we conclude as in the proof of Lemma 4.9.1 that $W^{\vec{e}}(C^{\text{sm-ch}}) \to \bar{\mathcal{H}}^{\text{sm-ch},\circ}_{k,g,2}$ is étale near X.

4.9.5 Flips in monodromy

Suppose that the flip F^iT is defined. Our deformation of X will be obtained by smoothing the node p^i. Such curves are parametrized by a component $\mathcal{H}^\circ_{k,2,2}$ of $\mathcal{H}_{k,2,2}$. The map $\iota\colon \mathcal{H}^\circ_{k,2,2} \hookrightarrow \bar{\mathcal{H}}^{\text{sm-ch}}_{k,g,2}$ sends (C, p, q) to the chain curve obtained by attaching $E^1 \cup_{p^1} \cdots \cup_{p^{i-2}} E^{i-1}$ to C so that p^{i-1} is identified with p, and attaching $E^{i+2} \cup \cdots \cup E^g$ so that p^{i+1} is identified with q.



Our original chain curve X is in the closure of $\iota(\mathcal{H}^\circ_{k,2,2})$.

Lemma 4.9.6. *Let L be a limit line bundle on $\iota(C, p, q)$ with $L^{E^t} = L^t_T$ for $t \notin \{i, i+1\}$, such that there exist points $x, y \in C$ with*

$$L^C \simeq \mathcal{O}((T[i] + i - 1)p + (d - T[i] - i)q + x) \simeq \mathcal{O}((T[i+1] + i - 1)p + (d - T[i+1] - i)q + y)$$

(which forces $\mathcal{O}_C(x - y) \simeq \mathcal{O}_C((T[i+1] - T[i])(p - q))$). Then L is limit $\vec{e}$-positive.

Proof. Because L_T is $\vec{e}$-positive, and $L^{E^t} = L^t_T$ for $t \notin \{i, i+1\}$, it suffices to show that for all $a, b \in \mathbb{Z}$ (with the notation of Definition 4.2.2):

$$h^0(L^C(-ap - bq)) \geq f(a, b) := \min_{d^i + d^{i+1} = d} h^0\big((L_T)^{E^i \cup E^{i+1}}_{(d^i, d^{i+1})}(-ap - bq)\big). \tag{4.9.1}$$

If $a + b \geq d$, then straight-forward casework (using our assumption that $T[i+1] \not\equiv T[i] - 1 \mod k$) implies

$$f(a, b) \leq h^0\big((L_T)^{E^i \cup E^{i+1}}_{(d-b, b)}(-ap - bq)\big) = 0,$$

and so (4.9.1) holds. Otherwise, if $a + b \leq d - 1$, then straight-forward casework (using our assumption that $T[i+1] \not\equiv T[i] + 1 \mod k$) implies

$$\begin{aligned} f(a, b) &\leq h^0\big((L_T)^{E^i \cup E^{i+1}}_{(d-b-1, b+1)}(-ap - bq)\big) \\ &= \begin{cases} 1 & \text{if } (a, b) = (T[i] + i - 1, d - T[i] - i); \\ 1 & \text{if } (a, b) = (T[i+1] + i - 1, d - T[i+1] - i); \\ d - a - b - 1 & \text{otherwise.} \end{cases} \end{aligned}$$

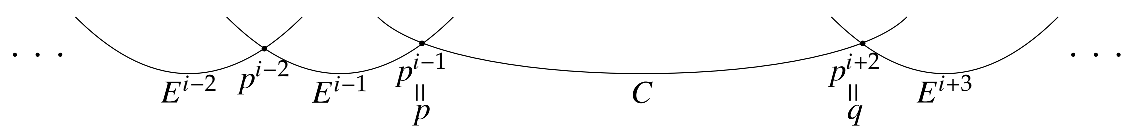
This immediately implies (4.9.1) in the "otherwise" case by Riemann–Roch. In the first two cases, this implies (4.9.1) by our assumption that $L^C(-ap - bq)$ is effective for these values of a and b. □

On a smooth curve of genus 2, the map $C \times C \to \mathrm{Pic}^0 C$ defined by $(x, y) \mapsto \mathcal{O}_C(x - y)$ is finite of degree 2 away from the diagonal (which is contracted to the identity in $\mathrm{Pic}^0 C$). Lemma 4.9.6 thus produces two limit $\vec{e}$-positive line bundles on the general curve in $\iota(\mathcal{H}^\circ_{k,2,2})$, i.e. a substack $Y_F(i,T)$ of the restriction of $W^{\vec{e}}(C^{\text{sm-ch}})$ to $\iota(\mathcal{H}^\circ_{k,2,2})$. Limiting to X, we obtain two $\vec{e}$-positive line bundles (which must be distinct because $W^{\vec{e}}(C^{\text{sm-ch}}) \to \bar{\mathcal{H}}^{\text{sm-ch}}_{k,g,2}$ is étale near X). By construction these correspond to tableaux T_1 and T_2 satisfying $T_j[t] = T[t]$ for $t \neq \{i, i+1\}$. But there are only two such tableaux: T itself and $F^i T$. Therefore the fiber of the closure of $Y_F(i,T)$ over X consists of L_T and L_{F^iT} as desired.

Note that, although the map $Y_F(i,T) \to W^{\vec{e}}(C^{\text{sm-ch}})$ depends on i and T, the stack $Y_F(i,T)$ and the map $Y_F(i,T) \to \mathcal{H}^\circ_{k,2,2}$ depend only on $n := T[i+1] - T[i]$, up to sign and modulo k. We therefore write $Y_F(n) = Y_F(i,T) \to \mathcal{H}^\circ_{k,2,2}$.

4.9.7 Shuffles in monodromy

Suppose that the shuffle $S^i T$ is defined, i.e., $T[i] = T[i+2]$ and $T[i+1] = T[i] \pm 1$. Without loss of generality, suppose that $T[i+1] = T[i] + 1$. Our deformation of X will be obtained by simultaneously smoothing the nodes p^i and p^{i+1}. Such curves are parametrized by a component $\mathcal{H}^\circ_{k,3,2}$ of $\mathcal{H}_{k,3,2}$. The map $\iota\colon \mathcal{H}^\circ_{k,3,2} \hookrightarrow \bar{\mathcal{H}}^{\text{sm-ch}}_{k,g,2}$ sends (C, p, q) to the chain curve obtained by attaching $E^1 \cup_{p^1} \cdots \cup_{p^{i-2}} E^{i-1}$ to C so that p^{i-1} is identified with p, and attaching $E^{i+3} \cup \cdots \cup E^g$ so that p^{i+2} is identified with q.

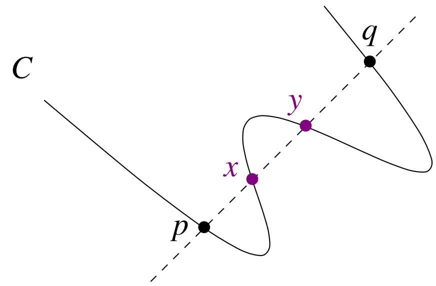


Our original chain curve X is in the closure of $\iota(\mathcal{H}^\circ_{k,3,2})$.

Lemma 4.9.8. *Suppose that C is a non-hyperelliptic curve of genus 3. Let L be a limit line bundle on $\iota(C,p,q)$ with $L^{E^t} = L^t_T$ for $t \notin \{i, i+1, i+2\}$, such that*

$$L^C \simeq \mathcal{O}_C(z + (T[i] + i)p + (d - T[i] - i - 1)q)$$

for $z \in \{x, y\}$, where $x, y \in C$ is the pair of points such that $p + q + x + y \sim K_C$ (i.e. the two points colinear with p and q in the canonical model of C as a plane quartic).



Then L is limit $\vec{e}$-positive.

Proof. Because L_T is $\vec{e}$-positive, and $L^{E^t} = L_T^t$ for $t \notin \{i, i+1, i+2\}$, it suffices to show that for all $a, b \in \mathbb{Z}$:

$$h^0(L^C(-ap - bq)) \geq f(a,b) := \min_{d^i + d^{i+1} + d^{i+2} = d} h^0\big((L_T)^{E^i \cup E^{i+1} \cup E^{i+2}}_{(d^i, d^{i+1}, d^{i+2})}(-ap - bq)\big). \quad (4.9.2)$$

If $a + b \geq d - 1$, then straight-forward casework implies

$$f(a,b) \leq h^0\big((L_T)^{E^i \cup E^{i+1} \cup E^{i+2}}_{(a, d-a-b, b)}(-ap - bq)\big) = \begin{cases} 1 & \text{if } (a,b) = (T[i] + i, d - T[i] - i - 1); \\ 0 & \text{otherwise.} \end{cases}$$

This immediately implies (4.9.2) in the "otherwise" case, and shows that in the first case, (4.9.2) follows from the condition

$$h^0(L^C(-(T[i] + i)p - (d - T[i] - i - 1)q)) \geq 1. \quad (4.9.3)$$

Otherwise, if $a + b \leq d - 2$, then straight-forward casework implies

$$f(a,b) \leq h^0\big((L_T)^{E^i \cup E^{i+1} \cup E^{i+2}}_{(a+1, d-a-b-2, b+1)}(-ap - bq)\big) = \begin{cases} 2 & \text{if } (a,b) = (T[i] + i - 1, d - T[i] - i - 2); \\ 1 & \text{if } (a,b) = (T[i] + i - 1, d - T[i] - i - 1); \\ 1 & \text{if } (a,b) = (T[i] + i, d - T[i] - i - 2); \\ d - a - b - 2 & \text{otherwise.} \end{cases}$$

This immediately implies (4.9.2) in the "otherwise" case by Riemann–Roch. Since vanishing at at p or q imposes at most one condition on sections of any line bundle, in the first three cases, (4.9.2) follows from the single condition

$$h^0(L^C(-(T[i] + i - 1)p - (d - T[i] - i - 2)q)) \geq 2. \quad (4.9.4)$$

We conclude by observing that L^C satisfies (4.9.3) and (4.9.4) by its definition. □

Lemma 4.9.8 thus produces two limit $\vec{e}$-positive line bundles on the general curve in $\iota(\mathcal{H}^\circ_{k,3,2})$, i.e. a substack $Y_S(i,T)$ of the restriction of $W^{\vec{e}}(C^{\text{sm-ch}})$ to $\iota(\mathcal{H}^\circ_{k,3,2})$. As in the previous "flip" case, the fiber of the closure of $Y_S(i,T)$ over X consists of L_T and L_{S^iT} as desired. Moreover, $Y_S(i,T) \to \mathcal{H}^\circ_{k,3,2}$ is independent of i and T. We therefore write $Y_S = Y_S(i,T) \to \mathcal{H}^\circ_{k,3,2}$.

4.9.9 Irreducibility of $Y_F(n)$ and Y_S

Our final task is to show that the following two double covers have a unique irreducible component dominating the desired component of the base:

$Y_F(n)$ The double cover of $\mathcal{H}_{k,2,2}$ parameterizing points x and y with $\mathcal{O}(x-y) \simeq \mathcal{O}(n(p-q))$ (where n is an integer not equal to $0, \pm 1 \bmod k$);

Y_S The double cover of the complement of the hyperelliptic locus in $\mathcal{H}_{k,3,2}$ parameterizing points x and y with $\mathcal{O}(x+y) \simeq \omega(-p-q)$ (where $k > 2$).

Notice that the definition of $Y_F(n)$ (respectively Y_S) extends naturally to the entire closure $\bar{\mathcal{H}}_{k,2,2}$ (respectively $\bar{\mathcal{H}}_{k,3,2}$), although it is not a priori finite flat of degree 2.

Proposition 4.9.10.

1. *$Y_F(n)$ is finite flat of degree 2 over the open substack of $\bar{\mathcal{H}}^{\text{ch}}_{k,2,2}$ defined by (C,p,q) satisfying the following conditions:*
    - *(a) C is irreducible,*
    - *(b) $p-q$ is exactly k-torsion on C,*
    - *(c) $p-q$ is exactly k-torsion on the partial normalization of C at any node (this condition is vacuous if C is smooth and implies the previous one if C is singular).*
2. *Y_S is finite flat of degree 2 over the open substack of $\bar{\mathcal{H}}^{\text{ch}}_{k,3,2}$ defined by (C,p,q) satisfying the following conditions:*
    - *(a) C is irreducible,*
    - *(b) p and q are not conjugate under the hyperelliptic involution if C is hyperelliptic (this condition is vacuous if C is not hyperelliptic),*
    - *(c) p and q are not conjugate under the hyperelliptic involution on the partial normalization of C at any node (this condition is vacuous if C is smooth and implies the previous one if C is singular).*

Proof. Write $U_2 \subseteq \bar{\mathcal{H}}^{\mathrm{ch}}_{k,2,2}$ (respectively $U_3 \subseteq \bar{\mathcal{H}}^{\mathrm{ch}}_{k,3,2}$) for the open substacks defined by the above conditions.

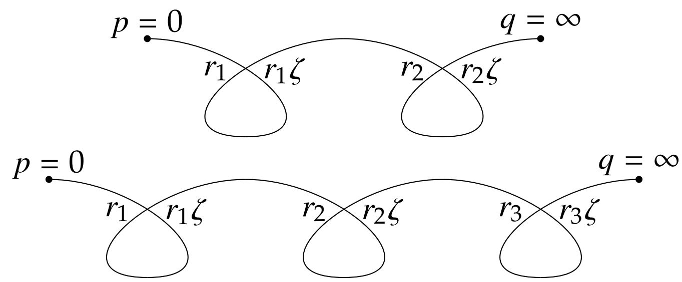
For $Y_F(n)$: Consider any $(C, p, q) \in U_2$. Since C is irreducible, the condition $\mathcal{O}(x - y) \simeq \mathcal{O}(n(p - q))$ is equivalent to $\mathcal{O}(x + \bar{y}) \simeq \omega(n(p - q))$, where $\bar{y}$ denotes the conjugate of y under the hyperelliptic involution. The line bundle $\omega(n(p - q))$ is of degree 2, and not isomorphic to ω because $p - q$ is exactly k torsion and $n \not\equiv 0 \bmod k$. Therefore, since C is irreducible, $\omega(n(p - q))$ has a unique section (up to scaling), vanishing on a Cartier divisor $D \subset C$ of degree 2. If D were supported at a node of C, consider the partial normalization C^ν of C at this node, and write s and t for the points on C^ν above this node. Then $\omega_C(n(p - q)) \simeq \mathcal{O}(s + t)$ as line bundles on C^ν. Since $\omega_C \simeq \omega_{C^\nu}(s + t) \simeq \mathcal{O}(s + t)$, we would have $\mathcal{O}_{C^\nu}(n(p - q)) \simeq \mathcal{O}_{C^\nu}$. But this is impossible since $p - q$ is exactly k-torsion on C^ν by assumption. Thus $D \subset C_{\mathrm{sm}}$.

It thus remains to see that these divisors D fit together to form a Cartier divisor $\mathcal{D}$ on the universal curve $\pi\colon \mathcal{C} \to U_2$ of relative degree 2 (which will then be supported in the smooth locus and identified with $Y_F(n)$). Write $\mathfrak{p}, \mathfrak{q}\colon U_2 \to \mathcal{C}$ for the universal sections. Observe that $\omega_{\mathcal{C}/U_2}(n(\mathfrak{p} - \mathfrak{q}))$ is a line bundle on $\mathcal{C}$, with a unique section up to scaling on every geometric fiber. Moreover, by Lemma 4.9.4, the base U_2 is smooth, and in particular reduced. Cohomology and base change thus implies that $\pi_*\omega_{\mathcal{C}/U_2}(n(\mathfrak{p} - \mathfrak{q}))$ is a line bundle on U_2. Working locally on U_2, we may trivialize it by picking a section, which gives a section of $\omega_{\mathcal{C}/U_2}(n(\mathfrak{p} - \mathfrak{q}))$, vanishing along a Cartier divisor $\mathcal{D}$.

For Y_S: Consider any $(C, p, q) \in U_3$. Since C is irreducible, and p and q are not conjugate under the hyperelliptic involution if C is hyperelliptic, we have $h^0(C, \mathcal{O}(p + q)) = 1$. By Serre duality, $\omega(-p - q)$ has a unique section (up to scaling), vanishing on a Cartier divisor $D \subset C$ of degree 2. If D were supported at a node of C, consider the partial normalization C^ν of C at this node, and write s and t for the points on C^ν above this node. Then $\omega_C(-p - q) \simeq \mathcal{O}(s + t)$ as line bundles on C^ν. Since $\omega_C \simeq \omega_{C^\nu}(s + t)$, we would have $\mathcal{O}_{C^\nu}(p + q) \simeq \omega_{C^\nu}$. But this is impossible since p and q are not conjugate under the hyperelliptic involution on C^ν by assumption. Thus $D \subset C_{\mathrm{sm}}$.

As in the previous case, these divisors D fit together to form a Cartier divisor $\mathcal{D}$ on the universal curve $\pi\colon \mathcal{C} \to U_3$ of relative degree 2. □

We now show that $Y_F(n)$ (respectively Y_S) has a unique irreducible component dominating $\bar{\mathcal{H}}^{\mathrm{ch},\circ}_{k,2,2}$ (respectively $\bar{\mathcal{H}}^{\mathrm{ch},\circ}_{k,3,2}$). To do this, we will restrict these double covers to certain schemes $R_h \to \bar{\mathcal{H}}^{\mathrm{ch},\circ}_{k,h,2}$ (with $h = 2$ respectively $h = 3$), where we can write down the equations of $Y_F(n)$ and Y_S explicitly and see that they are irreducible. The scheme R_2 is an open in $\{r_1, r_2\} \in \operatorname{Sym}^2 \mathbb{P}^1$, respectively R_3 is an open in $\{r_1, r_2, r_3\} \in \operatorname{Sym}^3 \mathbb{P}^1$. Let ζ denote a primitive kth root of unity (which exists by our assumption that the characteristic does not divide k). Our schemes R_h will parameterize stable curves of geometric genus 0 of the following forms:

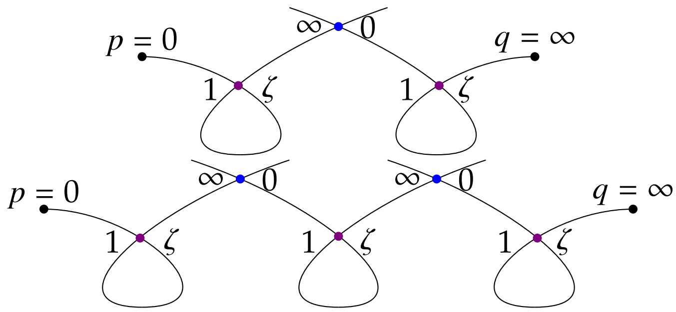


On such curves (and their normalizations at any one node), $p - q = 0 - \infty$ has order exactly k; the function t^k gives the linear equivalence between kp and kq. Moreover, any involution of $\mathbb{P}^1$ exchanging 0 and ∞ has the form $t \mapsto c/t$. Such an involution exchanges r_i and $r_i\zeta$ only if $c = r_i^2\zeta$. Therefore, if the $r_i^2\zeta$ are distinct, the points p and q are not conjugate under the hyperelliptic involution on the normalization of R_3 at any node. The general curves over R_h therefore satisfy the conditions of Proposition 4.9.10. A priori, however, R_h may not map to the desired component $\bar{\mathcal{H}}^{\mathrm{ch},\circ}_{k,h,2}$.

Proposition 4.9.11. *The image of R_h lies in the component $\bar{\mathcal{H}}^{\mathrm{ch},\circ}_{k,h,2}$.*

Remark 4.9.12. The authors conjecture $\bar{\mathcal{H}}_{k,h,2}$ is irreducible (when the characteristic does not divide k), which would immediately imply Proposition 4.9.11. Indeed, in characteristic zero, one can establish this using transcendental techniques. Moreover, techniques developed by Fulton in [30] show that irreducibility in characteristic zero implies irreducibility when the characteristic is greater than k. However, it seems difficult to extend this argument to small characteristics. Instead, we give here a direct algebraic proof of Proposition 4.9.11, which requires only our less strict hypothesis that the characteristic does not divide k.

Proof. By definition, $\bar{\mathcal{H}}^{\mathrm{ch},\circ}_{k,h,2}$ is the component of $\bar{\mathcal{H}}^{\mathrm{ch}}_{k,h,2}$ containing the locus of chains of elliptic curves. (Note that the locus of chains of elliptic curves is itself irreducible, as it is isomorphic to a g-fold product $X_1(k) \times \cdots \times X_1(k)$ of the classical modular curve $X_1(k)$, which is irreducible in characteristic not dividing k.) Consider the following two points of $\bar{\mathcal{H}}^{\mathrm{ch}}_{k,h,2}$:



By smoothing the top (blue) nodes, these curves are visibly in the closure of the image of R_h. Similarly, by smoothing the bottom (violet) nodes, these curves are visibly in the closure of the locus of chains of elliptic curves. Finally, by Lemma 4.9.4, these curves lie in a unique component. □

We finally compute explicitly the restriction of the covers $Y_F(n)$ and Y_S to R_2 and R_3 respectively. In particular, we will see that these covers are generically étale of degree 2 and have a unique irreducible component dominating R_2 and R_3 respectively. Therefore $Y_F(n)$ and Y_S have a unique irreducible component dominating $\bar{\mathcal{H}}^{\text{ch},\circ}_{k,2,2}$ and $\bar{\mathcal{H}}^{\text{ch},\circ}_{k,3,2}$ respectively (c.f. discussion after the proof of Lemma 4.9.1).

For $Y_F(n)$, we have $\mathcal{O}(x - y) \simeq \mathcal{O}(n(0 - \infty))$, so there is a function vanishing along $n \cdot 0 + y$, with a pole along $n \cdot \infty + x$. The only such function on the normalization is $t^n(t - y)/(t - x)$; this function must therefore descend to the nodal curve, i.e.:

$$\frac{r_1^n(r_1 - y)}{r_1 - x} = \frac{(r_1\zeta)^n(r_1\zeta - y)}{r_1\zeta - x} \quad \text{and} \quad \frac{r_2^n(r_2 - y)}{r_2 - x} = \frac{(r_2\zeta)^n(r_2\zeta - y)}{r_2\zeta - x}.$$

The first of these equations is linear in y; we may thus solve for y and substitute into the second equation. Clearing denominators, we obtain a quadratic equation for x, whose coefficients are symmetric in r_1 and r_2. Written in terms of the elementary symmetric functions $e_1 = r_1 + r_2$ and $e_2 = r_1 r_2$ on $\text{Sym}^2 \mathbb{P}^1$, this equation is:

$$(\zeta^{n+1} - 1) \cdot x^2 + (\zeta - \zeta^{n+1})e_1 \cdot x + (\zeta^{n+1} - \zeta^2)e_2 = 0. \tag{4.9.5}$$

This is linear in e_1 and e_2, so can only be reducible if it has a root $x \in \mathbb{P}^1$ which is constant (i.e. independent of e_1 and e_2). But upon setting $e_2 = \infty$, this quadratic has a double root at $x = \infty$ (note that $n \not\equiv 1 \bmod k$, so $\zeta^{n+1} - \zeta^2 \neq 0$). Similarly, upon setting $e_1 = e_2 = 0$, this quadratic has a double root at $x = 0$ (note that $n \not\equiv -1 \bmod k$, so $\zeta^{n+1} - 1 \neq 0$). Thus no such constant root exists, and (4.9.5) is irreducible as desired.

For Y_S, we have $\mathcal{O}(x + y) \simeq \omega(-p - q)$, so there is a section of the dualizing sheaf vanishing at x, y, 0, and ∞. When pulled back to the normalization, this gives a meromorphic 1-form with poles at the points lying above the nodes $(r_1, r_1\zeta, r_2, r_2\zeta, r_3, r_3\zeta)$ that vanishes at x, y, 0, and ∞. The only such 1-form is

$$\alpha = \frac{t(t-x)(t-y)\cdot dt}{(t-r_1)(t-r_1\zeta)(t-r_2)(t-r_2\zeta)(t-r_3)(t-r_3\zeta)}.$$

This 1-form must therefore descend to a section of the dualizing sheaf on the nodal curve, i.e.:

$$\mathrm{Res}_{t=r_1}\alpha + \mathrm{Res}_{t=r_1\zeta}\alpha = \mathrm{Res}_{t=r_2}\alpha + \mathrm{Res}_{t=r_2\zeta}\alpha = \mathrm{Res}_{t=r_3}\alpha + \mathrm{Res}_{t=r_3\zeta}\alpha = 0.$$

Since the sum of all residues ($\mathrm{Res}_{t=r_1}\alpha + \mathrm{Res}_{t=r_1\zeta}\alpha + \mathrm{Res}_{t=r_2}\alpha + \mathrm{Res}_{t=r_2\zeta}\alpha + \mathrm{Res}_{t=r_3}\alpha + \mathrm{Res}_{t=r_3\zeta}\alpha$) automatically vanishes, this is really just two conditions:

$$\begin{aligned}&\frac{r_1(r_1-x)(r_1-y)}{(r_1-r_1\zeta)(r_1-r_2)(r_1-r_2\zeta)(r_1-r_3)(r_1-r_3\zeta)}\\ &\qquad + \frac{(r_1\zeta)(r_1\zeta-x)(r_1\zeta-y)}{(r_1\zeta-r_1)(r_1\zeta-r_2)(r_1\zeta-r_2\zeta)(r_1\zeta-r_3)(r_1\zeta-r_3\zeta)}\\ &\qquad\qquad = \mathrm{Res}_{t=r_1}\alpha + \mathrm{Res}_{t=r_1\zeta}\alpha = 0.\end{aligned}$$

and

$$\begin{aligned}&\frac{r_2(r_2-x)(r_2-y)}{(r_2-r_2\zeta)(r_2-r_1)(r_2-r_1\zeta)(r_2-r_3)(r_2-r_3\zeta)}\\ &\qquad + \frac{(r_2\zeta)(r_2\zeta-x)(r_2\zeta-y)}{(r_2\zeta-r_2)(r_2\zeta-r_1)(r_2\zeta-r_1\zeta)(r_2\zeta-r_3)(r_2\zeta-r_3\zeta)}\\ &\qquad\qquad = \mathrm{Res}_{t=r_2}\alpha + \mathrm{Res}_{t=r_2\zeta}\alpha = 0.\end{aligned}$$

The first of these equations is linear in y; we may thus solve for y and substitute into the second equation. Clearing denominators, we obtain a quadratic equation for x, whose coefficients are symmetric in r_1, r_2, and r_3. Written in terms of the elementary symmetric functions $e_1 = r_1 + r_2 + r_3$ and $e_2 = r_1r_2 + r_2r_3 + r_3r_1$ and $e_3 = r_1r_2r_3$ on $\mathrm{Sym}^3\mathbb{P}^1$, this equation is:

$$(\zeta+1)e_2\cdot x^2 - [\zeta e_1e_2 + (\zeta^2+\zeta+1)e_3]\cdot x + (\zeta^2+\zeta)e_1e_3 = 0.$$

To see this is irreducible, it suffices to check irreducibility after specializing $e_1 = 1$, which yields the equation

$$(\zeta+1)e_2\cdot x^2 - [\zeta e_2 + (\zeta^2+\zeta+1)e_3]\cdot x + (\zeta^2+\zeta)e_3 = 0. \tag{4.9.6}$$

This is linear in e_2 and e_3, so can only be reducible if it has a root $x \in \mathbb{P}^1$ which is constant. But upon setting $e_2/e_3 = 0$, the roots are $x = \infty$ and $x = (\zeta^2 + \zeta)/(\zeta^2 + \zeta + 1) \neq 0$ (note that $\zeta^2 + \zeta = \zeta(\zeta + 1) \neq 0$ because ζ is a primitive kth root of unity with $k > 2$). Similarly, upon setting $e_2/e_3 = \infty$, the roots are $x = 0$ and $x = \zeta/(\zeta + 1) \neq \infty$ (again $\zeta + 1 \neq 0$). It thus remains to observe that $(\zeta^2 + \zeta)/(\zeta^2 + \zeta + 1) \neq \zeta/(\zeta + 1)$ because $\zeta \neq 0$.

Conclusion

We have now proved analogues of all of the main theorems of Brill–Noether theory for curves that are equipped with a fixed map to $\mathbb{P}^1$, answering Question 1.2.4 for general degree k covers. If k is taken to be suitably large relative to g (in which case the Hurwitz space $\mathcal{H}_{k,g}$ surjects onto $\mathcal{M}_g$), then our Theorems 1.3.7 and 1.3.9 in fact recover the classical Theorem 1.1.3.

Our techniques give us excellent control over $W^{\vec{e}}(C)$ in a degeneration as C specializes to the chain curve X of Section 3.1. Our regeneration theorem (Theorem 4.5.8) identifies the components of the limit of $W^{\vec{e}}(C)$ in terms of fillings of a Young diagram $\Gamma(\vec{e})$. Many geometric properties of $W^{\vec{e}}(C)$ can then be deduced from corresponding combinatorial questions involving these fillings. These fillings in turn correspond to reduced words for elements of the affine symmetric group.

In Chapter 2, we studied splitting loci more generally. The universal formulas we found there specialized to inform the shape of the class formula in Theorem 1.3.7 part 3. But our study of Brill–Noether splitting loci was also used to prove new facts about the geometry of splitting loci in general. Our smoothness result in Chapter 3 shows that the local geometry of Brill–Noether splitting loci reflect the local geometry of universal splitting loci. So, when our techniques of Chapter 4 established Cohen–Macaulayness of $W^{\vec{e}}(C)$, this implied Cohen–Macualayness of universal splitting loci (see Corollary 4.1.2.)

I would like to end with a final thank you to Ravi Vakil, who has been there for me, providing perspective, advice, and encouragement as every step of this story unfolded.

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