# Slopes and number-theoretic applications

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These are notes for a survey talk on slopes in Arakelov geometry and number-theoretic applications. In many ways one can think of Arakelov geometry as a recasting of Minkowski's geometry of numbers into more algebro-geometric language. In this talk we will introduce the language of *slopes* and see some powerful Diophantine applications (among many) of Bost's *slope method*, which is introduced in [Bos01, Section 4]. All errors and pedantry are due to me—please send me any comments or corrections.

## Contents



## 1 Basic definitions

We will introduce some basic definitions, which can be found in [Bos20, Chapter 1].

Fix $K$ a number field, $S \coloneqq \mathrm{Spec}(\mathcal{O}_K)$, and $K(\mathbf{C})$ the set of field embeddings $K \hookrightarrow \mathbf{C}$.

**Definition 1.1.** A *Hermitian vector bundle* $\overline{E}$ is a finite locally free (i.e. projective) $\mathcal{O}_K$-module $E$ along with a family $(\|\cdot\|_\sigma)_{\sigma \in K(\mathbf{C})}$ of Hermitian norms (i.e. induced by a Hermitian inner product) on the vector spaces $E_\sigma \coloneqq E \otimes_{\mathcal{O}_{K,\sigma}} \mathbf{C}$ that is invariant under complex conjugation. The last condition means that $\|e \otimes z\|_\sigma = \|e \otimes \overline{z}\|_{\overline{\sigma}}$ for all $e \in E, z \in \mathbf{C}, \sigma \in K(\mathbf{C})$.

We may define the *rank* of a Hermitian vector bundle in the obvious way, and a Hermitian line bundle is a Hermitian vector bundle with rank 1. We may also consider isomorphisms of Hermitian vector bundles, which must preserve the metrics.

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**Definition 1.2.** A *subbundle* (resp. *quotient bundle*) of a Hermitian vector bundle $\overline{E}$ is an $\mathcal{O}_K$-submodule (resp. torsion-free quotient module) $\overline{F}$ of positive rank, such that the Hermitian metrics of $\overline{F}$ are the restrictions (resp. the quotient metrics) of the metrics on $\overline{E}$.

Recall that in the quotient case, if $F = E/N$, then $\|x\|_{F,\sigma} \coloneqq \inf_{n\in N} \|x-n\|_{E,\sigma}$.

There are a bunch of operations one can do to Hermitian vector bundles; for example: direct sums, tensor products, duals, exterior powers, and pullback. We will not introduce these in detail.

**Definition 1.3.** A *morphism* of Hermitian vector bundles $\overline{E}$ and $\overline{F}$ is defined as an $\mathcal{O}_K$-module morphism $\varphi : E \to F$ such that for all $\sigma \in K(\mathbf{C})$, the induced $\mathbf{C}$-linear map $\varphi_\sigma : E_\sigma \to F_\sigma$ has operator norm at most 1.

**Definition 1.4.** Let $\overline{L}$ be a Hermitian line bundle. The *Arakelov degree* of $\overline{L}$ is

$$\widehat{\deg}(\overline{L}) \coloneqq \log|L/s\mathcal{O}_K| - \sum_{\sigma\in K(\mathbf{C})} \log\|s\|_\sigma = \sum_{\mathfrak{p}\in S} v_\mathfrak{p}(s)\log N_\mathfrak{p} - \sum_{\sigma\in K(\mathbf{C})} \log\|s\|_\sigma\,,$$

where $s$ is any nonzero section of $L$ (which can be rational for the second equality), "$\mathfrak{p} \in S$" means the maximal ideals of $\mathcal{O}_K$, and $N_\mathfrak{p}$ is the size of the residue field at $\mathfrak{p}$. It is independent of the choice of $s$ by the product formula.

**Definition 1.5.** If $\overline{E}$ is a Hermitian vector bundle of rank $r$, then the *Arakelov degree* of $\overline{E}$ is defined to be $\widehat{\deg}\left(\wedge^r\overline{E}\right)$.

More details on the above constructions can be found in [Mor14, Section 3.3] (in particular, Propositions 3.10 and 3.11) or in my previous notes [Hao25]. The above notions generalize the standard Minkowski geometry of numbers constructions, which occur in the case $K = \mathbf{Q}$. In that case, where Hermitian vector bundles are lattices in Euclidean inner product spaces, the Arakelov degree is simply $-\log$ of the covolume of the lattice.

**Proposition 1.6.** Let $\overline{E}_1, \overline{E}_2$ be two Hermitian vector bundles with ranks $r_1, r_2$. Then

(1) $\widehat{\deg}(\overline{E}_1 \otimes \overline{E}_2) = r_2\widehat{\deg}(\overline{E}_1) + r_1\widehat{\deg}(\overline{E}_2)$.

(2) $\widehat{\deg}(\overline{E}_1 \oplus \overline{E}_2) = \widehat{\deg}(\overline{E}_1) + \widehat{\deg}(\overline{E}_2)$.

(3) $-\widehat{\deg}(\overline{E}_1) = \widehat{\deg}(\overline{E}_1^\vee)$.

*Proof.* (1) The claim is obvious when the $\overline{E}_i$ are line bundles, and note that $\det(\overline{E}_1 \otimes \overline{E}_2) \cong (\det \overline{E}_1)^{r_2} \otimes (\det \overline{E}_2)^{\otimes r_1}$.

(2) Apply (1) upon replacing $\overline{E}_i$ with $\det(\overline{E}_i)$. Note that

$$\det(E_1 \oplus E_2) \cong \bigoplus_{i=1}^{r_1+r_2} \left( \bigwedge_{j=1}^{i} E_1 \otimes \bigwedge_{k=1}^{r_1+r_2-i} E_2 \right) = \det(E_1) \otimes \det(E_2).$$

(3) The claim is obvious when $\overline{E}_1$ is a line bundle, and then use that exterior power commutes with dual, along with (1).

□

We now come to the key definition.

**Definition 1.7.** Let $\overline{E}$ be a Hermitian line bundle of positive rank. The *slope* of $\overline{E}$ is defined as

$$\widehat{\mu}(\overline{E}) \coloneqq \frac{\widehat{\deg}(\overline{E})}{\operatorname{rank}(E)}.$$

We define the *maximal slope* of $\overline{E}$ to be

$$\widehat{\mu}_{\max}(\overline{E}) = \sup\{\widehat{\mu}(\overline{F}) : \overline{F} \text{ is a positive-rank Hermitian subbundle of } \overline{E}\}$$

and likewise the *minimal slope* of $\overline{E}$ to be

$$\widehat{\mu}_{\min}(\overline{E}) = \inf\{\widehat{\mu}(\overline{F}) : \overline{F} \text{ is a torsion-free Hermitian quotient bundle of } \overline{E}\}.$$

**Remark 1.8.** If $\overline{E}$ has rank 0, then its maximal (resp. minimal) slope is $-\infty$ (resp. $\infty$) by convention.

It turns out that the supremum and infimum in the definitions of the maximal and minimal slope are in fact attained. This is not immediately obvious, but not terribly hard to prove either; we omit the proof since it is not necessary for our purposes. Also, one can check that $\widehat{\mu}_{max}(\overline{E}) = -\widehat{\mu}_{min}(\overline{E}^{\vee})$.

By (1) of Proposition 1.6, we have

**Corollary 1.9.**

$$\widehat{\mu}(\overline{E}_1 \otimes \overline{E}_2) = \widehat{\mu}(\overline{E}_1) + \widehat{\mu}(\overline{E}_2).$$

We now define *heights* of generically defined morphisms of Hermitian vector bundles. These will play a role in the important slope inequality.

**Definition 1.10.** Let $\overline{E}, \overline{F}$ be Hermitian vector bundles and $\varphi : E_K \to F_K$ be a nonzero $K$-linear map. We define the *height* of $\varphi$ (with respect to $\overline{E}, \overline{F}$) to be

$$h(\varphi) \coloneqq \sum_{\mathfrak{p} \in S} \log \|\varphi\|_{\mathfrak{p}} + \sum_{\sigma \in K(\mathbf{C})} \log \|\varphi\|_{\sigma},$$

where $\|\varphi\|_{\mathfrak{p}}$ is defined to be given by

$$\inf\{\|r\|_{\mathfrak{p}}^{-1} : r \in K^{\times}, r\varphi \in (E^{\vee} \otimes_{\mathcal{O}_K} F)_{\mathcal{O}_{K,\mathfrak{p}}}\}$$

upon identifying $\operatorname{Hom}_K(E_K, F_K)$ with $(E^{\vee} \otimes_{\mathcal{O}_K} F)_K$, and we give $K$ the normalized $\mathfrak{p}$-adic absolute value sending a uniformizer to $1/N_{\mathfrak{p}}$. Also, $\|\varphi\|_{\sigma}$ is given by the operator norm of $\varphi_{\sigma} : E_{\sigma} \to F_{\sigma}$.

**Remark 1.11.** Note that $\|\varphi\|_{\mathfrak{p}}$ can also be thought of as an operator norm of $\varphi \in \mathrm{Hom}_{K_\mathfrak{p}}(E_{K_\mathfrak{p}}, F_{K_\mathfrak{p}})$, where $E_{K_\mathfrak{p}}, F_{K_\mathfrak{p}}$ have the $\mathfrak{p}$-adic norms induced by the $\mathcal{O}_\mathfrak{p}$ (completion of $\mathcal{O}_K$ at $\mathfrak{p}$) lattices $E_{\mathcal{O}_\mathfrak{p}}, F_{\mathcal{O}_\mathfrak{p}}$.

By convention we set $h(0) = -\infty$. It is clear that for morphisms $\varphi$ of Hermitian vector bundles (recall Definition 1.3), we have $h(\varphi) \leq 0$.

**Proposition 1.12.** Suppose $\overline{E}, \overline{F}$ are Hermitian line bundles. Then for a nonzero element $\varphi \in \mathrm{Hom}_K(E_K, F_K)$, we have

$$\widehat{\deg}(\overline{E}) - \widehat{\deg}(\overline{F}) = -\widehat{\deg}(\overline{E}^{\vee} \otimes \overline{F}) = h(\varphi).$$

Indeed, $\varphi$ is a rational section of $E^\vee \otimes F$, and the rest follows from the definitions.

Now we discuss the *slope inequality*. To motivate it, we make the trivial observation that if $\varphi : \overline{E} \to \overline{F}$ is an isometric injection, then one has $\widehat{\mu}(\overline{E}) \leq \widehat{\mu}_{max}(\overline{F})$. We would like to remove the rather strong hypothesis that $\varphi$ is an isometry. This leads to:

**Theorem 1.13.** Let $\overline{E}, \overline{F}$ be two Hermitian vector bundles with $r := \mathrm{rank}(E) \geq 1$, $\varphi_K : E_K \to F_K$ be an injective $K$-linear map. Then

$$\widehat{\mu}(\overline{E}) \leq \widehat{\mu}_{max}(\overline{F}) + h(\varphi). \tag{1.1}$$

The intuition for the theorem is simply that upon removing the isometry condition, we need a way of detecting how the metrics of $\overline{E}$ compare to those of $\overline{F}$, and the discrepancy must be reflected in any inequality relating any slopes of $\overline{E}$ and $\overline{F}$. This discrepancy is exactly furnished by the height $h(\varphi)$, which has the operator norms of $\varphi$ with respect to various places built in to its definition.

*Proof.* Let $F'$ be a $\mathcal{O}_K$-submodule of $F$ such that $F'_K = \varphi(E_K)$. Replace $F'$ with its saturation so that $F/F'$ is torsion-free. Hence we get a Hermitian vector bundle $\overline{F}'$ of rank upon taking the restrictions of the $F$-metrics, and $\varphi : E_K \to F'_K$ is a bijection. This induces a nonzero map $\wedge^r \varphi$ of the top exterior powers, and so gives a rational section of the Hermitian line bundle $\overline{L} := \wedge_{i=1}^r \overline{E}^\vee \otimes \wedge_{i=1}^r F'$. Apply Proposition 1.12:

$$\widehat{\deg}(\overline{E}) - \widehat{\deg}(\overline{F}') = -\widehat{\deg}(\overline{L}) = h(\wedge^r \varphi) = \sum_{\mathfrak{p} \in S} \log \|\wedge^r \varphi\|_{\mathfrak{p}} + \sum_{\sigma \in K(\mathbf{C})} \log \|\wedge^r \varphi\|_{\sigma}.$$

Note that $\|\wedge^r \varphi\|_v \leq \|\varphi\|_v^r$ for all places $v$ of $K$ by definition of operator norms (see Remark 1.11), and so

$$\widehat{\deg}(\overline{E}) - \widehat{\deg}(\overline{F}') \leq r h(\varphi).$$

Since $\widehat{\deg}(\overline{F}') \leq r \widehat{\mu}_{max}(\overline{F})$ by definition, we have the result upon dividing both sides in the above inequality by $r$. □

By dualizing the inequality (1.1), we can obtain a corresponding statement for surjective morphisms $\varphi$:

$$\widehat{\mu}(\overline{F}) \geq \widehat{\mu}_{min}(\overline{E}) - h(\varphi).$$

One application of Theorem 1.13 is to prove the classical Siegel's lemma:

**Theorem 1.14** (Siegel's lemma)**.** Consider a homogeneous linear system of $M$ equations in $N$ variables with integer coefficients $a_{ij}$ not all 0, such that $M < N$ and $B \coloneqq \max|a_{ij}|$. Then there is a nonzero integer solution $(x_1, \ldots, x_N)$ to the system where all $|x_i| \leq (NB)^{M/(N-M)}$ for all $i$.

In fact it might be more correct to think of the slope inequality as a reformulation of Siegel's lemma, but we won't explain this point further.

# 2 Bost's slope method

In Bost's work, he extends the above slope inequality into the case of *filtered vector bundles*. In this case, suppose $F_K$ is a finite-dimensional $K$-vector space with a filtration of $K$-vector subspaces

$$0 = F_K^{N+1} \subseteq F_K^N \subseteq \ldots \subseteq F_K^0 = F_K.$$

Also assume for all $0 \leq i \leq N$, $F_K^i/F_K^{i+1}$ is the $K$-vector space associated (after tensoring) a Hermitian vector bundle $\overline{G}^i$; that is, $G_K^i = F_K^i/F_K^{i+1}$. Now suppose $\overline{E}$ is a Hermitian vector bundle and $\varphi : E_K \to F_K$ is an injective $K$-linear map. We may then define $E_K^i \coloneqq \varphi^{-1}(F_K^i)$ and $E^i \coloneqq E_K^i \cap E$ to get a filtration

$$0 = E^{N+1} \subseteq E^N \subseteq \ldots \subseteq E^0 = E$$

of Hermitian vector subbundles of $E$ (give each $E^i$ the restricted norms from $E$). Set $\varphi^i$ to be the evident $K$-linear map $E_K^i \to G_K^i$.

**Theorem 2.1** (Proposition 4.6, [Bos01])**.** With the above notation, we have

$$\widehat{\deg}(\overline{E}) \leq \sum_{i=0}^{N} \operatorname{rank}(E^i/E^{i+1}) \left( \widehat{\mu}_{max}(\overline{G}^i) + h(\varphi^i) \right).$$

We will not prove this, but note that this reduces to Theorem 1.13 when $i = 0$. The proof is not hard, given our "baby" version of the slope inequality. The key observation is that $\widehat{\deg}(\overline{E}) = \sum_{i=0}^{N} \widehat{\deg}(\overline{E}_i/\overline{E}_{i+1})$, as well as applying Theorem 1.13 to the natural *injective* maps $\widetilde{\varphi}^i : E_K^i/E_K^{i+1} \to G_K^i$.

There are many applications of Bost's slope inequality (Theorem 2.1), most of them being geometric reworkings or strengthenings of results that can be proven by more classical methods. For example, it plays a role in bounding the ranks of certain $\mathbf{Z}$-modules of functions for application towards holonomy results in [CDT24] (see in particular Section 7). In other words, it shows that certain formal power series must satisfy an algebraic differential equation with some prescribed degree (i.e. a type of algebraization result). Bost himself used the slope framework towards algebraization results on foliations in the original paper [Bos01]. Note that all of these applications (including the next one) involve transcendental properties.

We will sketch an application of it towards the *Schneider–Lang theorem*, which has been vastly generalized to various algebraicity criteria of formal (analytic) maps in works such as [Gas10], [Gra05], and [Her12]. First, the classical statement of the Schneider–Lang theorem is as follows:

**Theorem 2.2.** Let $K$ be a number field with a fixed embedding $\sigma_0 : K \hookrightarrow \mathbf{C}$. Suppose $f_1, \dots, f_N$ are meromorphic functions, such that:

(1) At least two of the $f_i$ are algebraically independent over $\mathbf{C}$. Suppose these two algebraically independent functions have finite orders $\rho_1$ and $\rho_2$.[1]

(2) $f_j' \in K[f_1, \dots, f_N]$ for all $j$.

Then there are at most $(\rho_1 + \rho_2)[K : \mathbf{Q}]$ distinct complex numbers $\omega_1, \dots, \omega_m$ such that $f_i(\omega_j) \in K$ for all $i$ and $j$.

We state two fun corollaries.

**Corollary 2.3** (Lindemann–Weierstrass)**.** If $a \neq 0$ is an algebraic number, then $e^a$ is transcendental (in particular, $e$ and $\pi$ are transcendental).

*Proof.* Otherwise, there is some number field $K$ containing all $a, 2a, 3a, \dots$ along with all $e^a, e^{2a}, e^{3a}, \dots$, which contradicts Theorem 2.2 with $f_1(z) = z, f_2(z) = e^z$. □

**Corollary 2.4** (Gelfond–Schneider)**.** If $a, b$ are algebraic numbers with $a \neq 0, 1$ and $b \notin \mathbf{Q}$, then any value of $a^b$ is transcendental (this is possibly multivalued if $a$ is complex).

*Proof.* There is a contradiction to Theorem 2.2 with $f_1(z) = e^z, f_2(z) = e^{bz}$, since there is some number field $K$ containing the outputs of $f_1, f_2$ applied to the infinite sequence $\log a, 2\log a, 3\log a, \dots$. □

In the rest of these notes we will sketch the proof of Theorem 2.2 loosely following ideas from [Gra05, Section 3] and [Her12, Section 6]. We will focus on where the slope formalism is used and neglect some of the estimations and complex-analytic details that come up. To illustrate the ideas, we take $N = 2$ and $f_1, f_2$ to be entire, which is already enough to prove our "fun" corollaries.

[1] Recall that the *order* of an entire function $f$ is the infimum of all $\rho$ such that there exist constants $A, B > 0$ with $|f(z)| \leq Ae^{B|z|^\rho}$ for all $z \in \mathbf{C}$. In general for meromorphic functions, the order can be defined via Nevanlinna theory, or simply by asserting that a meromorphic function has order of growth $\rho$ if it can be written as a quotient of holomorphic functions with orders at most $\rho$.

## 2.1 Proof sketch of Theorem 2.2

*Sketch of Theorem 2.2.* Again, in this sketch we only take $N = 2$ and assume $f_1, f_2$ are entire. We will also only prove that there can only be finitely many distinct complex numbers $\omega_j$ such that $f_i(\omega_j) \in K$ for all $i$ and $j$, not that there are at most $(\rho_1 + \rho_2)[K : \mathbf{Q}]$ of them (i.e. we will not be so careful about the exact estimation). So suppose we have distinct $\omega_1, \dots, \omega_m$ such that each $f_i(\omega_j)$ is in $K$, and assume for convenience that $\omega_0 \coloneqq 0$ also has $f_i(0) \in K$ for all $i$ and $j$. In fact, upon scaling the $f_i$ by a common integer, we may and do assume that the $f_i(\omega_j)$ are in $\mathcal{O}_K$.

To begin, let $D$ be a positive integer. Note that if $f_1$ and $f_2$ are algebraically independent, then if the analytic map $z \mapsto Q(1, f_1(z), f_2(z))$ is identically 0 for $Q \in \mathbf{C}[X_0, X_1, X_2]_D$ a homogeneous degree-$D$ polynomial, then $Q \equiv 0$. In particular, for nonnegative integers $n$, if we consider the kernels of the maps $\theta_n : \mathbf{C}[X_0, X_1, X_2]_D \to \mathbf{C}[\![z]\!]/(z^{n+1})$ given by truncating the above analytic function at the $n$th power, they form a decreasing chain for $n = 0, 1, 2 \dots$ and their intersection is 0. By finite-dimensionality we must have $\ker(\theta_r) = 0$ for some large enough $r$.

Next, define $E$ to be the $\mathcal{O}_K$-module $H^0(\mathbf{P}^2_{\mathcal{O}_K}, \mathcal{O}(D))$, which is free of rank $N \coloneqq \binom{D+2}{2}$ with the standard monomial basis. We will equip $E$ with Hermitian metrics at each place $\sigma \in K(\mathbf{C})$ such that this standard monomial basis is orthonormal. From the definition we immediately have $\widehat{\deg}(\overline{E}) = 0$.

We now set up the $\overline{F}$ that will be used in the slope inequality. For $0 \le j \le m$, let $T_j \coloneqq z - \omega_j$ be a local coordinate at $\omega_j$ (with $\omega_0 = 0$). Consider

$$J_{q,j} \coloneqq \mathcal{O}_K[\![T_j]\!]/(T_j^{q+1}),$$

which is free of rank $q + 1$. We will use these to keep track of "higher derivatives/jets." We can also equip $J_{q,j}$ with Hermitian norms at each $\sigma \in K(\mathbf{C})$ by declaring the standard basis to be orthonormal. Now we define

$$\overline{F} \coloneqq \overline{J}_{r,0} \oplus \bigoplus_{j=1}^{m} \overline{J}_{q,j}$$

for some $q \in \mathbf{N}$ that will be chosen later. The point of this definition is to get a filtration by adding (at most) one line to the vector bundle at each step in a convenient way for the height estimates.

To define the desired map $\varphi : E_K \to F_K$, we need an algebraic (not analytic) way to encode the higher derivatives of the map $z \mapsto P(1, f_1(z), f_2(z))$ as elements of $K$ without first embedding $K$ into $\mathbf{C}$. Here $P(X_0, X_1, X_2) \in E$. Define the $K$-derivation $d : K[U_1, U_2] \to K[U_1, U_2]$ by $U_i \mapsto A_i(U_1, U_2)$, where $A_i \in K[U_1, U_2]$ satisfies $f_i' = A_i(f_1, f_2)$. We then define $g_P(X_1, X_2) \coloneqq P(1, X_1, X_2)$ and $K$-linear $\varphi_{q,j} : E_K \to J_{q,j,K}$ by

$$\varphi_{q,j}(P) = \sum_{k=0}^{q} \frac{(d^k g_P)(f_1(\omega_j), f_2(\omega_j))}{k!} \cdot T_j^k \in J_{q,j,K}.$$

Finally, we can define a $K$-linear map $\varphi : E_K \to F_K$ via the formula

$$P \mapsto (\varphi_{r,0}, \varphi_{q,1}, \ldots, \varphi_{q,m}).$$

By the choice of $r$, $\varphi$ is *injective*, because $\sigma_0(\varphi_{r,0}(P))$ (by abuse of notation) is precisely the Taylor expansion of $z \mapsto P(1, f_1(z), f_2(z))$, thought of as an entire function via $\sigma_0 : K \hookrightarrow \mathbf{C}$, about 0 truncated at the $r$th term.

Now we take the filtrations. We filter $F_K$, which has rank $(r+1) + m(q+1)$, by cutting off Taylor coefficients of $P(1, f_1(z), f_2(z))$ in an appropriate order after we impose enough vanishing conditions at the $\omega_j$ $(1 \leq j \leq m)$ via the $\overline{J}_{q,j}$. Let us take $q = r$, and filter $F_K$ via

$$F_K^{a(m+1)+b} = z^a J_{r,0,K} \oplus \bigoplus_{j=1}^{b} T_j^{a+1} J_{r,j,K} \oplus \bigoplus_{j=b+1}^{m} T_j^{a} J_{r,j,K}$$

for $0 \leq b \leq m$, $0 \leq a \leq r$. In other words, we impose order-$a$ vanishing at each $\omega_j \neq 0$ before imposing the same at 0, and we do this for all $0 \leq a \leq r$. So we have $F_K^0 = F_K$, $F_K^{r(m+1)+m+1} = 0$, and each $F^k/F^{k+1}$ is rank 1. Therefore with $\overline{E}^k$, $\overline{G}^k$ and $\varphi^k$ defined as in the first paragraph of this section, we may now apply Theorem 2.1:

$$0 = \widehat{\deg}(\overline{E}) \leq \sum_{k=0}^{r(m+1)+m} \operatorname{rank}(E^k/E^{k+1}) \left( \widehat{\mu}_{max}(\overline{G}^k) + h(\varphi^k) \right). \tag{2.1}$$

By construction, each $\operatorname{rank}(E^k/E^{k+1})$ is 0 or 1 because it injects into $F^k/F^{k+1}$ and each of those is rank 1; moreover exactly $\operatorname{rank}(E) = N = \binom{D+2}{2}$ of these ranks is 1. The fact that $F^k/F^{k+1}$ is rank 1 also means that $\widehat{\mu}_{max}(\overline{G}^k) = \widehat{\deg}(\overline{G}^k)$, and this is 0 for all $\overline{G}^k$ by construction of the metrics.

So we need to estimate the heights of the maps $\varphi^k : E_K^k \to G_K^k$. For $R > 0$ a positive radius to be optimized later, define $M(R) := \max_{|z|=R} \max(1, |f_1(z)|, |f_2(z)|)$. Let $k = a(m+1) + b$ for some $0 \leq a \leq r$, $0 \leq b \leq m$. By our algebraic construction $\varphi^k$ sends $P \in E_k$ to the $a$-th jet coefficient of $g_P : (X_1, X_2) \mapsto P(1, X_1, X_2)$ at $(f_1(\omega_b), f_2(\omega_b)) \in K$. For our distinguished embedding $\sigma_0$, the analytic function $g_{P,\sigma_0} : z \mapsto g_{\sigma_0(P)}(f_1(z), f_2(z))$ has zeroes of order at least $a$ at each $\omega_j$.[2] Omitting $\sigma_0$ from the notation for convenience, we may write

$$g_P(z) = \prod_{j=0}^{m} (z - \omega_j)^a h_P(z)$$

for some holomorphic $h_P$, and then

$$\frac{g_P^{(a)}(\omega_b)}{a!} = \prod_{j \neq b} (\omega_b - \omega_j)^a h_P(\omega_b).$$

[2] I apologize for the increasingly poor notation.

If $R \geq 2\max_{0\leq j\leq m}|\omega_j|$, we have

$$|h_P(\omega_b)| \leq \max_{|z|=R} \frac{|g_P(z)|}{\prod_{j=0}^{m}|z-\omega_j|^a} \leq \frac{\|P\|_{E,\sigma_0} \cdot M(R)^D \cdot N}{(R/2)^{a(m+1)}},$$

hence

$$\log|g_P^{(a)}(\omega_b)/a!|_{\sigma_0} \leq a\log V_b + \log\|P\|_{E,\sigma_0} + D\log M(R) - a(m+1)\log(R/2) + O(\log D)$$

for some positive constant $V_b$ depending only on $\omega_0,\ldots,\omega_m$ (which we will make negligible later). Here and thereafter, the big-O notation means that the implicit constant can depend on $K$, $f_1$ and $f_2$, but not other data (i.e. not the $\omega_i$'s, $D$, $R$, and certainly not $m$). Therefore

$$\log\left\|\varphi^{a(m+1)+b}\right\|_{\sigma_0} \leq a\log V_b + D\log M(R) - a(m+1)\log(R/2) + O(\log D)$$

For other $\sigma \in K(\mathbf{C})$ and any $R > 0$ (not the same as in the above case), we have a similar inequality, but we do not have any vanishing conditions (so it is valid to take any $R > 0$). By Cauchy's inequality,

$$\begin{aligned}\log|g_{P,\sigma}^{(a)}(\omega_b)|_\sigma &\leq \log(a!) - a\log R + \log\max_{|z-\omega_b|\leq R}|g_{P,\sigma}(z)| \\ &\leq \log(a!) - a\log R + \log\|P\|_{E,\sigma} + D\log M(R) + O(\log D).\end{aligned}$$

In particular,

$$\log|g_{P,\sigma}^{(a)}(\omega_b)/a!|_\sigma \leq -a\log R + \log\|P\|_{E,\sigma} + D\log M(R) + O(\log D),$$

which implies

$$\log\left\|\varphi^{a(m+1)+b}\right\|_\sigma \leq -a\log R + D\log M(R) + O(\log D).$$

Now we optimize $R$ for these $\sigma \neq \sigma_0$. By the finite order hypothesis, we have $\log M(R) = O(R^\rho)$ for some fixed $\rho > 0$. If $R$ is taken to be approximately $(a/D)^{1/\rho}$, then we have

$$\log\left\|\varphi^{a(m+1)+b}\right\|_\sigma \leq -\frac{a}{\rho}\log\frac{a}{D} + O(a+D).$$

For finite places, we choose nonzero $d \in \mathbf{Z}$ such that $df_i' = A_i(f_1, f_2)$ for some polynomial $A_i \in \mathcal{O}_K[f_1, f_2]$. Then for all primes $\mathfrak{p}$ of $\mathcal{O}_K$, due to the recursion on Taylor coefficients given by $df_i' = A_i(f_1, f_2)$ and the initial conditions on $f_1(\omega_b), f_2(\omega_b) \in \mathcal{O}_K$, we have

$$\log|g_P^{(a)}(\omega_b)/a!|_\mathfrak{p} \leq -\log|d^a|_\mathfrak{p} - \log|a!|_\mathfrak{p} = av_\mathfrak{p}(d)\log N_\mathfrak{p} + v_\mathfrak{p}(a!)\log N_\mathfrak{p}.$$

So the same inequality holds for $\log\left\|\varphi^{a(m+1)+b}\right\|_\mathfrak{p}$. Therefore

$$\sum_\mathfrak{p}\log\left\|\varphi^{a(m+1)+b}\right\|_\mathfrak{p} \leq \sum_\mathfrak{p} av_\mathfrak{p}(d)\log N_\mathfrak{p} + v_\mathfrak{p}(a!)\log N_\mathfrak{p} = a\log|N_{K/Q}(d)| + \log|N_{K/Q}(a!)|$$

$$= [K:\mathbf{Q}]O(a) + [K:\mathbf{Q}]O(a\log a).$$

Combining all the above estimates, we get

$$h(\varphi^{a(m+1)+b}) \leq [K:\mathbf{Q}]O(a) + [K:\mathbf{Q}]O(a\log a) - ([K:\mathbf{Q}]-1)\left(\frac{a}{\rho}\log\frac{a}{D} - O(a+D)\right)$$
$$+ a\log V_b + D\log M(R) - a(m+1)\log(R/2) + O(\log D).$$

This inequality is valid for all $D$, all $r$ large enough (compared to $D$), and all $R$ large enough (compared to the $|\omega_i|$). We would like $R$ to scale as approximately $D^{1/\rho}$, so take $D$ large enough such that $R = D^{1/\rho}$ is large enough in the above sense. The above inequality becomes

$$h(\varphi^{a(m+1)+b}) \leq [K:\mathbf{Q}]O(a\log a) - ([K:\mathbf{Q}]-1)\left(\frac{a}{\rho}\log\frac{a}{D} - O(a+D)\right)$$
$$+ a\log V_b + D^2 - \frac{a(m+1)}{\rho}\log(D) + O(\log D). \tag{2.2}$$

By (2.1), we have

$$0 \leq \sum_{a=0}^{r}\sum_{b=0}^{m} \operatorname{rank}(E^{a(m+1)+b}/E^{a(m+1)+b+1})h(\varphi^{a(m+1)+b}), \tag{2.3}$$

where exactly $N = \binom{D+2}{2}$ of the rank terms are 1, and the rest are 0. In our estimate for $h(\varphi^{a(m+1)+b})$, the dominant negative term in terms of $a$ is $a(m+1)\log(D)/\rho$. Therefore when all other parameters are held equal, the "worst-case scenario", i.e. when the right-hand side is maximized, occurs when these rank terms occur for as small of $a$ as possible (i.e. the nontrivial contributions to the sum occur at the smallest jet orders). Of course, this requires more careful analysis of the constants hidden in the big-O notations, but in the regime where $D$ is large (which is where we ultimately care about anyways, in the limit as $D \to \infty$), this is not hard to see.

So we only need to consider the case where the rank-1 terms in the sum (2.3) occur at filtration steps 0 to $N-1$, and derive a contradiction if $m$ is too large. We will do the estimation rather crudely as this is only a sketch, and leave it to the reader to actually work things out carefully.

Each of the inner sums has $m+1$ terms, so the index $a$ ranges from 0 to approximately $S := D^2/(2(m+1))$. Then if $D$ is large, we have

$$\sum_{a=0}^{S} a\log a \approx \int_0^S a\log a da = O(S^2\log S) \leq \frac{D^4 O(\log D)}{(m+1)^2}.$$

Therefore the positive contribution in (2.3) coming from the $[K:\mathbf{Q}]O(a\log a)$ term of (2.2) is approximately

$$\frac{D^4 O(\log D)}{(m+1)^2}\cdot(m+1) = \frac{D^4 O(\log D)}{m+1},$$

up to terms of smaller order in $D$. For the same reason, the negative contribution in (2.3) coming from the $-([K:\mathbf{Q}]-1)\left(\frac{a}{\rho}\log\frac{a}{D}\right)$ term of (2.2) is also approximately $\frac{D^4 O(\log D)}{\rho(m+1)}$, but this can actually be safely ignored as it will not be the dominant negative term due to the $m+1$ in the denominator. Next, the positive contribution from the $([K:\mathbf{Q}]-1)\,O(a+D)$ term is approximately

$$(m+1)\sum_{a=0}^{S} O(a+D) = (m+1)O(S^2) + O(DS) = \frac{O(D^4)}{m+1},$$

which is of smaller order in $D$. Similarly, the positive contributions from summing the $a\log V_b$ and $D^2$ terms are on the order of $\log V_b \cdot O(D^4)/(m+1)$ and $O(D^4)/(m+1)$, respectively. The $O(\log D)$ term is negligible.

It remains to look at the negative contribution from the $-\frac{a(m+1)}{\rho}\log(D)$ term. By similar reasoning it (negatively) contributes $\rho^{-1}D^4 O(\log D)$. Putting everything together, (2.3) gives

$$0 \leq \frac{C_1 D^4 \log D}{m+1} - \frac{C_2 D^4 \log D}{\rho} + C_3 D^4 \tag{2.4}$$

for some positive constants $C_1, C_2, C_3 > 0$, with $C_1$ and $C_2$ depending only on the data of $K$, $f_1$, $f_2$. This inequality must be true for all large enough $D$. It follows that if $m$ is too large with respect to $C_1, C_2, \rho$, then for $D$ large enough, we get a contradiction due to the $\frac{C_1 D^4 \log D}{m+1}$ scaling inversely with $m$. This gives an absolute bound on $m$, again depending only on the data of $K$, $f_1$, $f_2$, which is what we wanted. □

As mentioned at the beginning of the proof sketch, we did not compute the explicit bound $m \leq (\rho_1+\rho_2)[K:\mathbf{Q}]$, which requires more work, but the above suffices to show the application of Bost's slope method.

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