Wandering Intervals.

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Let I be the closed interval of real numbers fron -1 to +1. A differentiable function $F: I \to I$ is said to be convex if it has just one critical point at the origin 0, say, and if it is monotone decreasing to the left of 0 and monotone increasing to the right.

In this paper we construct a C^1 convex function F which has a "wandering" interval in the sense of Denjoy. That is, there exists a closed interval $J \subset I$ such that the set of forward and inverse images of J under F are disjoint and the complement of the union of their interiors is a Cantor set. This Cantor set is an exceptional minimal set since it is closed and invariant under F, contains no such proper subsets, and is neither periodic nor the entire interval I. (Coven and Nitecki [1] have recently constructed a related example with two turning points by adapting the Denjoy diffeomorphism of the circle.)

It turns out that F is not topologically conjugate to any C^2 convex function of I. In fact, if G is C^2 and topologically conjugate to F then G has a inflection point in its nonwandering set. It is not known if such a G exists or if there are any C^2 maps of the interval with exceptional minimal sets.

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§1. Basic Facts about Kneading.

Apart from Denjoy analysis, the main techniques we use are based on the kneading invariant of Milnor and Thurston [4]. This is a topological invariant which is defined in terms of the behaviour of the critical point of a convex function and characterises much of the dynamical behaviour of continuous families of C^1 functions such as $f(x) = x^2 - a$.

If f is convex and $x \in I$ let $\varepsilon_i(x)$ be -1, 0 or +1 according to whether $f^i(x) > 0$, $=0$, or <0. The sequence $\varepsilon_i(x)$ is called the itinerary of x. Let $\theta_i(x) = \prod_{j=0}^{i} \varepsilon_j(x)$. Then the formal power series $\theta(x) = \sum_{j=0}^{\infty} \theta_j(x)\, t^j$ is called the invariant coordinate of x. The map $x \mapsto \theta(x)$ is monotone decreasing if we endow the ring $\mathbb{Z}[[t]]$ with the lexicographical ordering. Let Λ denote the subset of $\mathbb{Z}[[t]]$ consisting of those formal power series whose coefficient lie in $\{-1, 0, +1\}$. Then it follows from the monotonicity of θ that

$$\theta(x^+) = \lim_{y \downarrow x} \theta(y) \quad \text{and} \quad \theta(x^-) = \lim_{y \uparrow x} \theta(y)$$

exist in the topology on Λ induced by the metric

$$\rho\left(\sum_{j=0}^{\infty} \theta_j t^j, \sum_{j=0}^{\infty} \theta'_j t^j\right) = \sum_{j=0}^{\infty} |\theta_j - \theta'_j| 2^{-j}$$

Definition. The kneading invariant V of f, denoted by V(f) is the formal power series $\theta(0^+)$.

The n'th lap number is defined as follows. Let $\ell_n - 1$ be the number of local maxima and minima of f^n within the interior of I. These points divide the interval into ℓ_n subintervals, each mapped homeomorphically by f^n. Milnor and Thurston proved that ℓ_n can be explicitly derived from the kneading invariant [4].

§2. Blowing up orbits.

Consider any convex function $f: I \to I$ such that $f(1) = f(-1)$. Each point $x \in (f(0), f(1)]$ has two inverse images, denoted a(x) and b(x) where $a(x) < b(x)$. Extend a and b to f(0) by letting $a(f(0)) = b(f(0)) = 0$. For $x \in I$, define G_x to be the semi-group consisting of all words of the form $\alpha f^n(x)$ where α is a word in the letters a and b, $n \geq 0$.

Call G_x the entire orbit of x. The set of points $\{f^n(x), n \geq 0\}$ is called the forward orbit of x, and the set of points $\{\alpha(x), \alpha$ a word in a and b$\}$ is called the backward orbit of x. If $p \in G_x$ let $\|P\|$ be the number of symbols in the word α plus n.

A new function g can be made from f, roughly speaking, by replacing the entire orbit of x by a set of disjoint intervals I_y and requiring $g(I_y) = I_{f(y)}$. The itinerary of any point remains unchanged. We call f the base function of g. The problem is to choose g to be as smooth as possible.

If the growth of the lap numbers is bounded by a polynomial then it is well known that the ω-limit set of $x \in I$ is a periodic point of period 2^n. In this case it is not too difficult to blow-up G_x and obtain a C^1 function g.

We examine the special case when the lap numbers grow faster than any polynomial and slower than any exponential.

Theorem 1. Let $f: I \to I$ be a convex function such that the lap numbers ℓ_n satisfy $\ell_{n+1}/\ell_n \to 1$. Then there exists a C^1 convex function g possessing an invariant set of disjoint intervals I_x with the critical point $c \in \text{Int } I_c$ such that $\nu f = \nu g$.

Proof. There are two possibilities corresponding to whether or not G_c is dense in I. When it is not, $I - \bar{G}_c$ contains a maximal open interval U. Since U does not contain c it is mapped homeomorphically onto its image. Also $f(U) \cap U = \emptyset$, otherwise a point $p \in G_c$ would be in the interior of one or the other which implies $U \cap G_c \neq \emptyset$. Hence the maximal connected intervals in the complement of $\bar{G}_c$ are mapped homeomorphically onto one another. We simultaneously crush these down to points and blow-up the orbit of c. If G_c is dense in I, our methods merely blow-up G_c.

Let $x \in G_c$. We construct disjoint intervals I_x with length

$$a_x = k/\|x\|^3 \ell^2_{\|x\|}$$

where k is a constant such that

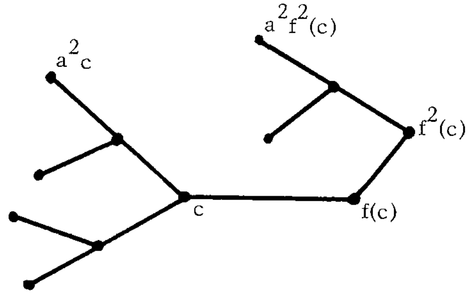
$$\text{(i)} \quad \sum_{x \in G_c} a_x = 1.$$

Such a k exists since

$$\sum_{x\in G_c} 1/\|x\|^3 \ell^2_{\|x\|} \leq \sum_{n=1}^{\infty} (\ell_1 + \ell_2 + \ldots + \ell_n)/n^3 \ell_n^2$$

$$< \sum_{n=1}^{\infty} n.\ell_n/n^3 \ell_n^2$$

$$< \sum_{n=1}^{\infty} 1/n^2 \quad .$$



Then

(ii) $\lim_{\|x\|\to\infty} a_{f(x)}/a_x$

$$= \lim_{\|x\|\to\infty} (\|x\|\pm 1)^3 \ell^2_{\|x\|\pm 1}/\|x\|^3 \ell^2_{\|x\|}$$

$$= \lim_{n\to\infty} (n\pm 1)^3 \ell^2_{n\pm 1}/n^3 \ell_n^2$$

$$= \lim_{n\to\infty} \ell^2_{n\pm 1}/\ell_n^2$$

$$= 1 \ .$$

Both of these conditions are useful in establishing the differentiability of our final function g. The first must hold in order that I remains compact, but, less obviously, it is useful to ensure that the invariant set of intervals have full measure in I. This will make it easier to calculate the derivative of g at a point not in $\cup I_x$ since lengths can be expressed as sums (possibly infinite) of the a_x.

The second condition (ii) is necessary for a canonical smoothing of g since, for some base functions f, arbitrarily close to an endpoint of I_x there will be both contracting and expanding intervals.

Define I_x by

$$I_x = [\sum_{\substack{y<x\\ y\in G_c}} a_y,\ 1 - \sum_{\substack{y>x\\ y\in G_c}} a_y] = [s_x, t_x].$$

Then $g_p = g|I_p$ is obtained by integrating g'_p which satisfies $\int_{I_x} g'_p = a_{f(x)}$ and $g'_p(x) = +1$ or -1 on the endpoints of I_p, $p \neq c$, depending on whether $p > c$ or $p < c$. For example if $p > c$, let

$$g'_p(x) = \begin{cases} 1 + R_p(s_p-x)(x-t_p)/a_p^2, & x \in I_p \\ 1 & , x \notin I_p \end{cases}$$

Then $\int_{s_p}^{t_p}[1 + R_p(s_p-x)(x-t_p)/a_p^2]dx = a_{f(p)}$. Hence $R_p = 6(a_{f(p)}/a_p - 1)$. Finally integrate $g'_p(x)$ and add a constant so that $g_p(s_p) = s_{f(p)}$.

Similarly construct g_p over I_p, $p < c$ by letting

$$g'_p(x) = \begin{cases} -1 + R_p(s_p-x)(x-t_p)/a_p^2, & x \in I_p \\ -1 & , x \notin I_p \end{cases} .$$

For $x = c$ let $g_c: I_c \to I_{f(c)}$ be any smooth function satisfying $g'_c(s_c) = -1$ and $g'_c(t_c) = +1$.

Define $g: I \to I$ by

$$g(x) = \begin{cases} g_p(x), & x \in I_p,\ p \in G_c \\ \lim g_p(s_p) & x = \lim s_p \end{cases} .$$

It follows from conditions (i) and (ii) above that g is C^1. For if $x \notin \cup I_p$ then

$$g'(x) = \lim_{y\to x} \frac{|g(x)-g(y)|}{|x-y|}$$

$$= \lim_{y\to x} \pm \sum_{x<p<y} a_{f(p)} \Big/ \sum_{x<p<y} a_p \quad \text{(the sign depending on whether } x \gtrless c.)$$

since $\cup I_p$ has full measure in I and $x \notin \cup I_p$. In general if b_k, $c_k \to 0$ are positive real numbers satisfying $b_k/c_k \to 1$ then

$$\lim \sum_{k=n}^{\infty} b_k \Big/ \sum_{k=n}^{\infty} c_k = 1 \quad .$$

Hence, for $x \notin \cup I_p$, $g'(x) = \begin{cases} 1 & x > c \\ -1 & x < c \end{cases}$. Thus g is continuously differentiable.

Finally observe that I_c is a wandering interval. □

§3. The Base Function.

Consider any continuous family of quadratics such as $f(x) = x^2 - a$.

Theorem 2. There exists a value A such that $f_A(x) = x^2 - A$ satisfies $\ell_{n+1}/\ell_n \to 1$ and $f_A^{2^k}(0)$ alternates on either side of 0, drawing monotonically and arbitrarily closer.

Proof. Let A be sup{a : critical point of f_a converges to a periodic orbit of period 2^n}. Milnor and Thurston discuss this in [4, §9.6] and prove that $\ell_{n+1}/\ell_n \to 1$. Using their notation, we verify the rest of the theorem.

Writing f for f_A, observe that the reciprocal zeta function

$$\hat{\zeta}(f,t)^{-1} = \prod_0^\infty (1 - t^{2^k}) = \Sigma(-1)^{\alpha(n)} t^n$$

where $\alpha(n)$ is the number of 1's in the diadic expansion of n. According to Milnor and Thurston, §9.6, in this case the kneading determinant

$$D(f,t) = \hat{\zeta}(f,t)^{-1} .$$

Hence

$$\theta_n(0^+) = (-1)^{\alpha(n)}$$

$$\Rightarrow \theta_n(f^p(0)) = \theta_{n+p}(0^+)/\theta_{p-1}(0^+) = (-1)^{\alpha(n-1)+\alpha(n+p)} .$$

Therefore if $k > 1$ $\theta_0(f^{2^{2k}}(0)) = (-1)^{\alpha(2^{2k}-1)+\alpha(2^{2k})} = (-1)^{2k+1} = -1$

and $\theta_0(f^{2^{2k+1}}(0)) = (-1)^{\alpha(2^{2k+1}-1)+\alpha(2^{2k+1})} = (-1)^{2k+2} = +1$.

Since $\theta_0(p) = \varepsilon_0$, the "address" of p,

$$f^{2^{2k}}(0) < 0 < f^{2^{2k+1}}(0) \quad .$$

We compare $\theta(f^{2^{2k}}(0))$ to $\theta(f^{2^{2(k+1)}}(0))$. Note that the first 2^{2k} terms are the same since for $n < 2^{2k}$,

$$\theta_n(f^{2^{2k}}(0)) = (-1)^{2k+2+\alpha(n)}$$

and

$$\theta^n(f^{2^{2(k+1)}}(0)) = (-1)^{2(k+1)+1+\alpha(n)} \quad .$$

But if $n = 2^{2k}$, $\quad \theta_n(f^{2^{2k}}(0)) = (-1)^{2k+1} = -1$

and $\quad \theta_n(f^{2^{2(k+1)}}(0)) = (-1)^{2(k+1)+2} = +1$.

Therefore $\quad f^{2^{2k}}(0) < f^{2^{2L}}(0)$, $L > k$. In a similar fashion, $f^{2^{2+1}}(0) > f^{2^{2L+1}}(0)$, $L > k$.

Finally, note that $\lim \theta(f^{2^{2k}}(0) = \theta(0^-)$. Hence $\theta(\lim(f^{2^{2k}}(0))) = \theta(0^-)$ since $\lim(f^{2^{2k}}(0)$ exists. In fact, $\lim(f^{2^{2k}}(0)) = 0$ as in this example 0 is the unique point p such that $\theta^-(p^{\mp}) = \theta(0^-)$. Similarly $\lim \theta(f^{2^{2(k+1)}}(0)) = \theta(0^+)$ so that $\lim f^{2^{2(k+1)}}(0) = 0$.□

§4. Unsmoothability of g.

With the results of Denjoy on the circle in mind, one might naturally suspect that a C^2 map f of I with growth rate faster than polynomial has no subinterval J such that $f^k|J$ is a homeomorphism for all $k \geq 0$. (See also the work of Guckenheimer and Misieurwicz for discussion of the case when f has negative Schwarzian derivative.) We apply Theorem 1 to f_A, creating g, and show that there is no C^2 convex map conjugate to g.

If $x, y \in I$ we denote the open interval with endpoints x and y by (x, y) regardless of the order.

Lemma 3. Let $f(x) = x^2 - A$. There exists an increasing sequence of positive integers n_i and points $c_{-1}, \ldots, c_{-n_i}$ such that $f(c_{-k}) = c_{-k+1}$, $c_0 = 0$, and the intervals (c_k, c_{-n+k}) are disjoint where $c_k = f^k(0)$.

Proof. Let $i \in z$. Consider all points x in both the forward and backward orbits of c_0 such that $\|x\| \leq 2^i$. Suppose x_i is a member of this set with minimum distance to c_0. Denote it by c_{n_i} if $f^{n_i}(c_0) = x_i$ and by c_{-n_i} if $f^{n_i}(x_i) = c_0$.

In the former case, we choose c_{-n_i} inductively. It helps to observe that if $c_1 \notin (x, y)$ and $f(x') = x$ is given, then there exists y' such that $f(y') = y$ and $f(x', y') \cong (x, y)$. Thus there exists c_{-1} such that $f(c_{-1}, c_{n_i-1}) \cong (c_0, c_{n_i}) = I_+$ since $c_1 \notin I_+$. Furthermore $c_1 \notin (c_{-1}, c_{n_i-1})$ otherwise $c_2 \in I_+$. Hence we can choose c_{-2} such that $f(c_{-2}, c_{n_i-2}) \cong (c_{-1}, c_{n_i-1})$ and inductively c_{-p} such that $f(c_{-p}, c_0) \cong (c_{-p+1}, c_1)$ $p \leq n_i$ since $c_p \notin I_+$.

In the latter case f maps $I_- = (c_0, c_{-n_i})$ diffeomorphically onto (c_1, c_{-n_i+1}) since $c_0 \notin I_-$. Inductively, $c_0 \notin f^n(I_-)$ otherwise $c_{-1} \in f^{n-1}(I_-) \Rightarrow c_{-n} \in I_-$. Hence $f(c_p, c_{-n_i+p}) \cong (c_{p+1}, c_{-n_i+p+1})$.

It is easy to check that in both cases, these intervals are disjoint by considering all three possible intersections : If $x_i = c_{-n}$,

(i) $c_p \in (c_m, c_{m-n})$ and $c_{m-n} \in (c_p, c_{p-n}) \Rightarrow c_{p-m}$ or c_{p-m} or $c_{m-p-n} \in I_-$

(ii) $c_{p-n} \in (c_m, c_{m-n})$ and $c_{m-n} \in (c_p, c_{m-n}) \Rightarrow c_{p-m-n}$ or $c_{m-p-n} \in I_-$

or

(iii) $c_p \in (c_m, c_{m-n})$ and $c_{p-n} \in (c_m, c_{m-n}) \Rightarrow c_{p-m}$ or $c_{p-m-n} \in I_-$.

None of these are possible since $|p-m| < n$. A similar argument holds if $x_i = c_n$. Note that Theorem 2 implies the n_i are unbounded so choose an increasing subsequence. □

We are now in a position to apply Denjoy analysis in which the following straightforward lemma is crucial.

Lemma 4. If f' has bounded variation and $|1/f'(x)|$ is bounded, then log f' has bounded variation.

If there is a critical point then the Denjoy analysis might not apply. However, the critical point of our blown-up function g is contained in the interior of a wandering interval I_0. This turns out to be strong enough to prove :

Theorem 5. The blown-up function g is not C^0 conjugate to any convex C^2 function G.

Proof. We construct a restricted function avoiding the critical point c . Let $U_0 \subset I_0$ be a small open interval containing c, and J_0 be a connected component of $I_0 - U_0$. Let J = I - (the entire orbit of U_0) and $F:J \to J$ be defined by F(x) = G(x). Let $J_n = f^n(J_0)$ and denote by J_{-n} the n'th inverse image of J_0 under consideration. Let a_n denote the length of J_n.

Consider the sequence $n_i \to \infty$ in Lemma 3. Then for each i there exists $x_0 \in J_0$ such that

$$\frac{a_{n_i}}{a_0} = (F^{n_i})'(x_0) = G'(x_{n_i-1})G'(x_{n_i-2}) - G'(x_0).$$

There also exists $\hat{x}_{-n_i} \in J_{-n_i} \subset I_{-n_i}$ (where I_{-n_i} is the blown-up interval corresponding to c_{-n_i} which, in turn, is defined in Lemma 3) such that

$$\frac{a_0}{a_{-n_i}} = (F^{n_i})'(\hat{x}_{-n_i}) = G'(\hat{x}_{-1})G'(\hat{x}_{-2}) \dots G'(\hat{x}_{-n_i})$$

$$\text{Hence} \quad \log \frac{a_0^2}{a_{n_i} a_{-n_i}} = \sum_{k=1}^{n_i} |\log G'(\hat{x}_{-k}) - \log G'(x_{n_i-k})|$$

$$\leq \sum_{k=1}^{\infty} |\log G'(\hat{x}_{-k}) - \log G'(x_{n_i-k})|$$

$$< \text{constant} \quad .$$

The last inequality holds since $\hat{x}_{-k}$ and x_{n_i-k}, $k = 1,\dots,n_i$, are pairs of points in disjoint intervals (according to Lemma 3), all derivatives are bounded away from 0 and G' has bounded variation.

Therefore
$$\frac{a_0^2}{a_{n_i} a_{-n_i}} < e^k$$
$$\Rightarrow a_{n_i} . a_{-n_i} > \text{constant}$$
$$\Rightarrow \sum_{i=-\infty}^{+\infty} a_{n_i} = \infty .$$
□

References.

1. E. Coven & Z. Nitecki, Nonwandering sets of the powers of maps of the interval, preprint.

2. A. Denjoy, Sur les courbes definies par les équations différentièlles à la surface du tore, J. Math. Pures Appl. [9], 11, 333-375, 1932.

3. H. Friedman, 102 problems in Math'l logic, J. of Symbolic Logic, vol. 40, No. 2, 1975, p. 113.

4. J. Milnor & W. Thurston, On iterated maps of the interval I and II, preprint, Princeton 1977.

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