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STOKES' THEOREM FOR NONSMOOTH CHAINS

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ABSTRACT. Much of the vast literature on the integral during the last two centuries concerns extending the class of integrable functions. In contrast, our viewpoint is akin to that taken by Hassler Whitney [*Geometric integration theory*, Princeton Univ. Press, Princeton, NJ, 1957] and by geometric measure theorists because we extend the class of integrable *domains*. Let ω be an n-form defined on $\mathbb{R}^m$. We show that if ω is sufficiently smooth, it may be integrated over sufficiently controlled, but nonsmooth, domains γ. The smoother is ω, the rougher may be γ. Allowable domains include a large class of nonsmooth chains and topological n-manifolds immersed in $\mathbb{R}^m$. We show that our integral extends the Lebesgue integral and satisfies a generalized Stokes' theorem.

1. INTRODUCTION

The standard version of Stokes' theorem:

$$\int_{\partial M} \omega = \int_M d\omega$$

requires both a smooth n-manifold M and a smooth $(n-1)$-form ω. It was realized at some point that one side of Stokes' formula could be used to *define* the other side in more general situations. In particular Whitney [14] used the right side to define the left in some cases where the interior of M is smooth, even though the boundary ∂M is not smooth. More generally the interior could be piecewise smooth or a suitable limit of simplicial chains. Whitney systematically developed this insight by defining his *flat norm* on chains, based on rectilinear subdivisions of simplices of one higher dimension of which the chain in question is a partial boundary. He treated forms as cochains—linear functions on the vector space of chains.

Stokes' theorem was thus extended by Whitney to integration of smooth forms over objects that were limits in the flat norm of chains. These include certain kinds of fractals. But other fractals escape Whitney's construction. Whitney's example of a function nonconstant on a connected arc of critical points [15] shows the limits to any generalization of Stokes' theorem. Stokes' theorem (which for arcs is just the fundamental theorem of calculus) must fail for such arcs. There exists a Jordan curve in 3-space that has Hausdorff dimension > 2, is not contained in any surface of finite area, and is not a limit in the flat norm of simplicial 1-chains. Whitney's methods to not define integration of forms over such a curve, nor do Lebesgue's. The methods for this paper accomplish exactly this.

In a recursive construction we define a family of norms: For each real λ with $n \leq \lambda \leq m$ the *λ-natural norm* $|A|_\lambda^\natural$ is defined on simplicial n-chains

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A in m-space. The definition utilizes projections of chains on hyperplanes, Whitney's method of subdivisions and partial boundaries, and λth powers of the n-masses of the projected chains. The recursion is on the integer part of $\lambda - n$.

The following theorem implies that this is indeed a norm on oriented simplicial chains in $\mathbb{R}^m$ (which is not obvious). Completion yields Banach spaces $X_{n,\lambda} = X_{n,\lambda}(\mathbb{R}^m)$. These spaces are nested, containing increasingly complex chains as λ increases. The parameter λ acts as a dimension; it bounds the fractal complexity of elements of $X_{n,\lambda}$. The boundary operator ∂ on simplices extends to a continuous operator $\partial : X_{n,\lambda} \to X_{n-1,\lambda}$.

Theorem 1. *Integration of an n-form ω defines a bounded linear functional L_ω on $X_{n,\lambda}$ if ω is of class $C^{\lambda-n}$. Furthermore, there is a constant $c > 0$ such that for a simplicial n-chain A:*

$$\left|\int_A \omega\right| \leq c|A|^\natural_\lambda \|\omega\|_{\lambda-n}$$

where $\|\omega\|_{\lambda-n}$ denotes the $\lambda - n$ norm on forms.

This theorem extends results in [14] and [7].

Denote the dual space of cochains by $X^{n,\lambda}$ and the product of a cochain $X \in X^{n,\lambda}$ with a chain $A \in X_{n,\lambda}$ by $X \cdot A$. Define $\Omega^{n,\lambda} = \{n\text{-forms } \omega$ of class $C^{\lambda-n}$ defined on $\mathbb{R}^m\}$. According to Theorem 1 the linear operator $A \mapsto \int_A \omega$ on simplicial n-chains in $\mathbb{R}^m$ is bounded and determines an element $L_\omega \in X^{n,\lambda}$. This results in a linear mapping $L : \Omega^{n,\lambda} \to X^{n,\lambda}$. For $A \in X_{n,\lambda}$ we define $\int_A \omega = L_\omega \cdot A$. Special cases: If A is a simplicial n-chain (finite) in $\mathbb{R}^m$, then $\int_A \omega$ is identical to the Lebesgue integral for simplicial n-chains. If A is an n-chain (possibly infinite) in $\mathbb{R}^n$, then $\int_A \omega$ is identical to the Lebesgue integral for infinite n-chains with finite n-mass. Since $L_{d\omega} = dL_\omega$, we obtain the following:

Theorem 2 (Stokes' theorem for nonsmooth chains). *If $A \in X_{n,\lambda}$ and $\omega \in C^{\lambda-n+1}$ is an $(n-1)$-form defined on $\mathbb{R}^m$, then $\int_A d\omega = \int_{\partial A} \omega$.*

Historical comments. In 1982 the author found a $C^{2+\alpha}$ counterexample to the Seifert Conjecture [2, 3]. At the heart of her construction is a diffeomorphism of the two-sphere with a fractal equator γ as an invariant set. The Hausdorff dimension of γ is precisely $1+\alpha$. Coarser relations between the differentiability class of a function and the topological dimension of a related set were well known in the theories of Sard [13], Denjoy [1], and Whitney [15]. Subsequently, these too were found to have fractal versions. (See [6, 11].) In each of these theories appears the same phenomenon—the smoother the function, the rougher an associated set. This paper is a result of the author's search for a general principle underlying this duality between dimension and differentiability.

Fractals are rife in many fields: geometric measure theory, dynamical systems, PDEs, function theory, foliations.... Except for various dimension theories, however, there are few available techniques for investigating them. Usually there is a large measure of geometric control over their formation that can be proved or reasonably postulated. In at least some interesting cases this ought to allow integration of sufficiently smooth forms over the fractals. For example, it would be very useful, and perhaps not too difficult, to show that certain solenoid-like invariant sets for flows are domains for integration of 1-forms and that certain closed unions of leaves of a foliation by k-manifolds are domains for integration of k-forms. Such sets could then be proved fairly directly to carry homology cases. Currently such proofs (due in various contexts to S. Schwartzmann, J. Plante, D. Sullivan, and others) are indirect and hard to come by. It may well be that these new discoveries in geometric integration theory will provide simple but powerful new tools to sharpen differentiability results and reduce technicalities in known proofs.

2. Norms on chains

We take Whitney's approach in [14] with simplicial n-chains. It provides an algebraic way of equating chains with common simplicial subdivisions. An *n-simplex* $\sigma = p_0 \cdots p_n$ in $\mathbb{R}^n$ is the convex hull of $n+1$ vertices p_i in $\mathbb{R}^n$, $i = 0, \ldots, n$. If the n vectors $\{p_0 - p_i\}$ are linearly dependent, then σ is *degenerate.* The order of the vertices p_i of a nondegenerate simplex σ determines an orientation on σ. The simplex $-\sigma$ is identified with the simplex with the same pointset as σ but the opposite orientation. Define a *simplicial n-chain A in $\mathbb{R}^n$* as an equivalence class of formal sums $\sum a_i\sigma_i$ with real coefficients and oriented simplices in $\mathbb{R}^n$ as follows: A formal sum $\sum a_i\sigma_i$ defines a function $A\colon \mathbb{R}^n \to \mathbb{R}$ by $A(x) = \sum \pm a_i$ for $x \in \operatorname{int}(\sigma_i)$ where $+a_i$ is used if σ_i is oriented in the same way as $\mathbb{R}^n$ and $-a_i$ is used otherwise. Set $A(x) = 0$ if x does not lie in the interior of any simplex σ_i. Two formal sums of oriented simplices are equivalent if the functions defined by them are identical except in sets with n-dimensional Lebesgue measure zero. (Degenerate simplices can be used but are equated with the zero n-chain.) The definition of a simplicial n-chain A in $\mathbb{R}^n$ is independent of the orientation chosen for $\mathbb{R}^n$.

More generally we can define a *simplicial n-chain in* $\mathbb{R}^m$ when $m \geq n$. For each affine n-plane P in $\mathbb{R}^m$ let $C_n(P)$ denote the linear space of simplicial n-chains in P. A *simplicial n-chain A in $\mathbb{R}^m$* is an element of the direct sum of the $C_n(P)$. Let A and B be two simplicial n-chains in $\mathbb{R}^m$. If the corresponding summands of A and B are equivalent, we write $A \sim B$.

Let $M_n(\sigma)$ denote the n-dimensional *mass* or volume of an n-simplex σ. Let A be a simplicial n-chain in $\mathbb{R}^m$ and π the orthogonal projection onto an affine subspace of $\mathbb{R}^m$. For $0 \leq \lambda \leq n$ define the *projected λ-mass* of A to be

$$M_{\lambda,\pi}(A) = \inf\left\{\sum |a_i|(M_n(\pi\sigma_i))^{\lambda/n} : \sum a_i\sigma_i \sim A\right\}.$$

If π is the identity, we write $M_\lambda(A) = M_{\lambda,\pi}(A)$ for simplicity of notation. Then $M_n(A)$ is the n-mass of A. It is a norm on simplicial n-chains in $\mathbb{R}^m$.

Integration of smooth integrands over smooth domains. The mass norm M_n is often used to estimate the integral of a continuous form over a domain with finite mass. For example, let A be a simplicial n-chain in $\mathbb{R}^m$ and ω a continuous n-form. Clearly,

$$\left|\int_A \omega\right| \leq M_n(A)\|\omega\|_0.$$

Whitney used a smaller norm on chains to find better estimates for simplicial n-chains with large mass.

Whitney's flat norm. Let A be a simplicial n-chain in $\mathbb{R}^m$. Whitney defined the *flat* norm of A as

$$|A|^\flat = \inf\{M_n(A - \partial C) + M_{n+1}(C)\},$$

where the infimum is taken over all simplicial $(n+1)$-chains C. For example, if A is a Jordan curve in $\mathbb{R}^2$, $|A|^\flat$ is bounded by the area of the Jordan domain spanning A. If A is an arc in $\mathbb{R}^2$, consider any simplicial 1-chain B so that $A + B$ is a Jordan curve. $|A|^\flat$ is bounded by the sum of the arc length of B plus the area of the Jordan domain spanning $A + B$.

The flat norm is used to complete the space of simplicial n-chains in $\mathbb{R}^m$. Denote the resulting Banach space of *flat n-chains* by X_n and the dual space of *flat n-cochains* by X^n. Whitney showed that the boundary operator ∂ defined on simplicial n-chains is continuous with respect to the flat norm and extends to $\partial\colon X_n \to X_{n-1}$.

Let A be a simplicial n-chain and C a simplicial $(n+1)$-chain in $\mathbb{R}^m$. Let ω be an n-form of class C^1 defined on $\mathbb{R}^m$. Whitney used Stokes' theorem for simplicial n-chains to show

$$(*) \qquad \left|\int_A \omega\right| \le \left|\int_{A-\partial C} \omega\right| + \left|\int_C d\omega\right| \le M_n(A - \partial C)\|\omega\|_0 + M_{n+1}(C)\|d\omega\|_0 .$$

Therefore, $|\int_A \omega| \le |A|^\flat \|\omega\|_1$. This estimate enabled Whitney to integrate over elements of X_n. Whitney believed he could generalize Stokes' theorem and wrote at the end of his first paper of geometric integration theory [16], "With the help of methods described above a very general form of Stokes' theorem may be proved. We shall not give details here." However, it turned out that his integral does not extend to the Lebesgue integral. Theorem 1 generalizes Whitney's theorem and leads to strict generalizations of the Lebesgue integral and Stokes' theorem.

The natural norms [5]. Let γ be a Jordan curve in $\mathbb{R}^3$ that has Hausdorff dimension greater than two. This curve not only has infinite arc length but has no spanning surface with finite area. Thus a sufficiently close simplicial 1-chain A will only bound 2-chains with huge area. Neither the arc length norm $M_1(A)$ nor the flat norm $|A|^\flat$ will give good estimates for integrals over A. The natural norms defined below give sharper estimates.

Since $\partial\partial = 0$, only simplicial n-chains without boundary have spanning $(n+1)$-chains. Yet all simplicial $(n+1)$-chains C can be viewed as "partial" spanning sets for an arbitrary simplicial n-chain A. The natural norms take into account the $(n+1)$-mass of these partial spanning sets C and the projected n-mass of what is left over, namely, $A - \partial C$. (See Figure 1 on page 240.)

List the n-dimensional coordinate planes of $\mathbb{R}^m$ from 1 to $N = \binom{m}{n}$. Let π_i be the projection onto the ith coordinate plane in this list.

Let A be an n-chain in $\mathbb{R}^m$. If $0 \le \lambda \le n$, define the *λ-natural norm* of A by

$$|A|^\natural_\lambda = \sum_{i=1}^{N} M_{\lambda,\pi_i}(A).$$

Then $|A|^\natural_n$ is proportional to the n-mass $M_n(A)$.

For $n < \lambda \leq m$, $\lambda \in \mathbb{R}$, define the λ-natural norm of A by recursion on the integer part of $\lambda - n$:

$$|A|_{\lambda}^{\natural} = \sum_{i=1}^{N} \inf_{C}\{M_{n,\pi_i}(A - \partial C) + |C|_{\lambda}^{\natural}\}.$$

Each term is infimized over all simplicial $(n+1)$-chains C in $\mathbb{R}^m$. The flat norm corresponds to $\lambda = n+1$. It is interesting to observe that λ may be an integer much larger than $n+1$. This allows us to work with domains that are curves in $\mathbb{R}^3$, for example. The λ-natural norm depends on the parameter λ, the topological dimension n of A, and the minimal ambient space dimension m of A. However, if A is an n-chain in $\mathbb{R}^p$ and in $\mathbb{R}^{p'}$, the norms defined are identical. For simplicity of notation the dependency on n and m are suppressed. The definition is independent of choice of coordinates. That is, under a $C^{\lambda-n+1}$ change of coordinates with bounded derivative, the λ-natural norm changes to an equivalent norm.

As does Whitney, we complete and obtain a Banach space $X_{n,\lambda}$ of *λ-natural n-chains* and its dual space $X^{n,\lambda}$ of *λ-natural n-cochains.*

Remark. There is a one-parameter family of homology and cohomology groups associated with the λ-natural norm, $H_{n,\lambda}$ and $H^{n,\lambda}$.

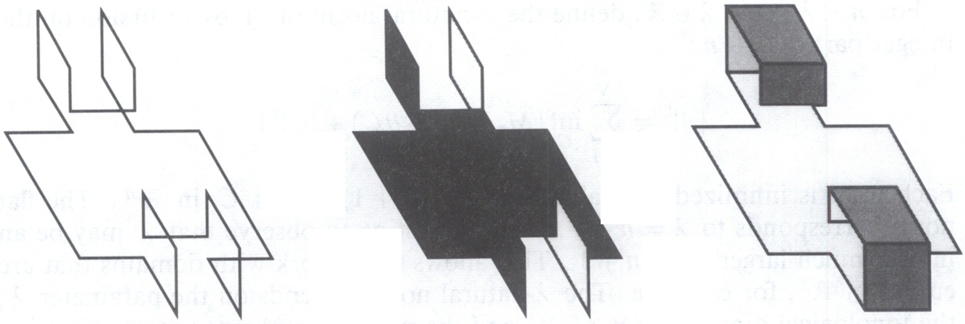
Now let ω be a 1-form defined on $\mathbb{R}^m$ of class C^1 and $A \in X_{1,\lambda}$. We have seen that the integral of ω over A can be estimated using either arc length or the flat norm, both of which may be very large. Theorem 1 shows that if ω is of class C^2, the integral can also be estimated using the 3-natural norm which may be much smaller.

$$(**) \qquad \left|\int_A \omega\right| \leq |A|_3^{\natural}\|\omega\|_z$$

To prove Theorem 1, we iterate a modified version of the Stokes' argument as seen in $(*)$ (see [5] for details; also see the example below). The reader should take note that Stokes' theorem, as such, may be applied only once since $dd\omega = 0$ and $\partial\partial A = 0$. However, the components of $d\omega$ may not be exact, and we may apply the exterior derivative to them. This means we may continue to use the exterior derivative and iterate the "Stokes' process" in a nontrivial fashion for as many times as ω is differentiable. We also use partial spanning sets of spanning sets.

Typically, an integral $\int_A \omega$ is treated with two basic methods: If $A = \partial B$ and ω is smooth, one may "go forward" and integrate $d\omega$ over B. Alternatively, if the form $\omega = d\nu$, one may "go backward" and integrate ν over ∂A. Sometimes neither of the equivalent new integrals is easier to calculate. Our methods allow one to go forward many times under suitable conditions, taking the exterior derivatives of forms in the hopes of finding an equivalent integral that may be calculated. It is worth mentioning that it is also possible, under suitable conditions, to go backward, taking antiderivatives of forms many times, and find an equivalent integral that may be calculated.

Example. There exists a self-similar Jordan curve γ in the unit cube with Hausdorff dimension > 2 so that each projection $\pi_i(\gamma)$, $i = 1, 2, 3$, onto the ith two-dimensional coordinate plane is an immersed curve bounding an infinite



α σ τ

FIGURE 1

2-chain with finite 2-mass < 1. (The curve γ is constructed by adding homothetic replicas of the curve α to itself in Figure 1 so that γ is embedded and has Hausdorff dimension $\sim \log 11/\log 3 > 2$.) A simplicial 1-chain A can be found approximating γ along with simplicial 2-chains C_i spanning A, $i = 1, 2, 3$, such that $M_{2,\pi_i}(C_i) < 1$. (Each surface C_i is a sum of homothetic replicas of the surfaces σ and τ in Figure 1, chosen to minimize the projected 2-mass of C_i.) Furthermore, the simplicial 3-chains D_{ij} spanning $C_i + C_j$ have $M_3(D_{ij}) < 1$. Even though the norms $M_1(A)$ and $|A|^\flat$ are both large, the 3-natural norm $|A|_3^\natural$ is not, for

$$|C_i^\natural|_3 \le \sum_{j=1}^{3}(M_{2,\pi_j}(C_j) + |D_{ij}|_3^\natural) < 12 \quad \text{and thus} \quad |A|_3^\natural \le \sum_{i=1}^{3}|C_i|_3^\natural < 36.$$

This curve γ cannot be a domain for the classical Lebesgue integral and cannot be treated using Whitney's methods. We show in §3 how to choose A_k approximating γ and define $\int_\gamma \omega = \lim \int_{A_k} \omega$.

We verify that Theorem 1 (see (**)) is valid for this example. It suffices to estimate $|\int_A \omega_i| = |\int_{C_i} d\omega_i|$ where ω_i is a component of ω. Note that $C_i + C_j = \partial D_{ij}$. Thus

$$\left|\int_{C_i} d\omega_i\right| \le \sum_{j=1}^{3}\left|\int_{C_j}(d\omega_i)_j\right| + \left|\int_{D_{ij}} d(d\omega_i)_j\right| \le |A|_3^\natural \|\omega\|_2.$$

Relation to the Lebesgue integral. Our integral of smooth forms contains as a special case the Lebesgue integral: First associate an L^1 function $f: \mathbb{R}^1 \to \mathbb{R}^1$ with an element of $X_{1,2}$ as follows. Assume for simplicity that f is the characteristic function of a bounded, measurable set E. Let S denote the region under the graph Γ of f and above the x-axis, oriented positively. Let m_n denote n-dimensional Lebesgue measure. Since $f \in L^1$, $m_2(S) < \infty$. For each k there exists a union P_k of disjoint intervals such that $E \subset P_k$ and $m_1(P_k) < \infty$. Furthermore, $\bigcap P_k = E$ except for a zero set. Subdivide each of the rectangles of $P_k \times I$ into finitely many 2-simplices and orient each positively. Denote the formal sum of these oriented 2-simplices by Q_k. Since $m_2(P_k \times I) < \infty$, each Q_k is an element of $X_{2,2}$. Since $m_2(S) < \infty$, the sequence $\{Q_k\}$ is Cauchy and thus converges to $A_f \in X_{2,2}$. It can be shown that $\partial A_f \in X_{1,2}$ is canonically associated with f. Now it is well known (see Saks [12]) that the Lebesgue integrals $\int_{\mathbb{R}} f$ and $\int_S dx\,dy$ are equal. Since S is canonically represented by $A_f \in X_{2,2}$, the Lebesgue integral $\int_S dx\,dy$ equals our integral $\int_{A_f} dx\,dy$. Since $y dx$ is analytic and $\partial A_f \in X_{1,2}$, Theorem 2 implies the Lebesgue integral $\int_{\mathbb{R}} f$ equals our integral $\int_\Gamma y\,dx$. (This example can be treated with the flat norm alone.)

3. Domains of integration

Which oriented n-dimensional topological submanifolds $M \subset \mathbb{R}^m$ with boundary can serve as domains for integrals of smooth forms ω of class $C^{\lambda-n}$, $n \le \lambda \le m$? The idea is to choose appropriate simplicial approximators A_k for M that converge in $X_{n,\lambda}$ for λ sufficiently large and define $\int_M \omega = \lim \int_{A_k} \omega$. (See [4] for some numerical methods.) But choosing A_k takes care. For example, let M_1 be a Jordan curve in $\mathbb{R}^2$ with positive Lebesgue area. Any Cauchy sequence of simplicial 1-chains inside M_1 will have different limit point in $X_{1,2}$ from a Cauchy sequence of 1-chains outside M_1. The area of M_1 itself contributes an unavoidable error.

Consider a compact oriented n-manifold $M \subset \mathbb{R}^m$ with boundary. There is an algorithm for constructing a particular sequence of simplicial n-chains in the nerves of coverings of M by boxes in cubic lattices, the inverse limit of these chains representing the fundamental Čech homology class of $(M, \partial M)$. If this sequence of *binary approximators* converges to some $\psi(M) \in X_{n,\lambda}(\mathbb{R}^m)$, then M is a *(λ, n)-set.* The algorithm commutes with the boundary operator on chains. Examples of (λ, n)-sets include some planar Jordan curves with positive Lebesgue area and the above example of a Jordan curve in 3-space with Hausdorff dimension > 2. If M is a (λ, n)-set and ω is an n-form of class $C^{\lambda-n}$, we can define $\int_M \omega = \int_{\psi(M)} \omega$ and apply Theorem 2 to conclude:

Theorem 3 (Stokes' theorem for (λ, n)-sets). *If M is a (λ, n)-set and $\omega \in C^{\lambda-n+1}$, then*

$$\int_{\partial M} \omega = \int_M d\omega .$$

Theorem 3 extends the classic Green's theorem for codimension one boundaries γ and C^1 forms ω. Hölder versions of the classic Green's theorem relating the box dimension of γ to the Hölder exponent of ω are proved in [7, 8, 10].

There are many examples of (λ, n)-sets:

(i) There exist Jordan curves in $\mathbb{R}^n$ with Hausdorff dimension d that are $(d, 1)$-sets for every $1 \le d \le n$.

(ii) Every hypersurface immersed in $\mathbb{R}^m$ with Hausdorff dimension $< m$ is an $(m, m-1)$-set.

(iii) Compact oriented n-manifolds immersed in $\mathbb{R}^m$ that are locally Lipschitz graphs are (n, n)-sets.

(iv) Compact oriented 1-manifolds immersed in $\mathbb{R}^2$ that are locally graphs of Hölder functions with exponent α are $(2/(1+\alpha), 1)$-sets.

Remark. Compactness in examples (ii), (iii), and (iv) can be relaxed. An arc in $\mathbb{R}^2$ that spirals to the origin passing through the x-axis at $x_n = 1/\sqrt{n}$, $n \ge 1$, is *not* a $(\lambda, 1)$-set for any $1 \le \lambda \le 2$. Other domains of integration include boundaries of open sets such as the topologist's since circle and the Denjoy Cantor set.

Integration on manifolds. We pass to smooth manifolds by way of local coordinates, since the λ-natural norm changes to an equivalent norm under a $(\lambda - n + 1)$-smooth change of coordinates with bounded derivative. One must also extend the definition of these norms to chains in proper open subsets of $\mathbb{R}^m$ as Whitney did for his flat norm. The integral can then be defined for smooth forms defined on smooth compact m-manifolds M over domains that are "rough" subsets of M.

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