RESEARCH SUMMARY

JENNY HARRISON

This summary of my research is in two parts. The first regards differential chains and their applications to pure and applied mathematics. The second part is a summary of my work in the calculus of variations.

1. Differential Chains

1.1. **Approximations by simplicial chains.** My papers during 1993-2005 used simplicial chains to approximate all sorts of domains of integration [Har93, Har94, Har97, Har98a, Har98b, Har99b, Har05a, Har05b, Har06]. They established an isomorphism between cochains and differential forms [Har98b] and provided continuous operators [Har98a] that yield concise generalizations of the classical integral theorems of calculus [Har93,Har99b,Har06], applying to both smooth and rough domains. There have been a number of attempts to do this, but an important operator, product, or relation is usually missing.

Whitney [Whi57] first thought his "sharp norm" on simplicial chains had a continuous boundary operator, but he withdrew that claim and introduced his "flat norm" to replace the sharp norm. The flat norm, by definition, has a continuous boundary operator, but no continuous Hodge star operator. Jim Eells was Whitney's student and told me, "Whitney spent the better part of twenty years trying to reconcile his two norms." His book [Whi57] was a disjoint amalgamation of his sharp and flat theories. In his thesis, Eells himself [Eel55] tried to work around these problems, but his approach also failed to have a continuous boundary operator. I built upon and modified Whitney's sharp norm to obtain both boundary and Hodge star continuous operators. The Russian analyst Boris Aleksandrovich Kats [Kat14] went so far as to say, "The idea of the approximation by polygonal constructions was essentially developed by Harrison [Har05a]." The approximations Kats refers to are within a topological vector space in which smooth and rough domains are represented as elements.

As further continuous operators were discovered and point masses were represented as limits of simplicial chains, the work evolved into a theory of generalized functions [Har06, Har11a, HP12, Har14a] which we now call "differential chains" (see below for further discussion.) Generalized functions of various types had been developed by Schwartz, de Rham, Sobolev, Sato, Colombeau, and others to make rigorous the celebrated "Dirac deltas" of Paul Dirac. A natural question was whether differential chains offer any advantages over prior theories, but that has now been answered by wide-ranging applications, e.g., [Seg03, Bos05, Mar05, Lyo06, Bos06, EF14, FS14, Kat14, Pug16, Sal16].

The space of differential k-chains is defined as the inductive limit (also called the direct limit) of a sequence of Banach spaces of simplicial k-chains and is endowed with a particular strong topology so that one can take limits and find new, non-pathological, and naturally arising models[1] amenable to classical methods of analysis. The topological dual of the space of differential k-chains is the space of differential k-forms each with uniform bounds on its derivatives. The Banach spaces are defined without reference to differential forms and we conclude that differential forms pair with differential chains to provide a non-trivial integral [Har98a, Har98b].

If an operator on differential chains is continuous and dualizes to an analytically defined classical operator on differential forms, then the resulting integral relation is also non-trivial. An example is a general form of Stokes' theorem [Har93]. This arises from a well-defined continuous boundary operator ∂ on differential chains whose topological dual is the exterior derivative d of forms ω. Thus the relation

$$d\omega = \omega\partial \tag{1}$$

is a proper theorem and not a "weak definition" as it is for currents.

The words of André Weil in his autobiography [Wei92] tell us some important history of the search for a general version of Stokes' theorem:

> *In his book on invariant integrals, Elie Cartan, following Poincaré in emphasizing the importance of this [Stokes'] formula, proposed to extend its domain of validity. Mathematically speaking, the question was of a depth that far exceeded what we [Henri Cartan and André Weil] were in a position to suspect…"Why don't we [with several friends] get together and settle such matters once and for all." Little did I know that at that moment Bourbaki was born.*

Another example is the geometric Hodge star operator $\perp$ which is a continuous operator on differential chains. Its topological dual is the classical Hodge star operator $\star$ on differential forms yielding the theorem

$$\star\,\omega = \omega\perp. \tag{2}$$

Combining these two results yields

$$d\star\omega = \omega\partial\perp \quad\text{and}\quad \star\, d\omega = \omega\perp\partial$$

which are concise and general versions of the Gauss-Green divergence theorem and the Kelvin-Stokes' curl theorem, respectively [Har99a, Har06]. These results apply equally well to classical smooth domains and rough domains with no tangent spaces defined anywhere. For example, the oriented graph of the Weierstrass nowhere differentiable function supports a differential chain. Even though this set is not rectifiable and has no measure theoretic tangent or normal vectors, one can still calculate net flux across a boundary which is a union of such curves.

The concise simplicity of these statements should not diminish their importance. Compare $d\star\omega = \omega\partial\perp$ [Har06] with the generalization of the Gauss-Green theorem due to Federer and De Giorgi

[1] Differential chains were originally called "chainlets." Both spaces are inductive limits of the same Banach spaces, but use different topologies on the inductive limit. Differential chains have better properties than chainlets if applications are sought on the full inductive limit. However, all applications to date have only required several of the initial Banach spaces in which chainlets, differential chains and their operators are identical. Spectral analysis requires the full inductive limit for some operators such as boundary.

[Fed69]. Their proof requires a good fraction of [Fed69]. Their result is not optimal, because their domains are assumed to have boundaries with measure theoretic exterior normals. Federer wrote

> *A striking application of our theory is the Gauss-Green type theorem discovered by E. De Giorgi and the author, which makes use of a measure theoretic concept of exterior normal.* (p. 4 [Fed69])

> *We present an elementary version of the Gauss-Green theorem in 4.1.31, and perhaps optimal results in 4.5.6,4.5.11. Research on the problem of finding the most natural and general form of this theorem has contributed greatly to the development of geometric measure theory.* (p. 343 [Fed69])

and

> *Using these concepts we prove an essentially optimal version of the Gauss-Green theorem in 4.5.6.* (p. 484 [Fed69])

1.2. **Approximations by monopole chains.** There are several dense subspaces of differential k-chains which provide methods of approximation with different flavors. In 2004, I began to transition from simplicial k-chains to "monopole k-chains." The latter are elements of the free space of the k^{th} power of the exterior algebra over $\mathbb{R}^n$ denoted by $\Lambda_k(\mathbb{R}^n)\langle\mathbb{R}^n\rangle$. Both simplicial k-chains and monopole k-chains are "discrete" since each is determined by a finite amount of data, but a monopole k-chain is also "infinitesimal" since its support is finite. That both subspaces are dense in the space of differential chains leads to a unification of discrete and continuum mathematics in what Ogus called "a novel way." The key idea is that a k-vector based at a point is represented as the limit of small simplexes containing the point, each with unit mass. This representation can be made rigorous in the space of differential k-chains. On the other hand, a k-simplex can be written as a limit of monopole k-chains.

Some might find it striking that the relations (1), (2) and their corollaries apply to monopole k-chains supported at a single point. One can define the boundary of a k-vector based at a point by taking suitable limits of $(k-1)$-dimensional simplexes, or even $(k-1)$-vectors, in the space of differential chains. Infinitesimal flux can then be calculated across the infinitesimal boundary of a k-vector. By taking limits of monopole approximators, one can immediately calculate flux across the boundary of a submanifold or across rough boundaries such as the lungs. In this way, the new domains of integration represented as differential chains become amenable to techniques of analysis and geometry based on classical differential calculus of smooth manifolds. As with the Riemann integral, all that one needs to do is approximate a domain with simplicial or monopole chains, calculate the integrals on the approximators and then take a limit. Tangent spaces are replaced by monopoles which opens the door to analysis on non-smooth domains. Examples include point masses, fractals, soap films with triple junctions, manifolds with singularities and surfaces with no tangent defined anywhere.

1.3. **Applications of differential chains.**

- **Continuum mechanics**
  (a) **Reynolds' transport**
  Fried and Seguin used differential chains to prove a broad generalization of Reynolds' transport theorem permitting domains of integration that might, at times, be highly irregular, split into pieces, develop holes or whose fractal dimension may evolve with time. They write that their results are "of potential value in continuum physics and the calculus of variations." They include a substantial review of the underlying theory of differential chains and point out that new computational methods arise which are applicable to the classical setting [EF14, FS14].
  (b) **Stress and strain**
  Alfredo Marzocchi's paper "Singular Stresses and Nonsmooth Boundaries in Continuum Mechanics" [Mar05] relies heavily on chainlets and contains numerous references to my papers.
  (c) Yevgeniy Guseynov wrote in [Gus16] "Whitneys approach was significantly extended in [24-26] where the surface integral was defined for a wide family of non-regular or fractal boundaries."
  Note that [24-26] consist of my 2005 arxiv post "Lecture notes on chainlet geometry" [Har05a] and two papers I published with Norton [Nor91, Nor92].
- **Electrical engineering** The French analysis Alain Bossavit, a student of Jacques-Louis Lions, became especially interested in the theory from the viewpoint of monopole chains because of his interest in electromagnetism. He wrote in his book [Bos05],

  We shall also need a topology on the space of p-chains, in order to define differential forms as continuous linear functionals on this space. As we shall argue later, physical observables such as electromotive force, flux, and so forth, can be conceived as the values of functionals of this kind, the chain operand being the idealization of some measuring device. Such values dont change suddenly when the measurement apparatus is slightly displaced, which is the rationale for continuity. But to make precise what slightly displaced means, we need a notion of nearness between chains - a topology. What follows is an attempt to bypass, rather than to face, this difficult problem, to which Harrisons work on "chainlet" spaces (nested Banach spaces which include chains and their limits with respect to various norms, Harrison [1998]), provides a much more satisfactory solution.

  When H. Pugh and I found a more useful topology on the inductive limit [HP12] and convinced Bossavit that this made no difference to applications in engineering, he wrote [Bos06],

  Remark. Not only linearity, but continuity, of maps e, b, h, etc., is an issue, especially in convergence proofs. Hence a need, fortunately satisfied by recent mathematical developments [4][2]*, for topologies on chain spaces, a subject computer users can safely ignore.*
- **Material science** The Israeli materials and civil engineer R. Segev based his paper [Seg03] entirely on my generalized Gauss-Green theorem [Har99a, Har06] which I told him about before publication. He remarked, "Harrison made important extensions to Whitneys work" Segev's paper has 21 citations.
- **Probabilty theory** Terry Lyons wrote in "On Gauss-Green theorem and boundaries of a class of Hölder domains" [Lyo06] "In conclusion, our work demonstrates an alternative

[2]Reference [4] refers to [Har].

approach, which is also complementary, to that by Harrison and Norton[3] on how to do calculus and geometric analysis on fractals"

- **Finite Element Method**
  Joe Salamon's 2016 Ph.D. thesis in Physics (UCSD, adviser Melvin Leok), [Sal16] contains a ten page summary of chainlets and cites several of my papers, including two arxiv papers [Har05a] [Har] and two published papers [Har06], [HP12].
- **Riemann boundary value problems**
  B.A. Kats, [Kat14] 2014 (cited by 10) This paper cites four of my papers [Nor91] [Nor92], [Har93], [Har05a].
- **Pugh's theorem**
  Notation: $B = B(M)$ is the topological vector space of smooth differential forms with bounded derivatives defined over a manifold M, $B' = B'(M)$ is its continuous dual, $'B = {}'B(M)$ is a continuous predual. (See [HP12], [Har14a] for more information about the specific predual we use and its topology.)

  H. Pugh [Pug16] has recently discovered a continuous completed coproduct
  $$\Delta : {}'B \to {}'B\hat{\otimes}{}'B$$
  which dualizes to wedge product on forms. In Sweedler notation: $\Delta(J) = \sum J_{(1)} \otimes J_{(2)}$ where $J_{(1)}, J_{(2)}, J \in {}'B$ and the sum may be an infinite series converging in the projective tensor product topology.

  Explicitly, the duality statement is

  Theorem 1.3.1 (H. Pugh 2016)**.** *Given $J \in {}'B(M)$ and differential forms $\alpha, \beta \in B(M)$,*
  $$\int_J \alpha \wedge \beta = \sum_{i,j} \int_{J_i} \alpha \int_{J_j} \beta.$$

  It is not obvious that the right hand side is well-defined when viewed alone since J_i and J_j are not uniquely determined and neither are α and β. Furthermore, Pugh shows that the boundary operator commutes with the completed coproduct, making $'B$ into a topological differential graded completed coalgebra. It dualizes to the classical differential graded algebra of differential forms. (See [Pug16] for a draft.)

1.4. **Postscript to differential chains.** The philosophical view of my postdoc adviser Hassler Whitney drove him to search for the most fundamental concepts and results of mathematics which would then yield powerful theories as a consequence. One might call these foundational concepts "ansatz of mathematics." In the introduction to Geometric Integration Theory [Whi57], Whitney wrote,

> *The discovery... that flat cochains ... correspond to differential forms caused a fundamental change in the point of view. The study [of flat chains] in Euclidean space now became primary; having found that the theory of cochains could be built on norms of chains, these norms became the basic tool.*

[3] My student Norton only worked on two simple early papers [Nor91, Nor92] establishing a Stokes' theorem for top dimensional domains. Kats proved the same results before us, but he wrote his papers in Russian and we learned about them years later. The higher codimensional theory was much deeper and Norton had no role in it.

Differential chains are of this nature. Whitney would have been pleased to see that our cochains correspond to differential forms, our coboundary operator corresponds to exterior derivative, there is a geometric predual to Hodge star, and the dual of completed coproduct corresponds to wedge product.

2. Plateau's problem

In its simplest form, Plateau's problem asks if there exists a surface of least area amongst all surfaces whose boundary is a given closed curve in $\mathbb{R}^3$. There are a variety of meanings of "surface," "area," and "boundary," giving rise to different statements of Plateau's problem in higher dimension and codimension. The word "area" most often refers to mass or Hausdorff measure. For those interested in soap films, size minimization[4] is more natural than mass minimization because multiplicities do not arise in physically realistic soap films. Furthermore, two soap films do not cancel if they overlap for films have no sense of orientation. In mass minimization problems, though, two surfaces with opposite orientation do cancel where they meet. The most progress in Plateau problems has been with regard to mass minimization problems. One reason is that oriented submanifolds have the cancellation property. Another is that mass is lower semicontinuous and Hausdorff measure is not which makes size minimization problems more difficult to solve. However, the bulk of physical applications of the calculus of variations do not exhibit the cancellation property. For example, when two roads coincide, they do not cancel.

2.1. **Early ideas.** After I found a concise statement of the Gauss-Green theorem, I wondered if other aspects of [Fed69] might be elucidated by the direct geometrical approach of differential chains. Plateau's problem using mass minimization is the other major application in [Fed69] and I started to work to understand what I could from the viewpoint of differential chains. My first papers [Har04a, Har04b] got me started. My first new idea used dipole surfaces to model films and solved what we call the "Y-problem." Simplicial chains introduce an extra boundary element where the three surfaces meet, but dipole chains do not (see Figure 1). They also model Moebius strips nicely so that the boundary operator is defined and produces a dipole chain whose support is the simple closed curve at the edge of the Moebius strip (see 2.) My papers were limited to codimension one and assumed a universal upper bound on the length of triple junctions. It was a good start, but I was not altogether satisfied with either assumption.

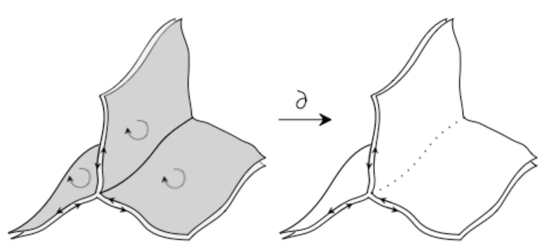

FIGURE 1. Triple junctions that do not contribute to the boundary of the film

[4]Hausdorff measure is used for size minimization.

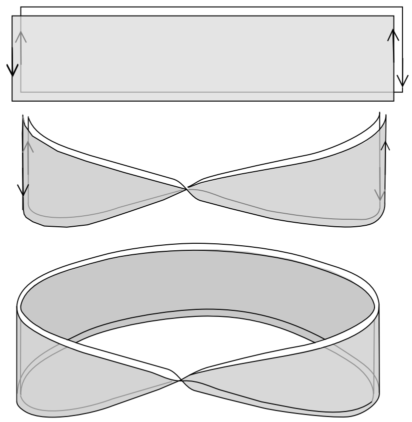

FIGURE 2. Dipole surfaces model Moebius strips and gave a well-defined dipole boundary curve

Meanwhile, I realized that the norm topology I had chosen for chainlets had every property that I had wanted, but its dual space of differential forms was too small[5]. I decided to try the inductive limit topology instead and Pugh offered to help prove that it had the properties we wanted. We learned from Grothendieck [Gro73] that the inductive limit topology, unlike the projective limit topology, is notoriously difficult to work with and establish fundamental properties. But if it is Hausdorff, complete, bornological, etc., it can have very nice applications because it behaves much like a finite dimensional vector space. For example, the Heine-Borel property holds. Much as with Schwartz distributions, one can use the properties of the space and ignore the particular topology chosen.

After absorbing several books on topological vector spaces, we knew enough to prove what we wanted and published [HP12]. Pugh and I turned back to Plateau's problem. We hoped we now had the tools to treat both of the limitations on my previous work. We had the old models of dipole surfaces in hand, and realized that differential chains had nice compactness properties since the linking maps between the Banach spaces were compact.

Around the same time as we wrote [HP12], I had the idea of using linking numbers to define spanning conditions in codimension one, and no longer made the artificial assumption on triple junctions. In [Har11b, Har14b] I proposed using simple links to test whether or not a closed set spans a given closed loop (although no regularity is addressed.) Assume for simplicity that B is a Jordan curve in $\mathbb{R}^3$. We say that a closed set X ***spans*** B if every simple link of B meets X. It is astonishing that no one ever thought of using this simple definition before. It has kindergarten simplicity and requires no boundary operator.

2.2. **Soap films via differential chains - concrete representations of films with a coherent boundary operator.** Pugh and I decided to try and tackle the size minimization problem together. In [HP13] we also used linking numbers to specify spanning conditions, but we developed them

[5]But this was not a mistake, as such. All of the applications which have relied on chainlets only made use of the first few Banach spaces used to define the inductive limit. Theses spaces have never changed since their inception [Har05a, Har05b].

more thoroughly. This time we proved a result that others are coming to appreciate as breaking new ground. It establishes existence and a.e. regularity of size minimizing Plateau problems in codimension one. It marked a breakthrough for the problem had not been previously solved for more than one prescribed boundary component, e.g., Borromean rings. In 2013 Ogus wrote, "We look forward to hearing news of its acceptance, which would signal a major achievement."

Theorem [HP13]

Let M be an oriented, compact $(n-2)$-dimensional submanifold of $\mathbb{R}^n$ and $\mathcal{S}$ the collection of compact sets spanning M. There exists an X_0 in $\mathcal{S}$ with smallest size. Any such X_0 contains a "core" $X_0^* \in \mathcal{S}$ with the following properties: It is a subset of the convex hull of M and is a.e. (in the sense of $(n-1)$-dimensional Hausdorff measure) a real analytic $(n-1)$-dimensional minimal submanifold.

2.3. **Plateau's problem for soap films via classical measure theory.** Higher codimension was much more difficult and required several new methods of our next paper [HP16b]. This paper generalizes [HP13] to arbitrary codimension and also permits boundaries with multiple components. In this paper we minimize with respect to a Hölder density function which appears to be new. Our methods are now developed in the language of classical measure theory and a few set theoretic constructions.

H. Pugh saw that linking number was a special case of cohomology (via Alexander duality) and proved about twenty necessary and nontrivial lemmas to establish closure properties. These are contained in the first part of our paper. However, there is much more. A second novel aspect is a newly identified concept we call "Reifenberg regularity" which we adapted from a technique deeply buried in a book-length proof of Reifenberg. This definition is a substantial improvement over the notion of quasiminimality (QM) of Almgren and promoted by a number of modern mathematicians. (QM is difficult to establish and has been never proven useful for solving Plateau problems, despite decades of effort by Almgren, David, and others. See below for more details.) Considerable work was needed from us both to define Reifenberg regularity, articulate and establish its basic properties, prove lower semicontinuity, rectifiability and tie everything together, once cohomology was set up properly. It is noteworthy that our methods only rely on some set theoretic constructions and classical measure theory. There is no need to introduce varifolds, currents or Whitney's flat norm, as required in Almgren's work.

Indeed, our review article on Plateau's problem [HP16a] demonstrates that Almgren's proof in [Alm68] is incorrect. At a key point in his proof he assumes that his minimizing sequence is uniformly quasiminimal in order to deduce rectifiability of his minimizing set. However, this step of his proof is unjustified and presents a gap that has not been filled to date. It is easy to produce examples of minimizing sequences that are not uniformly quasiminimal and there is no known method to convert such minimizing sequences into uniformly quasiminimal ones.

H. Pugh and I have thus published the first proof of existence and a.e. regularity of size minimizing solutions to Plateau's Problem for m-rectifiable surfaces in $\mathbb{R}^n$ that span an arbitrary prescribed compact set A. We used Čech cohomological spanning conditions [HP16b] since the original Čech homological spanning conditions of [Rei60] do not apply well to boundaries with more than one component. (See Theorem 2 of [Rei60]). The topological boundaries of our surfaces are only required to be contained in the compact set A and can slide around within A (see Figure 3).

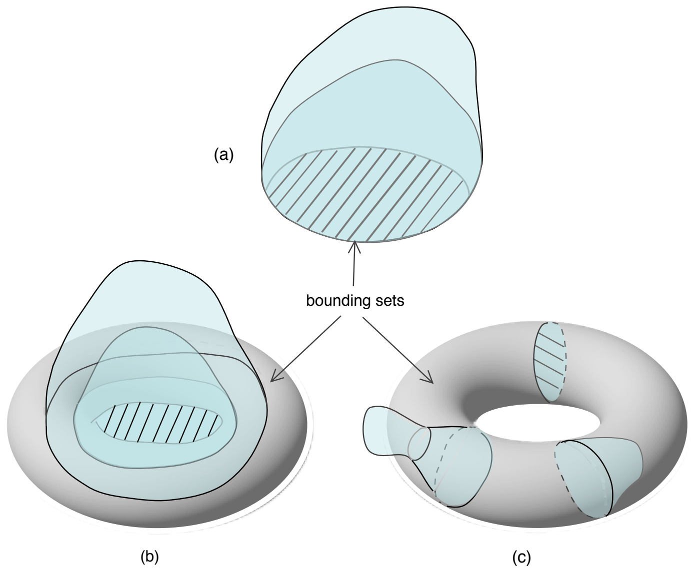

FIGURE 3. **Sliding boundaries**. In each of the three figures there are three competitors, including a shaded one with minimal area. Figure (a) depicts the classical Plateau problem where the bounding set is a prescribed curve. The bounding set of both Figures (b) and (c) is a 2-torus and boundaries of competing surfaces are not constrained, apart from being contained in the torus. This permits elliptic problems with "sliding boundaries" to be solved. In (a) we chose a spanning test using linking numbers. In (b) and (c) we chose a spanning test relative to one of the two generators of 1-dimensional homology.

3. Recent work

In our most recent paper [HP16c], which has not yet been formally accepted, we provide general methods in the calculus of variations for the anisotropic Plateau problem in arbitrary dimension and codimension. Given a collection of competing "surfaces," which span a given "bounding set" in an ambient metric space, we produce one minimizing an elliptic area functional. The collection of competing surfaces is assumed to satisfy a set of geometrically-defined axioms. These axioms hold for collections defined using any combination of homological, cohomological or linking spanning conditions. A variety of minimization problems can be solved, including sliding boundaries.

Our axioms are very simple: If our ambient space is a smooth manifold M and $A \subset M$ is a fixed bounding set, we assume that a collection C of compact rectifiable sets is closed under diffeomorphisms of M which keep A invariant and is also closed under rectifiable Hausdorff limits[6]. If f is an elliptic integrand, then there exists a minimizing solution in C with respect to $X \mapsto \int_X f(p, T_p X) d|H^m$.

[6]For technical reasons, we also assume that all sets are "reduced." Roughly speaking, there are no points where the local Hausdorff measure of a set is zero. In particular, there are no isolated points

H. Pugh [HP16c] generalized elliptic integrands to "elliptic measures" which are sensitive to the germ of a surface at a.e. point, not just the position and tangent direction. Interesting applications can take into account external forces such as gravity and variable wind. This extension was due solely to H. Pugh and is being developed by him in a separate new paper. This opens the door to solving Plateau problems, both mass and size minimizing, to take into account general types of external forces. This result has the potential for solving a number of problems in the physical sciences which have never been touched before.

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